INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Number 2, Pages 222–238

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Computing and Information

THE REGULARIZATION METHOD FOR A DEGENERATE PARABOLIC VARIATIONAL INEQUALITY ARISING FROM AMERICAN OPTION VALUATION

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Abstract. In this paper, we present a regularization method to a degenerate variational inequality of parabolic type arising from American option pricing. Main difficulty in actually analyzing this kind of problem is caused by the presence of a non-smoothing initial value function in the formulation of the problem. We first use a smoothing technique with small parameter $\varepsilon > 0$ to non-smoothing initial value function; and then we derive the error estimates for regularized continuous problem and regularized discrete problem, respectively. Numerical tests are given to confirm our theoretical results.

Key Words. regularization method, variational inequality, American Option valuation, finite element and error estimates.

1. Introduction

Option trading forms part of our financial markets. A traded option gives to its owner the right to buy (*call option*) or to sell (*put option*) a fixed quantity of assets of a specified stock at a fixed price (*exercise* or *strike price*). There are two major types of traded options. One is the *American option* that can be exercised at any time prior to its *expiry date*, and the other option, which can only be exercised on the expiry date, is called the *European option.* It was shown by Black and Scholes (cf. [3]) that the value of an European option is governed by a second order parabolic differential equation with respect to time and the underlying stock price. This is now referred to as the Black-Scholes equation. The value of an American option is governed by a more complex mathematical model due to the flexibility on exercise date. It can be shown that American option pricing is determined by a linear complementarity problem involving the Black-Scholes differential operator and a constraint on the value of the option (cf., for example, [20, 19]). This complementarity problem can also be formulated as a variational inequality (cf. [19]). The Black-Scholes equation is a degenerate partial differential as its coefficients of the first and second order spatial derivatives vanish as the underlying stock price approaches zero. A popular method of removing this difficulty is to introduce a new variable and transform the Black-Scholes equation into a heat equation defined on the whole real number set. This technique is used in many existing papers such as [1, 10, 20]. In this case, the degeneracy point is transformed to $-\infty$. However, when solve the resulting heat equation numerically,

Received by the editors November 1, 2006 and, in revised form, March 17, 2007.

2000 *Mathematics Subject Classification.* 35R35, 49J40 , 60G40.

This research was supported by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry (No.383(2005)) and North China Electric Power University (2005).

the infinite horizon is truncated to a finite region. Recently, a fitted finite volume method is proposed in [18] to handle the degeneracy, based on the idea in [13, 14]. This technique can also be used for solving the American option problem if it is used along with a power penalty method (cf., for example, [19]).

In this paper we shall discuss the regularization method [12] for solving the parabolic variational inequality with a degenerate partial differential operator governing American option valuation. To our best knowledge, there are relatively few papers in which numerical methods are studied for parabolic variational inequalities (cf, for example, [1, 7, 17] and references therein), let alone parabolic inequalities with degenerate partial differential operators (cf. [8]). The main difficulty is that solutions to parabolic variational inequalities normally less smooth than those of elliptic problems even all the data are smooth. Johnson [7] and Vuik [17] studied the finite element approximations of a variational inequality of parabolic type under some regularity assumptions on the exact solution. To bypass the difficulty, we shall construct a regularization method for the variational inequality involving the Black-Scholes operator, and derive the error bound in the weighted Sobolev space for the method.

The remainder of this paper is organized as follows. In the next section we will state the strong problem governing American put option pricing and some preliminaries. In Section3, we shall rewrite the problem as a more mathematical form, i.e., a variational inequality, and discuss the solvability of the resulting problem. In section4, we present the regularity problem of problem3.1, and prove its error bound with ε. We will present some numerical results to confirm the theoretical findings in Section5.

2. Preliminaries

Let V denote the value of an American put option with strike price K and expiry date T, and let x be the price of the underlying asset of the option. It is known (cf., for example, [20]) that V satisfies the following strong form of linear complementarity problem

$$LV(x,t) \geq 0, \tag{2.1}$$

$$V(x,t) - V^*(x) \geq 0, \tag{2.2}$$

$$LV(x,t) \cdot (V(x,t) - V^*(x)) = 0, \tag{2.3}$$

a.e. in $\Omega := I \times J$, where L is the Black-Scholes operator defined by

$$LV := -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 V}{\partial x^2} - r(t)x\frac{\partial V}{\partial x} + r(t)V, \tag{2.4}$$

$I = (0, X) \subset R$ and $J = (0, T)$ with positive constants X and T, $\sigma(t)$ denotes the volatility of the asset, $r(t)$ the interest rate, and V^* is the final (payoff) condition defined by

$$V(x,T) = V^*(x) = \max\{K - x, 0\}. \tag{2.5}$$

For clarity, we only consider American put options in this paper. Naturally, the theory developed applies to American call options and other complementarity problems of the form (2.1)–(2.3) arising in finance as well.

Some standard notation is to be used in the paper. For an open set $S \in R$ and $1 \leq p \leq \infty$, we let $L^p(S) = \{v : (\int_S |v(x)|^p dx)^{1/p} < \infty\}$ denote the space of all p-power integrable functions on S. The inner product and the norm on $L^2(S)$ are denoted respectively by $(\cdot,\cdot)_S$ and $\|\cdot\|_0$. We use $\|\cdot\|_{L^p(S)}$ to denote the norm on $L^p(S)$. For $m = 1, 2, ...$, we let $H^{m,p}(S)$ denote the usual Sobolev space with the norm $\|\cdot\|_{m,2,S}$. When $p=2$, we simply denote $H^{m,2}(S)$ and $\|\cdot\|_{m,2,S}$ by $H^m(S)$ and $\|\cdot\|_{m,S}$, respectively. Let $C^m(S)$ (respectively, $C^m(\bar{S})$ be the function set of which a function and its derivatives of up to order k are continuous on S (respectively, $\bar{S}$). When $S=I$, we omit the subscript S in the above notation. We put $H_0^m(I)=\{v\in H^m(I): v(0)=v(X)=0\}$. Finally, for any Hilbert space $H(I)$, we let $L^p(0,T;H(I))$ denote the space defined by

$$L^p(J;H(I))=\{v(\cdot,t): v(\cdot,t)\in H(I) \text{ a.e. } in\ J;\ \|v(\cdot,t)\|_H\in L^p(J)\}$$

where $1\le p<\infty$ and $\|\cdot\|_H$ denotes the natural norm on $H(I)$. The norm on this space is denoted by $\|\cdot\|_{L^p(J;H)}$, i.e.,

$$\|v\|_{L^p(J;H(I))}=\left(\int_0^T\|v(\cdot,t)\|_H^p dt\right)^{1/p}$$

When $p=\infty$, the norm is defined as follows

$$\|v\|_{L^\infty(J;H(I))}=\sup_{t\in[0,T]}\|v(\cdot,t)\|_H$$

Clearly, $L^p(J;L^p(I))=L^p(I\times(0,T))=L^p(\Omega)$.

