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AN EXTENDED DOMAIN METHOD FOR OPTIMAL BOUNDARY CONTROL FOR NAVIER-STOKES EQUATIONS

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Abstract. The matching velocity problem for the steady-state Navier-Stokes system is considered. We introduce an extended domain method for solving optimal boundary control problems. The Lagrangian multiplier method is applied to the extended domain with distributed controls and used to determine the optimality system and the control over the boundary of the inner domain. The existence, the differentiability and the optimality system of the optimal control problem are discussed. With this method inflow controls are shown to be numerical reliable over a large admissible control set. Numerical tests for steady-state solutions are presented to prove the effectiveness and robustness of the method for flow matching.

Key Words. optimal boundary control, optimal design, Navier-Stokes equations, velocity matching problem.

1. Introduction

Optimal boundary control problems associated with the Navier-Stokes equations have a wide and important range of applications such as the design of cars, airplanes and jet engines. Despite the fact that this field has been extensively studied, determining the best boundary control or even a simple effective boundary control for a system governed by the Navier-Stokes equations is still a difficult and time consuming task.

Early studies devoted to optimal boundary control problems for the Navier-Stokes equations can be found, for example, in [1, 8, 15, 16]. The optimal control of the Navier-Stokes equations shows many challenges and has been considered by numerous authors (see for exanple [4, 6, 9, 13, 35, 14, 20, 18, 19, 20, 21, 22, 23, 24, 25, 28, 38] and citations therein). The theoretical treatment of optimal boundary problems concerning with questions of existence, regularity of solutions, and differentiability properties is in some extent satisfactory but the numerical implementation, the analysis, and the consistency of discrete approximations still remain fundamental issues. Many results generally lack a coherent first-order necessary condition and often the regularity assumed cannot be used in numerical algorithms. Other papers deal with re-formulations of the problem, mainly to simplified situations with finite dimension controls.

In order to simplify the description of the problem we consider the two-dimensional steady-state incompressible flow of a viscous fluid with Dirichlet boundary conditions in a region Ω with boundary Γ as shown on the left of Figure 1. The velocity

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$\vec{u}$ and the pressure p satisfy the stationary Navier-Stokes system

$$-\nu\triangle\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \vec{h} \qquad \text{in } \Omega \tag{1}$$

$$\nabla\cdot\vec{u} = 0 \qquad \text{in } \Omega \tag{2}$$

along with the Dirichlet boundary conditions

$$\vec{u} = \vec{g} = \begin{cases} 0 & \text{on } \Gamma_1 \\ \vec{g} & \text{on } \Gamma_c\,, \end{cases} \tag{3}$$

where $\vec{h}$ is the given body force. In (1) ν denotes the inverse of the Reynolds number whenever the variables are appropriately nondimensionalized.

Along the uncontrolled part of boundary boundary Γ_1 the velocity vanishes and the function $\vec{g}$ must satisfy the compatibility condition

$$\int_\Gamma \vec{g}\cdot\vec{n}\,ds = 0 \tag{4}$$

where $\vec{n}$ is the unit normal vector along the surface Γ. If some other types of boundary conditions, e.g., natural boundary conditions or outflow boundary conditions, are specified along the left or right or bottom boundaries, the results given in this paper are formally valid but some technical details in the analysis should be carefully revised. There is a substantial literature discussing the set of all possible

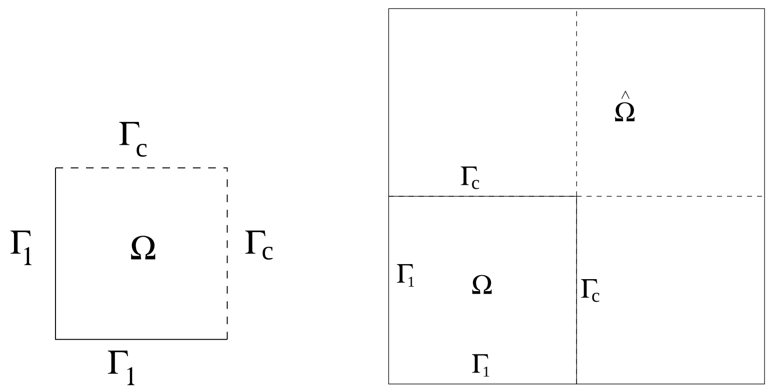


FIGURE 1. The flow domain Ω. Γ_c denotes the part of the boundary whose velocity is to be determined by the optimization process.

boundary controls. Clearly, the function $\vec{g}$ must belong to $H^{1/2}(\Gamma_c)$, the Sobolev space of order 1/2. However $H^{1/2}(\Gamma_c)$ or $H^1(\Gamma_c)$ may not be sufficient to enable one to explicitly derive a first-order necessary condition or implement numerically the boundary control. Thus in general the set of all admissible controls $\vec{g}$ must be restricted to more regular spaces, namely, to belong to $H^{3/2}(\Gamma_c)$.

One could examine several practical objective functionals for determining the boundary controls, e.g., the reduction of the drag due to viscosity or the identification of the velocity at a fixed vertical slit downstream. To fix ideas, we focus on the minimization of the cost functional that leads to matching velocity problems. In literature the steady optimal control problem is formulated by using the following functional (see for example [1, 25])

$$J(\vec{u},\vec{g}) = \frac{1}{2}\int_\Omega (\vec{u}-\widehat{U})^2\,d\vec{x} + \frac{\beta}{2}\int_{\Gamma_c} (\alpha\,\vec{g}_s^{\,2} + \vec{g}^{\,2})\,dx\,, \tag{5}$$

where $\vec{u}$ is the velocity field that solves the Navier Stokes system and $\vec{U}$ is the desired velocity field defined on the domain Ω. The vector $\vec{g}$ and $\vec{g}_s$ are respectively the velocity and the derivative along the boundary of the control. The minimization of the first term involving $(\vec{u} - \vec{U})$ is the real goal of the velocity matching problem and the other terms have been introduced in order to bound the control function and prove the existence of the solution of the optimal control problem and the optimality system. We may effectively limit the size of the control and prove the existence of the first order necessary condition for optimality through an appropriate choice of the positive coefficients β and α but the optimal control based on this admissible set of solutions and the choice of β and α is not very friendly from the numerical point of view and it turns out to be a very difficult task if injection or suction boundary velocity is required to satisfy the integral constraint (4). For $\beta = 0$, the functional (5) represents a sort of integral distance between the desired and the solution velocity field. The term with the parameter α is needed to impose regularity on the boundary control $\vec{g}$. The assumption α positive allows the application of the standard finite element theory for the Navier-Stokes system, impose continuity of the control field and differentiability of the optimal problem.

Formally speaking the control problem is to find $\vec{u}$ and $\vec{g}$ such that the functional (5) is minimized subject to the Navier-Stokes system (1)–(3). This optimal control approach is rarely used in practical problems since the corresponding numerical implementation is difficult. The resulting general optimality system includes the Navier-Stokes system, the adjoint system, the partial differential equation for the control and, for closed systems, the integral equation for the inflow boundary control (4). In this situation only tangential control may be implemented with a certain success.

In order to avoid these numerical problems we introduce an extended domain and transform the boundary control problem into a distributed control problem over the extended domain. Specifically, we extend the model domain of Figure 1 along the line of control. As in Figure 1 on the right we assume that all the controlled parts of Γ_c are contained in a simply connected extended domain $\widehat{\Omega}$. This allows the conservation of the mass in the extended domain which is a closed system. This strategy is used to avoid theoretical technicalities but it is not the obvious choice for many problems as we will discuss in the numerical section. Now we reformulate the two-dimensional problem for a viscous incompressible flow over the region $\widehat{\Omega}$ by using distributed controls. The velocity $\widehat{u}$ and the pressure $\widehat{p}$ satisfy the stationary Navier-Stokes system

$$-\nu\triangle\widehat{u} + (\widehat{u}\cdot\nabla)\widehat{u} + \nabla\widehat{p} = \chi_{\Omega_2}\left(\widehat{f} + \alpha\triangle\widehat{f}\right) \qquad \text{in } \widehat{\Omega} \tag{6}$$

$$\nabla\cdot\widehat{u} = 0 \qquad \text{in } \widehat{\Omega} \tag{7}$$

along with the Dirichlet boundary conditions $\widehat{u} = \widehat{g} = 0$ on Γ_1. The constant α is nonnegative. The function $\widehat{f}$ is now the control and χ_{Ω_2} is the characteristic function over $\Omega_2 = \widehat{\Omega}\backslash\Omega$. The vectors $\vec{g}$ on Γ is the trace of $\widehat{u}$ and satisfies automatically the compatibility condition (4). The idea is very simple. We substitute the boundary control with the trace of $\widehat{u}$ which is to be determined by the associated distributed optimal control over the extended domain. The new cost functional becomes

$$J(\vec{u},\widehat{f}) = \frac{1}{2}\int_\Omega (\widehat{u} - \vec{U})^2\, d\vec{x} + \frac{\beta}{2}\int_{\Omega_2} \widehat{f}^2 + \alpha\,|\nabla\widehat{f}|^2\, d\vec{x} \tag{8}$$

and the new control problem is to find $\widehat{u}$, $\widehat{f}$ and the trace of $\widehat{u}$ over Γ_c such that the functional (8) is minimized subject to the Navier-Stokes system (6)–(7).

This new approach is numerically more friendly than the previous one. The resulting optimality system includes only the Navier-Stokes system and its adjoint. The vector $\vec{g}$ obeys to the compatibility condition (4) and normal controls may be included reducing the computational load. For $\alpha > 0$ the set of admissible controls is large and include some non-continuous control functions which are not included in the set of admissible controls defined by the functional (5) and it may include the use of controlled boundary corners. The introduction of corners in the boundary Γ_c is an important issue for other optimal boundary control formulations since it can be treated only by imposing extra conditions.

Although we deal with a specific, two-dimensional matching minimization problem, the approach used here is discussed in general terms and can be used for many other optimal control problems involving different objective functionals and classes of controls. Furthermore, although the geometry is somewhat simple, our results can be extended to a general settings without further complications. Our aim here is to provide a systematic analysis for the problem in which the matching distance is minimized through the use of variational methods.

