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SUPERCONVERGENT TECHNIQUES IN MULTI-SCALE METHODS

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Abstract. It is well known that many problems of practical importance in science and engineering have multiple-scale solutions. Moreover, the calculations of numerical methods for these problems is very intensive, even if using some multi-scale proceedures. It is therefore important to seek efficient calculation methods. In this paper, superconvergent techniques are used in existing multi-scale methods to improve the calculation efficiency. Furthermore, based on comprehensive analysis, the order of the error estimates between the numerical approximation and the exact solution is verified to be improved.

Key Words. Elliptic equations, superconvergent technique, periodic microstructure, multi-scale methods, asymptotic expansion, homogenization.

1. Introduction

Multi-scale methods have been investigated for a long time in the mathematics and engineering literature. For example of these papers, we refer to [4], [9] and [11]. Early papers concentrated on multi-scale methods that are mainly based on the theory of asymptotic expansion and homogenization. Later, various different but related multi-scale methods were proposed, including the multigrid numerical homogenization method ([33], [34], [46], [47]), the multiscale finite element method (MsFEM) ([37], [38], [31]), the heterogeneous multiscale method (HMM) ([25], [26], [27], [28]), the finite element method based on the *Residual-Free Bubble* method ([12], [32], [35], [39]), the wavelet homogenization method ([22]) and so on. Each of these methods has advantages in some special cases. As is well known, the multi-grid method as a classical multi-scale technique achieves optimal efficiency by relaxing the errors at different scales on different grids. It can give an accurate approximation to the detailed solution of fine scale problems. HMM is a specific strategy to compute the macro-scale behavior of the system with a standard macro-scale scheme in which the missing micro-scale data can be evaluated concurrently by using the micro-scale model. It can deal with many multi-scale problems efficiently even for problems whose period is unknown. MsFEM can obtain the large scale solutions accurately and efficiently without resolving the small scale details. The main idea is to construct in each element finite base functions which can capture the small scale information. Such small-scale information is then brought to the large scales through the coupling of the global stiffness matrix.

Although the methods can deal efficiently with some practical problems, the computation cost may still be very large. For example, in order to simulate elliptic problems with non-uniformly oscillating coefficients by HMM, at least one unit cell

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in each element will be calculated to obtain the homogenized equation and obtain the information of the microstructure. This results in intensive calculations if the number of elements is large. In some cases where the domain and the solution are smooth enough, it is important to find a more efficient method or technique to reduce the calculations. It is known that, in [13], a fast post-processing algorithm which is based on asymptotic expansion used to analyze a multiscale method. But in [13], the authors just analyzed elliptic problems with uniformly highly oscillatory coefficients. In practice, there are many multiscale problems with non-uniformly oscillating coefficients, and by using the post processing technique directly, it is impossible to improve the order of the error estimate on the whole domain when one just uses linear interpolation for the unit cells that have been simulated. For instance, under the conditions above, the error estimate of the HMM for the H^1-broken norm is just $O(H)$. If we use a high order interpolation technique, then the number of unit cells needed in the calculations will increase greatly if the HMM method is employed. So, it is very important to reduce the number of unit cells needed in the calculations. In this paper, we show that it is not necessary to choose at least one unit cell in each element in which to calculate. We simulate unit cells on a new mesh, which is different from the partition of the whole domain. The size of the former is much bigger than that of the latter. This idea is different from that used in HMM and some other multiscale methods. By using high order interpolation techniques for the solved unit cells, we then successfully reduce the cost on unit cells. Moreover, we can use a superconvergent technique to deal with the numerical solution of the homogenized equation in order to improve its accuracy. Based on these ideas, some improved error estimates are given. In this paper, we just investigate the superconvergent techniques in the homogenized equations presented in [9] and [27]. In fact, superconvergent techniques can also be efficiently extended to some other multiscale methods. In addition, we just discuss elliptic problems. For parabolic multiscale problems with suitable conditions, the superconvergent technique is also valid.

In the past forty years, superconvergence finite element methods has been an active research field. Early papers concentrated on superconvergence at isolated points (see [23] *et al*). Later, various type of superconvergent techniques were established, either in the strict sense or in an approximate way (see [7], [8], [52], [53], [58], [59], [60], [40], [44] *et al*). In this paper, we merely give a framework to demonstrate that the superconvergent technique is suited to multi-scale methods and can efficiently improve the accuracy. Thus, we only employ certain postprocessing techniques proposed in [40] and [44] to improve the existing approximation accuracy. However some other superconvergent techniques, such as the *Zienkiewicz-Zhu superconvergent Patch Recovery* (ZZ-SPR), can also be used to improve the order of error estimates of multi-scale methods.

The outline of this paper is as follows. In the next section, we introduce the model problem and provide its two similar homogenized equations. Moreover, the error estimate between the exact solution of the original problem and the asymptotic expansion of order one is presented, and the estimates

$$\|u^\epsilon - u_1^\epsilon\|_{1,D} \le C\sqrt{\epsilon}\|U_0\|_{3,\infty,D}$$

$$\|u^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \le (Ch^k\|u_0\|_{1,D} + \sqrt{\epsilon}\|u_0\|_{3,\infty,D}),$$

are obtained.

Based on this result, we present the principal results of this paper in Section 3. The error estimate, between the exact solution and the numerical solution of the first order multiscale solution corrected by postprocessing, is shown to be

$$\|u^\epsilon - \overline{u}^\epsilon\|_{1,D} \le C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D} + h^k\|u_0\|_{1,D} + H^m\|\widetilde{u}_0\|_{m+1,D}),$$

and

$$\begin{aligned}\|u^\epsilon - \overline{u}^\epsilon\|_{1,D_0} \quad \le \quad & C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D_1} + h^k\|u_0\|_{1,D_1} \\ & + H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}).\end{aligned}$$

In Section 4 the superconvergent technique is extended to HMM and some useful error estimates are given. Moreover, from the analysis of the orders of the error estimates, we observe that the accuracy of the approximation is reasonably improved. In the last section, we discuss briefly some possible future work.

2. A model problem and its homogenized equations

In this paper, we adopt the standard notation of $W^{m,p}(D)$ for Sobolev spaces on D with norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. Set $W_0^{m,p} \equiv \{\omega \in W^{m,p}(D) : \omega|_{\partial D} = 0\}$ and denote $W^{m,2}(D)$ ($W_0^{m,2}(D)$) by $H^m(D)$ ($H_0^m(D)$) with norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$. In addition, c or C denotes a positive constant independent of the sizes of the finite elements and micro-structure size ϵ.

Consider the model problem:

$$\left\{\begin{array}{ll} -\nabla\cdot(A(x,\frac{x}{\epsilon})\nabla u^\epsilon) = f(x) & in \;\; D \\ u^\epsilon|_{\partial D} = 0 & \end{array}\right. , \tag{2.1}$$

where D is a bounded convex domain in R^2 with a Lipschitz boundary ∂D (for simplicity, we only discuss the model problem in R^2, in fact, the conclusions can be extended to R^d ($d > 2$ or $d = 1$), ϵ is a small positive number, and

$$A(x,Y) = \begin{pmatrix} a_{11}(x,Y) & a_{12}(x,Y) \\ a_{21}(x,Y) & a_{22}(x,Y) \end{pmatrix}$$

such that A is symmetric and

$$c\xi_i\xi_i \le |a_{ij}(x,Y)\xi_i\xi_j| \le C\xi_i\xi_i, \quad \forall\; \xi_i,\xi_j \in R^2, \quad i,j = 1,2. \tag{2.2}$$

Moreover, $a_{ij}(x,Y), f \in L^\infty(D)$ are all Q-periodic in Y, where $Y = x/\epsilon$, $Q = (0,1)\times(0,1)$.