To handle the degeneracy in the Black-Scholes equation, we introduce the following weighted L^2-norm

$$\|v\|_{0,w}:=\left(\int_0^X x^2v^2dx\right)^{1/2}.$$

The space of all weighted square-integrable functions is defined as

$$L_w^2:=\{v:\|v\|_{0,w}<\infty\}.$$

We also define a weighted inner product on $L_w^2(I)$ by $(u,v)_w:=\int_0^X x^2uvdx$. Using a standard argument (cf., for example, [5], Chapters 1 & 2) it is easy to show that the pair $(L_w^2(I),(\cdot,\cdot)_w)$ is a Hilbert space. For brevity, we omit this discussion. Using $L^2(I)$ and $L_w^2(I)$, we define the following weighted Sobolev spaces

$$H_w^1(I)=\{v|v\in L^2(I), v'\in L_w^2(I)\},\quad H_{0,w}^1(I)=\{v|v\in H_w^1(I),\text{and } v(X)=0\}$$
$$H_w^2(I)=\{v|v\in H_w^1, xv''\in L_w^2(I)\},\quad H_{0,w}^2(I)=\{v|v\in H_{0,w}^1(I), xv''\in L_w^2(I)\}.$$

where v', v'' denote the weak derivative of v. Let $\|\cdot\|_{1,w}$ and $\|\cdot\|_{2,w}$ be a functional on $H_w^1(I)$ and $H_w^2(I)$, respectively, defined by

$$\|v\|_{1,w}=(\|v\|_0^2+\|v'\|_{0,w}^2)^{1/2}=[(x^2v',v')+(v,v)]^{1/2}, \tag{2.6}$$

$$\|v\|_{2,w}=(\|v\|_{1,w}^2+\|xv''\|_{0,w}^2)^{1/2}=[(v,v)+(x^2v',v')+(x^4v'',v'')]^{1/2}. \tag{2.7}$$

It is easy to check that $\|\cdot\|_{1,w}$ and $\|\cdot\|_{2,w}$ are the weighted H^1- and H^2-norms on $H_{0,w}^1(I)$ and $H_{0,w}^2(I)$, respectively. Furthermore, using the inner products on $L^2(I)$ and $L_w^2(I)$, we define a weighted inner product on $H_{0,w}^1(I)$ by $(\cdot,\cdot)_H:=(\cdot,\cdot)+(\cdot,\cdot)_w$. It is also easy to prove that the pair $(H_{0,w}^1(I),(\cdot,\cdot)_H)$ is a Hilbert space. For brevity, we omit this proof, but refer the reader to a similar proof in [5], Chapter 2. More detailed discussions on (weighted) Sobolev spaces can be found in [2, 16].

We will often simply write $u(\cdot,t)$ as $u(t)$ when we regard $u(\cdot,t)$ as an element of $H_{0,w}^1(I)$. From time to time, we will also suppress the independent time variable t (or τ) when it causes no confusion in doing so.

We comment that it is unnecessary to impose the homogeneous boundary condition at $x=0$ in above weighted Sobolev space because of the weighting x^2 in the inner product. However, when we define the finite element problem in Section 4.2, we look for the solution satisfying the homogeneous boundary condition at $x = 0$.

3. The continuous problem

In this section we will outline the formulation of (2.1)–(2.5) as a variational inequality. We assume that $\sigma(t)$ and $r(t)$ satisfy respectively

$$\overline{\sigma} \geq \sigma(t) \geq \underline{\sigma} \quad \text{and} \quad \overline{r} \geq r(t) \geq \underline{r},$$

for some positive constants $\overline{\sigma}$, $\underline{\sigma}$, $\overline{r}$ and $\underline{r}$. We also assume that $X >> K$. It has been shown in [19] that the boundary conditions are

$$V(0,t) = K, \quad \text{and} \quad V(X,t) = 0. \tag{3.1}$$

for all $t \in [0, T)$.

Before reformulating the complementarity problem (2.1)–(2.3) as a variational problem, we first transform it into an equivalent standard form satisfying homogeneous Dirichlet boundary conditions.

Let V_0 be given by

$$V_0(x) = (1 - \frac{x}{X})K \tag{3.2}$$

and introduce a new variable

$$u(x,t) = e^{\beta t}(V_0(x) - V(x,t)) \tag{3.3}$$

where

$$\beta = \sup_{0<t<T} \sigma^2(t). \tag{3.4}$$

Using this transformation, it is easy to show (cf., [19]) that the complementarity problem (2.1)–(2.3) can be transformed into

$$\mathcal{L}u(x,t) \leq f(x,t), \tag{3.5}$$

$$u(x,t) - u^*(x,t) \leq 0, \tag{3.6}$$

$$(\mathcal{L}u(x,t) - f(x,t)) \cdot (u(x,t) - u^*(x,t)) = 0, \tag{3.7}$$

with the boundary and terminal conditions

$$u(0,t) = 0 = u(X,t), \quad t \in [0,T), \text{ and } u(x,T) = u^*(x,T), \quad x \in (0,X),$$

where

$$\mathcal{L}u = -\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}[a(t)x^2 \frac{\partial u}{\partial x} + b(t)xu] + c(t)u,$$

with

$$a = \frac{1}{2}\sigma^2, \quad b = r - \sigma^2,$$
$$c = r + b + \beta = 2r + \beta - \sigma^2,$$

and

$$f(t) = e^{\beta t} L V_0(x) = e^{\beta t} r(t) K \tag{3.8}$$

$$u^*(x,t) = \begin{cases} e^{\beta t}(1 - \frac{K}{X})x, & \text{if } 0 \leq x \leq K, \\ e^{\beta t}(1 - \frac{x}{X})K, & \text{if } K < x \leq X. \end{cases} \tag{3.9}$$

This is of the standard form for linear complementarity problems which can be cast into a variational inequality as given below.

Let $\mathcal{K} = \{v \in H^1_{0,w}(I) : v \leq u^*\}$. It is easy to verify that $\mathcal{K}$ is a convex and closed subset of $H^1_{0,w}(I)$. Using $\mathcal{K}$, we define the following problem

Problem 3.1. *Find $u(t) \in \mathcal{K}$ such that, for all $v \in \mathcal{K}$,*

$$\left(-\frac{\partial u(t)}{\partial t}, v - u(t)\right) + A(u(t), v - u(t); t) \geq (f(t), v - u(t)), \tag{3.10}$$

almost everywhere (a.e.) in J, where $A(\cdot,\cdot)$ be a bilinear form defined by

$$A(u, v; t) = (ax^2u' + bxu, v') + (cu, v), \qquad \forall u, v \in H^1_{0,w}(I). \tag{3.11}$$

For this variational inequality problem, we have

Theorem 3.1. *Problem3.1 is the variational form corresponding to the linear complementarity problem (3.5)–(3.7).*

PROOF. The proof is standard and thus it is omitted here. □

Lemma 3.1. *There exist positive constants C_0 and M_0, independent of u and v, such that for any $u, v \in H^1_{0,w}(I)$,*

$$A(u, u; t) \geq C_0\|u\|^2_{1,w}, \tag{3.12}$$

$$A(u, v; t) \leq M_0\|u\|_{1,w} \cdot \|v\|_{1,w} \tag{3.13}$$

for $t \in J$, where $\|\cdot\|_{1,w}$ is the norm defined in (2.6).

PROOF. For any $v \in H^1_{0,w}(I)$, it has been shown in Wang (2004) through using integration by parts that

$$\int_0^X b(t)xvv'dx = -\frac{1}{2}\int_0^X b(t)v^2dx$$

Therefore, using the above,we have

$$\begin{aligned} A(u, u; t) &= (ax^2u' + bxu, u') + (cu, u) \\ &= (ax^2u', u') + ((r + b + \beta - \frac{b}{2})u, u) \\ &= (ax^2u', u') + \frac{1}{2}((3r + 2\beta - \sigma^2)u, u) \\ &\geq C_0\|u\|^2_{1,w}. \end{aligned}$$

In addition

$$\begin{aligned} A(u, v; t) &= (ax^2u' + bxu, v') + (cu, v) \\ &= a(x^2u', v') + b(xu, v') + c(u, v) \\ &\leq M_0\|u\|_{1,w} \cdot \|v\|_{1,w}. \end{aligned}$$

Here $C_0 = C_0(\sigma, r) = \frac{1}{2}\min\{\underline{\sigma}^2, 3r\}$, $M_0 = M_0(a, b, c) = a + |b| + c$. □

Using above lemma, we have the following theorem.

