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NUMERICAL METHOD FOR THREE-POINT VECTOR DIFFERENCE SCHEMES ON INFINITE INTERVAL

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Abstract. A three-point vector difference scheme on a infinite interval is considered. Method of reduction of this scheme to a scheme with a finite number of nodes is proposed. Method is based on the extraction of sets of solutions of the difference equation, satisfying the limiting conditions at infinity. The method is applied for numerical solution of an elliptic singularly perturbed problem in a strip. Results of numerical experiments are discussed.

Key Words. difference scheme, infinite interval, transfer of boundary condition, singular perturbation.

1. Introduction

Some physical processes, as pollution transfer or chemical reactions, are often modelled by boundary value problems on unbounded domains. For computer computations one has to construct finite difference schemes with finite number of nodes. Two approaches are in use: transformation of a boundary value problem on an unbounded domain to one on a bounded domain and construction a formal difference scheme in an unbounded domain and then transformation it to a constructive scheme with a finite number of nodes. In this paper, we employ the second approach.

To solve these types of problems various numerical methods are proposed in the literature. We now discuss some results on this topic. Shishkin [11] considers an elliptic problem on the half-plane, where the difference between the solution and its limit condition at infinity is estimated. In the case of small difference, the constructive difference scheme is introduced for large enough finite domain. In Koleva and Vulkov [6], parabolic problems on unbounded domains with nonlinear boundary conditions are investigated. For artificial boundary condition, the integral relation of the solution and its derivatives is proposed. In Zaharov [16], Burger's equation on infinite interval is investigated. The boundary condition at a finite point is formulated as a result of integration of the differential equation from the given point to infinity. For construction of artificial boundary conditions, one can use the method of difference potentials [8].

In this article a three-point vector difference scheme with infinite number of nodes and zero boundary conditions at infinity is considered. It corresponds to approximation of a two-dimensional elliptic equation in an infinite strip with zero

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boundary condition at infinity. The difference scheme has infinite number of nodes and is not suitable for computer realization.

The main goal of the present paper is to develop the method of reduction of difference schemes with infinite number of nodes to constructive difference schemes with a finite number of nodes. It can be done by extracting sets of solutions satisfying the boundary conditions at plus and minus infinity. The extracted set will be given in the form of a two-point difference equation and can be used as a boundary condition for constructing a scheme with finite number of nodes.

The present paper extends [14], [15]. In [14] for vector difference schemes on a semi-infinite interval, the method of extraction of a stable set of solutions is proposed. Scalar difference schemes on an infinite interval are considered in [15]. The case of vector difference schemes on an infinite interval is discussed in this article. We note that for differential equations, the method of extraction of stable set of solutions to carry condition from a singular point was proposed by A.A. Abramov in [1] and has been worked out in many publications (see, for example, [2]).

We shall use the following notation for vector and consistent matrix norms:

$$\|Z\| = \max_j |Z^j|,\ 1 \le j \le N, \quad \|G\| = \max_i \sum_{j=1}^{N} |G_{ij}|.$$

A vector inequality should be considered as a system of componentwise inequalities. According to [13] D is scalar matrix, if D is diagonal matrix with equal diagonal elements.

2. Preliminary Analysis

Consider the original vector scheme

$$L_i U = C_i U_{i-1} - G_i U_i + D_i U_{i+1} = F_i,\ -\infty < i < \infty, \tag{1}$$

$$U_i \to 0 \text{ as } i \to \pm\infty. \tag{2}$$

For each i let U_i and F_i be N-dimensional vectors, C_i, D_i be positive diagonal matrices of order N and G_i be M-matrices ([13], p. 269). We assume that

$$C_i \to C_{+\infty},\ G_i \to G_{+\infty},\ D_i \to D_{+\infty},\ F_i \to 0,\ i \to +\infty, \tag{3a}$$

$$C_i \to C_{-\infty},\ G_i \to G_{-\infty},\ D_i \to D_{-\infty},\ F_i \to 0,\ i \to -\infty; \tag{3b}$$

$$\|G_i^{-1} C_i\| + \|G_i^{-1} D_i\| \le \sigma < 1, \tag{4a}$$

$$Q_i = G_i - C_i - D_i, \quad Q_i^{jj} \ge \sum_{k \ne j} |Q_i^{jk}| + \Delta,\ \Delta > 0, \tag{4b}$$

$$-\infty < i < \infty,\ 1 \le j \le N.$$

Our goal is to transform scheme (1)-(2) to a difference scheme with a finite number of nodes and estimate the accuracy of this operation. Firstly, we shall study the properties of scheme (1)-(2).

According to the next lemma, the inequality (4a) may be a corollary of (4b).

Lemma 1. *Let C_i, D_i be positive scalar matrices. Assume also G_i — M–matrices, the condition (4b) holds. Then there exists such $\sigma < 1$, that (4a) is fulfilled for any i.*

Proof. Suppose that condition (4a) is not valid for every i. Then for any $0 < \sigma < 1$ there exists an index i such that

$$\|G_i^{-1}C_i\| + \|G_i^{-1}D_i\| > \sigma.$$

Let $P_i = G_i^{-1}(C_i + D_i)$. Then

$$\|P_i\| = \|G_i^{-1}C_i\| + \|G_i^{-1}D_i\| > \sigma.$$

where G_i are M-matrices, so $P_i^{jk} \geq 0$ for each j, k. By j_0 we denote

$$\sum_k P_i^{j_0k} = \max_j \sum_k P_i^{jk}.$$

For each j, we have

$$\sum_k (I - P_i)^{jk} \geq \sum_k (I - P_i)^{j_0k}.$$

If $\|P_i\| > \sigma$, then

$$\sum_k (I - P_i)^{j_0k} < 1 - \sigma.$$

For $j = j_0$ from (4b), it follows that

$$\sum_{k=1}^{N} G_i^{j_0k} - C_i^{j_0j_0} - D_i^{j_0j_0} = \sum_k \{G_i - (C_i + D_i)\}^{j_0k} = \sum_k \{G_i(I - P_i)\}^{j_0k} =$$

$$= \sum_k \sum_t G_i^{j_0t}(I - P_i)^{tk} = \sum_t (G_i^{j_0t} \sum_k (I - P_i)^{tk}) =$$

$$\sum_{t \neq j_0} G_i^{j_0t} \left(\sum_k (I - P_i)^{tk} - \sum_k (I - P_i)^{j_0k} \right) + \left(\sum_t G_i^{j_0t} \right) \cdot \left(\sum_k (I - P_i)^{j_0k} \right) \leq$$

$$\leq \left(\sum_t G_i^{j_0t} \right) \cdot \left(\sum_k (I - P_i)^{j_0k} \right) \leq G_i^{j_0j_0} \left(\sum_k (I - P_i)^{j_0k} \right) \leq C(1 - \sigma),$$

where $C = \max_{i,j} G_i^{jj} > 0$. Since $C(1 - \sigma)$ can be made arbitrarily small, we have the contradiction with (4b). ♢

Lemma 2. *The following estimate holds true*

$$\max_i \|U_i\| \leq \Delta^{-1} \max_i \|F_i\|.$$

Proof. It is easy to see, that the maximum principle [10] holds for the operator L. Therefore, from the conditions

$$\lim_{i \to \pm\infty} \Psi_i \geq 0, \quad L_i\Psi \leq 0, \quad -\infty < i < \infty, \tag{5}$$

it follows that $\Psi_i \geq 0$ for any i. Conditions (5) are fulfilled for

$$\Psi_i^j = \Delta^{-1} \max_i \|F_i\| \pm U_i^j,$$

which completes the proof of the lemma. ♢

The uniqueness of the solution of problem (1)–(2) is a corollary of Lemma 2.