The plan of the rest of the paper is as follows. In the next section, we introduce some notations and consider distributed optimal control associated to the boundary value problem for the extended domain. We give a precise description of the model optimization problem and then state and prove some results concerning the optimal solutions and the relation with the original boundary problem. Finally in the section 3 we show numerical experiments and its multigrid implementation for which the fluid motion is controlled by velocity forcing, i.e., injection or suction, along a portion of the boundary, and the cost or objective functional is a matching type objective functional.

2. The stationary boundary control problem

2.1. Notations. We denote by $H^s(O)$, $s \in \Re$, the standard Sobolev space of order s with respect to the set O, which is either the flow domain Ω, or its boundary Γ, or part of its boundary. Whenever m is a nonnegative integer, the inner product over $H^m(O)$ is denoted by $(f,g)_m$ and (f,g) denotes the inner product over $H^0(O) = L^2(O)$. Hence, we associate with $H^m(O)$ its natural norm $\|f\|_{m,O} = \sqrt{(f,f)_m}$. Whenever possible, we will neglect the domain label in the norm.

For vector-valued functions and spaces, we use boldface notation. For example, $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$ denotes the space of $\Re^n$-valued functions such that each component belongs to $H^s(\Omega)$. Of special interest is the space

$$\mathbf{H}^1(\Omega) = \left\{ v_j \in L^2(\Omega) \;\Big|\; \frac{\partial v_j}{\partial x_k} \in L^2(\Omega) \quad \text{for } j,k = 1,2 \right\}$$

equipped with the norm $\|\vec{v}\|_1 = (\sum_{k=1}^{2} \|v_k\|_1^2)^{1/2}$.

For $\Gamma_s \subset \Gamma$ with nonzero measure, we also consider the subspace

$$\mathbf{H}^1_{\Gamma_s}(\Omega) = \{ \vec{v} \in \mathbf{H}^1(\Omega) \mid \vec{v} = \vec{0} \quad \text{on } \Gamma_s \} .$$

Also, we write $\mathbf{H}^1_0(\Omega) = \mathbf{H}^1_\Gamma(\Omega)$. For any $\vec{v} \in \mathbf{H}^1(\Omega)$, we write $\|\nabla\vec{v}\|$ for the seminorm. Let $(\mathbf{H}^1_{\Gamma_s})^*$ denote the dual space of $\mathbf{H}^1_{\Gamma_s}$. Note that $(\mathbf{H}^1_{\Gamma_s})^*$ is a subspace of $\mathbf{H}^{-1}(\Omega)$, where the latter is the dual space of $\mathbf{H}^1_0(\Omega)$. The duality pairing between $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}^1_0(\Omega)$ is denoted by $< \cdot, \cdot >$.

Let $\vec{g}$ be an element of $\mathbf{H}^{1/2}(\Gamma)$. It is well known that $\mathbf{H}^{1/2}(\Gamma)$ is a Hilbert space with norm

$$\|\vec{g}\|_{1/2,\Gamma} = \inf_{\vec{v}\in\mathbf{H}^1(\Omega);\, \gamma_\Gamma \vec{v}=\vec{g}} \|\vec{v}\|_1 \,,$$

where γ_Γ denotes the trace mapping $\gamma_\Gamma : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$. We let $(\mathbf{H}^{1/2}(\Gamma))^*$ denote the dual space of $\mathbf{H}^{1/2}(\Gamma)$ and $< \cdot, \cdot >_\Gamma$ denote the duality pairing between $(\mathbf{H}^{1/2}(\Gamma))^*$ and $\mathbf{H}^{1/2}(\Gamma)$. Let Γ_s be a smooth subset of Γ. Then, the trace mapping $\gamma_{\Gamma_s} : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma_s)$ is well defined and $\mathbf{H}^{1/2}(\Gamma_s) = \gamma_{\Gamma_s}(\mathbf{H}^1(\Omega))$.

Since the pressure is only determined up to an additive constant by the Navier-Stokes system with velocity boundary conditions, we define the space of square integrable functions having zero mean over Ω as

$$L^2_0(\Omega) = \{\, p \in L^2(\Omega) \mid \int_\Omega p \, d\vec{x} = 0 \,\} \,.$$

In order to define a weak form of the Navier-Stokes equations, we introduce the continuous bilinear forms

$$a(\vec{u},\vec{v}) = 2\nu \int_\Omega D(\vec{u}) : D(\vec{v}) \, d\vec{x} \qquad \forall\, \vec{u},\vec{v} \in \mathbf{H}^1(\Omega) \tag{9}$$

and

$$b(\vec{v},q) = -\int_\Omega q \,\nabla\cdot\vec{v} \, d\vec{x} \quad \forall\, q \in L^2_0(\Omega)\,, \quad \forall\, \vec{v} \in \mathbf{H}^1(\Omega) \tag{10}$$

and the trilinear form

$$c(\vec{w};\vec{u},\vec{v}) = \int_\Omega \vec{w}\cdot\nabla\vec{u}\cdot\vec{v}\, d\vec{x} = \sum_{i,j=1}^{2} \int_\Omega w_j \left(\frac{\partial u_i}{\partial x_j}\right) v_i \, d\vec{x} \quad \forall\, \vec{w},\vec{u},\vec{v} \in \mathbf{H}^1(\Omega)\,. \tag{11}$$

The tensor D is defined by $D_{ij} = 1/2(\partial u_i \partial x_j + \partial u_i \partial x_j)$. Obviously, $a(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ and $b(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega)\times L^2_0(\Omega)$; also $c(\cdot;\cdot,\cdot)$ is a continuous trilinear form on $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ which can be verified by the Sobolev embedding of $\mathbf{H}^1(\Omega) \subset \mathbf{L}^4(\Omega)$ and Holder's inequality. We also have the coercivity property

$$a(\vec{v},\vec{v}) \geq C\|\vec{v}\|_1^2 \qquad \forall\, \vec{v} \in \mathbf{H}^1_{\Gamma_s}(\Omega)$$

whenever $\Gamma_s \subset \Gamma$ has positive measure and the inf-sup condition

$$\inf_{p\in L^2_0(\Omega)} \sup_{\vec{v}\in\mathbf{H}^1_0} \frac{b(\vec{v},p)}{\|\vec{v}\|_1\|p\|} \geq K\,.$$

For details concerning the function spaces we have introduced, one may consult [2, 40, 11, 41] and for details about the bilinear and trilinear forms and their properties, one may consult [29, 40].

2.2. The optimal boundary value problem. We consider the formulation of the direct problem for the Navier-Stokes system (1)–(3) for which the boundary and all the data functions are known. A weak formulation of the Navier-Stokes system is given as follows:

Given $\vec{h} \in \mathbf{H}^{-1}(\Omega)$ and $\vec{g} \in \mathbf{H}^{1/2}(\Gamma)$, find $(\vec{u},p) \in \mathbf{H}^1(\Omega)\times L^2_0(\Omega)$ satisfying

$$
(12) \qquad \begin{cases} a(\vec{u},\vec{v}) + c(\vec{u};\vec{u},\vec{v}) + b(\vec{v},p) = <\vec{h},\vec{v}> \quad \forall\, \vec{v} \in \mathbf{H}^1_0(\Omega) \\ b(\vec{u},q) = 0 \quad \forall\, q \in L^2_0(\Omega) \\ <\vec{u},\vec{s}>_\Gamma = <\vec{g},\vec{s}>_\Gamma \quad \forall\, \vec{s} \in \mathbf{H}^{-1/2}(\Gamma)\,. \end{cases}
$$

Existence and uniqueness results for solutions of the system (12) are contained in the following theorems; see, e.g., [40].

Theorem 1. *Let Ω be an open, bounded set of $\Re^2$ with Lipschitz-continuous boundary Γ. Let $\vec{h} \in \mathbf{H}^{-1}(\Omega)$ and $\vec{g} \in \mathbf{H}^{1/2}(\Gamma)$ and let $\vec{g}$ satisfy the compatibility condition* (4)*. Then,*

: *i) there exists at least one solution $(\vec{u},p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ of* (12)*;*
: *ii) the set of velocity fields that are solutions of* (12) *is closed in $\mathbf{H}^1(\Omega)$ and is compact in $\mathbf{L}^2(\Omega)$; and*
: *iii) if*

$$
(13) \qquad \nu > \nu_0(\Omega,\vec{h},\vec{g})
$$

for some positive ν_0 whose value is determined by the given data, then the set of solutions of (12) *consists of a single element.*

Note that solutions of (12) exist for any value of the Reynolds number. However, iii) implies that uniqueness can be guaranteed only for "large enough" values of ν or for "small enough" data $\vec{h}$ and $\vec{g}$.

Theorem 2. *Let the hypotheses of Theorem* 1 *hold. Let Γ be piecewise $C^{1,1}$ with convex corners, $\vec{g} \in \mathbf{H}^{3/2}(\Gamma)$, and $\vec{h} \in \mathbf{L}^2(\Omega)$. Then,*

: *i) there exists at least one solution $(\vec{u},p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega) \cap L^2_0(\Omega)$;*
: *ii) the set of the velocity solutions is closed in $\mathbf{H}^2(\Omega)$ and compact in $\mathbf{H}^1(\Omega)$.*

Before continuing, we recall some notations and results about extended domains that will be of use in the sequel. We say that a domain Ω has a cusp at $x \in \Gamma$ if no affine image in $\overline{\Omega}$ of a finite cone has a vertex at x. If Ω is a Lipschitz continuous domain, the possibility of there being a cusp is excluded and therefore the domain Ω has the uniform extension property as the uniform Lipschitz sets are the open sets satisfying the cone property; see [2]. We recall the following extension theorem (Calderon's extension theorem); see [2].

Theorem 3. *For every uniform Lipschitz domain $\Omega \subset \Re^2$ and positive integer m, there exists a linear continuous extension operator*

$$
E : \mathbf{H}^m(\Omega) \to \mathbf{H}^m(\Re^2)
$$

such that for every $\vec{u} \in \mathbf{H}^m(\Omega)$ we have $\|E\widehat{u}\|_m \le C\|\vec{u}\|_m$, where the positive constant C depends only on the cone imbedded in Ω.