Theorem 3.2. *There exists a unique solution to Problem3.1.*

PROOF. This theorem is just a consequence of Lemma3.1 and Theorem 1.33 in [11], in which the unique solvability for an abstract variational inequality problem is established. □

The following lemma establishes the pointwise estimate and the regularities for the solution to Problem3.1 required for proving the regularities of the solution to Problem4.1 in Section 4.

Theorem 3.3. *The solution u to Problem3.1 satisfies*

$$u \in C([0,T], H^1_{0,w}(I)) \cap L^2(J; H^2_{0,w}(I)), \quad \frac{\partial u}{\partial t} \in L^2(J; L^2(I)), \tag{3.14}$$

and

$$\begin{aligned}\left(-\frac{\partial^+ u(t)}{\partial t}, v - u(t)\right) + A(u(t), v - u(t); t) \\ \geq (f(t), v - u(t)), \qquad \forall v \in \mathcal{K}, \quad t \in [0,T].\end{aligned} \tag{3.15}$$

Furthermore, we have the following pointwise relationships:

$$-\frac{\partial^+ u}{\partial t} = \mathcal{A}u + f \qquad a.e.\ on\ I^-(t), \tag{3.16}$$

$$-\frac{\partial^+ u}{\partial t} = -(\mathcal{M}u - f)^0 \qquad a.e.\ on\ I^0(t), \tag{3.17}$$

where $I^-(t) = \{x \in I : u(x,t) < u^*(x,t)\}$, $I^0(t) = \{x \in I : u(x,t) = u^*(x,t)\}$, $\partial^+ u/\partial t$ denotes the right-hand derivative of u with respect to t, and $\mathcal{A}$ denotes the operator associated with the bilinear form given in (3.11), i.e.,

$$\mathcal{A}u = -\frac{\partial}{\partial x}\left[a(t)x^2 \frac{\partial u}{\partial x} + b(t)xu\right] + c(t)u.$$

$\mathcal{M}u$ is a set defined as follows [6, pp.99]

$$\mathcal{M}u = \{y \in L^2(I) :\ A(u, v-u; t) \geq (y, v-u), \quad \forall v \in \mathcal{K},\ u = u^*(x,t)\}$$

$(\mathcal{M}u - f)^0 = g - f$, where $g \in \mathcal{M}u$, and satisfies

$$\inf_{y \in \mathcal{M}u} \|y - f\|_0 = \|g - f\|_0 \tag{3.18}$$

PROOF. In [6, pp.98-100] the authors proved that the solution to a general variational inequality problem of the form (3.10) satisfies (3.14)–(3.17), provided that the right-hand side function f satisfies $f \in L^2(J, L^2(I))$ and the initial(final) condition $u(T)$ satisfies $u(\cdot, T) \in \mathcal{K}$. Therefore, to prove this theorem, we need only to verify that f defined in (3.8) and $u(\cdot, T)$ have the required regularity.

From (3.8) we see that f is bounded function with respect to t and thus we have $f \in L^2(J, L^2(I))$. Furthermore, from (3.3) and (3.9) we see that $u(x,T) = -U(x,T) = u^*(x,T)$ which is differentiable. Therefore, we have $u(\cdot,T) \in \mathcal{K}$, and thus the solution u to Problem3.1 satisfies (3.14)–(3.17). □

4. Regularization of the variational inequality

In general, the analysis of the finite element method for Problem3.1 requires certain smoothness of the exact solution u. This requirement is guaranteed if $u^*(x,T)$ and $u^*(x,t) \in H^2_{0,w}(I)$ a.e. in J. However, from definition (3.9) we see that $u^*(x,t)$ is piecewise linear in x and thus not in $H^2_{0,w}(I)$. To overcome this difficulty, we smooth out the non-smooth point $x = K$ of $u^*(x,t)$ for any t using the following regularization technique with a parameter ε to form a new bound $u^*_\varepsilon(x,t)$ and using it to construct a regularized problem of Problem3.1. Finally, we will estimate the error of between the solutions to the regularized problem and Problem3.1.

To construct a smooth function approximating $u^*(x,t)$ locally near $x = K$, we let $\varepsilon > 0$ be a (small) parameter and consider the following polynomial

$$P_\varepsilon(x,t) = e^{\beta t}\left(a_1(x-K)^4 + a_2(x-K)^3 + a_3(x-K)^2 + a_4(x-K) + a_5\right) \tag{4.1}$$

interpolating $u^*(x,t)$ locally in $[K-\varepsilon, K+\varepsilon]$ satisfying

$$P_\varepsilon(K,t) = u^*(K,t), \quad P_\varepsilon(K\pm\varepsilon,t) = u^*(K\pm\varepsilon,t),$$

$$\frac{\partial}{\partial x}P_\varepsilon(K\pm\varepsilon,t) = \frac{\partial}{\partial x}u^*(K\pm\varepsilon,t).$$

It is easy to verify that P_ε is given by

$$\begin{aligned} P_\varepsilon(x,t) = \; & e^{\beta t}\left[\frac{1}{4\varepsilon^3}(x-K)^4 - \frac{3}{4\varepsilon}(x-K)^2\right] \\ & + e^{\beta t}\left[\left(\frac{1}{2}-\frac{K}{X}\right)(x-K) + \left(1-\frac{K}{X}\right)K\right]. \end{aligned} \tag{4.2}$$

Therefore, we define the following approximation $u^*_\varepsilon(x,t)$ to $u^*(x,t)$ on interval $[0,X]$

$$u^*_\varepsilon(x,t) = \begin{cases} P_\varepsilon(x,t) & \text{if } x\in[K-\varepsilon,K+\varepsilon], \\ u^*(x,t), & \text{if } x\in[0,K-\varepsilon)\cup(K+\varepsilon,X]. \end{cases} \tag{4.3}$$

We now put $\mathcal{K}_\varepsilon = \{v | v\in H^1_{0,w}(I),\ v\le u^*_\varepsilon\}$. It is easy to see that $\mathcal{K}_\varepsilon$ is a convex set approximating $\mathcal{K}$. In addition, we let

$$\begin{aligned} D_\varepsilon(x,t) &= u^*_\varepsilon(x,t) - u^*(x,t) \\ &= \begin{cases} P_\varepsilon(x,t) - e^{\beta t}(1-\frac{K}{X})x, & x\in[K-\varepsilon,K], \\ P_\varepsilon(x,t) - e^{\beta t}(1-\frac{x}{X})K, & x\in(K,K+\varepsilon], \\ 0. & \text{otherwise.} \end{cases} \end{aligned} \tag{4.4}$$

Lemma 4.1. *For any $\varepsilon>0$, $D_\varepsilon(x,t)\ge 0$ on the interval $[0,X]$ and $\mathcal{K}\subset\mathcal{K}_\varepsilon$.*

PROOF.
(i) If $x\in[0,K-\varepsilon]\cap[K+\varepsilon,X]$, according to the definition, evidently, $D_\varepsilon(x,t)=0$
(ii) If $x\in[K-\varepsilon,K]$

$$\begin{aligned} D_\varepsilon(x,t) &= e^{\beta t}\left(\frac{1}{4\varepsilon^3}(x-K)^4 - \frac{3}{4\varepsilon}(x-K)^2 - \frac{1}{2}(x-K)\right) \\ &= \frac{e^{\beta t}(x-K)}{4}\left(\frac{x-K}{\varepsilon}+1\right)^2\left(\frac{x-K}{\varepsilon}-2\right) \end{aligned}$$

Because $-\varepsilon\le x-K\le 0$, then $D_\varepsilon(x,t)\ge 0$
(iii) If $x\in(K,K+\varepsilon]$

$$\begin{aligned} D_\varepsilon(x,t) &= e^{\beta t}\left(\frac{1}{4\varepsilon^3}(x-K)^4 - \frac{3}{4\varepsilon}(x-K)^2 + \frac{1}{2}(x-K)\right) \\ &= \frac{e^{\beta t}(x-K)}{4}\left(\frac{x-K}{\varepsilon}-1\right)^2\left(\frac{x-K}{\varepsilon}+2\right) \end{aligned}$$