3. Construction of stable sets of solutions

By next difference equations we extract the sets of solutions of the difference equation (1), which (it will be shown later) satisfy the boundary conditions at $\pm\infty$:

$$U_i^+ = A_i^+ U_{i-1}^+ + B_i^+, \tag{6a}$$

$$U_i^- = A_i^- U_{i+1}^- + B_i^-, \tag{6b}$$

where $A_i^\pm$ and $B_i^\pm$ are matrices and vectors being solutions of the following problems:

$$A_i^+ = (G_i - D_i A_{i+1}^+)^{-1} C_i, \quad A_i^+ \to A_{+\infty},\ i \to \infty, \tag{7a}$$

$$A_i^- = (G_i - C_i A_{i-1}^-)^{-1} D_i, \quad A_i^- \to A_{-\infty},\ i \to -\infty, \tag{7b}$$

$$D_i B_{i+1}^+ - (G_i - D_i A_{i+1}^+) B_i^+ = F_i, \quad B_i^+ \to 0,\ i \to \infty, \tag{8a}$$

$$C_i B_{i-1}^- - (G_i - C_i A_{i-1}^-) B_i^- = F_i, \quad B_i^- \to 0,\ i \to -\infty. \tag{8b}$$

Matrices $A_{\pm\infty}$ from (7) with $\|A_{\pm\infty}\| < 1$ are the solutions of the quadratic matrix equations

$$D_{+\infty} A^2 - G_{+\infty} A + C_{+\infty} = 0, \tag{9a}$$

$$C_{-\infty} A^2 - G_{-\infty} A + D_{-\infty} = 0. \tag{9b}$$

Lemma 3. *There exist the solutions $A_{+\infty}$, $A_{-\infty}$ to (9a) and (9b), respectively, which satisfy the conditions*

$$\|A_{\pm\infty}\| < \sigma. \tag{10}$$

Proof. We consider only equation (9a). We seek solution $A_{+\infty}$ as the limit of the sequence

$$T_0 = Q, \quad Q^{ij} \geq 0, \quad T_{n+1} = P T_n^2 + Q,$$

where

$$P = G_{+\infty}^{-1} D_{+\infty},\ Q = G_{+\infty}^{-1} C_{+\infty}.$$

Obviously, if $\lim\limits_{n\to\infty} T_n = T_{+\infty}$ exists, then $T_{+\infty}$ is the solution of equation (9a).

It is easily to show that for each (i,j) the sequence $\{T_n^{ij}\}$ is nondecreasing and $T_n^{ij} < \sigma$. Therefore, there exists $\lim\limits_{n\to\infty} T_n = T_{+\infty}$. For each n, $\|T_n\| < \sigma$ and, hence, $\|T_{+\infty}\| \leq \sigma$. The equality $\|T_{+\infty}\| = \sigma$ is not possible since in that case

$$\sigma = \|T_{+\infty}\| = \|P T_{+\infty}^2 + Q\| \leq \|P\| \cdot \sigma^2 + \|Q\| < \sigma.$$

$\diamondsuit$

Introduce the notation

$$p_{\pm\infty} = \|G_{\pm\infty}^{-1} D_{\pm\infty}\|, \quad q_{\pm\infty} = \|G_{\pm\infty}^{-1} C_{\pm\infty}\|.$$

Lemma 4. *The following estimates hold true:*

$$\|A_{+\infty}\| \leq \frac{2q_{+\infty}}{p_{+\infty} + q_{+\infty} + 2(1-\sigma) + |p_{+\infty} - q_{+\infty}|}, \tag{11a}$$

$$\|A_{-\infty}\| \leq \frac{2p_{-\infty}}{p_{-\infty} + q_{-\infty} + 2(1-\sigma) + |p_{-\infty} - q_{-\infty}|}. \tag{11b}$$

Proof. We prove estimate (11a). It follows from (9a) that

$$p_{+\infty}\|A_{+\infty}\|^2 - \|A_{+\infty}\| + q_{+\infty} \geq 0. \tag{12}$$

Since $1 - 4p_{+\infty}q_{+\infty} > 0$, the equation

$$p_{+\infty}x^2 - x + q_{+\infty} = 0$$

has two roots

$$x_{\pm} = \frac{2q_{+\infty}}{1 \pm \sqrt{1 - 4p_{+\infty}q_{+\infty}}}.$$

Taking into account $\|A_{+\infty}\| < \sigma$ and using (12), we get

$$\|A_{+\infty}\| \leq x_+ = \frac{2q_{+\infty}}{1 + \sqrt{1 - 4p_{+\infty}q_{+\infty}}}.$$

The inequality $p_{+\infty} + q_{+\infty} \leq \sigma$ implies (11a). The argument for (11b) can be done in a similar way. ♢

We note that matrices $A_{+\infty}$, $A_{-\infty}$ are non-singular. This result follows from

$$C_{+\infty} = (G_{+\infty} - D_{+\infty}A_{+\infty})A_{+\infty},\ D_{-\infty} = (G_{-\infty} - C_{-\infty}A_{-\infty})A_{-\infty}$$

and from the fact that matrices $C_{+\infty}$, $D_{-\infty}$ are non-singular.

Lemma 5. *For each integer i, the two-sided inequalities*

$$0 < \|A_i^{\pm}\| < \sigma$$

hold true.