We recall also that a solenoidal extension to $\Re^2$ of a solenoidal function defined in Ω can be found as described in [7]. In the rest of the paper, whenever it is not confusing, we denote the function and its extension by the "hat" symbol.

We now formulate the mathematical model of the optimal boundary control problem over the extended domain. In order to simplify the notation in the rest of the paper we assume zero body force, i.e, $\vec{h} = 0$ over Ω. As shown in Figure 1 let $\widehat{\Omega}$ be an extended domain with boundary $\widehat{\Gamma}$ and Ω_2 be $\widehat{\Omega}\backslash\Omega$. The extended domain $\widehat{\Omega}$, which contains the controlled boundary $\Gamma_c = \Gamma\backslash\Gamma_1$, is assumed to be simply connected. Let $\mathbf{U_{ad}} \subseteq \mathbf{H}^1(\Omega)$ be the set of admissible desired velocity fields. The optimal boundary control problem can then be stated by using the extended domain $\widehat{\Omega}$ and the distributed extended force $\widehat{f}$ in the following way:

given $\vec{U} \in \mathbf{U_{ad}}$ *find* $(\vec{u}, p, \vec{g}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma)$ *solution of (12) such that* $\vec{g} = 0$ *on* Γ_1 *and* $\vec{u} = \widehat{u}$, $p = \widehat{p}$ *over* Ω *where* $(\widehat{u}, \widehat{p}, \widehat{f}) \in \mathbf{H}^1_0(\widehat{\Omega}) \times L^2_0(\widehat{\Omega}) \times \mathbf{L}^2(\Omega_2)$ *minimizes the functional*

$$J(\vec{u}, \widehat{f}) = \frac{1}{2} \int_{\Omega} |\vec{u} - \vec{U}|^2 \, d\vec{x} + \frac{\beta}{2} \int_{\Omega_2} \left(|\hat{f}\,|^2 + \alpha |\nabla \hat{f}\,|^2 \right) d\vec{x}\,, \tag{14}$$

and satisfies

$$\begin{cases} a(\widehat{u}, \widehat{v}) + c(\widehat{u}; \widehat{u}, \widehat{v}) + b(\widehat{v}, p) = \\ \qquad \displaystyle\int_{\Omega_2} \left(\widehat{f} \cdot \widehat{v} + \alpha \nabla \widehat{f} \cdot \nabla \widehat{v} \right) d\vec{x} \quad \forall\, \widehat{v} \in \mathbf{H}^1_0(\widehat{\Omega}) \\ b(\widehat{u}, \widehat{q}) = 0 \quad \forall\, \widehat{q} \in L^2_0(\widehat{\Omega}) \\ < \widehat{u}, \widehat{s} >_\Gamma = < \vec{g}, \widehat{s} >_\Gamma \quad \forall\, \widehat{s} \in \mathbf{H}^{-1/2}(\Gamma) \end{cases} \tag{15}$$

with $\alpha \geq 0$.

The corresponding boundary control $\vec{g}$ can be found after the solution of the above optimal control problem as the trace of the extended solution $\widehat{u}$ over Γ_c. We note that the boundary control $\vec{g}$ automatically satisfies the compatibility condition (4). For $\alpha > 0$ we have $\widehat{f} \in \mathbf{H}^1(\Omega_2)$ with $\vec{g} \in \mathbf{H}^{1/2}(\Gamma)$ and the admissible set of states and controls $\mathbf{A}_{\alpha+}$ is given by

$\mathbf{A}_{\alpha+} = \{(\vec{u}, p, \vec{g}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma)$
with $\vec{g} = 0$ *on* Γ_1 *and* $\vec{u} = \widehat{u}$, $p = \widehat{p}$ *over* Ω *such that* $(\widehat{u}, \widehat{p}, \widehat{f}) \in \mathbf{H}^1_0(\widehat{\Omega}) \times$
$L^2_0(\widehat{\Omega}) \times \mathbf{H}^1(\Omega_2)$ satisfies (15) with $J(\vec{u}, \widehat{f}) < \infty\}\,.$

For boundary which are regular and $\alpha = 0$ then $\widehat{f} \in \mathbf{L}^2(\Omega_2)$ with $\vec{g} \in \mathbf{H}^{3/2}(\Gamma)$ and the admissible set of states and controls $\mathbf{A}_0$ can be defined as

$\mathbf{A}_0 = \{(\vec{u}, p, \vec{g}) \in \mathbf{H}^2(\Omega) \cap \mathbf{V}(\Omega) \times H^1(\Omega) \cap L^2_0(\Omega) \times \mathbf{H}^{3/2}(\Gamma)$
with $\vec{g} = 0$ *over* Γ_1 *and* $\vec{u} = \widehat{u}$, $p = \widehat{p}$ *over* Ω *such that* $(\widehat{u}, \widehat{p}, \widehat{f}) \in \mathbf{H}^2(\widehat{\Omega}) \cap \mathbf{V}(\widehat{\Omega}) \times$
$H^1(\widehat{\Omega}) \cap L^2_0(\widehat{\Omega}) \times \mathbf{L}^2(\Omega_2)$ satisfies (15) with $J(\vec{u}, \widehat{f}) < \infty\}\,.$

In the rest of the paper we will denote by $\mathbf{A}_\alpha$ the set of admissible states and controls $\mathbf{A}_{\alpha+}$ or $\mathbf{A}_0$ as needed. The existence of optimal solutions in these admissible sets can be studied by using standard techniques (see for example [1, 22, 14, 23, 24, 25, 21, 27]).

Theorem 4. *i) For* $\alpha > 0$ *there exists at least one optimal solution* $(\widetilde{u}, \widetilde{p}, \widetilde{g}) \in \mathbf{A}_{\alpha+}$ *of the optimal control problem* (14-15).
ii) Let Γ *be piecewise* $C^{1,1}$ *with convex corners and* $\alpha = 0$. *Then there exists at least one optimal solution* $(\widetilde{u}, \widetilde{p}, \widetilde{g}) \in \mathbf{A}_0$ *of the optimal control problem* (14-15).

Proof: i) For $\alpha > 0$ then $\widehat{f} \in \mathbf{A}_{\alpha+}$ implies $\widehat{f} \in \mathbf{H}^1(\Omega_2)$. Since the control problem (15) over the domain $\widehat{\Omega}$ is a distributed optimal control problem the proof follows from standard techniques (see, e.g., [22] or [20]). The Laplacian in $\widehat{f}$ does not present problems and there exists a distributed optimal solution over the extended domain for $(\widetilde{u}, \widetilde{p}, \widetilde{f}) \in \mathbf{H}^1(\widehat{\Omega}) \times L^2_0(\widehat{\Omega}) \times \mathbf{H}^1(\Omega_2)$. Now it is clear that the restriction of $(\widetilde{u}, \widetilde{p})$ to the domain Ω is in $\mathbf{H}^1(\Omega) \times L^2_0(\widehat{\Omega})$ and $\widetilde{g} = \gamma_\Gamma\, \widetilde{u}$ is in $\mathbf{H}^{1/2}(\Gamma)$ and therefore $\widetilde{g} \in \mathbf{H}^{1/2}(\Gamma)$.

ii) If $\alpha = 0$ and the boundary is regular then $\widetilde{f} \in \mathbf{A}_0$ which implies $\widetilde{f} \in \mathbf{L}^2(\Omega_2)$ and therefore $(\widetilde{u}, \widetilde{p}) \in \mathbf{H}^2(\widehat{\Omega}) \times \mathbf{H}^1(\widehat{\Omega})$. Again from standard techniques ([22, 23, 24, 25, 21]) there exists an optimal solution $(\widetilde{u}, \widetilde{p}, \widetilde{f}) \in \mathbf{H}^2(\widehat{\Omega}) \times \mathbf{H}^1(\widehat{\Omega}) \times \mathbf{L}^2(\Omega_2)$. Their restriction $(\widetilde{u}, \widetilde{p})$ to the domain Ω gives the desired optimal solution. Since $\widetilde{u} \in \mathbf{H}^2(\widehat{\Omega})$ then its trace over Γ is in $\mathbf{H}^{3/2}(\Gamma)$. □

Finally we have two theorems on matching optimal solutions.

Theorem 5. *Let $(\widetilde{u}_1, \widetilde{p}_1, \widetilde{g}_1) \in \mathbf{A}_\alpha$ be an optimal solution for $\beta = \beta_1$. If $\beta_2 < \beta_1$ then there is an optimal solution $(\widetilde{u}_2, \widetilde{p}_2, \widetilde{g}_2) \in \mathbf{A}_\alpha$ for $\beta = \beta_2$ such that*

$$\|\widetilde{u}_2 - \vec{U}\|_\Omega \le \|\widetilde{u}_1 - \vec{U}\|_\Omega \,. \tag{16}$$

Proof: The result can be obtained by using the definition of minimum of the functional for $\beta = \beta_1$ and $\beta = \beta_2$. □

Theorem 6. *Let $(\widetilde{u}_1, \widetilde{p}_1, \widetilde{g}_1)$ be an optimal solution of the problem in (1–4). Then, for all $\epsilon > 0$ there is an optimal solution $(\widetilde{u}_2, \widetilde{p}_2, \widetilde{g}_2) \in \mathbf{A}_0$ of the problem in (14–15) with $\beta > \beta^\epsilon > 0$ such that*

$$\|\widetilde{u}_2 - \vec{U}\|_\Omega \le \|\widetilde{u}_1 - \vec{U}\|_\Omega + \epsilon \,. \tag{17}$$

Proof: Let $(\vec{u}_1, p_1, \vec{g}_1)$ be an optimal solution of the problem (1–4) then the solution is in $\mathbf{A}_0$. By using the extension theorem the corresponding $(\widehat{u}_1, \widehat{p}_1, \widehat{f}_1)$ is in $\mathbf{H}^2(\widehat{\Omega}) \cap \mathbf{H}^1_0(\widehat{\Omega}) \times H^1(\widehat{\Omega}) \cap L^2_0(\Omega_2) \times \mathbf{L}^2(\Omega_2)$. From the definition of optimal solution $(\widetilde{u}_2, \widetilde{p}_2, \widetilde{g}_2)$ we have

$$\|\widetilde{u}_2 - \vec{U}\|_\Omega + \beta \|\widehat{f}_2\|_{\Omega_2} \le \|\vec{u}_1 - \vec{U}\|_\Omega + \beta \|\widehat{f}_1\|_{\Omega_2} \,. \tag{18}$$

For $\beta^\epsilon = \epsilon / |\|\widehat{f}_1\|_{\Omega_2} - \|\widehat{f}_2\|_{\Omega_2}|$ we have

$$\|\widetilde{u}_2 - \vec{U}\|_\Omega \le \|\widetilde{u}_1 - \vec{U}\|_\Omega + \epsilon \,. \tag{19}$$

Theorem 5 completes the proof. □

2.3. The Euler condition for the optimal solution. In order to write the optimality system we must introduce the Lagrangian map of the extended problem ([5, 27]). The Lagrangian map can be shown to be strictly differentiable for all values of the distributed force over the extended domain and this allows us to apply the Lagrange multiplier method to find the optimality system for the optimal control solution.