We first introduce more notation. Let $m_Y(v)$ be the integral average of v on Q:

$$m_Y(v) = \frac{1}{|Q|}\int_Q v dY = \int_Q v dY \quad \forall v \in L^2(Q),$$

where $Q = (0,1)\times(0,1)$ is a unit cell which is the referred domain of the micro-structure Q_ϵ in D, and $|Q|$ is the area of Q.

Then, the homogenized bilinear equation of (2.1) reduces to finding $U_0(x) \in H_0^1(D)$ such that (see [9])

$$A_0(U_0, v) = (f, v) \qquad \forall\; v \in H_0^1(D), \tag{2.3}$$

where A_0 is defined by

$$A_0(v,\omega) = (\widetilde{A}\nabla v, \nabla\omega), \quad \forall\; v,\omega \in H_0^1(D), \tag{2.4}$$

with

$$\widetilde{A} = (\widetilde{A}_{ij})_{2\times 2}, \quad \widetilde{A}_{ij} = m_Y(a_{ij} + a_{ik}\frac{\partial N^j}{\partial Y_k}), \tag{2.5}$$

and N^j is the periodic solution of the equation:

$$\frac{\partial}{\partial Y_i}(a_{ik}(x,Y)\frac{\partial N^j(x,Y)}{\partial Y_k}) = -\frac{\partial}{\partial Y_i}a_{ij}(x,Y) \quad in \;\; Q, \qquad \int_Q N^j dY = 0. \tag{2.6}$$

For (2.6), we just want to obtain the solution $N^j(x, Y)$ in the Y-direction. But, unfortunately, there are two parameters x, Y in this equation. It is thus difficult to directly simulate the solution by any standard numerical method, since the coefficient matrix of any numerical scheme is not constant, but involves the parameter x. In order to resolve this difficulty, many earlier papers first gave a partition of the whole domain and then calculated (2.6) on some fixed points of the given mesh and finally derived a homogenized equation in the same partition. For instance, in ([25]), ([30]), ([27]), cell problems are solved at each quadrature point of every element. Similarly, in some other papers, the vertexes of each element are chosen as centers of unit cells in order to solve (2.6). For these examples, we note that the number of unit cells calculated in the whole domain is $O(n^2)$, if the number of elements in one direction of the partition is n. In this paper as mentioned earlier, we use the P_k-interpolation technique for the obtained unit cells in a new mesh, which is not necessarily the same as the partition of the homogenized equation. That is, we possibly use two different meshes to simulate multi-scale problems. The bigger one is for unit cells and the other one is for the homogenized equation. This idea is different from those in the papers we have mentioned above. Under the same accuracy as the method in ([25]), in the following Theorem 2.1, it is shown that the required number of unit cells is just $O(n)$ if we use P_2-interpolation. It is thus obvious that we can reduce the calculation on unit cells in a significant way.

Theorem 2.1. *Let $\rho(x)$ be a function satisfying $\rho(x) \in W^{k+1,\infty}(D)$, and let the P_k-interpolation of $\rho(x)$ be denoted by $\Pi_k\rho(x)$, then it can be shown that (see* [20]*),*

$$(2.7)\quad \|\rho(x) - \Pi_k\rho(x)\|_{s,\infty} = \begin{cases} O(H^{k+1-s-\delta}) & \text{for } \delta > 0, \quad \text{if } k = 1, \\ O(H^{k+1-s}) & \text{if } k \geq 2. \ (s = 0, 1) \end{cases}$$

Let T_h be a regular partition of D with elements e with size h_e, and define $h := \max_{e\in T_h} h_e$. Let P_k be the space of polynomials with degree no more than k. Then, from Theorem 2.1, we have that

$$(2.8)\qquad \|N^j(x, Y) - \Pi_k N^j(x_n, Y)\|_{1,\infty} \leq Ch^k, \qquad j = 1, 2, \quad k \geq 2,$$

where x_n is chosen point of T_h.

In addition, set T_H to be another regular partition of D with elements K with size h_K, and define $H := \max_{K\in T_H} h_K$. We define the finite element space to be

$$X_H := \{v \in H_0^1(D) : \quad v|_K \in P_1(K) \quad \forall\, K \in T_H\}.$$

From (2.8), the homogenized bilinear equation (2.3) can be written as

$$\int_D \widetilde{a}(x)\nabla U_0 \cdot \nabla v dx = \int_D f v dx$$

where $\widetilde{a}(x) = (\widetilde{a_{ij}}(x))$, and

$$(2.9)\qquad \widetilde{a_{ij}}(x)|_K = m_Y\left(a_{ij}(x, Y) + a_{im}\frac{\partial}{\partial Y_m}\Pi_k N^j(x_n, Y)\right),$$

For any $v, \omega \in X_H$, define the bilinear form:

$$(2.10)\qquad A_H(v, \omega) = \sum_{K\in T_H}\int_K\left(\widetilde{a_{ij}}(x)\frac{\partial v}{\partial x_j}\frac{\partial \omega}{\partial x_i}\right)dx$$

Then the homogenized numerical solution is to obtain $U_H \in X_H$ such that

$$(2.11)\qquad A_H(U_H, v) = (f, v), \quad \forall\ v \in X_H.$$

Remark 2.1. *Our main interest is the numerical approximation of (2.1). Therefore, we assume that the theoretical solution is reasonably regular in order for the estimates that follow to apply. In particular, it is convenient for our presentation to assume* $N^j(x,Y) \in W^{k+1,\infty}(D \times Q)$ *(j=1,2),* $(k \geq 2)$.

In the following, we give the main error estimates of this part.

Theorem 2.2. *Let* $u^\epsilon(x,Y)$ *be the solution of the equation (2.1), let* U_0 *be the solution of (2.3), and*

$$u_1^\epsilon = u_0 + \epsilon u_1 = U_0 + \epsilon N^k \frac{\partial U_0}{\partial x_k}.$$

Assume that D *is a smooth domain,* $a_{ij}(x,Y) \in W^{1,\infty}(D)$ *and* $f \in L^2(D)$. *Then (See* [56]*),*

$$\|u^\epsilon - u_1^\epsilon\|_{1,D} \leq C\sqrt{\epsilon}\|U_0\|_{3,\infty,D}. \tag{2.12}$$

Remark 2.2. *From Theorem 2.2, it is easy to see that*

$$\|u^\epsilon - U_0\|_{0,D} \leq \|u^\epsilon - u_1^\epsilon\|_{0,D} + \epsilon\|N^k \frac{\partial U_0}{\partial x_k}\|_{0,D} \leq C\sqrt{\epsilon}\|U_0\|_{3,\infty,D}. \tag{2.13}$$

Assume that $\widetilde{u}_0$ is the exact solution of the following equation:

$$\widetilde{a}(\widetilde{u}_0, v) := \sum_{K\in T_H} \int_K \Big(\widetilde{a}(x)\nabla\widetilde{u}_0 \cdot \nabla v\Big)dx = \int_D fvdx, \qquad \forall v \in H_0^1(D). \tag{2.14}$$

Let $N_{h_0}^j$ be the numerical approximation of N^j. It is known that the contribution of the error estimate $\|N^j - N_{h_0}^j\|_{1,D}$ to the error estimates of this part is small and can be neglected. Consequently, the error $\|N^j - N_{h_0}^j\|_{1,D}$ is not considered in the following part.

Next, we give the estimate of the error between u_0 and $\widetilde{u}_0$. First, we give some useful lemmas.

