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STABILIZATION OF NAVIER-STOKES EQUATIONS BY BOUNDARY FEEDBACK

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Abstract. In this paper, we consider the stabilization of steady state solutions to Navier-Stokes equations by boundary feedback control. The feedback control is determined by solving a linear quadratic regulator problem associated with the linearized Navier-Stokes equations. The control is effected through suction and blowing at the boundary. We show that the linear feedback control provides global exponential stabilization of the steady state solutions to the Navier-Stokes equations for arbitrary Reynolds number. This feedback is shown to provide global stability in both L^2 and H^1-norms.

Key Words. Navier-Stokes equation, Riccati equation, stabilization, feedback control, linear quadratic regulator.

1. Introduction

Control of fluid flows for the purpose of achieving some desired objective is crucial to many technological and scientific applications. The invention fast micro devises such as MEMS to actually implement these controls has increased the interest in this area. Control design for fluid dynamical systems is hindered by the intrinsic difficulties caused by the nonlinearity and infinite dimensionality of the Navier-Stokes equations that govern fluid flows. In the recent past, great advances have been made in theoretical and computational analysis of optimal control of fluids, see for e.g. [7, 21, 8, 9, 6, 12, 17, 11, 3, 16].

In this article, we address the stabilization problem for viscous flows modeled by the Navier-Stokes equations which has applications in turbulence and drag reduction. It is well-known that the steady state solutions to Navier-Stokes equations might be unstable for high Reynolds number. Our objective here is to develop a boundary feedback control to stabilize the steady solutions of Navier-Stokes equations in bounded domain. The control is effected through suction and blowing on the boundary and we do not make any distinction in our analysis here as to wall normal blowing and suction [16, 17] or tangential velocity actuation [18] as we did in those computational analysis works. We wish to find a boundary control in feedback form on a part of the boundary so that the corresponding system with this control substituted is globally exponentially stable for arbitrary Reynolds number. Motivated by the Lyapunov stability theory for finite dimensional nonlinear ordinary differential equations, we propose a linear feedback control using the algebraic Riccati equation associated with an infinite time horizon linear quadratic regulator (LQR) problem.

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In order to state our problem more precisely, consider the abstract evolution problem

$$\frac{d\mathbf{y}}{dt} = F(\mathbf{y}, g)\,, \quad \mathbf{y}(0) = \mathbf{y}_0\,, \tag{i}$$

where $g(t)$ is the control and $F : \mathbf{X} \times U \to V$ is a nonlinear mapping. The corresponding steady state problem is

$$F(\bar{\mathbf{y}}, \bar{g}) = 0\,. \tag{ii}$$

Suppose $(\bar{\mathbf{y}}, \bar{g}) \in \mathbf{X} \times U$ is a given steady state solution of (ii). The problem of stabilizing the unsteady solution $\mathbf{y}$ of (i) near the steady state solution $\bar{\mathbf{y}}$ with a prescribed rate $\sigma > 0$ is to find the control $g(t)$ such that the solution $\mathbf{y}(t)$ of the unsteady problem with this $g(t)$ satisfies

$$\|\mathbf{y}(t) - \bar{\mathbf{y}}\| \le ce^{-\sigma t}\,, \quad t \in (0, \infty)\,. \tag{iii}$$

The control $g(t)$ is called feedback if there is an operator $K : \mathbf{X} \to U$ such that $g(t) = K(\mathbf{y})$. The feedback stabilization problem that we consider here can be formulated, for the above abstract evolution equation (i), as follows:
Given a steady-state solution $(\bar{\mathbf{y}}, \bar{g})$ *of (ii), find an operator* $K : \mathbf{X} \to U$ *such that the solution* $\mathbf{y}(t)$ *of the problem*

$$\frac{d\mathbf{y}}{dt} = F(\mathbf{y}, K(\mathbf{y}))\,, \quad \mathbf{y}(0) = \mathbf{y}_0 \tag{iv}$$

satisfies (iii).
Our objectives are to first derive a feedback control using the theory of optimal linear quadratic regulator over an infinite time horizon for the linearized Navier-Stokes equations when the control is on the boundary and to show that the resulting linear feedback control globally stabilizes the nonlinear closed-loop problem in the sense stated above for arbitrary Reynolds number. In particular we will derive stability estimates in both L^2 and H^1-norms.

Other related works that use optimal feedback control theory to flow stabilization can be found in [3, 5]. In [3] robust feedback control is used for stabilization of Navier-Stokes equations by distributed control on the whole domain. In [5], stabilization by boundary control of two dimensional Euler equations for incompressible flow is considered.

The paper is organized as follows. In the rest of this section, we present the notations that we will use and the mathematical preliminaries we will need to present our results in the sequel. In Section 2, we formulate our stabilization problem. In Section 3, we present the feedback control design and study the stability of the nonlinear closed-loop system. The stability analysis is carried out with the help of Lyapunov techniques and Galerkin methods. In Section 4, we conclude the paper.

1.1. Notation and Preliminaries. We introduce the following standard notations over a bounded, connected, open set Ω in $\mathbb{R}^2$ with boundary $\Gamma \in C^2$. Let $\mathbf{n}$ denote the unit normal vector to Γ. For $p \in [1, \infty)$, let $L^p(\Omega)$ denote the measurable real-valued functions v on Ω for which $\int_\Omega |v(\mathbf{x})|^p d\mathbf{x} < \infty$. In addition, let $L^\infty(\Omega)$ denote the measurable real-valued functions that are bounded, or at least essentially bounded. For $v \in L^p(\Omega)$, we may define

$$\|v\|_p \equiv \left(\int_\Omega |v(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}\,, \quad \text{for } 1 \le p < \infty\,,$$

$$\|v\|_\infty \equiv \operatorname{ess\,sup}_{\mathbf{x}\in\Omega} |v(\mathbf{x})|\,.$$

For $p = 2$, $L^2(\Omega)$ is a Hilbert space under the scalar product

$$(u, v) \equiv \int_\Omega u\, v d\mathbf{x}, \quad u, v \in L^2(\Omega)$$

and the norm $\|u\| \equiv \sqrt{(u, u)}$. Let

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega) \text{ for } i = 1, 2 \right\},$$

$$H_0^1(\Omega) = \{ v \in H^1(\Omega) : v|_\Gamma = 0 \}$$

and

$$H^m(\Omega) = \left\{ v \in L^2(\Omega) : \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \in L^2(\Omega),\ \forall \alpha = (\alpha_1, \alpha_2),\ |\alpha| \le m \right\}.$$

Here $m > 0$ is an integer. For the definition of fractional order Sobolev spaces $H^s(\Omega)$ (s non-integer), see [1]. Negative ordered Sobolev spaces $H^{-s}(\Omega)$ ($s > 0$) are defined as the dual space, i.e., $H^{-s}(\Omega) = \left\{ H^s(\Omega) \right\}^*$. Vector-valued counterparts of these spaces are denoted by bold-face symbols, e.g., $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^d$ where $d = 2$. We introduce the solenoidal spaces $\mathbf{H}(\Omega)$ and $\mathbf{V}(\Omega)$ as

$$\mathbf{H}(\Omega) = \{ \mathbf{u} \in \mathbf{L}^2(\Omega) \,:\, \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega,\ \ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma \},$$

$$\mathbf{V}(\Omega) = \{ \mathbf{u} \in \mathbf{H}_0^1(\Omega) \,:\, \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \}.$$

For functions that also depend on time, we introduce the space $L^2(0, T; X)$ that consists of square integrable functions from $[0, T]$ into the space X and which is equipped with the norm

$$\left(\int_0^T \|f\|_X^2 dt \right)^{1/2}.$$

Similarly we introduce the space $C(0, T; X)$ that consists of continuous functions from $[0, T]$ into the space X and which is equipped with the norm

$$\sup_{t \in [0,T]} \|f\|_X .$$

We denote by A the Stokes operator, defined as an isomorphism from $\mathbf{V}$ onto the dual $\mathbf{V}^*$ of $\mathbf{V}$ such that, for $\mathbf{u} \in \mathbf{V}$, $A\mathbf{u}$ is defined by

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} d\mathbf{x} \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V},$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket between $\mathbf{V}^*$ and $\mathbf{V}$. It can be shown that $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}$. The Hodge orthogonal projector of the space $\mathbf{L}^2(\Omega)$ onto the divergence free space $\mathbf{H}(\Omega)$ is denoted by P_H. The Stokes operator is related to P_H by

$$A\mathbf{u} = -P_H(\Delta \mathbf{u}) \qquad \forall \mathbf{u} \in D(A).$$

Define a continuous trilinear form $b(\cdot, \cdot, \cdot)$ on $\mathbf{V}$ by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} d\mathbf{x}.$$

Then, by integration by parts, the following properties hold true

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \qquad \forall \mathbf{u} \in \mathbf{V}, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)$$

and

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \qquad \forall \mathbf{u} \in \mathbf{V},\ \forall \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega).$$