Let $\mathbf{B}_1 = \mathbf{H}^1_{\Gamma_1}(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma)$, $\mathbf{B}_2 = \mathbf{H}^{-1}(\Omega) \times L^2_0(\Omega) \times \mathbf{H}^{1/2}(\Gamma)$. We define the nonlinear mapping $M : \mathbf{B}_1 \to \mathbf{B}_2$ by $M(\vec{u}, p, \vec{g}) = (\vec{l}_1, l_2, \vec{l}_3)$ for $(\vec{u}, p, \vec{g}) \in \mathbf{B}_1$ and $(\vec{l}_1, l_2, \vec{l}_3) \in \mathbf{B}_2$ if and only if

$$\begin{cases} \nu a(\vec{u}, \vec{v}) + c(\vec{u}; \vec{u}, \vec{v}) + b(\vec{v}, p) = \displaystyle\int_\Omega \vec{l}_1 \cdot \vec{v}\, d\vec{x} \quad \forall\, \vec{v} \in \mathbf{H}^1_0(\Omega) \\ b(\vec{u}, z) = \displaystyle\int_\Omega l_2\, z\, d\vec{x} \quad \forall\, z \in L^2_0(\Omega) \\ \displaystyle\int_\Gamma (\vec{u} - \vec{g}) \cdot \vec{s}\, ds = \int_\Gamma \vec{l}_3 \cdot \vec{s}\, ds \qquad \forall \vec{s} \in \mathbf{H}^{-1/2}(\Gamma) \end{cases} . \tag{20}$$

The set of equations in the direct problem formulated in the Theorem 1 can be expressed as $M(\vec{u}, p, \vec{g}) = (\vec{0}, 0, \vec{0})$.

Let $\widehat{\Omega}$ be an open bounded extension of Ω with boundary $\widehat{\Gamma}$. In order to transform the boundary control problem into an extended distributed control problem we introduce the distributed control variable $\widehat{f}$. The function $\widehat{f}$ represents a distributed control over the extended domain Ω_2 and its contribution vanishes over Ω.

For $\alpha > 0$ let $\widehat{\mathbf{B}}_1 = \mathbf{H}^1_0(\widehat{\Omega}) \times L^2_0(\widehat{\Omega}) \times \mathbf{H}^1(\Omega_2) \times \mathbf{H}^{1/2}(\Gamma)$ and $\widehat{\mathbf{B}}_2 = \mathbf{H}^{-1}(\widehat{\Omega}) \times L^2_0(\widehat{\Omega}) \times \mathbf{H}^{1/2}(\Gamma)$. For $\alpha = 0$ we consider $\widehat{\Gamma}$ piecewise $C^{1,1}$ with convex corners and $\widehat{\mathbf{B}}_1 = \mathbf{H}^1_0(\widehat{\Omega}) \cap \mathbf{H}^2(\widehat{\Omega}) \times L^2_0(\widehat{\Omega}) \cap \mathbf{H}^1(\widehat{\Omega}) \times \mathbf{L}^2(\Omega_2) \times \mathbf{H}^{3/2}(\Gamma)$.

We define the nonlinear mapping $\widehat{M} : \widehat{\mathbf{B}}_1 \to \widehat{\mathbf{B}}_2$ by $\widehat{M}(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) = (\widehat{l}_1, \widehat{l}_2, \widehat{l}_3)$ for $(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \in \widehat{\mathbf{B}}_1$ and $(\widehat{l}_1, \widehat{l}_2, \widehat{l}_3) \in \widehat{\mathbf{B}}_2$ if and only if

$$(21) \qquad \begin{cases} \nu a(\widehat{u}, \widehat{v}) + c(\widehat{u}; \widehat{u}, \widehat{v}) + b(\widehat{v}, \widehat{p}) - \Big(\displaystyle\int_{\Omega_2} \widehat{f} \cdot \widehat{v}\, d\vec{x} + \alpha \int_{\Omega_2} \nabla \widehat{f} \cdot \nabla \widehat{v}\, d\vec{x} \Big) \\ \qquad\qquad = \displaystyle\int_{\widehat{\Omega}} \widehat{l}_1 \cdot \widehat{v}\, d\vec{x} \quad \forall\, \widehat{v} \in \mathbf{H}^1_0(\widehat{\Omega}) \\ b(\widehat{u}, \widehat{z}) = \displaystyle\int_{\widehat{\Omega}} \widehat{l}_2\, \widehat{z}\, d\vec{x} \quad \forall\, \widehat{z} \in L^2_0(\widehat{\Omega}) \\ \displaystyle\int_{\Gamma} (\widehat{u} - \vec{g}) \cdot \vec{s}\, ds = \int_{\Gamma} \widehat{l}_3 \cdot \vec{s}\, ds \qquad \forall \vec{s} \in \mathbf{H}^{-1/2}(\Gamma) \end{cases}$$

The constant α is non negative. The restriction of the map $\widehat{M}$ over the domain Ω gives the map M. We note that in the problem $\widehat{M}(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) = (\widehat{0}, \widehat{0}, \widehat{0})$ the third equation is independent and gives $\vec{g}$ as the trace of $\widehat{u}$ over the boundary Γ.

Also we consider the nonlinear map $\widehat{Q} : \widehat{\mathbf{B}}_1 \to \Re \times \widehat{\mathbf{B}}_2$ defined by

$$(22) \qquad \widehat{Q}(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) = \begin{pmatrix} J(\widehat{u}, \widehat{f}) - J(\widetilde{u}, \widetilde{f}) \\ \widehat{M}(\widehat{u}, \widehat{p}, \vec{g}) \end{pmatrix}$$

where $(\widetilde{u}, \widetilde{p}, \widetilde{f}, \widetilde{g})$ is an optimal solution. In order to find the explicit form of the optimal solution we must prove strict differentiability (see [12, 42]).

Let X and Y denote Banach spaces, then the mapping $\varphi : X \to Y$ is strictly differentiable at $x \in X$ if there exists a bounded, linear mapping D from X to Y such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\|x - x_1\|_X < \delta$ and $\|x - x_2\|_X < \delta$ for $x_1, x_2 \in X$, then

$$\|\varphi(x_1) - \varphi(x_2) - D(x_1 - x_2)\|_Y \le \epsilon \|x_1 - x_2\|_X\,.$$

The strict derivative D at the point $x \in X$, if it exists, will often be denoted by $D = \varphi'(x)$. The value of this mapping on an element $\widetilde{x} \in X$ will often be denoted by $\varphi'(x) \cdot \widetilde{x}$. In the next theorem we can identify $X = \widehat{\mathbf{B}}_1$ and $Y = \widehat{\mathbf{B}}_2$.

Lemma 1. *Let the nonlinear mappings $\widehat{M} : \widehat{\mathbf{B}}_1 \to \widehat{\mathbf{B}}_2$ and $Q : \widehat{\mathbf{B}}_1 \to \Re \times \widehat{\mathbf{B}}_2$ be defined by* (21) *and* (22)*, respectively. Then, these mappings are strictly differentiable at a point $(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \in \widehat{\mathbf{B}}_1$ and its strict derivative is given by the bounded linear operator $\widehat{M}'(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) : \widehat{\mathbf{B}}_1 \to \widehat{\mathbf{B}}_2$, where $\widehat{M}'(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \cdot (\widetilde{u}, \widetilde{p}, \widetilde{f}, \widetilde{g}) = (\bar{l}_1, \bar{l}_2, \bar{l}_3)$ for $(\widetilde{u}, \widetilde{p}, \widetilde{f}, \widetilde{g}) \in \widehat{\mathbf{B}}_1$ and $(\bar{l}_1, \bar{l}_2, \bar{l}_3) \in \widehat{\mathbf{B}}_2$ if and only if*

$$(23) \qquad \begin{cases} \nu a(\widetilde{u}, \widehat{v}) + c(\widetilde{u}; \widehat{u}, \widehat{v}) + c(\widehat{u}; \widetilde{u}, \widehat{v}) + b(\widehat{v}, \widetilde{p}) = \\ \qquad \displaystyle\int_{\Omega_2} \Big(\widetilde{f} \cdot \widehat{v} + \alpha \nabla \widetilde{f} \cdot \nabla \widehat{v} \Big)\, d\vec{x} + \int_{\widehat{\Omega}} \bar{l}_1 \cdot \widehat{v}\, d\vec{x} \quad \forall\, \widehat{v} \in \mathbf{H}^1_0(\widehat{\Omega}) \\ b(\widetilde{u}, \widehat{z}) = \displaystyle\int_{\widehat{\Omega}} \bar{l}_2\, \widehat{z}\, d\vec{x} \quad \forall\, \widehat{z} \in L^2_0(\widehat{\Omega}) \\ \displaystyle\int_{\Gamma} (\widetilde{u} - \widetilde{g}) \cdot \vec{s}\, ds = \int_{\Gamma} \bar{l}_3 \cdot \vec{s}\, ds \quad \forall \vec{s} \in \mathbf{H}^{-1/2}(\Gamma) \end{cases} .$$