Lemma 2.1. *Assume that* $N^j(x,Y)$ *is the solution of (2.6), satisfying* $N^j(x,Y) \in W^{k+1,\infty}(D)$ $(k \geq 2)$. *Let* $\widetilde{A}_{ij}(x)$ *and* $\widetilde{a_{ij}}(x)$ *be as defined in (2.5), and (2.9), respectively. Then, we have*

$$\|\widetilde{A}_{ij}(x) - \widetilde{a_{ij}}(x)\|_{0,\infty,D} \leq Ch^k. \tag{2.15}$$

Proof. From Theorem 2.1, it follows that there exists a positive constant C, such that

$$\max_{K\in T_H} \|N^j(x,Y) - \Pi_k N^j(x_n,Y)\|_{0,\infty,K} \leq \|N^j(x_n,Y) - \Pi_k N^j(x,Y)\|_{0,\infty,D} \leq Ch^k.$$

Using Minkowski's Integral Inequality, a direct calculation gives,

$$\begin{aligned}
& \|\widetilde{A}_{ij}(x) - \widetilde{a_{ij}}(x)\|_{0,\infty,D} \\
= \;& \max_{K\in T_H} \|m_Y(a_{ik}\frac{\partial}{\partial Y_k}(N^j(x,Y) - \Pi_k N^j(x_n,Y)))\|_{0,\infty,K} \\
\leq \;& \max_{K\in T_H} m_Y(\|a_{ik}\frac{\partial}{\partial Y_k}(N^j(x,Y) - \Pi_k N^j(x_n,Y))\|_{0,\infty,K}) \\
\leq \;& \max_{K\in T_H} m_Y(\|a_{ik}\|_{0,\infty,K} \cdot \frac{\partial}{\partial Y_k}\|N^j(x,Y) - \Pi_k N^j(x_n,Y)\|_{0,\infty,K}) \\
\leq \;& Ch^k.
\end{aligned}$$

So, (2.15) is proved. □

Lemma 2.2. *Assume that $A_0(u,v)$ is defined as (2.4) and satisfies the inf-sup condition, then for sufficiently small h, we have: $\widetilde{a}(u,v)$ also satisfies the inf-sup condition. That is, there exists a positive constant c, such that*

$$\sup_{0\neq v\in H_0^1}\frac{|\widetilde{a}(u,v)|}{\|v\|_{1,D}}\geq c\|u\|_{1,D}. \tag{2.16}$$

Proof. By using lemma 2.1, it can be easily shown that for all u, $v\in H_0^1(D)$,

$$\begin{aligned}
&|A_0(u,v)-\widetilde{a}(u,v)|\\
&=\sum_{K\in T_H}|A_0(u,v)-\widetilde{a}(u,v)|_K\\
&=\sum_{K\in T_H}|\int_K\Big((\widetilde{A}(x)-\widetilde{a}(x))\nabla u\cdot\nabla v\Big)dx|\\
&\leq\sum_{K\in T_H}Ch^k\|\nabla u\|_{0,K}\|\nabla v\|_{0,K}\\
&\leq Ch^k\|u\|_{1,D}\|v\|_{1,D}
\end{aligned}$$

Then, for any $v\in H_0^1(D)$ we have

$$\sup_{0\neq v\in H_0^1}\frac{|\widetilde{a}(u,v)|}{\|v\|_{1,D}}\quad\geq\quad\sup_{0\neq v\in H_0^1}\frac{|A_0(u,v)|}{\|v\|_{1,D}}-Ch^k\|u\|_{1,D} \tag{2.17}$$

But $A_0(u,v)$ satisfies the inf-sup condition; that is, there exists a constant $\widehat{C}>0$, such that

$$\sup_{0\neq v\in H_0^1}\frac{|A_0(u,v)|}{\|v\|_{1,D}}\geq\widehat{C}\|u\|_{1,D} \tag{2.18}$$

Combining the above inequality (2.17) and (2.18), we obtain that for sufficiently small h, there exists a positive constant c, such that

$$\sup_{0\neq v\in H_0^1}\frac{|\widetilde{a}(u,v)|}{\|v\|_{1,D}}\geq(\widehat{C}-Ch^k)\|u\|_{1,D}\geq c\|u\|_{1,D}.$$

Then, this lemma is proved. □

Based on the Lemma 2.1 and Lemma 2.2, we give an error estimate as follows.

Theorem 2.3. *Assume that the conditions of Lemma 2.2 are satisfied, u_0 is the solution of the homogenized equation (2.3) and $\widetilde{u}_0$ is the exact solution of (2.14), then for sufficiently small h, we have*

$$\|u_0-\widetilde{u}_0\|_{1,D}\leq Ch^k\|u_0\|_{1,D} \tag{2.19}$$

Proof. From Lemma 2.1 and Lemma 2.2, for sufficiently small h, we have

$$\begin{aligned}
c\|u_0-\widetilde{u}_0\|_{1,D}&\leq\sup_{0\neq v\in H_0^1}\frac{|\widetilde{a}(u_0-\widetilde{u}_0,v)|}{\|v\|_{1,D}}\\
&\leq\sum_{K\in T_H}\sup_{0\neq v\in H_0^1}\frac{|\int_K(\widetilde{A}(x)-\widetilde{a}(x))\nabla u_0\cdot\nabla v dx|}{\|v\|_{1,K}}\\
&\leq\sum_{K\in T_H}\widetilde{c}h^k\|u_0\|_{1,K}\\
&=\widetilde{c}h^k\|u_0\|_{1,D}
\end{aligned}$$

So, from the inequality above, we obtain that there exists a positive constant C, such that

$$\|u_0 - \widetilde{u}_0\|_{1,D} \le Ch^k \|u_0\|_{1,D}$$

This shows the result. □

Remark 2.3. *If all conditions in Theorem 2.3 are valid, then for sufficiently small h, from Remark 2.2 and Theorem 2.3, it follows that*

$$\|u^\epsilon - \widetilde{u}_0\|_{0,D} \le \|u^\epsilon - u_0\|_{0,D} + \|u_0 - \widetilde{u}_0\|_{0,D} \le (Ch^k \|u_0\|_{1,D} + \sqrt{\epsilon}\|u_0\|_{3,\infty,D}), \tag{2.20}$$

$$\|u^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \le \|u^\epsilon - u_1^\epsilon\|_{1,D} + \|u_1^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \le (Ch^k \|u_0\|_{1,D} + \sqrt{\epsilon}\|u_0\|_{3,\infty,D}), \tag{2.21}$$

where

$$\widetilde{u}_1^\epsilon = \widetilde{u}_0 + \epsilon \Pi_k N^j(x_n, Y) \frac{\partial \widetilde{u}_0}{\partial x_j}. \tag{2.22}$$

3. The Superconvergent techniques in multi-scale method

By the standard theory of the finite element method and the Nitsche technique, the following theorem follows:

Theorem 3.1. *Let U_H be the numerical solution of problem (2.11), and $\widetilde{u}_0$ be the exact solution of the equation (2.14). Then,*

$$\|\widetilde{u}_0 - U_H\|_{1,D} \le CH\|\widetilde{u}_0\|_{2,D}, \tag{3.1}$$

$$\|\widetilde{u}_0 - U_H\|_{0,D} \le CH^2\|\widetilde{u}_0\|_{2,D}.$$

In Theorem 3.1, the error estimate between the exact solution of (2.14) and its numerical approximation has been obtained. In the following, the postprocessing techniques of [40] and [44] are used to improve the accuracy of the multiscale method. In this part, for simplicity, we just give the superconvergent error estimate on a rectangular mesh. In fact, it can be extended successfully to triangular mesh.

Firstly, construct a postprocessing interpolation operator Π_{2H}^m, such that (see [40], [42], [44]),

1) Combining four neighboring elements into a big element, $\tilde{e} = \bigcup_{i=1}^4 e_i$, such that

$$\Pi_{2H}^m \omega \in Q_m(\tilde{e}), \quad \forall\, \omega \in C(\tilde{e}), \tag{3.2}$$

where Q_m is bi-P_m polynomial space.