We also define the bilinear mapping B by

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = b(\mathbf{u}, \mathbf{v}, \mathbf{w}).$$

The operator B is related to P_H by

$$B(\mathbf{u},\mathbf{v}) = P_H((\mathbf{u}\cdot\nabla)\mathbf{v}) \qquad \forall \mathbf{u},\mathbf{v}\in\mathbf{V}\,.$$

It is well known that $\mathbf{L}^2(\Omega) = \mathbf{H}\oplus\mathbf{H}^\perp$, where $\mathbf{H}^\perp$ denotes the orthogonal complement of $\mathbf{H}$ and characterized by

$$\mathbf{H}^\perp = \{\mathbf{u}\in\mathbf{L}^2(\Omega) : \mathbf{u} = \nabla p\,,\ \ p\in H^1(\Omega)\}\,.$$

1.2. Auxiliary Results. The estimates developed in this work involve several standard inequalities that we summarize in this section for clarity. The Cauchy-Schwarz inequality $|(u,v)|\le\|u\|\|v\|$; Young's inequality

$$ab \le \frac{\epsilon}{p}a^p + \frac{\epsilon^{-\frac{q}{p}}}{q}b^q\,, \qquad 1<p,q<\infty\,,\ \ \frac{1}{p}+\frac{1}{q}=1\,,\ \ a,b\ge 0\,.$$

Let Ω be any arbitrary two dimensional bounded domain with boundary Γ. If $u=0$ on Γ, then we have the Poincare inequality

$$\|u\| \le \lambda_1^{-1/2}\|\nabla u\| \qquad \forall u\in H_0^1(\Omega)\,,$$

where λ_1 is the smallest positive eigenvalue of

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega\,, \qquad u=0 \quad \text{on } \Gamma\,.$$

In addition, we will use the following generalized Sobolev's inequality [19, 15]:

$$\|\mathbf{w}\|_4^2 \le \frac{1}{\sqrt{2}}\|\mathbf{w}\|\|\nabla\mathbf{w}\|\,, \quad \forall \mathbf{w}\in\mathbf{V},$$

for any arbitrary two dimensional domain Ω.

We also recall the well known Gronwall's lemma:

Lemma 1. *If, for $t_0\le t\le t_1$, $\phi(t)\ge 0$ and $\psi(t)\ge 0$ are continuous functions such that the inequality*

$$\phi(t) \le L_1 + L_2\int_{t_0}^{t_1}\psi(s)\phi(s)ds$$

holds on $t_0\le t\le t_1$, with L_1 and L_2 positive constants, then

$$\phi(t) \le L_1 e^{\left(L_2\int_{t_0}^{t_1}\psi(s)ds\right)}$$

on $t_0\le t\le t_1$.

2. Formulation of Flow Stabilization Problem

We consider a viscous incompressible fluid in $\Omega\subset\mathbb{R}^2$, where Ω is a bounded open domain with boundary $\Gamma = \Gamma_C\cup\Gamma_D$. Let $\mathbf{u}(\mathbf{x},t)$ and $p(\mathbf{x},t)$ denote the velocity and pressure fields, respectively, and $\mathbf{u}_0$ the given initial velocity field. Moreover, let $\mathbf{b}$ denote a specified boundary velocity. The initial boundary value problem associated with the Navier-Stokes equations is then given by

$$\begin{aligned}
\frac{\partial\mathbf{u}}{\partial t} - \frac{1}{Re}\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= 0 \qquad \text{in } \Omega\times(0,\infty)\,,\\
\nabla\cdot\mathbf{u} &= 0 \qquad \text{in } \Omega\times(0,\infty)\,,\\
\mathbf{u} &= \mathbf{b} \qquad \text{on } \Gamma_D\times(0,\infty)\,,\\
\mathbf{u} &= l(t)\mathbf{h}(\mathbf{x}) \qquad \text{on } \Gamma_C\times(0,\infty)\,,\\
\mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega\,.
\end{aligned} \tag{1}$$

These equations are non-dimensional and the only non-dimensional parameter is the Reynolds number defined by $Re = \frac{U_0\ell_0}{\nu}$, where ν, ℓ_0 and U_0 are the kinematic viscosity, characteristic length and characteristic velocity, respectively. The function $l(t)\mathbf{h}(\mathbf{x})$ is the control input and the function $\mathbf{h}(\mathbf{x})$ is a distribution function of control input at Γ_C.

We consider the problem of stabilizing solutions near a given steady state solution $\mathbf{u}_d$ by means of feedback control defined on a part Γ_c of the boundary Γ. The control is effected by suction and blowing as defined in (1).

Suppose we would like to stabilize the steady state solutions $(\mathbf{u}_d, p_d)$ satisfying

$$
\begin{aligned}
-\frac{1}{Re}\Delta \mathbf{u}_d + \mathbf{u}_d \cdot \nabla \mathbf{u}_d + \nabla p_d = 0 \qquad & \text{in } \Omega\,, \\
\nabla \cdot \mathbf{u}_d = 0 \qquad & \text{in } \Omega\,, \\
\mathbf{u}_d = \mathbf{b} \qquad & \text{on } \Gamma_D\,, \\
\mathbf{u}_d = \mathbf{0} \qquad & \text{on } \Gamma_C\,.
\end{aligned} \tag{2}
$$

The task can be formulated as the following infinite time horizon optimal boundary control problem. That is to find $l(t)$, or, rather its time derivative $g(t) = \frac{dl}{dt}$ such that the cost functional

$$
\mathcal{J}(\mathbf{u}, g) = \int_0^\infty \left[\|\mathbf{u} - \mathbf{u}_d\|^2 + \gamma\, |g|^2\right] dt\,,
$$

is minimized. Here $\mathbf{u}_d$ is a smooth desired field, $\gamma > 0$ is some given number and the second term in the cost functional represents the cost of control forcing. We will next rewrite this control problem as one that is amenable to LQR designs and has the control in the right-hand side of the state equation.

We will use the following well-known boundary extension to handle the non-zero boundary conditions, see for e.g. [14, 22, 20].

Proposition 1. *Let the flux distribution satisfy* $\mathbf{h} \in \mathbf{H}^{3/2}(\Omega)$ *and* $\int_{\Gamma_c} \mathbf{h}\cdot\mathbf{n}ds = 0$. *Then* $\forall \delta > 0$, *there exists* $\mathbf{u}_l \in \mathbf{H}^2(\Omega)$ *such that* $\nabla \cdot \mathbf{u}_l = 0$, $\mathbf{u}_l|_{\Gamma_c} = \mathbf{h}$ *and*

$$
|b(\mathbf{z}, \mathbf{u}_l, \mathbf{z})| \le \delta \|\nabla \mathbf{z}\|^2\,, \tag{3}
$$

for all $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ *such that* $\nabla \cdot \mathbf{z} = 0$.

Let us now write the velocity field $\mathbf{u}(\mathbf{x}, t)$ in the Navier-Stokes equations (1) as $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_d(\mathbf{x}) + [\mathbf{w}(\mathbf{x}, t) + l(t)\mathbf{u}_l(\mathbf{x})]$. We will then get the following equations for $\mathbf{w}$ with homogeneous boundary condition

$$
\begin{aligned}
\frac{\partial \mathbf{w}}{\partial t} - \frac{1}{Re}\Delta \mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{w} + (\mathbf{u}_d\cdot\nabla\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{u}_d) + l(t)(\mathbf{w}\cdot\nabla\mathbf{u}_l + \mathbf{u}_l\cdot\nabla\mathbf{w}) \\
+\nabla(p - p_d) = -l(t)(\mathbf{u}_l\cdot\nabla\mathbf{u}_d + \mathbf{u}_d\cdot\nabla\mathbf{u}_l - \tfrac{1}{Re}\Delta\mathbf{u}_l) \\
-l^2(t)\mathbf{u}_l\cdot\nabla\mathbf{u}_l + \mathbf{u}_l\, g(t) \;\; \text{in } \Omega\times(0,\infty)\,, \\
\nabla\cdot\mathbf{w} = 0 \qquad \text{in } \Omega\times(0,\infty)\,, \\
\mathbf{w} = \mathbf{0} \qquad \text{on } \Gamma\times(0,\infty)\,, \\
\mathbf{w}(\mathbf{x}, 0) = \mathbf{u}_0 - l(0)\mathbf{u}_l(\mathbf{x}) \;\; \text{in } \Omega\,.
\end{aligned} \tag{4}
$$

In terms of the new variable $\mathbf{w}$, the cost functional $\mathcal{J}$ takes the form

$$
\mathcal{J}(\mathbf{w}, l, g) = \int_0^\infty \left[\|\mathbf{w}(\mathbf{x}, t) + l(t)\mathbf{u}_l(\mathbf{x})\|^2 + \gamma |g|^2\right] dt\,. \tag{5}
$$