Moreover, the strict derivative of Q at a point $(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \in \widehat{\mathbf{B}}_1$ is given by the bounded linear operator $\widehat{Q}'(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) : \widehat{\mathbf{B}}_1 \to \Re \times \widehat{\mathbf{B}}_2$, where $\widehat{Q}'(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \cdot (\widetilde{u}, \widetilde{p}, \widetilde{f}, \widetilde{g}) =$

$(\overline{a}, \overline{l}_1, \overline{l}_2, \overline{l}_3)$, *for* $(\widetilde{u}, \widetilde{p}, \widetilde{f}, \widetilde{g}) \in \widehat{\mathbf{B}}_1$ *and* $(\overline{a}, \overline{l}_1, \overline{l}_2, \overline{l}_3) \in \Re \times \widehat{\mathbf{B}}_2$ *if and only if*

$$\left(\begin{array}{c} J'(\widehat{u}, \widehat{f}) \cdot (\widetilde{u}, \widetilde{p}, \widetilde{f}, \widetilde{g}) \\ M'(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \cdot (\widetilde{u}, \widetilde{p}, \widetilde{f}, \widetilde{g}) \end{array} \right) = \left(\begin{array}{c} \overline{a} \\ (\overline{l}_1, \overline{l}_2, \overline{l}_3) \end{array} \right), \tag{24}$$

where

$$J'(\widehat{u}, \widehat{f}) \cdot (\widetilde{u}, \widetilde{f}) = \int_\Omega (\widehat{u} - \vec{U}) \cdot \widetilde{u} \, d\vec{x} + \beta \int_{\Omega_2} \left(\widehat{f} \, \widetilde{f} + \alpha (\nabla \widehat{f} \cdot \nabla \widetilde{f}) \right) d\vec{x} \, .$$

Proof: The linearity of the operator $\widehat{M'}(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g})$ is obvious and its boundedness follows from the continuity of the forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, and $c(\cdot, \cdot, \cdot)$ combined with the trace theorem for Sobolev spaces. The fact that $\widehat{M'}(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g})$ is the strict derivative of the mapping $\widehat{M}(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g})$ also follows from the continuity of the trilinear form $c(\cdot, \cdot, \cdot)$. Indeed, we have that for any $(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \in \widehat{\mathbf{B}}_1$ and for all $(\widehat{w}, \widehat{r}, \widehat{\theta}, \vec{\eta}) \in \widehat{\mathbf{B}}_2^*$,

$$\begin{aligned} &\Big\langle (\widehat{w}, \widehat{r}, \widehat{\theta}, \vec{\eta}), M(\widehat{u}_1, \widehat{p}_1, \widehat{f}_1, \vec{g}_1) - M(\widehat{u}_2, \widehat{p}_2, \widehat{f}_2, \vec{g}_2) \\ &-M'(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \cdot (\widehat{u}_1 - \widehat{u}_2, \widehat{p}_1 - \widehat{p}_2, \widehat{f}_1 - \widehat{f}_2, \vec{g}_1 - \vec{g}_2) \Big\rangle = \end{aligned}$$

$$\begin{aligned} &\nu a(\widehat{u}_1, \widehat{w}) + c(\widehat{u}_1; \widehat{u}_1, \widehat{w}) + b(\widehat{w}, \widehat{p}_1) - (\widehat{f}_1, \widehat{w})_{\Omega_2} - \alpha (\nabla \widehat{f}_1, \nabla \widehat{w})_{\Omega_2} + b(\widehat{u}_1, \widehat{r}) \\ &-\Big(\nu a(\widehat{u}_2, \widehat{w}) + c(\widehat{u}_2; \widehat{u}_2, \widehat{w}) + b(\widehat{w}, \widehat{p}_2) - (\widehat{f}_2, \widehat{w})_{\Omega_2} - \alpha (\nabla \widehat{f}_2, \nabla \widehat{w})_{\Omega_2} + b(\widehat{u}_2, \widehat{r}) \Big) \\ &-\Big(\nu a(\widehat{u}_1 - \widehat{u}_2, \widehat{w}) + c(\widehat{u}_1 - \widehat{u}_2; \widehat{u}, \widehat{w}) + c(\widehat{u}; \widehat{u}_1 - \widehat{u}_2, \widehat{w}) + b(\widehat{w}, \widehat{p}_1 - \widehat{p}_2) \\ &+b(\widehat{u}_1 - \widehat{u}_2, \widehat{r}) - (\widehat{f}_1 - \widehat{f}_2, \widehat{w})_{\Omega_2} - \alpha (\nabla (\widehat{f}_1 - \widehat{f}_2), \nabla \widehat{w})_{\Omega_2} \Big) \\ &+ \int_\Gamma \vec{\eta} \cdot (\widehat{u}_1 - \vec{g}_1) \, ds - \int_\Gamma \vec{\eta} \cdot (\widehat{u}_2 - \vec{g}_2) \, ds - \int_\Gamma \vec{\eta} \cdot (\widehat{u}_1 - \widehat{u}_2 - (\vec{g}_1 - \vec{g}_2)) \, ds \, . \end{aligned}$$

Therefore, we have

$$\begin{aligned} &|\Big\langle (\widehat{w}, \widehat{r}, \widehat{\theta}, \vec{\eta}), M(\widehat{u}_1, \widehat{p}_1, \widehat{f}_1, \vec{g}_1) - M(\widehat{u}_2, \widehat{p}_2, \widehat{f}_2, \vec{g}_2) \\ &-M'(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \cdot (\widehat{u}_1 - \widehat{u}_2, \widehat{p}_1 - \widehat{p}_2, \widehat{f}_1 - \widehat{f}_2, \vec{g}_1 - \vec{g}_2) \Big\rangle| \le \\ &|c(\widehat{u}_1 - \widehat{u}_2, \widehat{u} - \widehat{u}_1, \widehat{w}) + c(\widehat{u} - \widehat{u}_2, \widehat{u}_1 - \widehat{u}_2, \widehat{w})| \end{aligned}$$

Then, by using the continuity of the form $c(\cdot, \cdot, \cdot)$, the Sobolev imbedding theorem, and the trace theorem, we have, for some constants $C_1, C_2 > 0$, that

$$\begin{aligned} &\|\widehat{M}(\widehat{u}_1, \widehat{p}_1, \widehat{f}_1, \vec{g}_1) - \widehat{M}(\widehat{u}_2, \widehat{p}_2, \widehat{f}_2, \vec{g}_2) - \widehat{M'}(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \cdot (\widehat{u}_1 - \widehat{u}_2, \widehat{p}_1 - \widehat{p}_2, \widehat{f}_1 - \widehat{f}_2, \vec{g}_1 - \vec{g}_2)\|_{\widehat{\mathbf{B}}_1} \le \\ &\quad C_1 (\|\widehat{u}_1 - \widehat{u}_2\|_1 \, \|\widehat{u} - \widehat{u}_1\|_1 + \|\widehat{u}_1 - \widehat{u}_2\|_1 \, \|\widehat{u} - \widehat{u}_2\|) \le \\ &\quad C_2 \|(\widehat{u}_1 - \widehat{u}_2, \widehat{p}_1 - \widehat{p}_2, \widehat{f}_1 - \widehat{f}_2, \vec{g}_1 - \vec{g}_2)\|_{\widehat{\mathbf{B}}_1} \\ &\qquad \Big(\|(\widehat{u} - \widehat{u}_1, \widehat{p} - \widehat{p}_1, \widehat{f} - \widehat{f}_1, \vec{g} - \vec{g}_1)\|_{\widehat{\mathbf{B}}_1} + \|(\widehat{u} - \widehat{u}_2, \widehat{p} - \widehat{p}_2, \widehat{f} - \widehat{f}_2, \vec{g} - \vec{g}_2)\|_{\widehat{\mathbf{B}}_1} \Big) \, . \end{aligned}$$

Then, for any $\epsilon > 0$, by choosing $\delta = \epsilon C_2 / 2$, we have that, whenever $\|(\widehat{u} - \widehat{u}_1, \widehat{p} - \widehat{p}_1, \widehat{f} - \widehat{f}_1, \vec{g} - \vec{g}_1)\|_{\widehat{\mathbf{B}}_1} < \delta$ and $\|(\widehat{u} - \widehat{u}_2, \widehat{p} - \widehat{p}_2, \widehat{f} - \widehat{f}_2, \vec{g} - \vec{g}_2)\|_{\widehat{\mathbf{B}}_1} < \delta$,

$$\begin{aligned} &\|\widehat{M}(\widehat{u}_1, \widehat{p}_1, \widehat{f}_1, \vec{g}_1) - \widehat{M}(\widehat{u}_2, \widehat{p}_2, \widehat{f}_2, \vec{g}_2) \\ &\quad -\widehat{M'}(\widehat{u}, \widehat{p}, \widehat{f}, \vec{g}) \cdot (\widehat{u}_1 - \widehat{u}_2, \widehat{p}_1 - \widehat{p}_2, \widehat{f}_1 - \widehat{f}_2, \vec{g}_1 - \vec{g}_2)\|_{\widehat{\mathbf{B}}_2} \\ &\qquad \le \epsilon \|(\widehat{u}_1 - \widehat{u}_2, \widehat{p}_1 - \widehat{p}_2, \widehat{f}_1 - \widehat{f}_2, \vec{g}_1 - \vec{g}_2))\|_{\widehat{\mathbf{B}}_1} \, . \end{aligned}$$

Thus, the mapping $\widehat{M}$ is strictly differentiable on all of $\widehat{\mathbf{B}}_1$ and its strict derivative is given by M'.

Likewise, the linearity and boundedness of the operator $\widehat{Q}'(\widehat{u},\widehat{p},\widehat{f},\vec{g})$ are obvious. Using the strict differentiability of the mapping $\widehat{M}$ it is then easy to show that the mapping $\widehat{Q}$ is also strictly differentiable and that its strict derivative is given by $\widehat{Q}'$. □

Next, we prove some further properties of the derivatives of the mappings $\widehat{M}$ and $\widehat{Q}$ that are necessary to find the Euler-equations for the optimal solution.