Using $0\le x-K\le\varepsilon$, so $D_\varepsilon(x,t)\ge 0$ on $[K,K+\varepsilon]$. By above (i)-(iii), we obtain $D_\varepsilon(x,t)\ge 0$. By $D_\varepsilon(x,t)\ge 0$, contrasting to $\mathcal{K}=\{v|v\in H^1_{0,w}(I),\ v\le u^*\}$, we have $\mathcal{K}\subset\mathcal{K}_\varepsilon$. □

Lemma 4.2. *$\|\mathcal{A}u^*_\varepsilon(x,t)\|_{L^1(I)}$ is $\varepsilon-$uniformly bound on $[0,T]$, i.e. for any $\varepsilon>0$ and $t\in[0,T]$, $\|\mathcal{A}u^*_\varepsilon(x,t)\|_{L^1(I)}\le C$, where C is a constant, independent of ε.*

PROOF. Using the definition of $u_\varepsilon^*(x,t)$ and the fact that

$$\lim_{\varepsilon\to 0^+}\int_0^X \frac{\partial^2}{\partial x^2}P_\varepsilon(x,t)dx<\infty$$

on $[K-\varepsilon, K+\varepsilon]$, we obtain that $\|\mathcal{A}u_\varepsilon^*(x,t)\|_{L^1(I)}$ is $\varepsilon-$uniformly bound on $[0,T]$. □

4.1 The error estimates for regularized continuous problem

Using $\mathcal{K}_\varepsilon$, we define the regularized problem of Problem3.1 as follows:

Problem 4.1. *Find $u_\varepsilon(t)\in\mathcal{K}_\varepsilon$ such that, for all $v\in\mathcal{K}_\varepsilon$,*

$$\left(-\frac{\partial u_\varepsilon(t)}{\partial t}, v-u_\varepsilon(t)\right)+A(u_\varepsilon(t), v-u_\varepsilon(t);t)\geq (f(t), v-u_\varepsilon(t)). \tag{4.5}$$

a.e. in J, where $A(\cdot,\cdot)$ is the bilinear form defined in (3.11).

We comment that, similar to Problem3.1, Problem 4.1 also has a unique solution because the bilinear form $A(\cdot,\cdot)$ satisfies (3.12)–(3.13). The following lemma shows that the solution u_ε of Problem 4.1 has better regularity than that of Problem3.1.

Theorem 4.1. *If $r(t)\in C(0,T]$, then for any given $\varepsilon>0$ the solution u_ε to Problem 4.1 satisfies*

$$u_\varepsilon\in L^\infty(J;H_w^2(I)),\quad \frac{\partial u_\varepsilon}{\partial t}\in L^2(J;H_{0,w}^1(I))\cap L^\infty(J;L^\infty(I)) \tag{4.6}$$

and

$$\begin{aligned}&\left(\frac{\partial^+ u_\varepsilon(t)}{\partial t}, v-u_\varepsilon(t)\right)-A(u_\varepsilon(t), v-u_\varepsilon(t);t)\\ &\leq -(f(t), v-u_\varepsilon(t)),\qquad \forall v\in\mathcal{K}_\varepsilon,\quad t\in[0,T].\end{aligned}$$

Furthermore, we have the following pointwise relationships:

$$-\frac{\partial^+ u_\varepsilon}{\partial t}=\mathcal{A}u_\varepsilon+f\qquad \text{a.e. on } I_\varepsilon^-(t), \tag{4.7}$$

$$-\frac{\partial^+ u_\varepsilon}{\partial t}=\max(\mathcal{A}u_\varepsilon^*+f,0)\qquad \text{a.e. on } I_\varepsilon^0(t), \tag{4.8}$$

where $I_\varepsilon^-(t)=\{x\in I: u_\varepsilon(x,t)<u_\varepsilon^*(x,t)\}$, $I_\varepsilon^0(t)=\{x\in I: u_\varepsilon(x,t)=u_\varepsilon^*(x,t)\}$, $\partial^+u_\varepsilon/\partial t$ denotes the right-hand derivative of u with respect to t, and $\mathcal{A}$ denotes the operator associated with the bilinear form given in (3.11), i.e.,

$$\mathcal{A}u_\varepsilon=-\frac{\partial}{\partial x}\left[a(t)x^2\frac{\partial u_\varepsilon}{\partial x}+b(t)xu_\varepsilon\right]+c(t)u_\varepsilon.$$

PROOF. Because $u_\varepsilon(x,T)\in H_{0,w}^2(I)$ and $f\in C(J;L^\infty(I))$, $\frac{\partial f}{\partial t}\in L^2(J;L^\infty(I))$, the results of this lemma follow from the conclusions of [6, pp.98-100]. □

The following theorem establishes a computable upper bound for the difference between the solutions to Problem3.1 and Problem4.1.

Theorem 4.2. *Let $u(x,t)$ and $u_\varepsilon(x,t)$ be respectively the solutions to Problem3.1 and Problem4.1, then we have*

$$\|u(x,t)-u_\varepsilon(x,t)\|_{L^\infty(J;L^2(I))}\leq C_1(\varepsilon), \tag{4.9}$$

$$\|u(x,t)-u_\varepsilon(x,t)\|_{L^2(J;H_{0,w}^1(I))}\leq C_2(\varepsilon). \tag{4.10}$$

where $C_1(\varepsilon)$ and $C_2(\varepsilon)$, two computable positive constants, dependent of ε,and

$$C_2(\varepsilon) = \frac{C_1(\varepsilon)}{2C_0},$$
$$(C_1(\varepsilon))^2 = \frac{17}{630}e^{2\beta T}\varepsilon^3 + \left[\frac{17}{630C_0}\beta^2\varepsilon^3 + \frac{2M_0^2}{\beta C_0}\left(\frac{11}{504}\varepsilon^3 + \frac{3}{35}K^2\varepsilon\right)\right](e^{2\beta T} - 1).$$

PROOF. From (4.4) and $u + D_\varepsilon \in \mathcal{K}_\varepsilon$, taking $v = u + D_\varepsilon$ in (4.5), we have

$$\left(-\frac{\partial u_\varepsilon}{\partial t}, u + D_\varepsilon - u_\varepsilon\right) + A(u_\varepsilon, u + D_\varepsilon - u_\varepsilon) \geq (f, u + D_\varepsilon - u_\varepsilon). \tag{4.11}$$

Similarly, we see that $u_\varepsilon - D_\varepsilon \in \mathcal{K}$. Therefore, replacing v in (3.10) with $u_\varepsilon - D_\varepsilon$ we have

$$\left(-\frac{\partial u}{\partial t}, u_\varepsilon - D_\varepsilon - u\right) + A(u, u_\varepsilon - D_\varepsilon - u) \geq (f, u_\varepsilon - D_\varepsilon - u).$$

Adding up (4.11) and the above inequality gives

$$\begin{aligned}&\left(\frac{\partial}{\partial t}(u_\varepsilon - u), u - u_\epsilon\right) - A(u_\varepsilon - u, u - u_\varepsilon)\\ &\leq -\left(\frac{\partial}{\partial t}(u_\varepsilon - u), D_\varepsilon\right) - A(u - u_\varepsilon, D_\varepsilon).\end{aligned}$$

From this we have

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\|u - u_\varepsilon\|_0^2 + C_0\|u - u_\varepsilon\|_{1,w}^2\\ &\leq \left(\frac{\partial}{\partial t}(u - u_\varepsilon), D_\varepsilon\right) - A(u - u_\varepsilon, D_\varepsilon).\end{aligned} \tag{4.12}$$

i.e.