Let us now apply the Hodge orthogonal projection, $P_H : \mathbf{L}^2(\Omega) \to \mathbf{H}(\Omega)$ to the system (4) to get

$$\begin{aligned} \frac{d\mathbf{w}}{dt} + \frac{1}{Re} A\mathbf{w} + B(\mathbf{w},\mathbf{w}) + B_1(\mathbf{w}) + lB_2(\mathbf{w}) \\ = \mathbf{f}_1\, l + \mathbf{f}_2\, l^2 + \mathbf{f}_3\, g\,, \quad t \in (0,\infty), \\ \mathbf{w}(0) = \mathbf{w}_0\,, \end{aligned} \tag{6}$$

where $B_i \in \mathcal{L}(\mathbf{V},\mathbf{H})$, for $i = 1, 2$, are defined by

$$(B_1(\mathbf{w}),\mathbf{v}) = b(\mathbf{w},\mathbf{u}_d,\mathbf{v}) + b(\mathbf{u}_d,\mathbf{w},\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{H}(\Omega)\,,$$

$$(B_2(\mathbf{w}),\mathbf{v}) = b(\mathbf{w},\mathbf{u}_l,\mathbf{v}) + b(\mathbf{u}_l,\mathbf{w},\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{H}(\Omega)\,,$$

$\mathbf{f}_i \in \mathbf{H}(\Omega)$, for $i = 1, 2, 3$, are defined by

$$\mathbf{f}_1 = -P_H(\mathbf{u}_d \cdot \nabla\mathbf{u}_l - \frac{1}{Re}\Delta\mathbf{u}_l + \mathbf{u}_l \cdot \nabla\mathbf{u}_d)\,,$$

$$\mathbf{f}_2 = -P_H(\mathbf{u}_l \cdot \nabla\mathbf{u}_l)\,, \quad \mathbf{f}_3 = P_H(\mathbf{u}_l)\,,$$

and $\mathbf{w}_0 \in \mathbf{H}(\Omega)$ is defined by $\mathbf{w}_0 = P_H(\mathbf{u}_0 - l(0)\mathbf{u}_l(x))$. Setting $\mathbf{y} = \begin{pmatrix} \mathbf{w} \\ l \end{pmatrix}$ in (4) and (5), we obtain the desired infinite time horizon optimal control problem

$$\min_g \left\{ \mathcal{J}(\mathbf{y}_0, g) = \int_0^\infty [(\mathbf{y}, Q\mathbf{y}) + \gamma|g|^2]\, dt \right\}\,, \tag{7}$$

subject to the nonlinear equation

$$\begin{aligned} &\tfrac{d\mathbf{y}}{dt} + \mathcal{A}\mathbf{y} + N(\mathbf{y}) = \mathcal{B}\, g(t)\,, \quad t \in (0,\infty)\,, \\ &\mathbf{y}(0) = \mathbf{y}_0 \in \mathbf{H}, \end{aligned} \tag{8}$$

where

$$\mathcal{A} = \begin{pmatrix} \frac{1}{Re} A & 0 \\ 0 & 1 \end{pmatrix}, \; N(\mathbf{y}) = \begin{pmatrix} B(\mathbf{w},\mathbf{w}) + B_1(\mathbf{w}) + l\, B_2(\mathbf{w}) - \mathbf{f}_1\, l - \mathbf{f}_2\, l^2 \\ -l \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1 & \mathbf{u}_l \\ \mathbf{u}_l & \mathbf{u}_l^2 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} \mathbf{f}_3 \\ 1 \end{pmatrix}.$$

Note that the weighting function Q can be any positive definite self-adjoint operator. Suitable working function space for the problem is the Hilbert space $\mathbf{H} = \mathbf{H}(\Omega) \times \mathbb{R}$. The control set G is defined as

$$G = \left\{ g(t) \in \mathbb{R} : |l(t)| \le \hat{l} \right\}$$

for some $\hat{l} > 0$.

3. Feedback Control and Stability Results

3.1. LQR Feedback Control. The optimal linear quadratic regulator control problem for the infinite dimensional Navier-Stokes system is to minimize

$$\mathcal{J}(\mathbf{y}_0, g) \tag{9}$$

over all controls $g \in L^2(0,\infty; G)$ subject to the linear equation

$$\begin{aligned} &\frac{d\mathbf{y}}{dt} + \mathcal{A}\mathbf{y} = \mathcal{B}\, g(t)\,, \quad t \in (0,\infty)\,, \\ &\mathbf{y}(0) = \mathbf{y}_0\,. \end{aligned} \tag{10}$$

Note that the input operator $\mathcal{B} \in L(G, \mathbf{H})$ and $\mathcal{A} : D(\mathcal{A}) \to \mathbf{H}$ generates an analytic semi-group on $\mathbf{H}$.
Finite Cost Condition: For every $\mathbf{y}_0 \in \mathbf{H}$, there exists $g(t) \in L^2(0,\infty; G)$ such that the cost function defined in (7) is finite.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be the operators defined above. There exists a self-adjoint, nonnegative definite operator $\Pi \in \mathcal{L}(\mathbf{H},\mathbf{H})$ that satisfies the Algebraic Riccati Equation (ARE)*

$$\langle \Pi\mathbf{y}, \mathcal{A}\mathbf{w}\rangle + \langle \mathcal{A}\mathbf{y}, \Pi\mathbf{w}\rangle - \frac{1}{\gamma}\langle \mathcal{B}^*\Pi\mathbf{y}, \mathcal{B}^*\Pi\mathbf{w}\rangle + \langle Q\mathbf{y}, \mathbf{w}\rangle = 0 \,. \tag{11}$$

Moreover,
(i) $\frac{1}{\gamma}\mathcal{B}^*\Pi \in \mathcal{L}(\mathbf{H}, G)$,
(ii) $\mathcal{J}(\mathbf{y}_0, g_{opt}) = \langle \Pi\mathbf{y}_0, \mathbf{y}_0\rangle$.
The LQR problem has a solution of the form

$$g_{opt} = -\frac{1}{\gamma}\mathcal{B}^*\Pi\mathbf{y}_{opt},$$

where $\mathbf{y}_{opt}$ is the corresponding solution to (9) with $g = g_{opt}$.

Proof. We observe that since our problem satisfies the assumption (1.5) found on [13] and the finite cost condition, the results follow from Theorem 2.1 in [13]. □

3.2. Stability Results. By employing the linear feedback control $g_{opt} = -\frac{1}{\gamma}\mathcal{B}^*\Pi\mathbf{y}_{opt}$ in the nonlinear evolution equation (8), we obtain the nonlinear closed-loop system

$$\begin{aligned} &\frac{d\mathbf{y}}{dt} + (\mathcal{A} + \mathcal{B}K)\mathbf{y} + N(\mathbf{y}) = 0 \,, \quad t \in (0,\infty) \\ &\mathbf{y}(0) = \mathbf{y}_0 \,, \end{aligned} \tag{12}$$

where $K : \mathbf{H} \to G$ is a bounded linear operator defined by $K = \frac{1}{\gamma}\mathcal{B}^*\Pi$ and $\mathcal{B}K \in \mathcal{L}(\mathbf{H},\mathbf{H})$.

Definition 1. *A function* $\mathbf{y} = \begin{pmatrix} \mathbf{w} \\ l \end{pmatrix} \in L^2(0,T;\mathbf{V}) \times L^2(0,T;\mathbb{R})$ *is a weak solution of (12) if*

$$(\frac{d\mathbf{y}}{dt}, \boldsymbol{\phi}) + (\mathcal{A}\mathbf{y}, \boldsymbol{\phi}) + (\mathcal{B}K\mathbf{y}, \boldsymbol{\phi}) + (N(\mathbf{y}), \boldsymbol{\phi}) = 0 \tag{13}$$

is satisfied for all $\boldsymbol{\phi} \in \mathbf{V} \times \mathbb{R}$ and $\mathbf{y}(0) = \mathbf{y}_0 = \begin{pmatrix} \mathbf{w}(0) \\ l(0) \end{pmatrix}$.

Let us next choose $\Pi = \beta(\mathcal{B}\mathcal{B}^*)^{-1}$, where $\beta \in \mathbb{R}$ is a positive constant. It is easy to check to see that it is a solution to ARE (11) for a suitable weight function Q. For this choice of Π, the feedback control

$$g = -\frac{\beta}{\gamma}\mathcal{B}^*(\mathcal{B}\mathcal{B}^*)^{-1}\mathbf{y}$$

can be shown to be exponentially stabilizing. More precisely we will prove the following regarding the closed-loop system (12).