Lemma 2. *Let $(\widehat{u},\widehat{p},\widehat{f},\vec{g}) \in \widehat{\mathbf{B}}_1$ denote a solution of the optimal control problem. Then we have*

: *i) the operator $\widehat{M'}(\widehat{u},\widehat{p},\widehat{f},\vec{g})$ has closed range in $\widehat{\mathbf{B}}_2$;*
: *ii) the operator $\widehat{Q}'(\widehat{u},\widehat{p},\widehat{f},\vec{g})$ has closed range in $\Re \times \widehat{\mathbf{B}}_2$;*
: *iii) the operator $\widehat{Q}'(\widehat{u},\widehat{p},\widehat{f},\vec{g})$ is not onto $\Re \times \widehat{\mathbf{B}}_2$.*

Proof: First we prove the theorem for $\alpha = 0$. In order to show i), we shall show that the range of the map $\widehat{M'}$ is close. The question of the closedness of the range of the operator $\widehat{M'} : \widehat{\mathbf{B}}_1 \to \widehat{\mathbf{B}}_2$ reduces to the like question for the inhomogeneous Stokes operator $\widetilde{S} : \mathbf{H}^2(\widehat{\Omega}) \cap \mathbf{H}^1_0(\widehat{\Omega}) \times H^1(\widehat{\Omega}) \cap L^2_0(\widehat{\Omega}) \to \mathbf{H}^{-1}(\widehat{\Omega}) \times L^2_0(\widehat{\Omega})$ defined as follows: $\widetilde{S} \cdot (\widetilde{w},\widetilde{p}) = (\widetilde{l}_1,\widetilde{l}_2,\widetilde{l}_2)$ if and only if

$$
\begin{cases}
\nu a(\widetilde{w},\vec{v}) + b(\vec{v},\widetilde{p}) - (\vec{v},\widetilde{f}) = <\widetilde{l}_1,\vec{v}> \quad \forall\, \vec{v} \in \mathbf{H}^1_0(\widehat{\Omega}) \\
b(\widetilde{w},z) = (\widetilde{l}_2,z) \quad \forall\, z \in L^2(\widehat{\Omega}) \,,
\end{cases}
\tag{25}
$$

where $\widetilde{f} \in \mathbf{L}^2(\Omega_2)$, $\vec{g} \in \mathbf{H}^{3/2}(\Gamma)$, and Γ is piecewise $C^{1,1}$ with convex corners. The fact that the operator $\widetilde{S}$ has closed range in $\mathbf{H}^{-1}(\widehat{\Omega}) \times L^2_0(\widehat{\Omega})$ follows easily from well-known results for the Stokes equations; see, e.g., [40]. We can then conclude that the operator $\widetilde{S}$ has closed range in $\widehat{\mathbf{B}}_2$, and, since the operator $\widehat{M'}(\widehat{u},\widehat{p},\widehat{f},\vec{g})$ is a compact perturbation of the operator $\widetilde{S}$, we have, from the Fredholm theory, that $\widehat{M'}(\widehat{u},\widehat{p},\widehat{f},\vec{g})$ itself has closed range in $\widehat{\mathbf{B}}_2$. Starting from i), the proof of ii) and iii) can be found easily by using the standard techniques in [18, 19, 20, 25].

For $\alpha > 0$ the operator $\widehat{M'}(\widehat{u},\widehat{p},\widehat{f},\vec{g})$ is a compact perturbation of the corresponding operator $\widetilde{S} : \mathbf{H}^1_0(\widehat{\Omega}) \times L^2_0(\widehat{\Omega}) \to \mathbf{H}^{-1}(\widehat{\Omega}) \times L^2_0(\widehat{\Omega})$ and the proof can be found in a similar way. □

The first-order necessary condition follows easily from the fact that the operator $\widehat{Q}'(\widehat{u},\widehat{p},\widehat{f},\vec{g})$ is not onto $\Re \times \widehat{\mathbf{B}}_2$; see, e.g., [18, 24, 25].

Theorem 7. *Given $(\widehat{u},\widehat{p},\widehat{f},\vec{g}) \in \mathbf{A}_\alpha$. If $(\widehat{u},\widehat{p},\widehat{f},\vec{g}) \in \widehat{\mathbf{B}}_1$ is a solution of the optimal control problem, then there exists a nonzero Lagrange multiplier $(\lambda,\widehat{w},\widehat{r},\vec{\theta},\vec{\eta}) \in \Re \times \widehat{\mathbf{B}}^*_2$ satisfying the Euler equations*

$$
\begin{gathered}
\lambda\, J'(\widehat{u},\widehat{f}) \cdot (\widetilde{u},\widetilde{f}) + \Big\langle (\widehat{w},\widehat{r},\vec{\theta},\vec{\eta}), \widehat{M'}(\widehat{u},\widehat{p},\widehat{f},\widehat{g}) \cdot (\widetilde{u},\widetilde{p},\widetilde{f},\widetilde{g}) \Big\rangle = 0 \\
\forall\, (\widetilde{u},\widetilde{p},\widetilde{f},\widetilde{g})) \in \widehat{\mathbf{B}}_1 \,,
\end{gathered}
\tag{26}
$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $\widehat{\mathbf{B}}_2$ and $\widehat{\mathbf{B}}^*_2$.

2.4. The optimality system. In this section we compare our optimality system for boundary control with those available in literature. The non-extended domain solution $(\vec{u},p,\vec{g},\vec{w},\sigma,k) \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_{\Gamma_1}(\Omega) \times L^2_0(\Omega) \cap \mathbf{H}^1(\Omega) \times \mathbf{H}^{3/2}(\Omega) \times \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \cap \mathbf{H}^1(\Omega) \times \Re$ of the problem in (1-4) with the functional (5) solves the following optimality system which consists of the Navier-Stokes system [20, 18, 19]

$$
\begin{cases}
\nu a(\vec{u},\vec{v}) + c(\vec{u};\vec{u},\vec{v}) + b(\vec{v},p) = 0 \quad \forall\, \vec{v} \in \mathbf{H}^1_0(\Omega) \\
b(\vec{u},q) = 0 \quad \forall\, q \in L^2_0(\Omega) \\
<\ (\vec{u} - \vec{g}) \cdot \vec{s}\ >= 0 \qquad \forall \vec{s} \in \mathbf{H}^{-1/2}(\Gamma)
\end{cases} , \tag{27}
$$

the adjoint system

$$
\begin{cases}
\nu a(\vec{w},\vec{v}) + c(\vec{w};\vec{u},\vec{v}) + c(\vec{u};\vec{w},\vec{v}) + b(\vec{v},\sigma) = \\
\qquad\qquad \displaystyle\int_\Omega (\vec{u} - \vec{U}) \cdot \vec{v}\, d\vec{x} \quad \forall\, \vec{v} \in \mathbf{H}^1_0(\Omega) \\
b(\vec{w},q) = 0 \quad \forall\, q \in L^2_0(\Omega)\,,
\end{cases} \tag{28}
$$

with the control equation

$$
<\vec{g},\vec{s}>_\Gamma +\alpha <\vec{g}_s,\vec{s}_s>_\Gamma +k <\vec{n},\vec{s}>_\Gamma = \frac{1}{\beta} <\gamma_1 \vec{w},\vec{s}>_\Gamma \qquad \forall \vec{s} \in \mathbf{H}^{1/2}(\Gamma)\,, \tag{29}
$$

and the compatibility equation

$$
\int_\Gamma \vec{g} \cdot \vec{n}\, ds = 0\,. \tag{30}
$$

The operator γ_1 is the trace operator of order one. The use of standard finite element approximation in $\mathbf{H^1}$ gives $\gamma_1 \vec{w} \in \mathbf{H^{-1/2}}$. In order to compute numerically $\vec{g}$ we must choose α positive. This determines the admissible set of $\vec{g}$ in $H^{3/2}(\Gamma)$ for the problem (1-4) with the functional (5). If $\vec{g}$ is desired in $H^{1/2}(\Gamma)$ then the equation (30) must be solved with non standard methods or $\nabla\vec{w}$ must be approximated in $\mathbf{H}^1(\Omega)$ with a mixed finite element method. This implementation is rather demanding and therefore the control $\vec{g}$ must be assumed in $H^{3/2}(\Gamma)$. Also we remark that, in some cases of Dirichlet boundary conditions and normal control the constant k should be computed by enforcing the compatibility condition (30).

Now we focus on our new extended approach. The Theorem 7 implies that optimal control solutions must satisfy a first-order necessary condition. The solutions $(\widehat{u},\widehat{p},\widehat{g},\widehat{f},\widehat{w},\widehat{\sigma}) \in \mathbf{H}^1_0(\widehat{\Omega}) \times L^2_0(\widehat{\Omega}) \times \mathbf{H}^{1/2}(\Gamma) \times \mathbf{L}^2(\widehat{\Omega}) \times \mathbf{H}^1_0(\widehat{\Omega}) \times L^2_0(\widehat{\Omega})$ must satisfy the following Navier-Stokes system

$$
\begin{cases}
\nu a(\widehat{u},\widehat{v}) + c(\widehat{u};\widehat{u},\widehat{v}) + b(\widehat{v},\widehat{p}) = \\
\qquad\qquad <\widehat{f},\widehat{v}> +\alpha <\nabla\widehat{f},\nabla\widehat{v}> \quad \forall\, \widehat{v} \in \mathbf{H}^1_0(\widehat{\Omega}) \\
b(\widehat{u},\widehat{q}) = 0 \quad \forall\, \widehat{q} \in L^2_0(\widehat{\Omega})
\end{cases} \tag{31}
$$

and the adjoint system

$$
\begin{cases}
\nu a(\widehat{w},\widehat{v}) + c(\widehat{w};\widehat{u},\widehat{v}) + c(\widehat{u};\widehat{w},\widehat{v}) + b(\widehat{v},\widehat{\sigma}) \\
\qquad\qquad = \displaystyle\int_{\Omega_2} (\vec{u} - \vec{U}) \cdot \widehat{v}\, d\vec{x} \quad \forall\, \widehat{v} \in \mathbf{H}^1_0(\widehat{\Omega}) \\
b(\widehat{w},\widehat{q}) = 0 \quad \forall\, \widehat{q} \in L^2_0(\widehat{\Omega})\,,
\end{cases} \tag{32}
$$

with

$$
<\widehat{u},\widehat{s}>_\Gamma = <\widehat{g},\widehat{s}>_\Gamma \qquad \forall \widehat{s} \in \mathbf{H}^{-1/2}(\Gamma)\,, \tag{33}
$$

and the control equation

$$
\beta\Big(<\widehat{f},\widehat{v}> +\alpha <\nabla\widehat{f},\nabla\widehat{v}> \Big) = \\
<\widehat{w},\widehat{v}> +\alpha <\nabla\widehat{w},\nabla\widehat{v}> \quad \forall\, \widehat{v} \in \mathbf{H}^1(\Omega_2)\,. \tag{34}
$$

The equations (33-35) imply $\widehat{f} = \widehat{w}/\beta$ over Ω_2 and $\vec{g} = \gamma_\Gamma \widehat{u}$. The optimality system for the boundary control is reduced to a distributed optimal control problem which requires much less computational resources than the optimality system in (27-30) .