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\|u - u_\varepsilon\|_0^2 + C_0\|u - u_\varepsilon\|_{1,w}^2\\ &\leq \frac{\partial}{\partial t}(u - u_\varepsilon, D_\varepsilon) - (u - u_\varepsilon, \frac{\partial}{\partial t}D_\varepsilon) - A(u - u_\varepsilon, D_\varepsilon).\end{aligned}$$

Integrating from t to T, and using (3.14) and (4.6), we have

$$\begin{aligned}&\frac{1}{2}\|u(t) - u_\varepsilon(t)\|_0^2 + C_0\int_t^T \|u(\tau) - u_\varepsilon(\tau)\|_{1,w}^2 d\tau\\ &\leq \frac{1}{2}\|u(T) - u_\varepsilon(T)\|_0^2 + (u(T) - u_\varepsilon(T), D_\varepsilon(T)) - (u(t) - u_\varepsilon(t), D_\varepsilon(t))\\ &\quad - \int_t^T (u - u_\varepsilon, \frac{\partial}{\partial t}D_\varepsilon)d\tau + M_0\int_t^T \|u - u_\varepsilon\|_{1,w}\|D_\varepsilon\|_{1,w}d\tau\end{aligned}$$

Note that $u(x,T) = u^*(x,T)$ and $u_\varepsilon(x,T) = u_\varepsilon^*(x,T)$. We obtain from the above

$$\begin{aligned}&\frac{1}{2}\|u(t) - u_\varepsilon(t)\|_0^2 + C_0\int_t^T \|u(\tau) - u_\varepsilon(\tau)\|_{1,w}^2 d\tau\\ &\leq -\frac{1}{2}\|D_\varepsilon(T)\|_0^2 + \frac{1}{4}\|u(t) - u_\varepsilon(t)\|_0^2 + \|D_\varepsilon(t)\|_0^2 + \frac{1}{2}C_0\int_t^T \|u(\tau) - u_\varepsilon(\tau)\|_{1,w}^2 d\tau\\ &\quad + \frac{1}{C_0}\int_t^T \|\frac{\partial}{\partial \tau}D_\varepsilon(\tau)\|_0^2 d\tau + \frac{M_0^2}{C_0}\int_t^T \|D_\varepsilon(\tau)\|_{1,w}^2 d\tau\end{aligned}$$

That is

$$\begin{aligned} \|u(t)-u_\varepsilon(t)\|_0^2 &+ 2C_0\int_t^T \|u(\tau)-u_\varepsilon(\tau)\|_{1,w}^2 d\tau \\ &\le 4\|D_\varepsilon(t)\|_0^2 + \frac{4}{C_0}\int_t^T \|\frac{\partial}{\partial\tau}D_\varepsilon(\tau)\|_0^2 d\tau + \frac{4M_0^2}{C_0}\int_t^T \|D_\varepsilon(\tau)\|_{1,w}^2 d\tau \end{aligned} \tag{4.13}$$

In the following, we shall compute $\|D_\varepsilon(t)\|_0^2$, $\int_t^T \|\frac{\partial}{\partial\tau}D_\varepsilon(\tau)\|_0^2 d\tau$ and estimate $\int_t^T \|D_\varepsilon(\tau)\|_{1,w}^2 d\tau$.

From (4.4) and (4.2), it is easy to obtain that

$$\|D_\varepsilon(\tau)\|_0^2 = \frac{17}{2520}e^{2\beta\tau}\varepsilon^3, \qquad \|\frac{\partial}{\partial\tau}D_\varepsilon(\tau)\|^2 = \frac{17}{2520}\beta^2 e^{2\beta\tau}\varepsilon^3$$

Therefore,

$$\begin{aligned} \|D_\varepsilon(\tau)\|_{1,w}^2 &= \int_{K-\varepsilon}^{K+\varepsilon}\left(|D_\varepsilon(x,\tau)|^2 + x^2\left|\frac{\partial}{\partial x}D_\varepsilon(x,\tau)\right|^2\right)dx \\ &= \frac{11}{504}e^{2\beta\tau}\varepsilon^3 + \frac{3}{35}K^2\varepsilon e^{2\beta\tau}. \end{aligned}$$

Hence

$$\begin{aligned} \int_t^T \|D_\varepsilon(\tau)\|_{1,w}^2 d\tau &\le \frac{1}{2\beta}\left[\frac{11}{504}\varepsilon^3 + \frac{3}{35}K^2\varepsilon\right](e^{2\beta T}-1), \\ \int_t^T \|\frac{\partial}{\partial t}D_\varepsilon(\tau)\|_0^2 d\tau &\le \frac{17}{2520}\beta^2\varepsilon^3(e^{2\beta T}-1). \end{aligned}$$

Combining the above estimates and (4.13), we finally have (4.9)and (4.10). □

In addition to the error estimates above, we can prove the following properties of $u_\varepsilon(x,t)$ and $\partial u_\varepsilon(x,t)/\partial t$.

Theorem 4.3. *Let $u_\varepsilon(x,t)$ be the solution to Problem4.1 and $r(t) \in C(0,T]$. Then, $u_\varepsilon(x,t)$ and $\partial u_\varepsilon(x,t)/\partial t$ are ε-uniformly bounded. i.e.*

$$\|u_\varepsilon\|_{L^\infty(J;H_{0,w}^1(I))} \le C, \qquad \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^2(J;H_{0,w}^1(I))} \le C$$

for any $0<\varepsilon\le\varepsilon_0$, where ε_0 and C are positive constants, independent of ε.

PROOF. The proof is similar to Theorem II.4 and Proposition II.1 of [6]. Let C denote a generic positive constant, independent of ε. By Theorem3.3 and the fact that $u_\varepsilon \to u$, weakly in $H_{0,w}^1(I)$, we have proved the first inequality. The proof of the second inequality is stated as follows: using the conclusion of Theorem II.4 of [6], we have

$$\left\|\frac{\partial u_\varepsilon}{\partial t}(t)\right\|_{L^p(I)} \le \|\mathcal{A}u_\varepsilon(T)+f(T)\|_{L^p(I)} + \int_0^T\left\|\frac{\partial f}{\partial t}(s)\right\|_{L^p(I)} ds \quad (1<p<\infty)$$

Owing to $\mathcal{A}u_\varepsilon(T)+f(T)\in L^1(I)$ for any fixed $\varepsilon\in(0,\varepsilon_0]$, taking limit $p\to 1$, we obtain

$$\left\|\frac{\partial u_\varepsilon}{\partial t}(t)\right\|_{L^1(I)} \le \|\mathcal{A}u_\varepsilon(T)+f(T)\|_{L^1(I)} + \int_0^T\left\|\frac{\partial f}{\partial t}(s)\right\|_{L^1(I)} ds. \tag{4.14}$$

Because $\|\mathcal{A}u_\varepsilon(T)\|_{L^1(I)}$, $\|f(T)\|_{L^1(I)}$ and $\int_0^T \left\|\frac{\partial f}{\partial t}(s)\right\|_{L^1(I)} ds$ are ε-uniformly bounded, so $\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^1(I)}$ is ε-uniformly bounded for any $t \in [0,T]$, where $\mathcal{A}$ is the operator defined in Theorem4.1.