Theorem 2. *The followings results hold true for the closed-loop system (12) with* $K = \frac{\beta}{\gamma}\mathcal{B}^*(\mathcal{B}\mathcal{B}^*)^{-1}$.
(i) For arbitrary initial data $\mathbf{y}_0 \in \mathbf{H} \times \mathbb{R}$, there exists a unique weak solution

$\mathbf{y} \in [L^2(0,\infty;\mathbf{V}) \cap L^\infty(0,\infty;\mathbf{H})] \times L^2(0,\infty;\mathbb{R})$ *that satisfies the following stability estimate*

$$\|\mathbf{y}\| \le \|\mathbf{y}_0\| e^{-\sigma t}\,. \tag{14}$$

(ii) For arbitrary initial data $\mathbf{y}_0 \in \mathbf{V} \times \mathbb{R}$*, there exists a unique weak solution* $\mathbf{y} \in [L^2(0,\infty;\mathbf{H}^2 \cap \mathbf{V}) \cap L^\infty(0,\infty;\mathbf{V})] \times L^2(0,\infty;\mathbb{R})$ *that satisfies the following stability estimate*

$$\|\nabla \mathbf{y}\| \le M(\sigma, Re, \|\mathbf{y}_0\|, \|\nabla \mathbf{y}_0\|) e^{-\sigma t}\,, \tag{15}$$

where $\sigma > 0$*.*
The solutions in parts (i)-(ii) depend continuously on the initial data in the L^2- *norm.*

Proof. Let us begin with the proof of the stability estimates followed by the existence and uniqueness results.
Proof of estimate (14):
The weak form equations (13) for $\mathbf{y}$, taking as the test function $\boldsymbol{\phi} = \mathbf{y}$, have the form

$$(\frac{d\mathbf{y}}{dt}, \mathbf{y}) + (\mathcal{A}\mathbf{y}, \mathbf{y}) + (\mathcal{B}K\mathbf{y}, \mathbf{y}) + (N(\mathbf{y}), \mathbf{y}) = 0\,. \tag{16}$$

There are several alternative forms for the time derivative term,

$$(\frac{d\mathbf{y}}{dt}, \mathbf{y}) = \frac{1}{2}\frac{d}{dt}\|\mathbf{y}\|^2 = \|\mathbf{y}\|\frac{d}{dt}\|\mathbf{y}\|\,.$$

So by the definition of the operators $\mathcal{A}$ and A, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{y}\|^2 + \frac{1}{Re}\|\nabla \mathbf{w}\|^2 + l^2 + \frac{\beta}{\gamma}\|\mathbf{y}\|^2 = -(N(\mathbf{y}), \mathbf{y})\,. \tag{17}$$

Using the definition of the nonlinear mapping $N(\mathbf{y})$, it holds that

$$(N\mathbf{y}, \mathbf{y}) = (B(\mathbf{w}, \mathbf{w}), \mathbf{w}) + (B_1(\mathbf{w}), \mathbf{w}) + l(B_2(\mathbf{w}), \mathbf{w}) - l(\mathbf{f}_1, \mathbf{w}) - l^2(\mathbf{f}_2, \mathbf{w}) - l^2\,.$$

Due to the properties of the trilinear form $b(\cdot,\cdot,\cdot)$, we obtain

$$(N(\mathbf{y}), \mathbf{y}) = lb(\mathbf{w}, \mathbf{u}_l, \mathbf{w}) - l(\mathbf{f}_1, \mathbf{w}) - l^2(\mathbf{f}_2, \mathbf{w}) - l^2 + b(\mathbf{w}, \mathbf{u}_d, \mathbf{w})\,, \tag{18}$$

since $b(\mathbf{w}, \mathbf{w}, \mathbf{w}) = b(\mathbf{u}_d, \mathbf{w}, \mathbf{w}) = b(\mathbf{u}_l, \mathbf{w}, \mathbf{w}) = 0$. Let us next estimate the terms in (18). Using the inequality (3), we obtain

$$|b(\mathbf{w}, \mathbf{u}_l, \mathbf{w}) \le \frac{1}{4Re}\|\nabla \mathbf{w}\|^2\,.$$

Now using the well-known generalized Sobolev's inequality,

$$\|\mathbf{w}\|_4^2 \le \frac{1}{\sqrt{2}}\|\mathbf{w}\|\|\nabla \mathbf{w}\|\,, \quad \forall \mathbf{w} \in \mathbf{V},$$

for any arbitrary two dimensional domain Ω, we obtain

$$|b(\mathbf{w}, \mathbf{u}_d, \mathbf{w})| \le \sqrt{3}\|\mathbf{w}\|_4^2\|\nabla \mathbf{u}_d\| \le \sqrt{3/2}\|\mathbf{w}\|\|\nabla \mathbf{w}\|\|\nabla \mathbf{u}_d\|\,.$$

By Cauchy-Schwarz inequality, we obtain

$$|l(\mathbf{f}_1, \mathbf{w}) + l^2(\mathbf{f}_2, \mathbf{w})| \le |l|\|\mathbf{f}_1\|\|\mathbf{w}\| + |l^2|\|\mathbf{f}_2\|\|\mathbf{w}\|\,.$$

Using all these estimates in (18), it holds that

$$\begin{aligned}|(N(\mathbf{y}),\mathbf{y})| &\le \sqrt{3/2}\|\nabla \mathbf{u}_d\|\|\mathbf{w}\|\|\nabla \mathbf{w}\| + \tfrac{1}{4Re}\|\nabla \mathbf{w}\|^2 + |l|\|\mathbf{f}_1\|\|\mathbf{w}\| \\ &+ l^2\|\mathbf{f}_2\|\|\mathbf{w}\| + l^2 \\ &\le 3Re\|\nabla \mathbf{u}_d\|^2\|\mathbf{w}\|^2 + \tfrac{1}{4Re}\|\nabla \mathbf{w}\|^2 + \tfrac{1}{4Re}\|\nabla \mathbf{w}\|^2 + |l|\|\mathbf{f}_1\|\|\mathbf{w}\| \\ &+ l^2\|\mathbf{f}_2\|\|\mathbf{w}\| + l^2\end{aligned}$$

Using this in (17), it holds that

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|\mathbf{y}\|^2 + \tfrac{1}{2Re}\|\nabla \mathbf{w}\|^2 + l^2 + \tfrac{\beta}{\gamma}\|\mathbf{y}\|^2 &\le 3Re\|\nabla \mathbf{u}_d\|^2\|\mathbf{w}\|^2 + |l|\|\mathbf{f}_1\|\|\mathbf{w}\| \\ &+ l^2\|\mathbf{f}_2\|\|\mathbf{w}\| + l^2 \\ &\le 3Re\|\nabla \mathbf{u}_d\|^2\|\mathbf{y}\|^2 + \|\mathbf{f}_1\|\|\mathbf{y}\|^2 \\ &+ \hat{l}\|\mathbf{f}_2\|\|\mathbf{y}\|^2 + l^2\,.\end{aligned} \tag{19}$$

That is

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{y}\|^2 + \frac{1}{2Re}\|\nabla \mathbf{w}\|^2 + \left[\frac{\beta}{\gamma} - (3Re\|\nabla \mathbf{u}_d\|^2 + \|\mathbf{f}_1\| + \hat{l}\|\mathbf{f}_2\|)\right]\|\mathbf{y}\|^2 \le 0\,.$$

Let us choose β large enough and define a constant $\sigma = [\frac{\beta}{\gamma} - (3Re\|\nabla \mathbf{u}_d\|^2 + \|\mathbf{f}_1\| + \hat{l}\|\mathbf{f}_2\|)] > 0$. For this σ, we have

$$\tfrac{1}{2}\tfrac{d}{dt}\|\mathbf{y}\|^2 + \tfrac{1}{2Re}\|\nabla \mathbf{w}\|^2 + \sigma\|\mathbf{y}\|^2 \quad \le 0\,. \tag{20}$$

Dropping $\frac{1}{2Re}\|\nabla \mathbf{w}\|^2$ from the left-side of inequality (20) and integrating it from 0 to t yields

$$\|\mathbf{y}\| \le \|\mathbf{y}_0\| - \sigma\int_0^t \|\mathbf{y}\|ds\,.$$

Finally by Gronwall's lemma, we obtain the desired stability estimate (14) in L^2-norm:

$$\|\mathbf{y}\| \le \|\mathbf{y}_0\|e^{-\sigma t}\,.$$

Proof of estimate (15):
We begin with the derivation of some auxiliary estimates which we will need in the sequel. Dropping $\sigma\|\mathbf{y}\|^2$ from the left-side of equation (20), we obtain the differential inequality

$$\tfrac{1}{2}\tfrac{d}{dt}\|\mathbf{y}\|^2 + \tfrac{1}{2Re}\|\nabla \mathbf{w}\|^2 \quad \le 0\,. \tag{21}$$

Integrating this from 0 to t, we obtain

$$\|\mathbf{y}\|^2 + \tfrac{1}{Re}\textstyle\int_0^t \|\nabla \mathbf{w}\|^2 ds \le \|\mathbf{y}_0\|^2\,.$$