Proof. Since $\|A_{+\infty}\| < \sigma$, then the inequality

$$\|A_i^+\| < \sigma$$

holds for all large $i > K$. Consider the case $i \leq K$ and assume that

$$\|A_{i+1}^+\| < \sigma.$$

From (7a),

$$\|A_i^+\| \leq \|(I - G_i^{-1}D_iA_{i+1}^+)^{-1}\| \cdot \|G_i^{-1}C_i\|.$$

From here and applying the inequality ([13], p. 110)

$$\|(I - Z)^{-1}\| \leq \frac{1}{1 - \|Z\|} \text{ for } \|Z\| < 1,$$

we get

$$\|A_i^+\| \leq \frac{\|G_i^{-1}C_i\|}{1 - \|G_i^{-1}D_iA_{i+1}^+\|} < \frac{\|G_i^{-1}C_i\|}{1 - \sigma\|G_i^{-1}D_i\|}.$$

From (4a), it follows that

$$\|A_i^+\| < \sigma.$$

The case of (7b) can be investigated in a similar way. ♢

Non-singularity of the matrices $A_i^{\pm}$ follows from the inequality $\|A_i^{\pm}\| < \sigma$ and equations (7).

Lemma 6. *All solutions to equations (6a), (6b) satisfy the boundary conditions (2) at $\pm\infty$.*

Proof. Let U_i^+ be a solution to (6a). Whenever $i > 0$, we have

$$\|U_i^+\| \le \sigma^i \|U_0^+\| + \sum_{k=1}^{i} \sigma^{i-k} \|B_k\|.$$

Taking into account that $\|B_k\| \to 0$ as $k \to \infty$ and $\sigma < 1$, it can be shown that $\|U_i^+\| \to 0$, $i \to \infty$. In the same way, we conclude that $\|U_i^-\| \to 0$, $i \to -\infty$. ♢

Thus, equations (6a), (6b) give the sets of solutions to (1), which satisfy the boundary conditions at $\pm\infty$.

4. Existence and uniqueness of solutions of auxiliary problems

To use equations (6a), (6b) as the sets of solutions, the coefficients $A_i^\pm$, $B_i^\pm$ must be uniquely defined. Thus, we have to investigate existence and uniqueness of solutions to problems (7), (8). We shall use the results from [3], where difference matrix Riccati equations with infinite number of nodes are investigated.

Let $\lambda^i_{\pm\infty}$, $\nu^j_{+\infty}$ and $\nu^j_{-\infty}$ be the eigenvalues of $A_{\pm\infty}$, $B_{+\infty} = D^{-1}_{+\infty} G_{+\infty} - A_{+\infty}$ and $B_{-\infty} = C^{-1}_{-\infty} G_{-\infty} - A_{-\infty}$, respectively. In order to prove existence and uniqueness of solutions to (7) and (8), we need the following separation conditions for all j

$$|\nu^j_{+\infty}| > 1, \tag{13a}$$

$$|\nu^j_{-\infty}| > 1. \tag{13b}$$

Since all eigenvalues $\lambda^i_{\pm\infty}$ satisfy the inequations

$$|\lambda^i_{\pm\infty}| \le \max\left(\|A_{+\infty}\|, \|A_{-\infty}\|\right) < \sigma < 1,$$

then from (13) we conclude that for all i, j

$$|\lambda^i_{+\infty}| < |\nu^j_{+\infty}|, \tag{14a}$$

$$|\lambda^i_{-\infty}| < |\nu^j_{-\infty}|. \tag{14b}$$

Lemma 7. *If (13a) and (13b) hold true, then there exist unique solutions to (7a), (8a) and (7b), (8b).*

Proof. Firstly, we consider the case of (7a), (8a). We rewrite (7a) in the form

$$(G_i - D_i A^+_{i+1}) A_i^+ = C_i.$$

Denoting $Z_i = A_i^+ - A_{+\infty}$, we have

$$Z_{i+1} A_{+\infty} = (D_i^{-1} G_i - A_{+\infty}) Z_i - Z_{i+1} Z_i + (D_i^{-1} G_i A_{+\infty} - A^2_{+\infty} - D_i^{-1} C_i). \tag{15}$$

According to Theorem 2 from [3], for all large enough $i \ge K$ there exists unique solution of the difference matrix Riccati equation

$$x_{i+1} a_i = b_i x_i + x_{i+1} d_i x_i + g_i,$$

which vanishes at infinity, if the matrices d_i are bounded

$$|d_i^{kl}| \le d, \ i \ge 1, \ 1 \le k, l \le N.$$

The matrices a_i are nonsingular, the limits

$$\lim_{i\to\infty} g_i = 0, \quad \lim_{i\to\infty} a_i = a_{+\infty}, \quad \lim_{i\to\infty} b_i = b_{+\infty},$$

exist and the separation conditions (14a) are fulfilled for the eigenvalues $\lambda^i_{+\infty}$ of $a_{+\infty}$ and the eigenvalues $\nu^j_{+\infty}$ of $b_{+\infty}$.

It is easily to see, that from (15)

$$d_i \equiv -I, \quad \lim_{i\to\infty} a_i = A_{+\infty}, \quad \lim_{i\to\infty} b_i = D_{+\infty}^{-1}G_{+\infty} - A_{+\infty} = B_{+\infty},$$

In view of (9a),

$$\lim_{i\to\infty} g_i = D_{+\infty}^{-1}G_{+\infty}A_{+\infty} - A_{+\infty}^2 - D_{+\infty}^{-1}C_{+\infty} = 0.$$

Since (14a) follows from (13a), then, according to Theorem 2 of [3], the equation (15) has unique solution, which vanishes at infinity. This solution is defined for $i \geq K$. But from (4) and Lemma 5, we conclude that the matrices $G_i - D_iA_{i+1}^+$ are nonsingular for any i and, hence, the solution to (7a) is also defined for $i < K$.

Similarly, equation (8a) can be rewritten in the form

$$B_{i+1}^+ = (D_i^{-1}G_i - A_{i+1}^+)B_i^+ + D_i^{-1}F_i. \tag{16}$$

In this case

$$d_i \equiv 0, \quad \lim_{i\to\infty} a_i = I, \quad \lim_{i\to\infty} b_i = D_{+\infty}^{-1}G_{+\infty} - A_{+\infty} = B_{+\infty}, \quad \lim_{i\to\infty} g_i = 0.$$

According to (13a), equation (16) has unique solution which vanishes at infinity.

The case (7b), (8b) can be considered in a similar way. ♢

The next lemma states some requirements under which the separation conditions (13) are fulfilled.