2)

$$\|\Pi_{2H}^m \omega - \omega\|_l \le CH^{r+1-l}\|\omega\|_{r+1}, \quad 0 \le r \le m, \quad l = 0, 1; \tag{3.3}$$

3)

$$\|\Pi_{2H}^m v\|_l \le C\|v\|_l, \quad \forall\, v \in V^H(D), \quad l = 0, 1, \tag{3.4}$$

where $V^H(D)$ is finite element space.

4)

$$\Pi_{2H}^m \omega^I = \Pi_{2H}^m \omega, \tag{3.5}$$

where $\omega^I \in V^H$ is the finite element interpolation of ω.

In the following, the result of the superconvergence in the whole domain is given based on the theory of high order interpolation operator.

Theorem 3.2. *(See [42]) Let $\widetilde{u}_0$ be the exact solution of equation (2.14), let U_H, u^I be the finite element solution and finite element interpolation of $\widetilde{u}_0$, respectively, and satisfy:*

$$\|U_H - u^I\|_l \le CH^{\alpha+1-l}\|\widetilde{u}_0\|_{m+1}, \quad \alpha > p, \quad m \ge \alpha, \quad l = 0, 1,$$

where p is the order of the finite element polynomial space. Then,

$$\|\Pi_{2H}^m U^H - \widetilde{u}_0\|_l \le CH^{\alpha+1-l}\|\widetilde{u}_0\|_{m+1},$$

where Π_{2H}^m satisfies (3.2), (3.3), (3.4) and(3.5).

The conditions assumed in these techniques depend on the regularity of the partition and the smoothness of the solution. In many practical problems it is useful and often necessary to give some superconvergent error estimates in a local domain.

Theorem 3.3. *(See [44]) Let $\widetilde{u}_0$ be the exact solution of the equation (2.14), let U_p^H, u_p^I be the finite element solution and finite element interpolation of $\widetilde{u}_0$, respectively. Assume that $D_0 \subset\subset D_1 \subset\subset D$. If $\widetilde{u}_0$ is smooth enough and the mesh in D_1 is almost uniform, then,*

$$\|U_p^H - u_p^I\|_{1,D_0} \le C(H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1});$$

$$\|U_p^H - u_p^I\|_{1,\infty,D_0} \le C(H^{p+1}|\ln H|^\lambda\|\widetilde{u}_0\|_{p+2,\infty,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}),$$

where p is the order of the finite element polynomial space, s is any non-negative integer, and

$$\lambda = \begin{cases} 1, & if \ \ p = 1 \\ 0, & if \ \ p \ge 2. \end{cases}$$

By using the postprocessing interpolation operator, we have

Theorem 3.4. *(See [44]) Under the condition of Theorem 3.3, then*

$$\|\Pi_{2H}^{p+1}U_p^H - \widetilde{u}_0\|_{1,D_0} \le C(H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1});$$

$$\|\Pi_{2H}^{p+1}U_p^H - \widetilde{u}_0\|_{1,\infty,D_0} \le C(H^{p+1}|\ln H|^\lambda\|\widetilde{u}_0\|_{p+2,\infty,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}).$$

We next retrieve the microscopic information in whole domain from $\Pi_{2H}^m U_H$ and give the most important results of this section.
Assume that

$$R(v) = v + \epsilon\Pi_k N^j(x_n, Y)\frac{\partial v}{\partial x_j}, \tag{3.6}$$

Define

$$\overline{u}^\epsilon|_K = R(\Pi_{2H}^m U_H)|_K. \tag{3.7}$$

Theorem 3.5. *Let u^ϵ be the solution of (2.1), $\overline{u}^\epsilon$ be given by (3.7). Assume that all conditions of Theorem 3.1 are valid. Then,*

$$\|u^\epsilon - \overline{u}^\epsilon\|_{1,D} \le C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D} + h^k\|u_0\|_{1,D} + H^m\|\widetilde{u}_0\|_{m+1,D}). \tag{3.8}$$

Proof. Note that on each element K,

$$\begin{aligned} \frac{\partial \overline{u}^\epsilon}{\partial x_i} \quad &= \quad \frac{\partial}{\partial x_i}\Pi_{2H}^m U_H + (\frac{\partial}{\partial Y_i} + \epsilon\frac{\partial}{\partial x_i})\Pi_k N^j(x_n, Y) \cdot \frac{\partial}{\partial x_j}\Pi_{2H}^m U_H \\ &\quad + \epsilon\Pi_k N^j(x_n, Y) \cdot \frac{\partial^2}{\partial x_i \partial x_j}\Pi_{2H}^m U_H. \end{aligned} \tag{3.9}$$

Furthermore,

$$\frac{\partial \widetilde{u}_1^\epsilon}{\partial x_i} = \frac{\partial \widetilde{u}_0}{\partial x_i} + (\epsilon \frac{\partial}{\partial x_i} + \frac{\partial}{\partial Y_i})\Pi_k N^j(x_n, Y)\frac{\partial \widetilde{u}_0}{\partial x_j} + \epsilon \Pi_k N^j(x_n, Y)\frac{\partial^2 \widetilde{u}_0}{\partial x_i \partial x_j}. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}\frac{\partial}{\partial x_i}(\overline{u}^\epsilon - \widetilde{u}_1^\epsilon) \quad = \quad & \frac{\partial}{\partial x_i}(\Pi_{2H}^m U_H - \widetilde{u}_0) + (\frac{\partial}{\partial Y_i} + \epsilon\frac{\partial}{\partial x_i})\Pi_k N^j(x_n, Y)\cdot\frac{\partial}{\partial x_j}(\Pi_{2H}^m U_H - \widetilde{u}_0) \\ & + \epsilon \Pi_k N^j(x_n, Y)\cdot \frac{\partial^2}{\partial x_i \partial x_j}(\Pi_{2H}^m U_H - \widetilde{u}_0).\end{aligned}$$

From Theorem 3.2, we can obtain that

$$\begin{aligned}\|\nabla(\overline{u}^\epsilon - \widetilde{u}_1^\epsilon)\|_{0,D} \quad \le \quad & C\|\nabla(\Pi_{2H}^m U_H - \widetilde{u}_0)\|_{0,D} + C\epsilon\|\widetilde{u}_0\|_{2,D} \\ & \le CH^m\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{2,D}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|\overline{u}^\epsilon - \widetilde{u}_1^\epsilon\|_{0,D} \quad \le \quad & C\|\Pi_{2H}^m U_H - \widetilde{u}_0\|_{0,D} + C\epsilon\|\widetilde{u}_0\|_{1,D} \\ & \le CH^{m+1}\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{1,D}.\end{aligned}$$

From the inequalities above, it follows that

$$\|\overline{u}^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \le CH^m\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{2,D}.$$

Combining with (2.21) yields (3.8). This proves Theorem 3.5. □

Remark 3.1. *Sometimes in applications the superconvergent error estimate in a local domain is more important. By the same method used in proving Theorem 3.5 and from Theorem 3.4, it follows that*

$$\begin{aligned}\|u^\epsilon - \overline{u}^\epsilon\|_{1,D_0} \quad \le \quad & C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D_1} + h^k\|u_0\|_{1,D_1} \\ & + H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}),\end{aligned} \tag{3.11}$$

where, $\overline{u}^\epsilon = \Pi_{2H}^{p+1}U_p^H$.

4. The Superconvergent technique for HMM

In this section, the superconvergent technique will be successfully applied to the HMM to reduce its calculation. First, we recall the HMM scheme as follows (see [30]).