Because of this inequality, it holds that

$$\int_0^t \|\nabla \mathbf{w}\|^2 ds \le Re\|\mathbf{y}_0\|^2\,. \tag{22}$$

Multiplying (21) by $e^{\sigma t}$ and using the stability estimate (14), we obtain

$$\tfrac{d}{dt}(e^{\sigma t}\|\mathbf{y}\|^2) + \tfrac{1}{Re}e^{\sigma t}\|\nabla \mathbf{w}\|^2 \quad \le \sigma e^{\sigma t}\|\mathbf{y}\|^2 \le \sigma e^{-\sigma t}\|\mathbf{y}_0\|^2\,.$$

Integrating this inequality from 0 to t, it holds that

$$e^{\sigma t}\|\mathbf{y}\|^2 + \frac{1}{Re}\int_0^t e^{\sigma s}\|\nabla \mathbf{w}\|^2 ds \le (2 - e^{-\sigma t})\|\mathbf{y}_0\|^2 .$$

Therefore we get the a priori estimate

$$\int_0^t e^{\sigma s}\|\nabla \mathbf{w}\|^2 ds \le Re\|\mathbf{y}_0\|^2 . \tag{23}$$

Inner-product between (12) and $(A\mathbf{w}, 0)^T$ yields

$$(\frac{d\mathbf{w}}{dt}, A\mathbf{w}) + \frac{1}{Re}\|A\mathbf{w}\|^2 + \frac{\beta}{\gamma}(\mathbf{w}, A\mathbf{w}) + (N(\mathbf{y}), (A\mathbf{w}, 0)^T) = 0 .$$

It follows from the identity

$$(\frac{d\mathbf{w}}{dt}, A\mathbf{w}) = \frac{1}{2}\frac{d}{dt}(\mathbf{w}, A\mathbf{w}) = \frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{w}\|^2$$

that

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{w}\|^2 + \frac{1}{Re}\|A\mathbf{w}\|^2 + \frac{\beta}{\gamma}\|\nabla \mathbf{w}\|^2 = -(N(\mathbf{y}), (A\mathbf{w}, 0)^T) . \tag{24}$$

Using the definition of the nonlinear map $N(\cdot)$, we can write the right-hand side as

$$\begin{aligned}(N(\mathbf{y}), (A\mathbf{w}, 0)^T) &= b(\mathbf{w}, \mathbf{w}, A\mathbf{w}) + b(\mathbf{w}, \mathbf{u}_d, A\mathbf{w}) + b(\mathbf{u}_d, \mathbf{w}, A\mathbf{w}) \\ &\quad + l[b(\mathbf{w}, \mathbf{u}_l, A\mathbf{w}) + b(\mathbf{u}_l, \mathbf{w}, A\mathbf{w})] - l(\mathbf{f}_1, A\mathbf{w}) \\ &\quad - l^2(\mathbf{f}_2, A\mathbf{w}) .\end{aligned} \tag{25}$$

Let us estimate the terms in the right-hand side of equation (25). Using Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}|l(\mathbf{f}_1, A\mathbf{w}) + l^2(\mathbf{f}_2, A\mathbf{w})| &\le \|\mathbf{f}_1\||l|\|A\mathbf{w}\| + |l|^2\|\mathbf{f}_2\|\|A\mathbf{w}\| \\ &\le C_0(Re)(|l|^2 + |l|^4) + \tfrac{1}{12Re}\|A\mathbf{w}\|^2 .\end{aligned}$$

For the other terms, we have the following estimates. First we estimate the term

$$|b(\mathbf{w}, \mathbf{w}, A\mathbf{w})| \le \|\mathbf{w}\|_4\|\nabla \mathbf{w}\|_4\|A\mathbf{w}\| .$$

We have the following estimate, see [14]:

$$\|\nabla \mathbf{w}\|_4 \le C\|\mathbf{w}\|^{\frac{1}{2}}\{\|\nabla \mathbf{v}\| + \|D^2\mathbf{w}\|\}^{\frac{1}{2}} , \quad \forall \mathbf{w} \in \mathbf{H}^2(\Omega) .$$

Therefore, by the properties of the Stokes operator, we obtain

$$\|\nabla \mathbf{w}\|_4 \le C\|\nabla \mathbf{w}\|^{\frac{1}{2}}\{\|\nabla \mathbf{w}\| + \|A\mathbf{w}\|\}^{\frac{1}{2}} , \quad \forall \mathbf{w} \in \mathbf{H}^2(\Omega) \cap \mathbf{V} .$$

We thus obtain

$$|b(\mathbf{w}, \mathbf{w}, A\mathbf{w})| \le C\|\mathbf{w}\|^{\frac{1}{2}}\|\nabla \mathbf{w}\|^{\frac{3}{2}}\|A\mathbf{w}\| + C\|\mathbf{w}\|^{\frac{1}{2}}\|\nabla \mathbf{w}\|\|A\mathbf{v}\|^{\frac{3}{2}} .$$

Now, using Young's inequality, we obtain

$$|b(\mathbf{w}, \mathbf{w}, A\mathbf{w})| \le C_1(Re)\|\mathbf{w}\|\|\nabla \mathbf{w}\|^3 + C_2(Re)\|\mathbf{w}\|^2\|\nabla \mathbf{w}\|^4 + \frac{1}{12Re}\|A\mathbf{w}\|^2 .$$

Similarly, we obtain the estimates

$$|b(\mathbf{w}, \mathbf{u}_d, A\mathbf{w})| \le C_3(Re)\|\mathbf{w}\|^2 + \frac{1}{12Re}\|A\mathbf{w}\|^2 .$$

$$|b(\mathbf{w}, \mathbf{u}_l, A\mathbf{w})| \le C_4(Re)\|\mathbf{w}\|^2 + \frac{1}{12Re}\|A\mathbf{w}\|^2 ,$$

$$|b(\mathbf{u}_l,\mathbf{w},A\mathbf{w})| \le C_5(Re)\|\nabla\mathbf{w}\|^2 + \frac{1}{12Re}\|A\mathbf{w}\|^2\,,$$

and

$$|b(\mathbf{u}_d,\mathbf{w},A\mathbf{w})| \le C_6(Re)\|\nabla\mathbf{w}\|^2 + \frac{1}{12Re}\|A\mathbf{w}\|^2\,.$$

Using all these estimates in (25) yields

$$\begin{aligned} |(N(\mathbf{y}),(A\mathbf{w},0)^T)| \quad &\le C_1(Re)\|\mathbf{w}\|\|\nabla\mathbf{w}\|^3 + C_2(Re)\|\mathbf{w}\|^2\|\nabla\mathbf{w}\|^4 \\ &+C_3(Re)\|\mathbf{w}\|^2 + C_4(Re)\|\mathbf{w}\|^2 \\ &+(C_5(Re)+C_6(Re))\|\nabla\mathbf{w}\|^2 \\ &+C_0(Re)\{|l|^2+|l|^4\} + \tfrac{1}{2Re}\|A\mathbf{w}\|^2\,. \end{aligned}$$

With the help of this estimate, it follows from (24) that

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\|\nabla\mathbf{w}\|^2 \quad &+\tfrac{1}{2Re}\|A\mathbf{w}\|^2 + [\tfrac{\beta}{\gamma} - (C_5(Re)+C_6(Re))]\|\nabla\mathbf{w}\|^2 \\ &\le \{C_1(Re)\|\mathbf{w}\|\|\nabla\mathbf{w}\| + C_2(Re)\|\mathbf{w}\|^2\|\nabla\mathbf{w}\|^2\}\|\nabla\mathbf{w}\|^2 \\ &+(C_3(Re)+C_4(Re))\|\mathbf{w}\|^2 + C_0(Re)\{|l|^2+|l|^4\}. \end{aligned}$$

Let us choose β such that $(\frac{\beta}{\gamma} - (C_5(Re)+C_6(Re)) > 0$. Then, we obtain

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\|\nabla\mathbf{w}\|^2 \quad &\le \{C_1(Re)\|\mathbf{w}\|\|\nabla\mathbf{w}\| + C_2(Re)\|\mathbf{w}\|^2\|\nabla\mathbf{w}\|^2\}\|\nabla\mathbf{w}\|^2 \\ &+(C_3(Re)+C_4(Re))\|\mathbf{w}\|^2 + C_0(Re)\{|l|^2+|l|^4\}. \end{aligned}$$

Multiplying this by $e^{\sigma s}$ and integrating from 0 to t yields

$$\begin{aligned} \textstyle\int_0^t e^{\sigma s}\tfrac{1}{2}\tfrac{d}{ds}\|\nabla\mathbf{w}\|^2 ds \quad &\le \textstyle\int_0^t e^{\sigma s}\|\nabla\mathbf{w}\|^2\{C_1(Re)\|\mathbf{w}\|\|\nabla\mathbf{w}\| \\ &+C_2(Re)\|\mathbf{w}\|^2\|\nabla\mathbf{w}\|^2\}ds \\ &+\textstyle\int_0^t e^{\sigma s}(C_3(Re)+C_4(Re))\|\mathbf{w}\|^2 \\ &+e^{\sigma s}C_0(Re)\{|l|^2+|l|^4\}ds\,. \end{aligned} \tag{26}$$