Lemma 8. *If $C_{+\infty}^{ii} \geq D_{+\infty}^{ii}$, $C_{-\infty}^{ii} \leq D_{-\infty}^{ii}$ $\quad 1 \leq i \leq N$, or matrices $C_{\pm\infty}$, $D_{\pm\infty}$ are scalar, then (13a) and (13b) hold true.*

Proof. We prove (13a). Let $C_{+\infty}^{ii} \geq D_{+\infty}^{ii}$ for all i. In view of (9a)

$$B_{+\infty} = D_{+\infty}^{-1}G_{+\infty} - A_{+\infty}D_{+\infty}^{-1}C_{+\infty}A_{+\infty}^{-1}.$$

Since $B_{+\infty}$ is nonsingular, then for all eigenvalues $\nu_{+\infty}^j$

$$|\nu_{+\infty}^j| \geq \frac{1}{\|B_{+\infty}^{-1}\|} \geq \frac{1}{\|A_{+\infty}\|\cdot\|C_{+\infty}^{-1}D_{+\infty}\|} > \frac{1}{\sigma}\cdot\frac{1}{\|C_{+\infty}^{-1}D_{+\infty}\|}.$$

The inequality $C_{+\infty}^{ii} \geq D_{+\infty}^{ii}$ implies $\|C_{+\infty}^{-1}D_{+\infty}\| \leq 1$ and, hence,

$$|\nu_{+\infty}^j| > \frac{1}{\sigma} > 1.$$

Consider the case when $C_{+\infty}$, $D_{+\infty}$ are scalar. In this case

$$|\nu_{+\infty}^j| \geq \frac{1}{\|B_{+\infty}^{-1}\|} \geq \frac{1}{\|A_{+\infty}\|\cdot\|C_{+\infty}^{-1}D_{+\infty}\|} = \frac{1}{\|A_{+\infty}\|}\cdot\frac{\|C_{+\infty}\|}{\|D_{+\infty}\|}.$$

Since $C_{+\infty}$ and $D_{+\infty}$ are scalar, we have

$$\frac{\|C_{+\infty}\|}{\|D_{+\infty}\|} = \frac{\|G_{+\infty}^{-1}\|\cdot\|C_{+\infty}\|}{\|G_{+\infty}^{-1}\|\cdot\|D_{+\infty}\|} = \frac{\|G_{+\infty}^{-1}C_{+\infty}\|}{\|G_{+\infty}^{-1}D_{+\infty}\|} = \frac{q_{+\infty}}{p_{+\infty}}.$$

According to lemma 4,

$$\|A_{+\infty}\| \leq \frac{2q_{+\infty}}{p_{+\infty} + q_{+\infty} + 2(1-\sigma) + |p_{+\infty} - q_{+\infty}|} < \frac{q_{+\infty}}{p_{+\infty}},$$

and, hence, $|\nu_{+\infty}^j| > 1$. The proof of (13b) can be done in a similar way. ♢

Further on for (7), (8) we will suppose existence and uniqueness of solution.

5. Construction of the scheme with a finite number of nodes

Using equations (6), we introduce the problem:

$$U_i = A_i^+ U_{i-1} + B_i^+, \quad i \geq 1,$$
$$U_0 = (G_0 - C_0 A_{-1}^- - D_0 A_1^+)^{-1}(C_0 B_{-1}^- + D_0 B_1^+ - F_0),$$
$$U_i = A_i^- U_{i+1} + B_i^-, \quad i \leq -1. \tag{17}$$

Since the coefficients $A_i^{\pm}$, $B_i^{\pm}$ are uniquely defined from their problems (7), (8), the matrix $G_0 - C_0 A_{-1}^- - D_0 A_1^+$ is nonsingular and, hence, there exists unique solution of (17). The problem (1), (2), as it was shown before, has at most one solution. From Lemma 6, we can conclude that the solution of (17) satisfies the difference equation (1) and the boundary conditions (2). Hence, the problem (1), (2) has unique solution that coincides with the solution of (17).

Using (6), for some integers $M < 0$, $K > 0$, we introduce the problem with finite number of nodes:

$$L_i U = C_i U_{i-1} - G_i U_i + D_i U_{i+1} = F_i, \quad M < i < K,$$
$$U_M = A_M^- U_{M+1} + B_M^-, \qquad U_K = A_K^+ U_{K-1} + B_K^+, \tag{18}$$

which corresponds to the original problem (1), (2) with an infinite number of nodes. The solution of (18) can be found by using the Gauss elimination method or some other methods. Under conditions (4), problem (18) has unique solution ([9], p. 106). Since the solution of problem (17) satisfies (18), from the uniqueness property, we conclude that the solutions of problems (1), (2) and (18) coincide for all $M \leq i \leq K$.

Thus, the difference scheme (18) with a finite number of nodes is the exact transformation of the difference scheme (1), (2) with an infinite number of nodes.

6. Stability to perturbation of coefficients

Since the coefficients A_M^-, B_M^-, A_K^+, B_K^+ in (7), (8) can be found only approximately, we have to estimate a computational error in the solution of (18). Consider the difference scheme (18) with perturbed coefficients $\tilde{A}_M^-$, $\tilde{B}_M^-$, $\tilde{A}_K^+$, $\tilde{B}_K^+$:

$$L_i \tilde{U} = C_i \tilde{U}_{i-1} - G_i \tilde{U}_i + D_i \tilde{U}_{i+1} = F_i, \quad M < i < K,$$
$$\tilde{U}_M = \tilde{A}_M^- \tilde{U}_{M+1} + \tilde{B}_M^-, \qquad \tilde{U}_K = \tilde{A}_K^+ \tilde{U}_{K-1} + \tilde{B}_K^+. \tag{19}$$

Theorem 1. *Let $\tilde{U}$ be the solution of (19) and*

$$\|\tilde{A}_M^- - A_M^-\| \leq \Delta_1,\ \|\tilde{A}_K^+ - A_K^+\| \leq \Delta_1,\ \|\tilde{A}_M^-\| < 1,\ \|\tilde{A}_K^+\| < 1,$$
$$\|\tilde{B}_M^- - B_M^-\| \leq \Delta_2,\ \|\tilde{B}_K^+ - B_K^+\| \leq \Delta_2.$$

Then

$$\max_i \|\tilde{U}_i - U_i\| \leq \frac{1}{1 - \max(\|\tilde{A}_M^-\|, \|\tilde{A}_K^+\|)} \{\Delta_1(\|U_{M+1}\| + \|U_{K-1}\|) + \Delta_2\}.$$

Proof. For $Z_i = U_i - \tilde{U}_i$, then we have the problem:

$$L_i Z = C_i Z_{i-1} - G_i Z_i + D_i Z_{i+1} = 0, \quad M < i < K,$$
$$Z_M = \tilde{A}_M^- Z_{M+1} + (A_M^- - \tilde{A}_M^-) U_{M+1} + B_M^- - \tilde{B}_M^-,$$
$$Z_K = \tilde{A}_K^+ Z_{K-1} + (A_K^+ - \tilde{A}_K^+) U_{K-1} + B_K^+ - \tilde{B}_K^+.$$

We now define a grid vector-function Ψ :

$$\Psi_i^j = \Delta_1(\|U_{M+1}\| + \|U_{K-1}\|) + \Delta_2 + \max(\|\tilde{A}_M^-\|, \|\tilde{A}_K^+\|) \cdot \max_i \|Z_i\| \pm Z_i^j,$$

such that

$$\Psi_M \geq 0, \quad \Psi_K \geq 0, \quad L_i\Psi \leq 0, \; M \leq i \leq K.$$

From the maximum principle for the difference operator L, it follows that $\Psi_i \geq 0$ for all $M \leq i \leq K$, and we proved the theorem. ♢

Thus, the solution of problem (18) is stable to perturbations of the coefficients in the the boundary conditions.