The following proof is similar to that of proposition II.1 of [6], we introduce a function F: if $u_\varepsilon \in L^p(I)$, $(1 < p < \infty)$, then there exist linear functional F such that $Fu_\varepsilon \in L^q(I)$, $(1/p + 1/q = 1)$, here F is determined by the following relation

$$(Fu_\varepsilon, u_\varepsilon) = \|Fu_\varepsilon\|^2_{L^q(I)} = \|u_\varepsilon\|^2_{L^p(I)} \tag{4.15}$$

specially, when $p = 2$, $\|Fu_\varepsilon\|^2_{L^2(I)} = \|u_\varepsilon\|^2_{L^2(I)}$. Assuming $\delta t > 0$, on $(\delta t, T)$, we have

$$-\frac{\partial}{\partial t}u_\varepsilon(t) + \frac{\partial}{\partial t}u(t-\delta t) + \mathcal{M}u_\varepsilon(t) - \mathcal{M}u_\varepsilon(t-\delta t) \ni f(t) - f(t-\delta t) \tag{4.16}$$

Where $\mathcal{M}$ is defined similarly for $L^p(I)$ space as that of Theorem3.3. Using Rize–representation theorem for $L^p(I)$ and the coerciveness of $A(.,.;,)$; and then integrating parts over I, we have

$$\begin{aligned}
&(\mathcal{M}u_\varepsilon(t) - \mathcal{M}u_\varepsilon(t-\delta t), F(u_\varepsilon(t) - u_\varepsilon(t-\delta t)))\\
&\geq A(F(u_\varepsilon(t) - u_\varepsilon(t-\delta t)), F(u_\varepsilon(t) - u_\varepsilon(t-\delta t)))\\
&= \|F(u_\varepsilon(t-\delta t) - u_\varepsilon(t))\|_0^2 + \|x\frac{\partial}{\partial x}F(u_\varepsilon(t-\delta t) - u_\varepsilon(t))\|_0^2\\
&\geq \alpha\|u_\varepsilon(t) - u_\varepsilon(t-\delta t)\|^2_{1,w}, \quad (\alpha > 0)
\end{aligned}$$

and

$$\begin{aligned}
&-\left(\frac{\partial}{\partial t}(u_\varepsilon(t) - u_\varepsilon(t-\delta t)), F(u_\varepsilon(t) - u_\varepsilon(t-\delta t))\right)\\
&= -\frac{1}{2}\|u_\varepsilon(t-\delta t) - u_\varepsilon(t)\|^2_{L^p}
\end{aligned}$$

Multiplying (4.16) by $F(u_\varepsilon(t) - u_\varepsilon(t-\delta t))$ and integrating on $(\delta t, T)$, we obtain

$$\begin{aligned}
&\alpha\int_{\delta t}^T \|u_\varepsilon(t) - u_\varepsilon(t-\delta t)\|^2_{1,w}dt\\
&\leq \|u_\varepsilon(T) - u_\varepsilon(T-\delta t)\|^2_{L^p(I)} + \int_{\delta t}^T \|f(t) - f(t-\delta t)\|_{L^p}\|u_\varepsilon(t) - u_\varepsilon(t-\delta t)\|_{L^p}dt
\end{aligned}$$

Dividing both sides of above inequality by $(\delta t)^2$; and then letting $\delta t \to 0$ and using (4.3), we have

$$\alpha\int_0^T \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2_{1,w} dt \leq \frac{1}{2}\|\mathcal{A}u_\varepsilon(T) + f(T)\|^2_{L^p(I)} + \int_0^T \left\|\frac{\partial f}{\partial t}\right\|_{L^p}\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^p} dt$$

i.e.

$$\begin{aligned}
\left\|\tfrac{\partial u_\varepsilon}{\partial t}\right\|^2_{L^2(J;H^1_{0,w}(I))} \leq\ & C\|\mathcal{A}u_\varepsilon(T) + f(T)\|^2_{L^p(I)}\\
&+C\int_0^T \left\|\frac{\partial f(t)}{\partial t}\right\|_{L^p(I)}\left\|\frac{\partial u_\varepsilon(t)}{\partial t}\right\|_{L^p(I)} dt
\end{aligned}$$

Similar to getting (4.14), for any given $\varepsilon \in (0, \varepsilon_0]$, taking limit $p \to 1$, we have

$$\begin{aligned}
\left\|\tfrac{\partial u_\varepsilon}{\partial t}\right\|^2_{L^2(J;H^1_{0,w}(I))} \leq\ & C\|\mathcal{A}u_\varepsilon(T) + f(T)\|^2_{L^1(I)}\\
&+C\int_0^T \left\|\frac{\partial f(t)}{\partial t}\right\|_{L^1(I)}\left\|\frac{\partial u_\varepsilon(t)}{\partial t}\right\|_{L^1(I)} dt
\end{aligned}$$

Using (4.14) and Lemma4.2, for any $\varepsilon \in (0, \varepsilon_0]$, we obtain $\|\frac{\partial u_\varepsilon}{\partial t}\|_{L^2(J,H_0^1(I))}$ is ε-uniformly bound. Therefore, we have finished the proof of the Theorem4.3. $\square$

4.2 The error estimates for regularized discrete problem

In actual computations, the Problem3.1 is first discretized, e.g. by the finite element method. Similar to section4.1, we derive an error upper bound for the difference between the solution to finite element discrete problem and its regularized problem.

Let the interval $I = (0, X)$ be divided into M sub-intervals

$$I_i := (x_i, x_{i+1}),\ i = 0, 1, ..., M - 1.$$

with $0 = x_0 < x_1 < \cdots < x_M = X$ and let $x = K$ be a finite element net node. For each $i = 0, 1, ..., M - 1$, we put

$$h_i = x_{i+1} - x_i, \quad \text{and} \quad h = \max_{0 \le i \le M-1} h_i.$$

Define

$$\begin{aligned} &M_h = \{v \in C([0, X]) : v \text{ is linear on each } I_i,\ v(0) = v(X) = 0\}, \\ &\mathcal{K}_h = M_h \cap \mathcal{K}, \qquad \mathcal{K}_{h,\varepsilon} = M_h \cap \mathcal{K}_\varepsilon. \end{aligned}$$

We have $M_h \subset H_{0,w}^1(I)$. let us first construct a set of standard piecewise linear basis functions for M_h. For $i = 0, 1, ..., M$, we let $\psi_i(x)$ be the piecewise linear basis functions on I defined by

$$\psi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in I_{i-1}, \\ \frac{x - x_{i+1}}{x_i - x_{i+1}}, & x \in I_i, \\ 0 & \text{otherwise} \end{cases} \tag{4.17}$$

with the convention that $I_{-1} = I_{M+1} = \emptyset$. It is easy to show that these basis functions satisfy, for any $i = 0, 1, ..., M - 1$,

$$\sum_{j=i}^{i+1} \psi_j(x) v(x) = v_I(x), \qquad \forall x \in I_i. \tag{4.18}$$

For the time domain, we let $t_n = (N - n)\Delta t$ for $n = 0, 1, ..., N$, where N denote a positive integer and $\Delta t = T/N$. We put $J_n = (t_n, t_{n+1})$ for $n = 0, 1, ..., N - 1$. For any admissible function $w(t)$, we let $w^n = w(t_n)$ for $n = 0, 1, ..., N$. Using this notation and the above partition, we define following finite element problem for finding an approximation to the solution of Problem3.1.

Problem 4.2. *Find a map $u_h(\cdot) : \{t_0, ..., t_N\} \to \mathcal{K}_h$, such that for $n = 0, 1, ..., N-1$*

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v - u_h^{n+1}\right) + A(u_h^{n+1}, v - u_h^{n+1}; t^{n+1}) \tag{4.19}$$

$$\ge (f^{n+1}, v - u_h^{n+1}), \qquad \forall v \in \mathcal{K}_h,$$

$$\|u_h^0 - u^0\|_0 \le Ch \tag{4.20}$$

for a positive constant C, independent of u_h.