Consider the classical problem

$$\begin{cases} -\nabla\cdot(a^\epsilon(x)\nabla u^\epsilon(x)) = f(x) & in \;\; D \subset \mathcal{R}^d, \\ u^\epsilon(x) = 0 & on \;\; x \in \partial D. \end{cases} \tag{4.1}$$

In this part, a conventional P_k finite element method on a triangulation T_H of element size H is chosen and we just consider the case $d = 2$. Let A_H be defined as

$$A_H(V, V) = \sum_{K\in T_H}|K|\sum_{x_l\in K}\omega_l(\nabla V\cdot \mathcal{A}_H\nabla V)(x_l), \tag{4.2}$$

where x_l and w_l are the quadrature points and weights in K, $K \in T_H$. In the absence of an explicit knowledge of $\mathcal{A}_H(x)$, let

$$(\nabla V\cdot\mathcal{A}_H\nabla V)(x_l) = \frac{1}{\delta^d}\int_{I_\delta(x_l)}\nabla v_l^\epsilon(x)\cdot a^\epsilon(x)\nabla v_l^\epsilon dx, \tag{4.3}$$

where $I_\delta(x_l) = x_l + \delta I$, $I = [0,1]^2$. Here δ is chosen such that a^ϵ restricted to $I_\delta(x_l)$ gives an accurate enough representation of the local variations of a^ϵ. Let $v_l^\epsilon(x)$ be the solution of the problem:

$$\left\{ \begin{array}{ll} -\nabla \cdot (a^\epsilon(x) \nabla v_l^\epsilon(x)) = 0 & in \;\; I_\delta(x_l), \\ v_l^\epsilon(x) = V_l(x) & on \;\; \partial I_\delta(x_l), \end{array} \right. \tag{4.4}$$

where V_l is the linear approximation of V at x_l.

Then, the HMM solution $u_H \in X_H$ is defined by

$$A_H(u_H, V) = (f, V), \qquad \forall\, V \in X_H. \tag{4.5}$$

For problem (4.4), set $w_l^\epsilon(x) = v_l^\epsilon(x) - V_l(x)$. Then we have

$$\left\{ \begin{array}{ll} -\nabla \cdot (a^\epsilon(x) \nabla w_l^\epsilon(x)) = \nabla \cdot (a^\epsilon(x) \nabla V_l(x)) & in \;\; I_\delta(x_l), \\ w_l^\epsilon(x) = 0 & on \;\; \partial I_\delta(x_l), \end{array} \right. \tag{4.6}$$

Since $\nabla V_l(x)$ is constant, if $N_j^\epsilon(x)$ satisfies:

$$\left\{ \begin{array}{ll} -\nabla \cdot (a^\epsilon(x) \nabla N_j^\epsilon(x)) = \frac{\partial}{\partial x_i}(a_{ij}^\epsilon)(x), & in \;\; I_\delta(x), \\ N_j^\epsilon(x) = 0 & on \;\; \partial I_\delta(x), \end{array} \right. \tag{4.7}$$

where $I_\delta(x) = x + \delta I$, then

$$w_l^\epsilon(x) = N_j^\epsilon(x) \frac{\partial V_l(x)}{\partial x_j}.$$

It follows that

$$v_l^\epsilon(x) = V_l(x) + N_j^\epsilon(x) \frac{\partial V_l(x)}{\partial x_j} \tag{4.8}$$

Let T_h, h be as defined in Section 2. Assume that $N_j^\epsilon(x) \in W^{k+1,\infty}(D)$. Then from Theorem 2.1, we have

$$\|N_j^\epsilon(x) - \Pi_k N_j^\epsilon(x)\|_{1,\infty} \le Ch^k, \qquad j = 1,2, \quad k \ge 2. \tag{4.9}$$

Set

$$\widetilde{A}_H(V, V) = \sum_{K \in T_H} |K| \sum_{x_l \in K} \omega_l (\nabla V \cdot \widetilde{\mathcal{A}}_H \nabla V)(x_l), \tag{4.10}$$

where

$$(\nabla V \cdot \widetilde{\mathcal{A}}_H \nabla V)(x_l) = \frac{1}{\delta^d} \int_{I_\delta(x_l)} \nabla \widetilde{v}_l^\epsilon(x) \cdot a^\epsilon(x) \nabla \widetilde{v}_l^\epsilon(x) dx, \tag{4.11}$$

and

$$\widetilde{v}_l^\epsilon(x) = V_l(x) + \Pi_k N_j^\epsilon(x) \frac{\partial V_l(x)}{\partial x_j}. \tag{4.12}$$

Then, the revised HMM solution $U_H \in X_H$ is defined by

$$\widetilde{A}_H(U_H, V) = (f, V), \qquad \forall\, V \in X_H. \tag{4.13}$$

Theorem 4.1. *Let $\mathcal{A}_H$ and $\widetilde{\mathcal{A}}_H$ are defined as (4.3) and (4.11), respectively. Then, we have*

$$\max_{x_l \in K} \|\mathcal{A}_H - \widetilde{\mathcal{A}}_H\| \le Ch^k. \tag{4.14}$$

Proof. From inequalities (4.8), (4.9) and (4.12), it follows easily that

$$\|\nabla v_l^\epsilon(x) - \nabla \widetilde{v}_l^\epsilon(x)\|_{0,I_\delta(x_l)} \leq Ch^k \|\nabla V_l(x)\|_{0,I_\delta(x_l)}.$$

So, from (4.3) and (4.11), we obtain that

$$\begin{aligned}
&|\nabla W(x_l)(A_H - \widetilde{\mathcal{A}}_H)\nabla V(x_l)| \\
&= |\frac{1}{\delta^d}\int_{I_\delta(x_l)} \Big(\nabla w_l^\epsilon(x)\cdot a^\epsilon(x)\nabla v_l^\epsilon(x) - \nabla \widetilde{w}_l^\epsilon(x)\cdot a^\epsilon(x)\nabla \widetilde{v}_l^\epsilon(x)\Big)dx| \\
&= |\frac{1}{\delta^d}\int_{I_\delta(x_l)} (\nabla w_l^\epsilon(x) - \nabla \widetilde{w}_l^\epsilon(x))\cdot a^\epsilon(x)(\nabla v_l^\epsilon(x) - \nabla \widetilde{v}_l^\epsilon(x))dx \\
&+\frac{1}{\delta^d}\int_{I_\delta(x_l)} (\nabla w_l^\epsilon(x) - \nabla \widetilde{w}_l^\epsilon(x))\cdot a^\epsilon(x)\nabla \widetilde{v}_l^\epsilon(x)dx \\
&+\frac{1}{\delta^d}\int_{I_\delta(x_l)} \nabla \widetilde{w}_l^\epsilon(x)\cdot a^\epsilon(x)(\nabla v_l^\epsilon(x) - \nabla \widetilde{v}_l^\epsilon(x))dx| \\
&\leq C\Big(h^{2k}\|\nabla W(x_l)\|_{0,I_\delta(x_l)}\|\nabla V(x_l)\|_{0,I_\delta(x_l)} + h^k\|\nabla W(x_l)\|_{0,I_\delta(x_l)}\|\nabla \widetilde{v}_l^\epsilon\|_{0,I_\delta(x_l)} \\
&+h^k\|\nabla \widetilde{w}_l^\epsilon(x_l)\|_{0,I_\delta(x_l)}\|\nabla V_l(x)\|_{0,I_\delta(x_l)}\Big) \\
&\leq Ch^k\|\nabla W(x_l)\|_{0,I_\delta(x_l)}\|\nabla V(x_l)\|_{0,I_\delta(x_l)}.
\end{aligned}$$

This inequality gives the desired result (4.14). □

Set the homogenized equation of (4.1) to be as follows (see [30]).

$$(4.15)\qquad \begin{cases} -\nabla\cdot(\mathcal{A}(x)\nabla U(x)) = f(x) & in\ \ D\subset \mathcal{R}^d, \\ U(x) = 0 & on\ \ x\in\partial D. \end{cases}$$

where $\mathcal{A}(x)$ is the homogenized coefficient.