There are many advantages in this extended formulation. In fact the optimality system is a full distributed control which is less sensitive to numerical errors. The optimality sistem (31-35) can be regularized by changing the single parameter β and the compatibility condition is automatically satisfied. The tangential control can be numerically achieved by using non-embedded techniques but in some cases the compatibility constraint may be a limit to the feasibility of the normal boundary controls. In fact in order to use standard finite element methods, solve (27-30) and prove differentiability necessary to apply the first order condition the function $\vec{g}$ must belong to $C^0(\Gamma_c)$ with piecewise $C^{1,1}$ boundary.

3. Numerical computation of the boundary control problem

3.1. The finite element optimality system. Let Ω_h be the square domain in Figure 1 with boundary Γ_h which consists of Γ_{1h}, Γ_{2h} and Γ_{ch}. We impose Dirichlet boundary conditions over Γ_{1h}, homogeneous Neumann boundary conditions over Γ_{2h} and boundary controls over Γ_{ch}. Also let $\widehat{\Omega}_h$ be the extended domain and $\widehat{\Omega}_{1h}$ be the controlled domain. In this paper computations of the optimality system (31-35) are performed by using a new multigrid approach. The implementation is based on a local Vanka-type solver for the Navier-Stokes and the adjoint system where solution is achieved by solving and relaxing local optimal control problems. The multigrid smoother operator is constructed directly from the optimal control formulation and requires the iterative exact solution of the optimality system over a limited number of unknowns. Now, by starting at the multigrid coarse level l_0 we subdivide $\widehat{\Omega}_h$ into triangles or rectangles by unstructured families of meshes T_h^{i,l_0}. Based on the simple element midpoint refinement different multigrid levels can be built to reach a complete unstructured mesh $T_h^{i,l}$ for finite element over the entire domain Ω_h at the top finest multigrid level l_{n_t}.

We introduce the approximation spaces $\mathbf{X}_{h_l} \subset \mathbf{H}^1(\widehat{\Omega})$ and $S_{h_l} \subset L^2(\widehat{\Omega})$ for the extended velocity and pressure respectively at the multigrid level l. The approximate function obeys to the standard approximation properties including the LBB-condition. Let $P_{h_l} = X_{h_l}|_{\partial\widehat{\Omega}}$, i.e., P_{h_l} consists of all the restrictions, to the boundary $\partial\widehat{\Omega}$, of functions belonging to X_{h_l}. For all choices of conforming finite element space X_h we then have that $P_{h_l} \subset H^{-\frac{1}{2}}(\partial\widehat{\Omega})$. See [24] for details concerning these approximation spaces. The extended velocity and pressure fields $(\widehat{u}_{h_l}, \widehat{p}_{h_l}) \in \mathbf{X}_{h_l}(\widehat{\Omega}_h) \times S_{h_l}(\widehat{\Omega}_h)$ at the level l satisfy the Navier-Stokes equations

$$
(35)\begin{cases}
a(\widehat{u}_{h_l}, \widehat{v}_{h_l}) + c(\widehat{u}_{h_l}; \widehat{u}_{h_l}, \widehat{v}_{h_l}) + b(\widehat{v}_{h_l}, \widehat{p}_{h_l}) = \\
< \widehat{f}_{h_l}, \widehat{v}_{h_l} >_{\Omega_2} + \alpha a(\widehat{f}_{h_l}, \widehat{v}_{h_l})_{\Omega_2} \quad \forall \widehat{v}_{h_l} \in \mathbf{X}_{h_l}(\widehat{\Omega}_h) \cap \mathbf{H}^1_{\widehat{\Gamma}_h - \Gamma_{2h}}(\widehat{\Omega}_h) \\
b(\widehat{u}_{h_l}, \widehat{r}_{h_l}) = 0 \quad \forall \widehat{r}_{h_l} \in S_{h_l}(\widehat{\Omega}_h) \\
< \widehat{u}_{h_l}, \widehat{s}_{h_l} >_{\widehat{\Gamma}_h - \Gamma_{2h}} = < \vec{g}, \widehat{s}_{h_l} >_{\widehat{\Gamma}_h - \Gamma_{2h}} \quad \forall \widehat{s}_{h_l} \in \mathbf{P}_{h_l}(\widehat{\Gamma}_h)
\end{cases}
$$

and the adjoint system

$$(36)\quad \begin{cases} a(\widehat{w}_{h_l}, \widehat{v}_{h_l}) + c(\widehat{w}_{h_l}; \widehat{u}_{h_l}, \widehat{v}_{h_l}) + c(\widehat{u}_{h_l}; \widehat{w}_{h_l}, \widehat{v}_{h_l}) + b(\widehat{v}_{h_l}, \widehat{\sigma}_{h_l}) \\ \qquad = \displaystyle\int_{\Omega_1} (\vec{u}_{h_l} - \vec{U}) \cdot \widehat{v}_{h_l} \, d\vec{x} \quad \forall \widehat{v}_{h_l} \in \mathbf{X}_{h_l}(\widehat{\Omega}_h) \cap \mathbf{H}^1_{\widehat{\Gamma}_h - \Gamma_{2h}}(\widehat{\Omega}_h) \\ b(\widehat{w}_{h_l}, \widehat{q}_{h_l}) = 0 \quad \forall \widehat{q}_{h_l} \in S_{h_l}(\widehat{\Omega}_h) \\ < \widehat{w}_{h_l}, \widehat{s}_{h_l} >_{\widehat{\Gamma_h} - \Gamma_{2h}} = 0 \qquad \forall \widehat{s}_{h_l} \in \mathbf{P}_{h_l}(\widehat{\Gamma}_h) \ , \end{cases}$$

with

$$(37)\qquad \vec{g}_{ch_l} = \gamma_{\Gamma_c} \widehat{u}_{h_l}$$

and $\widehat{f}_{h_l} = \widehat{w}_{h_l}/\beta$ over $\widehat{\Omega}_{2h_l}$. Existence and uniqueness results for finite element solutions of (35) are well known; see, e.g., [29, 23].

The unique representations of $\widehat{u}_{h_l}$,$\widehat{w}_{h_l}$ and $\widehat{p}_{h_l}$, $\widehat{\sigma}_{h_l}$ as a function of the nodal point values $\widehat{u}_l(k_1)$, $\widehat{w}_l(k_1)$ and $\widehat{p}_l(k_2)$, $\widehat{\sigma}_l(k_2)$ ($k_1 = 1, 2, ...nvt$ with $nvt =$ number of vertex velocity points and $k_2 = 1, 2, ...npt$ with $npt =$ number of vertex pressure points) define the finite element isomorphisms $\Phi_l : U_l \to X_{hl}$, $\Phi_l^+ : W_l \to X_{hl}$, $\Psi_l : \Pi_l \to S_{hl}$ $\Psi_l^+ : \Sigma_l \to S_{hl}$ between the vector spaces U_l, W_l, Π_l, Σ_l of *nvt*-dimension and *npt*-dimension vectors and the finite element spaces X_{h_l}, S_{h_l}(see [10, 33] for details) .

Essential elements of a multigrid algorithm are the velocity and pressure prolongation maps

$$P_{l,l-1}(u) : U_{l-1} \to U_l \qquad P_{l,l-1}(p) : \Pi_{l-1} \to \Pi_l$$

and the velocity and restriction operators

$$R_{l-1,l}(u) = P^*_{l,l-1}(u) : U_l \to U_{l-1} \qquad R_{l-1,l}(u) = P^*_{l,l-1}(u) : \Pi_l \to \Pi_{l-1} \, .$$

Since we would like to use conforming Taylor-Hood finite element approximation spaces we have the nested finite element hierarchies $X_{h_0} \subset X_{h_1} \subset ... \subset X_{h_l}$ and $S_{h_0} \subset S_{h_1} \subset ... \subset S_{h_l}$ and the canonical prolongation maps $P_{l,l-1}(u)$, $P_{l,l-1}(p)$ can be obtained simply by

$$(38)\qquad P_{l,l-1}(u) = \Phi_{l-1}(\Phi_l^{-1}(u))$$

$$(39)\qquad P_{l,l-1}(p) = \Psi_{l-1}(\Psi_l^{-1}(p)) \, .$$

For details and properties one can consult [33, 37] and citations therein.