Like for the continuous problem, let us replace the Problem4.2 by a sequence of regularized problems

Problem 4.3. *Find a map* $u_{h,\varepsilon}(\cdot) : \{t_0, ..., t_N\} \to \mathcal{K}_{h,\varepsilon}$, *such that for* $n = 0, 1, ..., N-1$

$$\left(\frac{u_{h,\varepsilon}^{n+1} - u_{h,\varepsilon}^{n}}{\Delta t}, v - u_{h,\varepsilon}^{n+1}\right) + A(u_{h,\varepsilon}^{n+1}, v - u_{h,\varepsilon}^{n+1}; t^{n+1}) \geq (f^{n+1}, v - u_{h,\varepsilon}^{n+1}), \qquad \forall v \in \mathcal{K}_{h,\varepsilon}, \tag{4.21}$$

$$\|u_{h,\varepsilon}^{0} - u_{h}^{0}\|_0 \leq Ch \tag{4.22}$$

for a positive constant C*, independent of* u_h *and* ε

The existence and uniqueness of the solution to Problem4.3 for a given $u_{h,\varepsilon}^0$ can be proved using a standard argument. For a detailed discussion, we refer the reader to [9, pp.425]. Similarly, we can obtain the following upper bound of the difference between $u_h(x,t)$ and $u_{h,\varepsilon}(x,t)$.

Theorem 4.4. *Let* $u_h(x,t)$ *and* $u_{h,\varepsilon}(x,t)$ *be respectively the solutions Problem4.2 and Problem4.3. Then we have*

$$\|u_h(x,t) - u_{h,\varepsilon}(x,t)\|_{L^\infty(J;L^2(I))} \leq C_1(\varepsilon), \tag{4.23}$$

$$\|u_h(x,t) - u_{h,\varepsilon}(x,t)\|_{L^2(J;H^1_{0,w}(I))} \leq C_2(\varepsilon). \tag{4.24}$$

where $C_1(\varepsilon)$ and $C_2(\varepsilon)$, two computable positive constants, dependent of ε, and

$$C_2(\varepsilon) = \frac{C_1(\varepsilon)}{2C_0},$$

$$(C_1(\varepsilon))^2 = \frac{17}{630}e^{2\beta T}\varepsilon^3 + \left[\frac{17}{630C_0}\beta^2\varepsilon^3 + \frac{2M_0^2}{\beta C_0}\left(\frac{11}{504}\varepsilon^3 + \frac{3}{35}K^2\varepsilon\right)\right](e^{2\beta T} - 1).$$

After regularizing Problem3.1, $u_\varepsilon^*(x,t) \in H^2_{0,w}(I)$, according to the method of [9], we can derive the following finite element error estimates for regularity Problem4.1

Theorem 4.5. *Let* u_ε *and* $u_{\varepsilon,h}$ *be, respectively, the solutions to Problem4.1 and Problem4.3. If* $\in r(t) \in C(0,T]$*; then, for any fixed* $\varepsilon > 0$*, there exists a constant* $C(\varepsilon) > 0$*, such that*

$$\max_{1\leq n\leq N} \|u_\varepsilon^n - u_{\varepsilon,h}^n\|_0 + \left(\sum_{n=1}^{N} \|u_\varepsilon^n - u_{\varepsilon,h}^n\|^2_{1,w}\Delta t\right)^{1/2} \leq C(\varepsilon)(h + \Delta t^{1/2}). \tag{4.25}$$

where $C(\varepsilon)$ is a constant, dependent of ε.

5. Numerical Experiments

In this section, basing on finite element discrete schemes, we demonstrate the convergence of regularized solution by solving the following model test problem.

Test. American Put Option with parameter: $X = 100$, $T = 1.5$, $r = 0.06$, $\sigma = 0.4$ and $K = 50$. The space and time intervals are respectively $I = (0, 100)$ and $J = (0, 1.5)$.

To solve this problem we divide I and J uniformly into into M and N subintervals, respectively so that $h = 1/M$ and $\Delta t = 1/N$. The mesh points are

$$x_i = ih, \quad i = 0, 1, ..., M \quad \text{and} \quad t_n = (N-n)\Delta t, \quad n = 0, 1, ..., N.$$

For each i and n, we let

$$u_h^n = \sum_{i=1}^{M-1} u_i^n \psi_i(x),$$

ε	$E_1(u_h,\varepsilon)$	$C_1(\varepsilon)$	order in C_1	$E_2(u_h,\varepsilon)$	$C_2(\varepsilon)$	order in C_2
0.1	0.0231	10.9002	——	0.3884	27.2505	——
0.01	0.0023	3.4469	0.5	0.0388	8.6174	0.5
0.001	2.3312e-4	1.0900	0.5	0.0039	2.7250	0.5
0.0001	2.3189e-5	0.3447	0.5	4.0406e-4	0.8617	0.5

TABLE 5.1. Computed errors in the two different norms using various parameter ε when $M = 320$ and $N = 160$.

where ψ_i's are basis functions defined in (4.17). We let

$$d(i,n) = e^{\beta t_n}[(1 - x_i/X)K - \max\{K - x_i, 0\}, \quad f^n = rKe^{\beta t_n}$$

for $i = 0, 1, ..., M$ and $n = 0, 1, ..., N$. Clearly, at each time step Problem4.2 becomes a linear inequality system in the unknown coefficients $\{u_i^{n+1}\}_{i=1}^{M-1}$. To solve these linear inequalities, we use the following project scheme used in [9, pp.433]:

$$\left(\frac{u_h^{n+1/2} - u_h^n}{k}, \psi_j(x)\right) + \left(ax^2\frac{\partial u_h^{n+1/2}}{\partial x} + bxu_h^{n+1/2}, \frac{\partial \psi_j(x)}{\partial x}\right) + (cu_h^{n+1/2}, \psi_j(x))$$
$$= (f_h^{n+1}, \psi_j(x)), \qquad j = 1, 2, ..., M - 1,$$
$$u_i^{n+1} = \min\{u_i^{n+1/2}, d(i, n + 1)\}, \quad i = 1, 2, ..., M - 1.$$

The computed option value V, the derivative $\Delta = \partial V/\partial x$ (which is often used for the so-called Δ-hedging in option trading) and the constraint $V - V^*$ are depicted in Figure 5.1. From these figures we see that the solutions are qualitatively very good with option value V always being not less than the lower bound V^*. The free boundary of the problem is also displayed clearly in the computed Δ.

Similar to the method for solving Problem4.2, we solve Problem4.3. To exhibit the difference with Figure 5.1, we take $\varepsilon = 5$. Correspondingly, option value V, the derivative $\Delta = \partial V/\partial x$ and the constraint $V - V^*$ are depicted in Figure 5.2.

To test the theoretical results of Theorem4.4, we compute the errors in two different discrete norms on a number of different parameter ε under a fixed finite element partition. The two norms are

$$E_1(u_h, \varepsilon) = \max_{1\le n\le N} \|u_h(\cdot, t_n) - u_{h,\varepsilon}^n\|_0$$

and

$$E_2(u_h, \varepsilon) = \left(\sum_{n=1}^{N} \|u_h(\cdot, t_n) - u_{h,\varepsilon}^n\|_{1,w}^2 \Delta t\right)^{1/2}.$$

We use the numerical solution on the uniform mesh with $M = 320$, $N = 160$ as an example, and computed convergence history about ε in the two norms are given in Table 5.1. From the table we see that the rate of convergence in $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are equal to 0.5. Same results can be derived from the expression form of $C_1(\varepsilon)$ and $C_2(\varepsilon)$, this confirms our theoretical results. In addition, there is an evident fact in Table5.1, i.e., the rates of convergence in $E_1(u_h, \varepsilon)$ and $E_2(u_h, \varepsilon)$ are equal to 1. This demonstrates that in actual computation, the rates of convergence of $E_1(u_h, \varepsilon)$ and $E_2(u_h, \varepsilon)$ are greater than those of $C_1(\varepsilon)$ and $C_2(\varepsilon)$.