To obtain stability estimates of solutions to problems (7), (8) with respect to perturbations of the coefficients of the original difference scheme (1), we need the inequalities

$$G_i^{jj} - \sum_{k\neq j} |G_i^{jk}| - D_i^{jj} - D_i^{jj}\|A_{i+1}^+\| \geq \Delta, \; i \geq K > 0, \tag{20a}$$

$$G_i^{jj} - \sum_{k\neq j} |G_i^{jk}| - C_i^{jj} - C_i^{jj}\|A_{i-1}^-\| \geq \Delta, \; i \leq M < 0. \tag{20b}$$

The following two lemmas give conditions for inequalities (20) to be valid.

Lemma 9. *If for $i \geq K$*

$$C_i^{jj} \geq D_i^{jj}, \; 1 \leq j \leq N,$$

then (20a) holds true for $i \geq K$.

If for $i \leq M$

$$D_i^{jj} \geq C_i^{jj}, \; 1 \leq j \leq N.$$

then (20b) holds true for $i \leq M$.

Proof. Since $\|A_l^{\pm}\| < 1$ for each l and taking into account (4), we get the required inequalities. ♢

Lemma 10. *If for $i \geq K$ the scalar matrices C_i, D_i satisfy*

$$\sigma\frac{\|D_i\|}{\|D_{i+1}\|} - \frac{\|C_i\|}{\|C_{i+1}\|} \leq (1-\sigma)\frac{\|C_i\|}{\|D_{i+1}\|}, \tag{21a}$$

then for $i \geq K$ (20a) holds true.

If for $i \leq M$ the scalar matrices C_i, D_i satisfy

$$\sigma\frac{\|C_i\|}{\|C_{i-1}\|} - \frac{\|D_i\|}{\|D_{i-1}\|} \leq (1-\sigma)\frac{\|D_i\|}{\|C_{i-1}\|}, \tag{21b}$$

then for $i \leq M$ (20b) holds true.

Proof. To prove (20a), we show that in the case of the scalar matrices C_i, D_i the following estimate holds:

$$\|A_{i+1}^+\| \leq \frac{\|C_i\|}{\|D_i\|}. \tag{22}$$

In this case the required inequality follows from (4).

We can observe that

$$\|A_{+\infty}\| < \frac{\|C_{+\infty}\|}{\|D_{+\infty}\|}.$$

Thus, (22) is fulfilled for all large enough $i \geq \tilde{K}$. We now check that (22) holds true for $K \leq i < \tilde{K}$. Assume that

$$\|A_{i+2}^+\| \leq \frac{\|C_{i+1}\|}{\|D_{i+1}\|}.$$

By using the inequality

$$\|(I-Z)^{-1}\| \le \frac{1}{1-\|Z\|} \text{ for } \|Z\|<1,$$

condition (4) and taking into account that the matrices C_{i+1}, D_{i+1} are scalar, we get

$$\|A^+_{i+1}\| = \|(I-G^{-1}_{i+1}D_{i+1}A^+_{i+2})^{-1}(G^{-1}_{i+1}C_{i+1})\| \le \frac{\sigma\|C_{i+1}\|}{\|C_{i+1}\|+\|D_{i+1}\|-\sigma\|C_{i+1}\|}.$$

The use of (21a) completes the proof of (20a). The arguments for $i \le M$ can be done in a similar way. ♢

Consider the linear operators:

$$S^+_i Z = D_i(Z_{i+1}-Z_i) - M_i Z_i, \quad S^-_i Z = C_i(Z_{i-1}-Z_i) - M_i Z_i,$$

where matrices M_i are strict diagonally dominant

$$M^{jj}_i(1-\eta) \ge \sum_{k\ne j} |M^{jk}_i|, 0<\eta<1,\ M^{jj}_i \ge \theta > 0, 1 \le j \le N. \tag{23}$$

Lemma 11. *Let conditions (23) be fulfilled and Z_i be an arbitrary grid function. If there exists $\lim\limits_{i\to+\infty} Z_i$, then for $i \ge K$ the estimate*

$$\max_{i\ge K}\|Z_i\| \le \eta^{-1}\{\theta^{-1}\max_{i\ge K}\|S^+_i Z\| + \|\lim_{i\to\infty} Z_i\|\}, K>0, \tag{24a}$$

holds true.

If there exists $\lim\limits_{i\to-\infty} Z_i$, then for $i \le M$ the estimate

$$\max_{i\le M}\|Z_i\| \le \eta^{-1}\{\theta^{-1}\max_{i\le M}\|S^-_i Z\| + \|\lim_{i\to-\infty} Z_i\|\}, M<0, \tag{24b}$$

holds true.

Proof. To prove (24a), we define for an arbitrary j

$$T_i Z^j = D^{jj}_i(Z^j_{i+1}-Z^j_i) - M^{jj}_i Z^j_i (S^+_i Z)^j + \sum_{k\ne j} M^{jk}_i Z^k_i.$$

If for a grid function Ψ the inequalities

$$T_i\Psi \le 0,\ i\ge K, \quad \lim_{i\to\infty}\Psi_i \ge 0 \tag{25}$$

are fulfilled, then from the maximum principle for operator T, we conclude

$$\Psi_i \ge 0,\ i \ge K.$$

For an arbitrary j, the grid function

$$\Psi_i = \theta^{-1}\max_{i\ge K}\|S^+_i Z\| + (1-\eta)\max_{i\ge K}\|Z_i\| + \|\lim_{i\to\infty} Z_i\| \pm Z^j_i.$$

satisfies (25), and, hence, $\Psi_i \ge 0$ for $i \ge K$. This proves the first part of lemma.

The arguments for (24b) can be done in a similar manner. ♢

Lemma 12. *For some positive constant C, independent of Δ from (4b) the estimates*

$$\max_{i\le M}\|B^-_i\| \le \frac{C}{\Delta}\max_{i\le M}\|F_i\|,\ M<0,$$

$$\max_{i\ge K}\|B^+_i\| \le \frac{C}{\Delta}\max_{i\ge K}\|F_i\|,\ K>0,$$

are valid for the solutions of (8).