We now estimate the last two terms in the right-hand side of inequality (26). Because of the stability estimate (14), it holds that

$$\begin{aligned} \textstyle\int_0^t e^{\sigma s}(C_3(Re)+C_4(Re))\|\mathbf{w}\|^2 ds \quad &\le \textstyle\int_0^t e^{-\sigma s}(C_3(Re)+C_4(Re))\|\mathbf{y}_0\|^2 ds \\ &= (C_3(Re)+C_4(Re))\tfrac{\|\mathbf{y}_0\|^2}{\sigma}(1-e^{-\sigma t}) \\ &\le (C_3(Re)+C_4(Re))\tfrac{\|\mathbf{y}_0\|^2}{\sigma}\,. \end{aligned}$$

Similarly the right-most term in inequality (26) can be estimated as

$$\begin{aligned}\int_0^t e^{\sigma s} C_0(Re)\{|l|^2+|l|^4\}ds &\le \int_0^t e^{\sigma s} C_0(Re)\{\|\mathbf{y}\|^2+\|\mathbf{y}\|^4\}ds \\ &\le C_0(Re)\int_0^t\{\|\mathbf{y}_0\|^2 e^{-\sigma s}+\|\mathbf{y}_0\|^4 e^{-3\sigma s}\}ds \\ &\le \tfrac{C_0(Re)}{\sigma}[\|\mathbf{y}_0\|^2(1-e^{-\sigma t})+\tfrac{1}{3}\|\mathbf{y}_0\|^4(1-e^{-3\sigma t})] \\ &\le \tfrac{C_0(Re)}{\sigma}\{\|\mathbf{y}_0\|^2+\tfrac{1}{3}\|\mathbf{y}_0\|^4\}\,.\end{aligned}$$

Using these two estimates in (26), we obtain

(27)
$$\begin{aligned}\int_0^t e^{\sigma s}\tfrac{1}{2}\tfrac{d}{ds}\|\nabla\mathbf{w}\|^2 ds &\le \int_0^t e^{\sigma s}\|\nabla\mathbf{w}\|^2\{C_1(Re)\|\mathbf{w}\|\|\nabla\mathbf{w}\| \\ &+C_2(Re)\|\mathbf{w}\|^2\|\nabla\mathbf{w}\|^2\}ds+(C_3(Re)+C_4(Re))\tfrac{\|\mathbf{y}_0\|^2}{\sigma} \\ &+\tfrac{C_0(Re)}{\sigma}\{\|\mathbf{y}_0\|^2+\tfrac{1}{3}\|\mathbf{y}_0\|^4\}\,.\end{aligned}$$

Denoting

$$h(s)\equiv 2(C_1(Re)\|\mathbf{w}\|\|\nabla\mathbf{w}\|+C_2(Re)\|\mathbf{w}\|^2\|\nabla\mathbf{w}\|^2)$$

and

$$h_0(\sigma,Re,\|\mathbf{y}\|_0)\equiv 2((C_3(Re)+C_4(Re))\frac{\|\mathbf{y}_0\|^2}{\sigma}+\frac{C_0(Re)}{\sigma}\{\|\mathbf{y}_0\|^2+\frac{1}{3}\|\mathbf{y}_0\|^4\})\,,$$

we can rewrite equation (27) as

$$\int_0^t e^{\sigma s}\tfrac{d}{ds}\|\nabla\mathbf{w}\|^2 ds \quad \le h_0+\int_0^t e^{\sigma s}\|\nabla\mathbf{w}\|^2 h(s)ds\,. \tag{28}$$

Integrating the identity

$$e^{\sigma t}\frac{d}{dt}\|\mathbf{w}\|^2=\frac{d}{dt}(e^{\sigma t}\|\nabla\mathbf{w}\|^2)-\sigma e^{\sigma t}\|\nabla\mathbf{w}\|^2\,.$$

from 0 to t yields

$$\int_0^t e^{\sigma s}\frac{d}{ds}\|\nabla\mathbf{w}\|^2 ds=e^{\sigma t}\|\nabla\mathbf{w}\|^2-\|\nabla\mathbf{w}_0\|^2-\sigma\int_0^t e^{\sigma s}\|\nabla\mathbf{w}\|^2 ds\,. \tag{29}$$

Combining equations (28) and (29), we get

$$e^{\sigma t}\|\nabla\mathbf{w}\|^2\le h_0+\|\nabla\mathbf{w}_0\|^2+\sigma\int_0^t e^{\sigma s}\|\nabla\mathbf{w}\|^2 ds+\int_0^t e^{\sigma s}\|\nabla\mathbf{w}\|^2 h(s)ds\,. \tag{30}$$

Using the estimate (23) in the last estimate, it holds that

$$e^{\sigma t}\|\nabla\mathbf{w}\|^2\le\left[h_0+\|\nabla\mathbf{w}_0\|^2+\sigma Re\|\mathbf{y}_0\|^2\right]+\int_0^t e^{\sigma s}\|\nabla\mathbf{w}\|^2 h(s)ds\,. \tag{31}$$

This is an integral inequality, so that by Gronwall's lemma, it holds that

$$e^{\sigma t}\|\nabla\mathbf{w}\|^2\le\left[h_0+\|\nabla\mathbf{w}_0\|^2+\sigma Re\|\mathbf{y}_0\|^2\right]e^{\int_0^t h(s)ds}\,. \tag{32}$$

Let us next estimate $\int_0^t h(s)ds$. By the definition of $h(s)$, we obtain with the help Poincare inequality that

$$\int_0^t h(s)ds=2\int_0^t(C_1(Re)\lambda_1^{-1/2}+C_2(Re)\|\mathbf{w}\|^2)\|\nabla\mathbf{w}\|^2\,ds$$

Using the estimate (14), we obtain

$$\begin{aligned}\int_0^t h(s)ds &\le 2\int_0^t (C_1(Re)\lambda_1^{-1/2} + C_2(Re)\|\mathbf{y}_0\|^2)\|\nabla \mathbf{w}\|^2 ds \\ &= 2(C_1(Re)\lambda_1^{-1/2} + C_2(Re)\|\mathbf{y}_0\|^2)\int_0^t \|\nabla \mathbf{w}\|^2 ds\,. \end{aligned} \tag{33}$$

Now applying the estimate (22), in the last equality we obtain

$$\int_0^t h(s)ds \le h_1(\lambda_1, Re, \|\mathbf{y}_0\|)\,, \tag{34}$$

where

$$h_1(\lambda_1, Re, \|\mathbf{y}_0\|) \equiv 2Re(C_1(Re)\lambda_1^{-1/2} + C_2(Re)\|\mathbf{y}_0\|^2)\|\mathbf{y}_0\|^2\,.$$

Employing the estimate (34) in (32), we obtain

$$e^{\sigma t}\|\nabla \mathbf{w}\|^2 \le \left[h_0(\sigma, Re, \|\mathbf{y}_0\|) + \|\nabla \mathbf{w}_0\|^2 + \sigma Re\|\mathbf{y}_0\|^2\right] e^{h_1(\lambda_1, Re, \|\mathbf{y}_0\|)}\,.$$

Therefore we obtain the desired estimate (15)

$$\|\nabla \mathbf{w}\|^2 \le e^{-\sigma t}\left[h_0(\sigma, Re, \|\mathbf{y}_0\|) + \|\nabla \mathbf{w}_0\|^2 + \sigma Re\|\mathbf{y}_0\|^2\right] e^{h_1(\lambda_1, Re, \|\mathbf{y}_0\|)}\,.$$

or by setting

$$\begin{aligned} M(\sigma, \lambda_1, Re, \|\mathbf{y}_0\|, \|\nabla \mathbf{y}_0\|) &\equiv [h_0(\sigma, Re, \|\mathbf{y}_0\|) + \|\nabla \mathbf{w}_0\|^2 \\ &\quad + \sigma Re\|\mathbf{y}_0\|^2] e^{h_1(\lambda_1, Re, \|\mathbf{y}_0\|)} \end{aligned}$$

that

$$\|\nabla \mathbf{w}\|^2 \le M(\sigma, \lambda_1, Re, \|\mathbf{y}_0\|, \|\nabla \mathbf{y}_0\|)e^{-\sigma t}\,.$$