Lemma 4.1. *Let*

$$e(\text{HMM}) = \max_{x_l\in K}\|\mathcal{A}(x_l) - \mathcal{A}_H(x_l)\|,$$

then for the periodic homogenization problems (see [30]*),*

$$(4.16)\qquad e(\text{HMM}) \leq \begin{cases} C\delta, & if\ \delta\ is\ an\ interger\ multiple\ of\ \epsilon, \\ C(\epsilon/\delta+\delta), & if\ \delta\ is\ not\ an\ interger\ multiple\ of\ \epsilon. \end{cases}$$

Theorem 4.2. *Assume that u^ϵ is the exact solution of the problem (4.1), that U_0 is the exact solution of equation (4.15), while $\widetilde{U}_0$ is the exact solution of (4.13) with the space X_H replaced by H_0^1. Moreover, set $a^\epsilon(x) = a(x, x/\epsilon)$. Then we have*

$$(4.17)\qquad \|u^\epsilon - \widetilde{U}_0\|_{0,D} \leq C(\sqrt{\epsilon} + h^k + e(\text{HMM})),$$

$$(4.18)\qquad \|u^\epsilon - \widetilde{u}_1^\epsilon\|_{1,D} \leq C(\sqrt{\epsilon} + h^k + e(\text{HMM})),$$

where $\widetilde{u}_1^\epsilon = \widetilde{U}_0 + \Pi_k N_j^\epsilon \frac{\partial \widetilde{U}_0}{\partial x_j}$.

Proof. From (4.14) and (4.16), it follows that

$$(4.19)\qquad \max_{x_l\in K}\|\mathcal{A}(x_l) - \widetilde{\mathcal{A}}_H(x_l)\| \leq C(h^k + e(\text{HMM})).$$

In view of (4.11), (4.15) and (4.19), we have

$$\begin{aligned} c\|U_0-\widetilde{U}_0\|_{1,D}\|W\|_{1,D} &\le |A(U_0-\widetilde{U}_0,W)| \\ &= |A(U_0,W)-A(\widetilde{U}_0,W)| \\ &= |A(U_0,W)-(A-\widetilde{A}_H)(\widetilde{U}_0,W)-\widetilde{A}_H(\widetilde{U}_0,W)| \\ &= |(f,W)-(A-\widetilde{A}_H)(\widetilde{U}_0,W)-(f,W)| \\ &= |(A-\widetilde{A}_H)(\widetilde{U}_0,W)| \\ &\le C(h^k+e(\mathrm{HMM}))\|\widetilde{U}_0\|_{1,D}\|W\|_{1,D}. \end{aligned}$$

So,

$$\|U_0-\widetilde{U}_0\|_{1,D}\le C(h^k+e(\mathrm{HMM}))\|\widetilde{U}_0\|_{1,D}$$

Hence,

$$\|u^\epsilon-\widetilde{U}_0\|_{0,D}\le\|u^\epsilon-U_0\|_{0,D}+\|U_0-\widetilde{U}_0\|_{0,D}\le C(\sqrt{\epsilon}+h^k+e(\mathrm{HMM})).$$

In addition, if $a^\epsilon(x)=a(x,x/\epsilon)$, then we have $N_j^\epsilon(x)=\epsilon N^j(x)$.

So, we can obtain,

$$\begin{aligned} \|u_1^\epsilon-\widetilde{u}_1^\epsilon\|_{1,D} &\le \|U_0-\widetilde{U}_0\|_{1,D}+\|N_j^\epsilon\frac{\partial U_0}{\partial x_j}-\Pi_kN_j^\epsilon\frac{\partial\widetilde{U}_0}{\partial x_j}\|_{1,D} \\ &\le \|U_0-\widetilde{U}_0\|_{1,D}+\|(N_j^\epsilon-\Pi_kN_j^\epsilon)\frac{\partial U_0}{\partial x_j}\|_{1,D}+\|\Pi_kN_j^\epsilon\cdot\frac{\partial}{\partial x_j}(U_0-\widetilde{U}_0)\|_{1,D} \\ &\le C(\sqrt{\epsilon}+h^k+e(\mathrm{HMM})). \end{aligned}$$

Then this theorem is proved. □

As theorem 3.1, we can obtain

Theorem 4.3. *Let U_H be the numerical solution of problem (4.13), and $\widetilde{u}_0$ be the exact solution of equation (4.13) with X_H replaced by $H_0^1(D)$. Then (see* [20]*),*

$$\|\widetilde{u}_0-U_H\|_{1,D}\le CH\|\widetilde{u}_0\|_{2,D}, \tag{4.20}$$
$$\|\widetilde{u}_0-U_H\|_{0,D}\le CH^2\|\widetilde{u}_0\|_{2,D}.$$

Next, superconvergent techniques are applied to HMM to improve its accuracy.

First, assume that a postprocessing interpolation operator Π_{2H}^m satisfies all of the conditions (3.2), (3.3), (3.4) and (3.5). Then it follows from the result of superconvergence in the whole domain that

Theorem 4.4. *(see* [42]*) Let $\widetilde{u}_0$ be the exact solution of equation (4.13) with X_H replaced by $H_0^1(D)$, let U_H, u^I be the finite element solution and finite element interpolation of $\widetilde{u}_0$, respectively, and satisfy:*

$$\|U_H-u^I\|_l\le CH^{\alpha+1-l}\|\widetilde{u}_0\|_{m+1},\quad \alpha>p,\ \ m\ge\alpha,\ \ l=0,1,$$

where p is the order of the finite element polynomial space. Then,

$$\|\Pi_{2H}^mU^H-\widetilde{u}_0\|_l\le CH^{\alpha+1-l}\|\widetilde{u}_0\|_{m+1}.$$

Concurrently, we have some superconvergent error estimates in a local domain.

Theorem 4.5. *(See* [44]*) Let $\widetilde{u}_0$ be the exact solution of equation (4.13) with X_H replaced by $H_0^1(D)$, let U_p^H, u_p^I be the finite element solution and finite element interpolation of $\widetilde{u}_0$, respectively. Assume that $D_0\subset\subset D_1\subset\subset D$. If $\widetilde{u}_0$ is smooth enough and the mesh in D_1 is almost uniform, then,*

$$\|U_p^H-u_p^I\|_{1,D_0}\le C(H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1}+\|\widetilde{u}_0-U_p^H\|_{-s,D_1});$$

$$\|U_p^H - u_p^I\|_{1,\infty,D_0} \leq C(H^{p+1}|\ln H|^{\lambda}\|\widetilde{u}_0\|_{p+2,\infty,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}),$$

where p is the order of the finite element polynomial space, s is any non-negtive integer, and

$$\lambda = \begin{cases} 1, & if \;\; p = 1 \\ 0, & if \;\; p \geq 2. \end{cases}$$

By using the postprocessing interpolation operator, we have

Theorem 4.6. *(See* [44]*) Under the condition of Theorem 4.5, then*

$$\|\Pi_{2H}^{p+1}U_p^H - \widetilde{u}_0\|_{1,D_0} \leq C(H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1});$$
$$\|\Pi_{2H}^{p+1}U_p^H - \widetilde{u}_0\|_{1,\infty,D_0} \leq C(H^{p+1}|\ln H|^{\lambda}\|\widetilde{u}_0\|_{p+2,\infty,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}).$$

We next retrieve the microscopic information in whole domain from $\Pi_{2H}^m U_H$ and give the most important results of this section.
Assume that

$$R(v) = v + \Pi_k N_j^{\epsilon}(x)\frac{\partial v}{\partial x_j}, \tag{4.21}$$