Proof. Let $i \geq K$. Consider matrices $P_i = G_i - D_i - D_i A^+_{i+1}$. Taking into account (20a), we can show that (23) holds true for P_i if

$$\theta = \min_j P_i^{jj}, \quad \eta = \Delta / \max_j P_i^{jj}.$$

Now the second estimate follows from Lemma 11. ◇

We now prove that the solutions of (7), (8) are stable with respect to perturbations of the coefficients of the difference scheme (1). Consider problem (1), (2) with perturbed coefficients such that for $i \leq M$ and $i \geq K$

$$\|C_i - \tilde{C}_i\| \leq \delta,\ \|G_i - \tilde{G}_i\| \leq \delta,\ \|D_i - \tilde{D}_i\| \leq \delta,\ \|F_i - \tilde{F}_i\| \leq \delta.$$

Theorem 2. *Let the coefficients $\tilde{C}_i$, $\tilde{G}_i$, $\tilde{D}_i$, $\tilde{F}_i$ satisfy the conditions (3), (4) and $\tilde{A}^\pm_i$, $\tilde{B}^\pm_i$ be the solutions of (7), (8) in the case of the perturbed coefficients $\tilde{C}_i$, $\tilde{G}_i$, $\tilde{D}_i$, $\tilde{F}_i$. Suppose also that the conditions of Lemma 9 or Lemma 10 are fulfilled for the coefficients of the original and perturbed problems, respectively. Then for some constant C independent of δ and σ for $i \leq M$, $i \geq K$ the estimates*

$$\|A^\pm_i - \tilde{A}^\pm_i\| \leq \frac{C}{1-\sigma}\delta, \quad \|B^\pm_i - \tilde{B}^\pm_i\| \leq C\delta \tag{26}$$

hold true.

Proof. We start with the first estimate in (26). Let $i \geq K$ and $Z_i = A^+_i - \tilde{A}^+_i$. We can show that

$$Z_i = G_i^{-1} D_i A^+_{i+1} Z_i + G_i^{-1} D_i Z_{i+1} \tilde{A}^+_i + P_i, \tag{27}$$

where

$$P_i = G_i^{-1}(C_i - \tilde{C}_i) + G_i^{-1}(D_i - \tilde{D}_i)\tilde{A}^+_{i+1}\tilde{A}^+_i + G_i^{-1}(\tilde{G}_i - G_i)\tilde{A}^+_i.$$

Thus,

$$\|Z_i\| \leq \|G_i^{-1} D_i A^+_{i+1}\| \cdot \|Z_i\| + \|G_i^{-1} D_i\| \cdot \|Z_{i+1}\| + \|P_i\|.$$

According to Lemmas 9 and 10,

$$\|G_i^{-1} D_i A^+_{i+1}\| \leq \|G_i^{-1} C_i\|.$$

For

$$R = \max_{i \geq K} \|Z_i\|,$$

we have

$$R \leq \max_i (\|G_i^{-1} D_i\| + \|G_i^{-1} C_i\|) \cdot R + \max_i \|P_i\| \leq \sigma R + C\delta.$$

From here, it follows the first estimate in (26).

We now prove the second estimate in (26). Let $i \geq K$ and $Z_i = B^+_i - \tilde{B}^+_i$. Hence,

$$\tilde{S}^+_i Z = \tilde{D}_i(Z_{i+1} - Z_i) - \tilde{P}_i Z_i = \tilde{R}_i,$$

where

$$\tilde{P}_i = \tilde{G}_i - \tilde{D}_i - \tilde{D}_i \tilde{A}^+_{i+1},$$

$$\tilde{R}_i = (\tilde{D}_i - D_i) B^+_{i+1} + (F_i - \tilde{F}_i) + (G_i - \tilde{G}_i) B^+_i$$
$$+ (\tilde{D}_i - D_i) A^+_{i+1} B^+_i + \tilde{D}_i(\tilde{A}^+_{i+1} - A^+_{i+1}) B^+_i.$$

Since conditions (23) are fulfilled for matrices $\tilde{P}_i$ and $\lim\limits_{i \to +\infty} Z_i = 0$, we conclude the required estimate in Lemma 11. The arguments for $i \leq M$ can be done in a similar way. ◇

7. Computation of the coefficients in the boundary conditions

In this section we compute the coefficients in the boundary conditions of the transformed problem (18) with finite number of nodes. Suppose that for large enough $i \geq K$, the coefficients of scheme (1) can be expanded in the following series of i:

$$C_i = \sum_{k=0}^{r} \frac{C_+^{(k)}}{i^k} + O\left(\frac{1}{i}\right)^{r+1}, \quad G_i = \sum_{k=0}^{r} \frac{G_+^{(k)}}{i^k} + O\left(\frac{1}{i}\right)^{r+1},$$

$$D_i = \sum_{k=0}^{r} \frac{D_+^{(k)}}{i^k} + O\left(\frac{1}{i}\right)^{r+1}, \quad F_i = \sum_{k=0}^{r} \frac{F_+^{(k)}}{i^k} + O\left(\frac{1}{i}\right)^{r+1},$$

where $r > 0$, $O(i^{-k})$ stands for matrices or vectors with norms of order $O(i^{-k})$.

We seek the coefficients A_i^+ and B_i^+, $i \geq K$, from (7a), (8a) in the form

$$\tilde{A}_i^+(p) = \sum_{k=0}^{p} \frac{A_+^{(k)}}{i^k}, \quad \tilde{B}_i^+(p) = \sum_{k=0}^{p} \frac{B_+^{(k)}}{i^k}, \quad p \leq r. \tag{28}$$

Substituting these expansions into (7a), (8a), we get the recurrence formulas for $A_+^{(k)}$, $B_+^{(k)}$:

$$\begin{aligned}(G_{+\infty} - D_{+\infty}A_{+\infty})A_+^{(k)} - D_{+\infty}A_+^{(k)}A_{+\infty} = C_+^{(k)} + \\ + \sum_{m=1}^{k-1}\sum_{l=0}^{k-m}\sum_{j=0}^{k-m-l} \gamma_{k-m-l}^{j} D_+^{(l)} A_+^{(j)} A_+^{(m)} - \sum_{j=1}^{k} G_+^{(j)} A_+^{(k-j)} + \\ + (D_{+\infty}\sum_{j=0}^{k-1}\gamma_k^j A_+^{(j)} + \sum_{l=1}^{k}\sum_{j=0}^{k-l}\gamma_{k-l}^{j} D_+^{(l)} A_+^{(j)})A_{+\infty}, \quad k \geq 1, \quad A_+^{(0)} = A_{+\infty},\end{aligned} \tag{29}$$

$$\begin{aligned}(D_{+\infty} + D_{+\infty}A_{+\infty} - G_{+\infty})B_+^{(k)} = F_+^{(k)} - D_{+\infty}\sum_{m=1}^{k-1}\gamma_k^m B_+^{(m)} - \\ - \sum_{l=1}^{k-1}\sum_{m=1}^{k-l}\gamma_{k-l}^{m} D_+^{(l)} B_+^{(m)} + \sum_{l=1}^{k-1} G_+^{(k-l)} B_+^{(l)} - \\ - \sum_{j=1}^{k-1}\sum_{l=0}^{k-j}\sum_{m=0}^{k-j-l}\gamma_{k-j-l}^{m} D_+^{(l)} A_+^{(m)} B_+^{(j)}, \quad k \geq 1, \quad B_+^{(0)} = 0,\end{aligned} \tag{30}$$

where the coefficients γ_l^m are defined in the form

$$\gamma_0^0 = 1,\ \gamma_l^0 = 0, \ \ \gamma_0^1 = 0,\ \gamma_l^1 = (-1)^{l-1},\ l > 0,$$

$$\gamma_l^{m+1} = 0 \text{ for } l \leq m,\ \gamma_{m+l}^{m+1} = \sum_{j=1}^{l}(-1)^{j-1}\gamma_{m+l-j}^{m} \text{ for } l > 0,\ m > 0.$$