Existence: We prove the existence of a weak solution by invoking the method of Galerkin to approximate (13) by a finite dimensional problem. Let $\{\boldsymbol{\psi}_i\}_{i=1}^{\infty}$ be a complete orthonormal set of eigenfunctions of $\mathcal{A}$, i.e., the set $\{\boldsymbol{\psi}_i\}_{i=1}^{\infty}$ forms a Riesz basis in $D(\mathcal{A})$. Let $\mathbf{Y}^l$ denote the finite dimensional space spanned by $\{\boldsymbol{\psi}_i\}_{i=1}^{l}$. We now seek an approximate solution of (13) in the form

$$\mathbf{y}^l = \sum_{j=1}^{l} \alpha_j^l(t)\boldsymbol{\psi}_j\,.$$

We require $\mathbf{y}^l$ to satisfy (13) restricted to $\mathbf{Y}^l$, i.e.

$$\begin{aligned} &(\tfrac{d\mathbf{y}^l}{dt}, \boldsymbol{\phi}) + (\mathcal{A}\mathbf{y}^l, \boldsymbol{\phi}) + (\mathcal{B}K\mathbf{y}^l, \boldsymbol{\phi}) + (N(\mathbf{y}^l), \boldsymbol{\phi}) = 0\,, \quad \forall \boldsymbol{\phi} \in \mathbf{Y}^l \\ &\mathbf{y}^l(0) = \mathbf{y}_0^l\,, \end{aligned} \tag{35}$$

where $\mathbf{y}_0^l$ denotes the projection of $\mathbf{y}_0$ on $\mathbf{Y}^l$, i.e., $\mathbf{y}_0^l = \sum_{j=1}^{l} \mathbf{y}_{0j}\boldsymbol{\psi}_j$ for $\mathbf{y}_0 = \sum_{j=1}^{\infty} \mathbf{y}_{0j}\boldsymbol{\psi}_j$. Taking $k \le l$ and $\boldsymbol{\phi} = \boldsymbol{\psi}_k$ in (35), we obtain

$$\frac{d\alpha_k^l}{dt}(t) + \sum_{i=1}^{l} a_{ki}\alpha_i^l(t) + \frac{\beta}{\gamma}\alpha_k^l(t) + \sum_{i,j=1}^{l} a_{k,i,j}\alpha_i^l(t)\alpha_j^l(t) = 0\,, \quad k = 1, \ldots, l,$$

$$\alpha_k^l(0) = y_{0k}\,.$$

This is a nonlinear system of ordinary differential equations for the functions $\{\alpha_k^l(t)\}_{k=1}^{l}$ and by the standard existence theory, there exists a unique solution that exists on some time interval $[0, T_l)$. In order to show that we can take $T = T_l$ and that we can let $l \to \infty$, we need to show the boundedness of $\alpha_k^l(t)$. Replacing

$\boldsymbol{\phi}$ by $\mathbf{y}^l$ in (35) and arguing as we did for the derivation of the a priori estimates (14)-(15), we obtain

$$\|\mathbf{y}^l(t)\| \le \|\mathbf{y}_0^l\| e^{-\sigma t} \le \|\mathbf{y}_0\| e^{-\sigma t}\,, \tag{36}$$

and

$$\int_0^T e^{\sigma s}\|\nabla \mathbf{y}^l(s)\|^2 ds \le M\|\mathbf{y}_0^l\| \le M\|\mathbf{y}_0\|$$

for $t \in [0,T]$. The next step in showing the existence is to show that a subsequence of approximating solutions $\{\mathbf{y}^l\}_{l=1}^{\infty}$ converges to $\mathbf{y}$ as $l \to \infty$. As a consequence of the a priori estimates (36), $\{\mathbf{y}^l\}_{l=1}^{\infty}$ is bounded in $L^2(0,T;\mathbf{V})$ and $L^\infty(0,T;\mathbf{H})$. Therefore there exists a subsequence $\{\mathbf{y}^l\}_{l=1}^{\infty}$ such that $\mathbf{y}^l$ converges to $\mathbf{y} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H})$, albeit weakly in $L^2(0,T;\mathbf{V})$ and weak* in $L^\infty(0,T;\mathbf{H})$. But by compactness [14], we can obtain a subsequence that converges to $\mathbf{y}$ strongly in $L^2(0,T;\mathbf{H})$. We can use these convergence properties to prove that the limiting function $\mathbf{y}$ is indeed a weak solution of (12). In fact we can show using the standard arguments in the theory of Navier-Stokes equations that each term of equation

$$(\tfrac{d\mathbf{y}^l}{dt}, \boldsymbol{\phi}) + (\mathcal{A}\mathbf{y}^l, \boldsymbol{\phi}) + (\mathcal{B}K\mathbf{y}^l, \boldsymbol{\phi}) + (N(\mathbf{y}^l), \boldsymbol{\phi}) = 0\,, \tag{37}$$

converges to the corresponding term of equation

$$(\tfrac{d\mathbf{y}}{dt}, \boldsymbol{\phi}) + (\mathcal{A}\mathbf{y}, \boldsymbol{\phi}) + (\mathcal{B}K\mathbf{y}, \boldsymbol{\phi}) + (N(\mathbf{y}), \boldsymbol{\phi}) = 0\,, \tag{38}$$

as $l \to \infty$, except for the term $\frac{\beta}{\gamma}(\mathbf{y}^l, \boldsymbol{\phi})$. But by convergence of $\mathbf{y}^l$ to $\mathbf{y}$ in $L^2(0,T;\mathbf{H})$ we have that

$$\frac{\beta}{\gamma}\int_0^T (\mathbf{y}^l, \boldsymbol{\phi})dt \to \frac{\beta}{\gamma}\int_0^T (\mathbf{y}, \boldsymbol{\phi})dt$$

as $l \to \infty$. This proves the existence of a weak solution to (12).

Continuous Dependence and Uniqueness: Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two solutions of (13) corresponding to initial conditions $\mathbf{y}_1^0$ and $\mathbf{y}_2^0$, respectively. Let $\widetilde{\mathbf{y}} = \mathbf{y}_1 - \mathbf{y}_2$ then $\widetilde{\mathbf{y}}$ satisfies

$$\begin{aligned} &(\tfrac{d\widetilde{\mathbf{y}}}{dt}, \boldsymbol{\phi}) + (\mathcal{A}\widetilde{\mathbf{y}}, \boldsymbol{\phi}) + (\mathcal{B}K\widetilde{\mathbf{y}}, \boldsymbol{\phi}) + (N(\mathbf{y}_1) - N(\mathbf{y}_2), \boldsymbol{\phi}) = 0\,, \ \ \forall \boldsymbol{\phi} \in \mathbf{V} \times \mathbb{R}\,, \\ &\widetilde{\mathbf{y}}(0) = \mathbf{y}^0\,. \end{aligned} \tag{39}$$

Setting $\boldsymbol{\phi} = \widetilde{\mathbf{y}}$ in (39) and proceeding as before, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\mathbf{y}}\|^2 + \frac{1}{Re}\|\nabla\widetilde{\mathbf{w}}\|^2 + \tilde{l}^2 + \frac{\beta}{\gamma}\|\widetilde{\mathbf{y}}\|^2 = -(N(\mathbf{y}_1) - N(\mathbf{y}_2), \widetilde{\mathbf{y}})\,. \tag{40}$$

By the definition of the nonlinear map $N(\cdot)$, we have

$$\begin{aligned} (N(\mathbf{y}_1) - N(\mathbf{y}_2), \widetilde{\mathbf{y}}) \ &= b(\mathbf{w}_1, \mathbf{w}_1, \widetilde{\mathbf{w}}) - b(\mathbf{w}_2, \mathbf{w}_2, \widetilde{\mathbf{w}}) + b(\mathbf{w}_1, \mathbf{u}_d, \widetilde{\mathbf{w}}) \\ &+ b(\mathbf{u}_d, \mathbf{w}_1, \widetilde{\mathbf{w}}) - b(\mathbf{w}_2, \mathbf{u}_d, \widetilde{\mathbf{w}}) - b(\mathbf{u}_d, \mathbf{w}_2, \widetilde{\mathbf{w}}) \\ &+ l_1[b(\mathbf{w}_1, \mathbf{u}_l, \widetilde{\mathbf{w}}) + b(\mathbf{u}_l, \mathbf{w}_1, \widetilde{\mathbf{w}})] \\ &- l_2[b(\mathbf{w}_2, \mathbf{u}_l, \widetilde{\mathbf{w}}) + b(\mathbf{u}_l, \mathbf{w}_2, \widetilde{\mathbf{w}})] \\ &- \tilde{l}(\mathbf{f}_1, \widetilde{\mathbf{w}}) - l_1^2(\mathbf{f}_2, \widetilde{\mathbf{w}}) + (\mathbf{f}_2, \widetilde{\mathbf{w}})l_2^2 + \tilde{l}^2\,. \end{aligned} \tag{41}$$