Define

$$\overline{u}^{\epsilon}|_K = R(\Pi_{2H}^m U_H)|_K. \tag{4.22}$$

Theorem 4.7. *Let u^{ϵ} be the solution of (4.1), $\overline{u}^{\epsilon}$ be given by (4.22). Assume that all conditions of Theorem 4.3 are valid. Then,*

$$\|u^{\epsilon} - \overline{u}^{\epsilon}\|_{1,D} \leq C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D} + h^k\|u_0\|_{1,D} + H^m\|\widetilde{u}_0\|_{m+1,D}). \tag{4.23}$$

Proof. Note that on each element K,

$$\begin{aligned} \frac{\partial \overline{u}^{\epsilon}}{\partial x_i} &= \frac{\partial}{\partial x_i}\Pi_{2H}^m U_H + \frac{\partial}{\partial x_i}\Pi_k N_j^{\epsilon}(x)\cdot\frac{\partial}{\partial x_j}\Pi_{2H}^m U_H \\ &\quad + \Pi_k N_j^{\epsilon}(x)\cdot\frac{\partial^2}{\partial x_i \partial x_j}\Pi_{2H}^m U_H. \end{aligned} \tag{4.24}$$

Furthermore,

$$\frac{\partial \widetilde{u}_1^{\epsilon}}{\partial x_i} = \frac{\partial \widetilde{u}_0}{\partial x_i} + \frac{\partial}{\partial x_i}\Pi_k N_j^{\epsilon}(x)\frac{\partial \widetilde{u}_0}{\partial x_j} + \Pi_k N_j^{\epsilon}(x)\frac{\partial^2 \widetilde{u}_0}{\partial x_i \partial x_j}. \tag{4.25}$$

It follows from (4.24) and (4.25) that

$$\begin{aligned} \frac{\partial}{\partial x_i}(\overline{u}^{\epsilon} - \widetilde{u}_1^{\epsilon}) &= \frac{\partial}{\partial x_i}(\Pi_{2H}^m U_H - \widetilde{u}_0) + \frac{\partial}{\partial x_i}\Pi_k N_j^{\epsilon}(x)\cdot\frac{\partial}{\partial x_j}(\Pi_{2H}^m U_H - \widetilde{u}_0) \\ &\quad + \Pi_k N_j^{\epsilon}(x)\cdot\frac{\partial^2}{\partial x_i \partial x_j}(\Pi_{2H}^m U_H - \widetilde{u}_0). \end{aligned}$$

From Theorem 4.4 and $N_{lj}^{\epsilon}(x) = O(\epsilon)$, it follows that

$$\begin{aligned} \|\nabla(\overline{u}^{\epsilon} - \widetilde{u}_1^{\epsilon})\|_{0,D} &\leq C\|\nabla(\Pi_{2H}^m U_H - \widetilde{u}_0)\|_{0,D} + C\epsilon\|\widetilde{u}_0\|_{2,D} \\ &\leq CH^m\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{2,D}. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\overline{u}^{\epsilon} - \widetilde{u}_1^{\epsilon}\|_{0,D} &\leq C\|\Pi_{2H}^m U_H - \widetilde{u}_0\|_{0,D} + C\epsilon\|\widetilde{u}_0\|_{1,D} \\ &\leq CH^{m+1}\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{1,D}. \end{aligned}$$

From the inequalities above, it follows that

$$\|\overline{u}^{\epsilon} - \widetilde{u}_1^{\epsilon}\|_{1,D} \leq CH^m\|\widetilde{u}_0\|_{m+1,D} + C\epsilon\|\widetilde{u}_0\|_{2,D}.$$

Combining with (4.18), it is easy to obtain (4.23). This proves Theorem 4.7. □

Remark 4.1. *In some cases, the superconvergent error estimate in a local domain is more important. By the same method used in showing Theorem 4.7 and from Theorem 4.6, it follows that*

$$\begin{aligned} \|u^\epsilon - \overline{u}^\epsilon\|_{1,D_0} \leq\ & C(\sqrt{\epsilon}\|u_0\|_{3,\infty,D_1} + h^k\|u_0\|_{1,D_1} \\ & + H^{p+1}\|\widetilde{u}_0\|_{p+2,D_1} + \|\widetilde{u}_0 - U_p^H\|_{-s,D_1}), \end{aligned} \tag{4.26}$$

where, $\overline{u}^\epsilon = \Pi_{2H}^{p+1} U_p^H$.

5. Discussion

In this paper, we discussed superconvergent techniques in multi-scale methods, especially in HMM. For simplicity, in order to derive error estimates we assumed that the conditions in the model problems were such that the solutions were smooth enough. In many problems, such conditions may not be satisfied. We can still possibly use this method by retrieving techniques, such as error expansion and defect correction. In future work, we plan pursue research based on the ideas presented here on relevant problems in engineering.

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References

[1] A. Abdulle and W. E, *Finite difference heterogeneous multi-scale method for homogenization problems*, J. Comput. Phys., **191**, (2003) 18-39.
[2] R. A. Adams, *Sobolev Spaces*, Academic press, 1975.
[3] R. E. Alcouffe, A. Brandt, J. E. Dendy and J. W. Painter, *The multigrid method for the diffusion equation with strongly discontinuous coefficients*, SIAM J. Sci. Statist. Comput., 2, (1981) 430-454.
[4] I. Babuška, *Homogenization and its applications, mathematical and computational problems,* Numerical Solutions of Partial Differential Equations-III, (SYNSPADE 1975, College Park MD, May 1975), (B. Hubbard ed.), Academic Press, New York, 1976, pp. 89-116.
[5] I. Babuška and A. K. Aziz, *Survey Lectures on Mathematical Foundations of the Finite Element Method*, in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, ed. by A.Aziz, Academic Press New York and London, 1972, PP. 5-359.
[6] I. Babuška, G. Caloz and J. Osborn, *Special finite element methods for a class of second order elliptic problems with rough coefficients*, SIAM J. Numer. Anal., 31, (1994) 945-981.
[7] I. Babuška, T. Strouboulis, S. K. Gangaraj, and C. S. Upadhyay $\eta\%$ *superconvergence in the interior of locally refined meshes of quadrilaterals: superconvergence of the gradient in finite element solutions of Laplace's and Poisson's equations*, Applied Numerical Mathematics, 122, (1994) 273-305.
[8] I. Babuška, T. Strouboulis, and C. S. Upadhyay, $\eta\%$ *superconvergence of finite element approximations in the interior of meshes of triangles,* Computer Methods in Applied Mechanics and Engineering, 16, (1994) 3-49.
[9] A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, New-York, Oxford, 1978.
[10] J. F. Bourgat, *Numerical experiments of the homogenization method for operators with periodic coefficients*, Lecture Notes in Mathematics, Vol. 707. 330-356.
[11] A. Brandt, *Multi-level adaptive solutions to boundary-value problems*, Math. Comp., 31, (1977) 333-390.
[12] F. Brezzi, L. P. Franca, T. J. R. Hughes and A. Russo, $b = \int gdx$, Comput. Meth. Appl. Mech. Engrg., 145, (1997) 329-339.
[13] L. Cao, J. Cui and J. Luo, *Multiscale asymptotic expansion and a post-processing algorithm for second-order elliptic problems with highly oscillatory coefficients over general convex domains*, Journal of Computational and Applied Mathematics, 157, (2003) 1-29.

[14] L. Cao, J. Cui, D. Zhu and J. Luo, *Multiscale Finite Element Method for Subdivided Periodic Elastic Structures of Composite materials,* Journal of Computational Mathematics, Vol.19, No.2, (2001) 205-212.

[15] C. M. Chen, and Y. Q. Huang, *high accuracy theory of finite element methods*, Hunan Science and Technology Press, Changsha (in Chinese), 1995.

[16] C. Chen, *Superconvergence for triangular finite elements*, Science in China (Ser A), 42, (1999) 917-934.