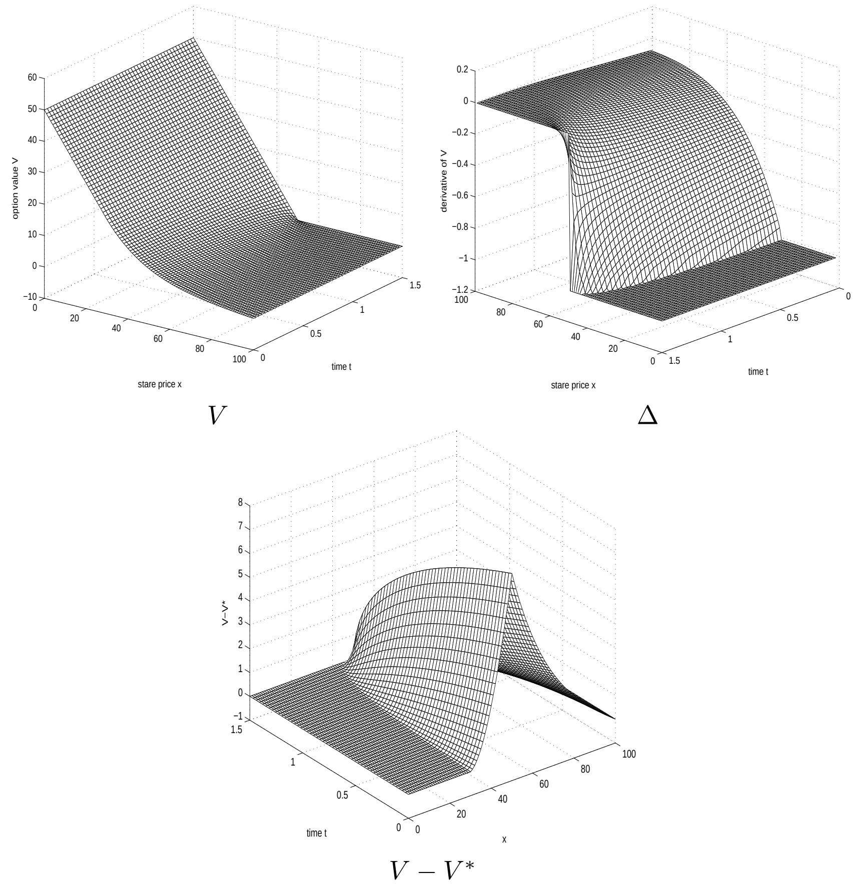

FIGURE 5.1. Computed V, Δ and $V - V^*$ for Problem4.2

To examine the convergence of the solutions to Problem4.2 and Problem4.3, we define the following two norms

$$\|u - u_h\|_{0,h} = \left(\sum_{n=1}^{N+1} \sum_{i=1}^{M+1} |u_i^n - u_h(x_i, t_n)|^2 h \Delta t \right)^{1/2}$$

and

$$\|u_\varepsilon - u_{h,\varepsilon}\|_{1,h} = \max_{1 \le n \le N} \|u_\varepsilon(\cdot, t_n) - u_{h,\varepsilon}^n\|_0 + \left(\sum_{n=1}^{N} \|u_\varepsilon(\cdot, t_n) - u_{h,\varepsilon}^n\|_{1,w}^2 \Delta t \right)^{1/2}.$$

We use the numerical solutions on the uniform mesh with $M = 2560$, $N = 1280$ as the 'exact solutions' of Problem4.2 and Problem4.3($\varepsilon = 0.1$), respectively. Table5.2 and Table5.3 respectively give the convergence history in the two norms of the finite element solutions for the original problem and the regularized problem. From Table5.2 and Table5.3, we see not only the two finite element solutions are convergent, but also the rate of convergence in $\| \cdot \|_{0,h}$ and $\| \cdot \|_{1,h}$ are greater than 0.5. This fact shows that real rate of convergence of the solution to Problem4.3 is greater than that of theoretical result in Threorem4.5. (Note that it is known that

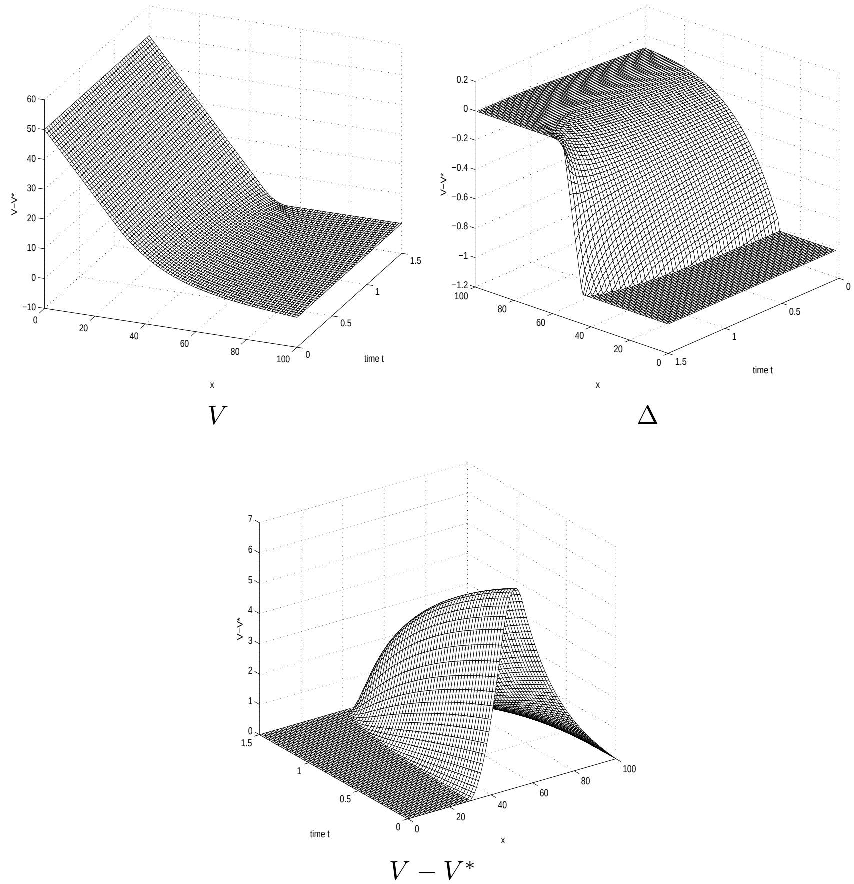

FIGURE 5.2. Computed V, Δ and $V - V^*$ for Problem4.3($\varepsilon = 5$)

M	N	$\|\| \cdot \|\|_{0,h}$	order in $\|\| \cdot \|\|_{0,h}$	$\|\| \cdot \|\|_{1,h}$	order in $\|\| \cdot \|\|_{1,h}$
20	10	2.2069	—	10.7517	—
40	20	1.1797	0.9036	6.4825	0.7299
80	40	0.6194	0.9204	3.8858	0.7383
160	80	0.3161	0.9668	2.2866	0.7650

TABLE 5.2. Computed errors for the solution to Problem4.2(the finite element solution of original problem) .

M	N	$\|\| \cdot \|\|_{0,h}$	order in $\|\| \cdot \|\|_{0,h}$	$\|\| \cdot \|\|_{1,h,\varepsilon}$	order in $\|\| \cdot \|\|_{1,h}$
20	10	2.1662	—	10.5562	—
40	20	1.1480	0.9160	6.3138	0.7415
80	40	0.5921	0.9552	3.7253	0.7612
160	80	0.2977	0.9920	2.1245	0.8102

TABLE 5.3. Computed errors for the solution to Problem4.3 (the finite element solution of the regularized problem, $\varepsilon = 0.1$).

the rate of convergence are normally over-estimates when the mesh approaches to the one used for the 'exact solution'). In addition, we find that the corresponding rates of convergence are greater in Table5.3 than in Table5.2. In contrast to this case, the two errors in Table5.3 are smaller than those in Table5.2, correspondingly. This is because for any fixed $\varepsilon > 0$, $u_\varepsilon(x,t)$ has a better regularity than $u(x,t)$, so the regularization is rewarding.

Acknowledgments

The authors thank the referees for their valuable comments and suggestions.

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