Because of the properties of the trilinear form $b(\cdot,\cdot,\cdot)$, it follows that

$$\begin{aligned}(N(\mathbf{y}_1)-N(\mathbf{y}_2),\widetilde{\mathbf{y}}) &= b(\widetilde{\mathbf{w}},\mathbf{w}_2,\widetilde{\mathbf{w}})+b(\widetilde{\mathbf{w}},\mathbf{u}_d,\widetilde{\mathbf{w}})\\ &+\hat{l}b(\widetilde{\mathbf{w}},\mathbf{u}_l,\widetilde{\mathbf{w}})-\tilde{l}(\mathbf{f}_1,\widetilde{\mathbf{w}})-l_1^2(\mathbf{f}_2,\widetilde{\mathbf{w}})+(\mathbf{f}_2,\widetilde{\mathbf{w}})l_2^2+\tilde{l}^2\,.\end{aligned}$$

The right-hand side of this equality can be mojorized as follows

$$\begin{aligned}|(N(\mathbf{y}_1)-N(\mathbf{y}_2),\widetilde{\mathbf{y}})| &\le |b(\widetilde{\mathbf{w}},\mathbf{w}_2,\widetilde{\mathbf{w}})|+|b(\widetilde{\mathbf{w}},\mathbf{u}_d,\widetilde{\mathbf{w}})|+\hat{l}|b(\widetilde{\mathbf{w}},\mathbf{u}_l,\widetilde{\mathbf{w}})|\\ &+|\tilde{l}|\|\mathbf{f}_1\|\|\widetilde{\mathbf{w}}\|+\|\mathbf{f}_2\|\|\widetilde{\mathbf{w}}\||\hat{l}||\tilde{l}|+|\tilde{l}|^2.\end{aligned}\tag{42}$$

Let us next estimate the terms in the right-hand side of this inequality. Because of the generalized Sobolev's inequality and the Young's inequality the first term can be majorized as

$$\begin{aligned}|b(\widetilde{\mathbf{w}},\mathbf{w}_2,\widetilde{\mathbf{w}})| &\le \sqrt{3/2}\|\widetilde{\mathbf{w}}\|\|\nabla\mathbf{w}_2\|\|\nabla\widetilde{\mathbf{w}}\|\\ &\le \tfrac{9}{4}Re\|\widetilde{\mathbf{w}}\|^2\|\nabla\mathbf{w}_2\|^2+\tfrac{1}{6Re}\|\nabla\widetilde{\mathbf{w}}\|^2\,.\end{aligned}$$

Similarly, we obtain the estimates

$$\begin{aligned}|b(\widetilde{\mathbf{w}},\mathbf{u}_d,\widetilde{\mathbf{w}})| &\le \sqrt{3/2}\|\widetilde{\mathbf{w}}\|\|\nabla\widetilde{\mathbf{w}}\|\|\nabla\mathbf{u}_d\|\\ &\le \tfrac{9}{4}Re\|\widetilde{\mathbf{w}}\|^2\|\nabla\mathbf{u}_d\|^2+\tfrac{1}{6Re}\|\nabla\widetilde{\mathbf{w}}\|^2\,,\end{aligned}$$

and

$$\hat{l}|b(\widetilde{\mathbf{w}},\mathbf{w}_l,\widetilde{\mathbf{w}})|\le\frac{1}{6Re}\|\nabla\widetilde{\mathbf{w}}\|^2\,.$$

Using these estimates in (42) we obtain

$$\begin{aligned}|(N(\mathbf{y}_1)-N(\mathbf{y}_2),\widetilde{\mathbf{y}})|\le\ &\tfrac{9}{4}Re\|\widetilde{\mathbf{w}}\|^2\|\nabla\mathbf{w}_2\|^2+\tfrac{9}{4}Re\|\nabla\mathbf{u}_d\|^2\|\widetilde{\mathbf{w}}\|^2+\|\mathbf{f}_1\|\|\widetilde{\mathbf{y}}\|^2\\ &+\hat{l}\|\mathbf{f}_2\|\|\widetilde{\mathbf{y}}\|^2+\tilde{l}^2+\tfrac{1}{2Re}\|\nabla\widetilde{\mathbf{w}}\|^2\,.\end{aligned}$$

Employing this in the right-hand side of the identity (40), we obtain

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|\widetilde{\mathbf{y}}\|^2+\tfrac{1}{2Re}\|\nabla\widetilde{\mathbf{w}}\|^2+\tilde{l}^2+\tfrac{\beta}{\gamma}\|\widetilde{\mathbf{y}}\|^2 &\le \tfrac{9}{4}Re\|\widetilde{\mathbf{w}}\|^2\|\nabla\mathbf{w}_2\|^2\\ &+\tfrac{9}{4}Re\|\nabla\mathbf{u}_d\|^2\|\widetilde{\mathbf{w}}\|^2+\|\mathbf{f}_1\|\|\widetilde{\mathbf{y}}\|^2\\ &+\hat{l}\|\mathbf{f}_2\|\|\widetilde{\mathbf{y}}\|^2+\tilde{l}^2\,.\end{aligned}\tag{43}$$

It follows from this inequality that

$$\tfrac{1}{2}\tfrac{d}{dt}\|\widetilde{\mathbf{y}}\|^2+\left[\tfrac{\beta}{\gamma}-(\tfrac{9}{4}Re\|\nabla\mathbf{u}_d\|^2+\|\mathbf{f}_1\|+\hat{l}\|\mathbf{f}_2\|)\right]\|\widetilde{\mathbf{y}}\|^2 \le \tfrac{9}{4}Re\|\nabla\mathbf{w}_2\|^2\|\widetilde{\mathbf{y}}\|^2\,.\tag{44}$$

Now choosing β such that $[\frac{\beta}{\gamma}-(\frac{9}{4}Re\|\nabla\mathbf{u}_d\|^2+\|\mathbf{f}_1\|+\hat{l}\|\mathbf{f}_2\|)]>0$, we arrive at the inequality

$$\tfrac{d}{dt}\|\widetilde{\mathbf{y}}\|^2\le\tfrac{9}{2}Re\|\nabla\mathbf{w}_2\|^2\|\widetilde{\mathbf{y}}\|^2\,.$$

Integrating this from 0 to t we get

$$\|\widetilde{\mathbf{y}}\|^2\le\|\widetilde{\mathbf{y}}_0\|^2+\frac{9}{2}Re\int_0^t\|\nabla\mathbf{w}_2\|^2\|\widetilde{\mathbf{y}}\|^2ds\,.$$

Finally applying the Gronwall's lemma and estimate (22), we obtain the desired continuous dependence inequality

$$\|\widetilde{\mathbf{y}}\|^2\le\|\widetilde{\mathbf{y}}_0\|^2e^{\frac{9}{2}Re\int_0^t\|\nabla w_2\|^2ds}\le\|\widetilde{\mathbf{y}}_0\|^2e^{(\frac{9}{2}Re^2\|\mathbf{y}_2^0\|^2)}\,.\tag{45}$$

This proves the continuous dependence of solutions on the initial data in L^2-norm.

Remark 1. *It is possible to show that* $\widetilde{\mathbf{y}} \to 0$ *as* $t \to \infty$ *by taking the parameter* β *large enough that*

$$\frac{\beta}{\gamma} > \frac{9}{4} Re \|\nabla \mathbf{u}_d\|^2 + \|\mathbf{f}_1\| + \hat{l}\|\mathbf{f}_2\| + \frac{9}{4} Re\, M(\sigma, Re, \|\mathbf{y}_0\|, \|\nabla \mathbf{y}_0\|)\,.$$

In fact because of the estimate (15), the inequality (44) becomes

$$\frac{d}{dt}\|\widetilde{\mathbf{y}}\|^2 + 2\widetilde{\sigma}\|\widetilde{\mathbf{y}}\|^2 \leq 0\,, \tag{46}$$

where

$$\widetilde{\sigma} \equiv \frac{\beta}{\gamma} - \left(\frac{9}{4} Re \|\nabla \mathbf{u}_d\|^2 + \|\mathbf{f}_1\| + \hat{l}\|\mathbf{f}_2\| + \frac{9}{4} Re\, M(\sigma, Re, \|\mathbf{y}_0, \|\nabla \mathbf{y}_0\|)\right) > 0\,.$$

Therefore by Gronwall's lemma applied to (46), we obtain

$$\|\widetilde{\mathbf{y}}\| \leq \|\widetilde{\mathbf{y}}(0)\| e^{-\widetilde{\sigma} t}\,.$$

□

4. Concluding Remarks

In this paper, we considered the problem of stabilizing the steady states of Navier-Stokes equations in bounded domains using boundary feedback control. Inspired by the Lyapunov stability theory for finite dimensional nonlinear systems, we proposed a linear feedback control using the algebraic Riccati equation associated with an infinite time horizon linear quadratic regulator problem. The resulting feedback control was proven to be globally exponentially stabilizing the steady states of the Navier-Stokes equations for arbitrary Reynolds number. This feedback control was shown to guarantee global stability in both L^2 and H^1 -norms.

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