[17] P. Chen and N. Yan, *Heterogeneous multi-scale method for Helmholtz equation with period micro-structure*, Chinese Journal of Engineering Mathematics, Vol. 21, No. 8 (2004) 145-149.

[18] Z. M. Chen and T. Y. Hou, *A mixed multiscale finite element method for elliptic problems with oscillating coefficients*, Math. Comp. 72, (2002) 541-576.

[19] L. T. Cheng and W. E, *The heterogeneous multiscale method for interface dynamics*, to appear in Contemporary Mathematics: A Special Volume in Honour of Stan Osher, S. Y. Cheng, C. W. Shu and T. Tang eds.

[20] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[21] J. Z. Cui and L. Q. Cao, *The two-scale analysis methods for woven composite materials*, Engineering Computation and Computer Simulation I, (Zhi-hua Zhong eds.), Hunan University Press, 1995, 203-212.

[22] M. Dorobantu and B. Engquist, *Wavelet-based numerical homogenization*, SIAM J. Numer. Anal., 35, (1998) 540-559.

[23] J. Douslas and T. Dupont, *Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary value problems*, Topics in Numerical Analysis, Academic Press, (1973) 89-92.

[24] L. J. Durlofsky, *Numerical-calculation of equivalent grid block permeability tensors for heterogeous porous media*, Water Resour. Res., 27, (1991) 699-708.

[25] W. E, B. Engquist, X. Li, W. Ren, E. Vanden-Eijnden, *The Heterogeneous Multiscale Method: A Review.* http://www.math.princeton.edu/multiscale/

[26] W. E, *Analysis of the heterogeneous multiscale method for ordinary differential equations*, Comm. Math. Sci., Vol. 1 (3), (2003) 423-436.

[27] W. E and B. Engquist, *The Heterogeneous Multi-scale Methods,* Comm. Math. Sci., 1, (2003) 87-133.

[28] W. E, B. Engquist, *Multiscale modeling and computation*, Notice Amer. Math. Soc., 50, (2003) 1062-1070.

[29] W. E, B. Engquist and Z. Huang, *Heterogeneous multi-scale method – a general methodology for multi-scale modeling*, Phys. Rev. B, 67(9), (2003) 092101 .

[30] Weinan E, Pingbing Ming, Pingwen Zhang *Analysis of the heterogeneous multiscale method for elliptic homogenization problems.* J. Amer. Math. Soc. Vol 18, No 1, (1985) 121–156.

[31] Y. R. Efendiev, T. Hou and X. Wu, *Convergence of a nonconforming multiscale finite element method*, SIAM J. Numer. Anal., 37, (2000) 888-910.

[32] C. Farhat. I. Harari and L. P. Franca, *The discontinuous enrichment method*, Comput. Meth. Appl. Mech. Engrg., 190, (2001) 6455-6479.

[33] J. Fish and V. Belsky, *Multigrid method for a periodic heterogeneous medium, Part I: Convergence studies for one-dimensional case*, Comput. Meth. Appl. Mech. Engrg., 126, (1995) 1-16.

[34] J. Fish and V. Belsky, *Multigrid method for a periodic heterogeneous medium, Part I: Multiscale modeling and quality in multi-dimensional case*, Comput. Meth. Appl. Mech. Engrg., 126, (1995)17-38.

[35] J. Fish and Z. Yuan, *Multiscale enrichment based on partition of unity*, Inter. J. Numer. Meth. Engrg., 62, (2005) 1341-1359.

[36] L. Greengard and V. Rokhlin, *A fast algorithm for particle simulations*, J. Comput. Phys., 73, (1987) 325-348.

[37] T. Y. Hou and X. H. Wu, *A multiscale finite element method for elliptic problems in composite materials and porous media,* J. Comput. Phys., 134, (1997) 169-189.

[38] T. Y. Hou, X. H. Wu and Z. Cai, *Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating cofficients,* Math. Comput., 68, (1999) 913-943.

[39] T. J. R. Hughes, *Multiscale phenomena: Green's functions, the Dirichlet to Neumann formulation, subgrid scale models, bubbles and the origin of stabilized methods*, Comput. Meth. Appl. Mech. Engrg., 127, (1995) 387-401.

[40] Q. Lin, *Global error expansion and superconvergence for higher order interpolation of finite elements*, J. Comp. Math., Supplementary Issue, (1992) 286-289.

[41] Q. Lin, *A rectangle test for FEM*, ibid, 213-216.
[42] Q. Lin, N. N. Yan, A. Zhou, *A rectangle test for interpolated finite elements* Proc. Sys. Sci. & Sys. Engrg., Great wall Culture Publ. Co., Hong Kong, (1991) 217-229.
[43] Q. Lin, N. N. Yan, *A rectangle test for singular solution with irregular meshes* Proc. Sys. Sci. & Sys. Engrg., Great Wall Culture Publ. Co., Hong Kong, (1991) 236-237.
[44] Q. Lin, N. N. Yan, *Construction and Analysis for Efficient Finite Element Methods*, Publishing of Hebei University (in Chinese), 1996.
[45] Pingbing Ming and Xingye Yue, *Numerical methods for multiscale elliptic problems*, to appear.
[46] J. D. Moulton, J. E. Dendy and J. M. Hyman, *The black box multigrid numerical homogenization algorithm*, J. Comput. Phys., 141, (1998) 1-29.
[47] N. Neuss, W. Jäger and G. Wittum, *Homoginization and multigrid*, Computing., 66, (2001) 1-26.
[48] J. T. Oden and K. S. Vemaganti, *Estimation of local modeling error and global-oriented adaptive modeling of heterogeneous materials. I: Error estimates and adaptive algorithms*, J. Comput. Phys., 164, (2000) 22-47.
[49] O. A. Oleinik, A. S. Shamaev and G. A. Yosifian, *Mathematical Problems in Elasticity and Homogenization*, North-Holland, Amsterdam, 1992.
[50] W. Ren and W. E, *Heterogeneous multiscale method for the modeling of complex fluids and micro-fluidics*, J. Comput. Phys., in press.
[51] G. Sangalli, *Capturing small scales in elliptic problems using a Residual-Free Bubbles finite element methods*, Multiscale Model. Simul., 1, (2003) 485-503.
[52] A. H. Schatz and L. B. Wahlbin, *Superconvergence in finite element methods and meshes which are locally symmetric with respect to a point*, SIAM J. on Numer. Anal, 33, (1996) 505-521.
[53] A. H. Schatz, *Pointwise error estimates and asymptotic expansion error inequalities for the finite element mehtod on irregualr grids, Part I: global estimates*, Math of Compt, 67, (1998) 877-899.
[54] Z. Zhang, *Ultraconvergence of patch recovery technique*, Math of Compt, 65, (1996) 1431-1437.
[55] Z. Zhang, *Ultraconvergence of patch recovery technique II.*, Math of Compt, 69, (2000) 141-168.
[56] V. V. Zhikov, S. M. Kozlov and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functions,* Springer-verlag Berlin Heidelberg, 1994.
[57] Qiding Zhu, *Superconvergence analysis of cubic triangular elements in the finite element method*, J. of Comput Math, 18, (2000) 545-550.
[58] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergent patch recovery and a posteriori error estimates. Part 1: The recovery technique*, Internation J. of Numer. Methods in Engineering, 33, (1992) 1331-1364.
[59] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergent patch recovery and a posteriori error estimates. Part 2: Error estimates and adaptivity*, Internation J. of Numer. Methods in Engineering, 33, (1992) 1365-1382.
[60] O. C. Zienkiewicz and J. Z. Zhu, *The superconvergent patch recovery(SPR) and adaptive finite element refinement*, Computer Methods in Applied Methanics and Engineering, 101, (1992) 207-224.

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