At each step k equation (29) has the form

$$BX + XA = F. \tag{31}$$

This is the continuous Silvester equation with unique solution ([5], p. 92), if the matrix spectrums of A and B are separated. According to (14a), the spectrums of $A_{+\infty}$ and $D_{+\infty}^{-1}(G_{+\infty} - D_{+\infty}A_{+\infty})$ are separated, hence, the equation (29) has unique solution.

The matrix $G_{+\infty} - D_{+\infty} - D_{+\infty}A_{+\infty}$ is strict diagonally dominant, which can be shown in a similar way as in Lemma 12 for the matrices P_i. Thus, the equation (30) has unique solution.

Similarly, we write down the coefficient expansions for $i \le M < 0$ in the forms:

$$C_i = \sum_{k=0}^{r} \frac{C_-^{(k)}}{i^k} + O\left(\frac{1}{i}\right)^{r+1}, \quad G_i = \sum_{k=0}^{r} \frac{G_-^{(k)}}{i^k} + O\left(\frac{1}{i}\right)^{r+1},$$

$$D_i = \sum_{k=0}^{r} \frac{D_-^{(k)}}{i^k} + O\left(\frac{1}{i}\right)^{r+1}, \quad F_i = \sum_{k=0}^{r} \frac{F_-^{(k)}}{i^k} + O\left(\frac{1}{i}\right)^{r+1}.$$

If we take the coefficients A_i^- and B_i^- from (7b), (8b) for $i \le M$ in the forms

$$\tilde{A}_i^-(p) = \sum_{k=0}^{p} \frac{A_-^{(k)}}{i^k}, \quad \tilde{B}_i^-(p) = \sum_{k=0}^{p} \frac{B_-^{(k)}}{i^k}, \quad p \le r, \tag{32}$$

then

$$\begin{gathered}(G_{-\infty} - C_{-\infty}A_{-\infty})A_-^{(k)} - C_{-\infty}A_-^{(k)}A_{-\infty} = D_-^{(k)} + \\ + \sum_{m=1}^{k-1}\sum_{l=0}^{k-m}\sum_{j=0}^{k-m-l} \omega_{k-m-l}^{j} C_-^{(l)} A_-^{(j)} A_-^{(m)} - \sum_{j=1}^{k} G_-^{(j)} A_-^{(k-j)} + \\ + (C_{-\infty}\sum_{j=0}^{k-1}\omega_k^j A_-^{(j)} + \sum_{l=1}^{k}\sum_{j=0}^{k-l}\omega_{k-l}^j C_-^{(l)} A_-^{(j)})A_{-\infty}, \quad k \ge 1, \quad A_-^{(0)} = A_{-\infty},\end{gathered} \tag{33}$$

$$\begin{gathered}(C_{-\infty} + C_{-\infty}A_{-\infty} - G_{-\infty})B_-^{(k)} = F_-^{(k)} - C_{-\infty}\sum_{m=1}^{k-1}\omega_k^m B_-^{(m)} - \\ - \sum_{l=1}^{k-1}\sum_{m=1}^{k-l}\omega_{k-l}^m C_-^{(l)} B_-^{(m)} + \sum_{l=1}^{k-1} G_-^{(k-l)} B_-^{(l)} - \\ - \sum_{j=1}^{k-1}\sum_{l=0}^{k-j}\sum_{m=0}^{k-j-l}\omega_{k-j-l}^m C_-^{(l)} A_-^{(m)} B_-^{(j)}, \quad k \ge 1, \quad B_-^{(0)} = 0,\end{gathered} \tag{34}$$

where ω_l^m are defined by

$$\omega_0^0 = 1,\ \omega_l^0 = 0,\ \omega_0^1 = 0,\ \omega_l^1 = 1,\ l > 0,$$

$$\omega_l^{m+1} = 0 \text{ for } l \le m,\ \omega_{m+l}^{m+1} = \sum_{j=1}^{l} \omega_{m+l-j}^m \text{ for } l > 0,\ m > 0.$$

The equations (33), (34) have unique solutions, it can be shown in the same way as for (29), (30).

Thus, if the coefficients $\tilde{A}_K^+$, $\tilde{B}_K^+$, $\tilde{A}_M^-$, $\tilde{B}_M^-$ are found by (28) and (32), then from Theorem 2, for $i \le M$, $i \ge K$ and some constant C, we have

$$\|A_i^\pm - \tilde{A}_i^\pm\|, \|B_i^\pm - \tilde{B}_i^\pm\| \le \delta,\ \delta = \frac{C}{1-\sigma}\max\left(|M|^{-(p+1)}, K^{-(p+1)}\right).$$

From Theorem 1, for $M \le i \le K$ and some constant C_1 we deduce

$$\|U_i - \tilde{U}_i\| \le \frac{C_1}{1-\sigma}\max\left(|M|^{-(p+1)}, K^{-(p+1)}\right). \tag{35}$$

Choosing a number of terms in (28) and (32), one can transform the difference scheme (1), (2) to (18) with the required number of nodes and accuracy.

8. Application to an elliptic problem in a strip

In this section we consider a singular perturbed elliptic equation in a strip. Numerical methods for singular perturbed elliptic equations in a finite domain were investigated in many works. We use known approach for construction of difference scheme in a strip with the property of uniform convergence. Our purpose is application of developed above method to transform formal scheme with infinite number of nodes to the scheme with a finite number of mesh nodes.

Consider a problem:

$$\varepsilon\frac{\partial^2 u}{\partial x^2}+\varepsilon\frac{\partial^2 u}{\partial y^2}-a(x)\frac{\partial u}{\partial x}-b(x)u(x,y)=f(x,y),\ |x|<\infty,\ y\in(0,1), \tag{36}$$

with boundary conditions

$$u(x,0)=\varphi_1(x),\quad u(x,1)=\varphi_2(x),\quad u(x,y)\to 0,\ x\to\pm\infty. \tag{37}$$

Suppose that functions in (36), (37) are smooth enough and

$$\varepsilon>0,\quad a(x)\ge\alpha>0,\quad b(x)\ge\beta>0,$$

$$a(x)\to a_{\pm\infty},\quad b(x)\to b_{\pm\infty},\quad f(x,y)\to 0,\quad \varphi_i(x)\to 0,\quad x\to\pm\infty.$$

According to [12], the solution of problem (36), (37) has parabolic boundary layers along the strip and the derivatives have estimates:

$$\left|\frac{\partial^{j+k}}{\partial x^k\partial y^j}u(x,y)\right|\le C\left[1+\varepsilon^{-j/2}\exp\{-qy\}+\varepsilon^{-j/2}\exp\{q(y-1)\}\right],$$

$$0\le j,k\le 4,\quad q=\left(m\varepsilon^{-1}\right)^{1/2},\ \beta/2<m<\beta, \tag{38}$$

were constant C independent of ε. We use a uniform mesh in the x-direction and the nonuniform Shishkin mesh [12] in the y-direction:

$$y_j=\begin{cases}4\sigma_1 j/N, & 0\le j\le N/4,\\ \sigma_1+2(j-N/4)(1-2\sigma_1)/N, & N/4\le j\le 3N/4,\\ 1-\sigma_1+4(j-3N/4)\sigma_1/N, & 3N/4\le j\le N,\end{cases}$$

$$\sigma_1=\min\{1/4,\ \sqrt{\varepsilon}\ln N\}.$$

Consider the upwind difference scheme on x:

$$\Lambda_{yy}^{i,j}u^h+\varepsilon\frac{u^h_{i+1,j}-2u^h_{i,j}+u^h_{i-1,j}}{h^2}-a_i\frac{u^h_{i,j}-u^h_{i-1,j}}{h}-b_iu^h_{i,j}=f_{i,j},$$

$$u^h_{i,0}=\varphi_1^i,\ u^h_{i,N}=\varphi_2^i,\ \ 0<j<N,\ \ -\infty<i<+\infty,\quad u^h_{i,j}\to 0,\ i\to\pm\infty, \tag{39}$$

where $a_i=a(x_i),\ b_i=b(x_i)$,

$$\Lambda_{yy}^{i,j}u^h=2\varepsilon\frac{h^y_j(u^h_{i,j+1}-u^h_{i,j})-h^y_{j+1}(u^h_{i,j}-u^h_{i,j-1})}{h^y_jh^y_{j+1}(h^y_j+h^y_{j+1})},$$

$$f_{i,j}=f(x_i,y_j),\ \varphi_k^i=\varphi_k(x_i),\ k=1,2,\ h^y_j=y_j-y_{j-1}.$$

Using the derivative estimates (38) and the properties of Shishkin's mesh [12], ([7], p. 64), one can prove

$$\max_{i,j}|u^h_{i,j}-u(x_i,y_j)|\le C\left[\frac{1}{N}\ln^2(N)+h\right], \tag{40}$$

were constant C independents of ε, N, h. If the monotone Samarskii scheme ([10], p. 169), [4])

$$\Lambda_{yy}^{i,j} u^h + \varepsilon_i \frac{u_{i+1,j}^h - 2u_{i,j}^h + u_{i-1,j}^h}{h^2} - a_i \frac{u_{i,j}^h - u_{i-1,j}^h}{h} - b_i u_{i,j}^h = f_{i,j},$$

$$u_{i,0}^h = \varphi_1^i,\ u_{i,N}^h = \varphi_2^i,\quad 0 < j < N,\quad -\infty < i < +\infty,\quad u_{i,j}^h \to 0,\ i \to \pm\infty, \tag{41}$$

where $\varepsilon_i = \varepsilon(1 + (a_i h)(2\varepsilon)^{-1})^{-1}$, is in use, then we can increase the accuracy of the approximation with respect to x:

$$\max_{i,j} \left| u_{i,j}^h - u(x_i, y_j) \right| \le C \left[\frac{1}{N} \ln^2(N) + \frac{h^2}{h + \varepsilon} \right]. \tag{42}$$

The schemes (39) and (41) can be written in the vector form (1), (2). Since the required conditions on the matrices of the schemes are fulfilled, the vector difference schemes can be reduced to a scheme with a finite number of grid nodes (18).

We used Gauss elimination method ([9], p. 106) to solve a three-point vector difference scheme (18). Conditions (4a), (4b), $||A_i^{\pm}|| < \sigma < 1$ are fulfilled, therefore Gauss elimination method is correct and stable ([9], p. 107). Since matrix of the difference scheme is strict diagonally dominant, iterative block Gauss-Seidel method has the property of convergence ([13], p. 259) and may be used too.

Consider the test problem (36), (37) with

$$u(x,y) = \frac{\exp\left(-y/\sqrt{\varepsilon}\right) + \exp\left((y-1)/\sqrt{\varepsilon}\right) + x}{x^2 + 1},\quad a(x) = 1,\quad b(x) = 1 + \frac{1}{(x^2+1)^2}.$$

We compare some approaches for reduction of scheme (39) to a scheme with a finite number of nodes. For this purpose, we rewrite difference scheme (39) with an infinite number of nodes in vector form (1),(2). For numerical experiments we use transformed problem (18) with $M = -K$. For all $|i| \le K$ define vector of error $Z_i = U_i - \tilde{U}_i$, where U is solution of the scheme (1),(2) and $\tilde{U}$ is solution of the scheme (18), when coefficients in boundary conditions are found according to formulas (28) and (32). Solution of problem (1),(2) for $|i| \le K$ can be found with given accuracy, if we use scheme (18) with large enough number of nodes $\tilde{K}$. We used $\tilde{K} = 10000$.

In Table 1, we present $\|Z\|$ for different approaches and different values of K, where $h = 1, N = 10$ and $\varepsilon = 0.01$. First and second approaches correspond to cases, when one introduces artificial condition $U_K = 0$ or $U_K = U_{K-1}$ instead given condition $\lim\limits_{K\to\infty} U_K = 0$. Values of p correspond to boundary conditions (18) with approximate values (28),(32). The numerical results confirm the accuracy estimate (35). Note, that we got similar numerical results for other values of parameter ε.

TABLE 1. Error of boundary condition transfer from infinity depending on approach and number of remained nodes

Approach	$K = 10$	$K = 100$	$K = 1000$
$U_{\pm K} = 0$	0.1	$0.1E-1$	$0.1E-2$
$U_{\pm K} = U_{\pm K \mp 1}$	$0.22E-1$	$0.2E-3$	$0.2E-5$
$p = 0$	$0.99E-1$	$0.1E-1$	$0.1E-2$
$p = 1$	$0.54E-2$	$0.5E-4$	$0.5E-6$
$p = 2$	$0.75E-3$	$0.84E-6$	$0.93E-9$
$p = 3$	$0.29E-3$	$0.38E-7$	$0.23E-9$

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