We solve the coupled system by using an iterative method. Multigrid solvers for coupled velocity/pressure system compute simultaneously the solution for both pressure and velocity and they are known to be ones of the best class of solvers for laminar Navier-Stokes equations (see [34, 39]). An iterative coupled solution of the linearized and discretized incompressible Navier-Stokes equations requires the approximate solution of sparse saddle point problems. In this multigrid approach the most suitable class of solvers is the Vanka-type smoothers. They can be considered as block Gauss-Seidel methods where one block consists of a small number of degrees of freedom (for details see [39, 33, 34]). The characteristic feature of this type of smoother is that in each smoothing step a large number of small linear systems of equations has to be solved. In the Vanka-type smoother, a block consists of all degrees of freedom which are connected to few neighboring elements. As shown in Fig.2 for conforming finite elements the block could consist of all the elements containing a pressure vertex or four pressure nodes , namely 21 velocity nodes (circles and squares) with one pressure node (square) or 16 velocity nodes (circles and squares) with four pressure nodes (squares) respectively. Thus, in the

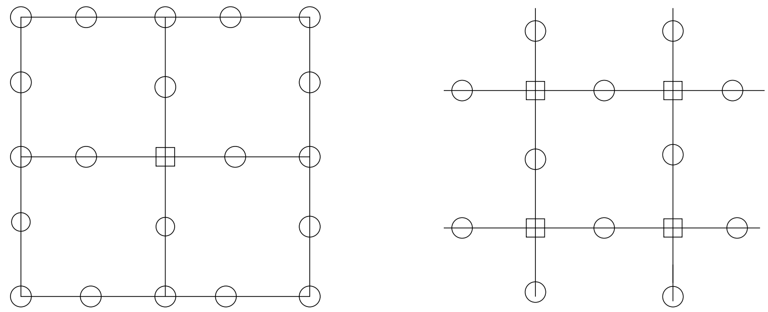


FIGURE 2. Blocks of unknowns: $21V+1P$ (on the left) and $16V+4P$ (on the right)

first case a relaxation step with this Vanka-type smoother consists of the iterative solution of the corresponding block of equations over all the pressure nodes. In the second case a relaxation step consists of the solution of the block of equations over all the elements where the velocity and pressure variables are updated iteratively. Different blocks of unknowns can be solved including local constraints as they arise from the optimal control problem. For convergence and properties of this class of smoothers one can consult [39, 33, 34] and citations therein.

3.2. Boundary control test 1. We consider the square domain $\Omega = [0,.5]\times[0,.5]$ in Figure 1 with boundary Γ. The boundary Γ consists of Γ_1 and Γ_c where the homogeneous Dirichlet boundary conditions and controls are applied respectively. We set $\widehat{\Omega} = [0,1]\times[0,1]$, $1/\nu = 100$ and $\alpha = 1\times 10^{-3}$. The target velocity $\vec{U} = (\frac{1}{2}(cos(4\pi x)-1)\,sin(4\pi y), \frac{1}{2}(cos(4\pi y)-1)\,sin(4\pi x))$ is enforced over the domain $\Omega_1 = [0,.35]\times[0,.35]$.

In Figure 3 the solution of the optimality system is obtained as described in the previous section and shown in Figure 3 for the parameter $\beta = 1\times 10^{-4}$. The velocity and the adjoint field are shown over the extended domain $\widehat{\Omega}$ on the top left and top right respectively. The velocity field $\vec{u}$ is shown over the domain Ω and over the matching domain Ω_1 on the bottom left and center respectively together with the target velocity $\vec{U}$ (bottom right). We note that the boundary control achieves some matching of the desired flow if the normal and the tangential control are combined. This embedded method can handle the normal control in a relative straightforward manner, satisfies the compatibility constraint and improves the effectiveness of the control. In these computations $\Omega_1 \neq \Omega$ since a better matching can be reached over small controlled area Ω_1. In the Figure 4 we show the velocity over Ω_1 for decreasing values of the penalty parameter β. From the top left to the bottom left β is equal to 1×10^{-2},1×10^{-3} and 1×10^{-4}. The target velocity $\vec{U}$ is shown on the bottom right over $\Omega_1 = [0,.35]\times[0,.35]$. The controlled boundary Γ_{ch} consists of a vertical and an horizontal part. Figure 5 shows the boundary control on the horizontal part of Γ_c for $\beta = 1\times 10^{-4}$ (A), 1×10^{-3} (B) and 1×10^{-2} (C). The u-component is shown on the left and the v-component on the right. In a similar way Figure 6 shows the u-component (on the left) and the v-component (on the right) of the vertical part of the controlled boundary Γ_c for $\beta = 1\times 10^{-4}$ (A), 1×10^{-3} (B) and 1×10^{-2} (C). We note that the controlled normal component of the boundary control may be positive and negative, namely there is injection and suction along the same portion of the boundary. If a standard non-embedded method is used the normal component of the control must satisfy the integral equation (4) and this

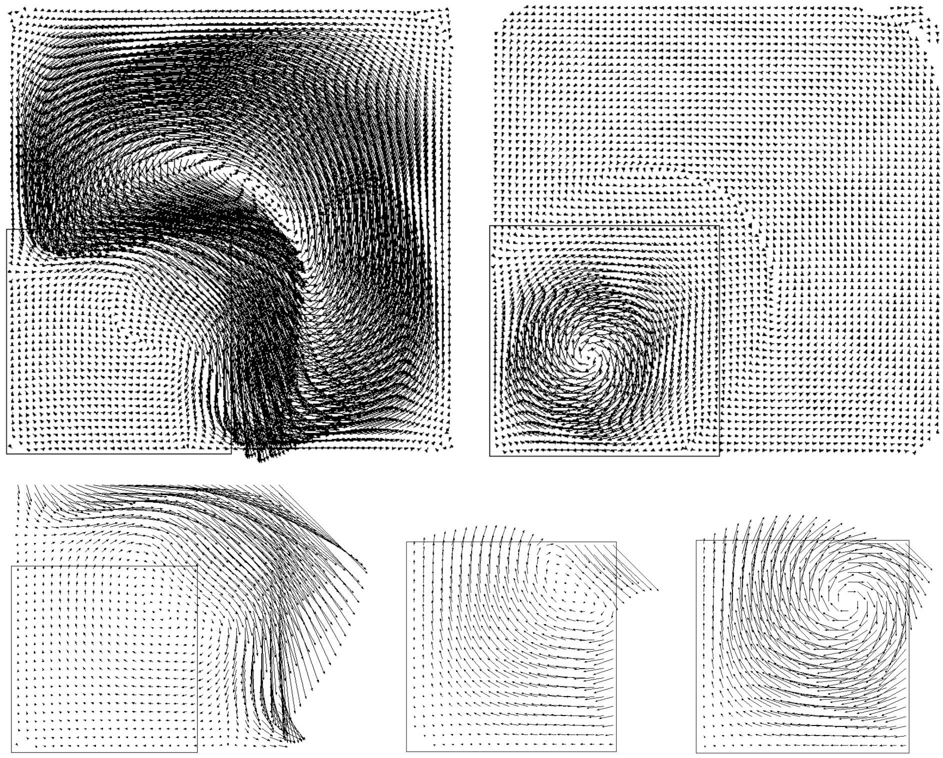


FIGURE 3. Velocity (top left) and adjoint field (top right) over the extended domain $\widehat{\Omega} = [0,1]\times[0,1]$. Controlled flow for $\beta = 1\times10^{-4}$ and $\alpha = 1\times10^{-3}$ over Ω (bottom right), over Ω_1 (bottom middle) and the target velocity $\vec{U}$ over Ω_1 (bottom left).

may be numerically very challenging. Also this technique solves the corner point, intersection between the horizontal and the vertical part of Γ_c, in a straightforward manner while in the boundary control (27-30) it must be fixed by an artificial boundary condition which may limit the strength of the control. The boundary control is in some way effective as we can see in Figure 7. For different value of β ($\beta = 1\times10^{-4}$ (A), $\beta = 1\times10^{-3}$ (B) $\beta = 1\times10^{-2}$ (C)) we can see the value of the v-component (on the left) and of the u-component (on the right) with the target velocity along the horizontal line of coordinates $y = 0.2$ with $0 \le x \le .35$. For decreasing values of the parameters β is possible to decrease the matching distance between $\vec{u}$ and $\vec{U}$ as shown in Table 1.

β	$\Vert\vec{u}-\vec{U}\Vert_{\Omega_1}/\Vert\vec{U}\Vert_{\Omega_1}$
0.01	4.48e-1
0.001	2.34e-1
0.0001	1.32e-1
0.00001	1.31e-1

TABLE 1. Values of $\Vert\vec{u}-\vec{U}\Vert_{\Omega_1}/\Vert\vec{U}\Vert_{\Omega_1}$ for different β.

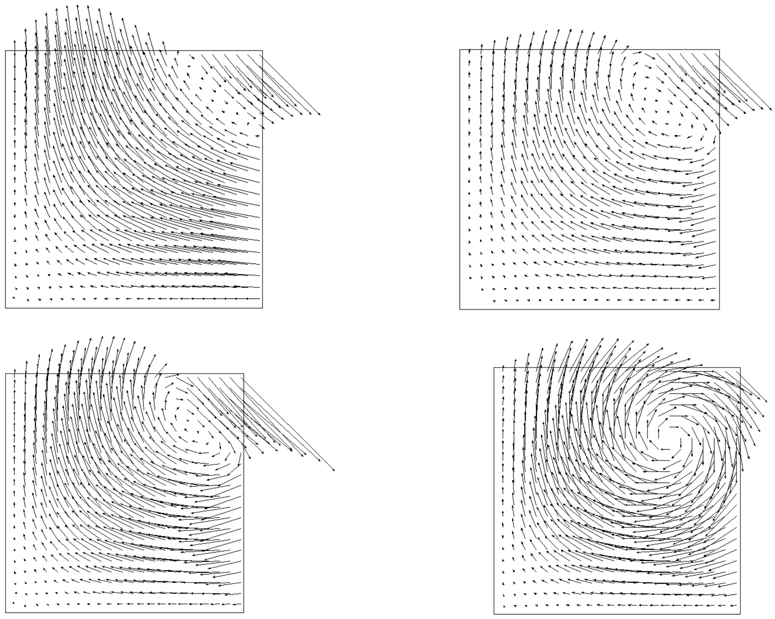


Figure 4. Controlled flow over $\widehat{\Omega}_1$. Controlled flow for $\beta = 1 \times 10^{-2}$ (top left), 1×10^{-3} (top right) and 1×10^{-4} (bottom left) and the desired flow (bottom right).

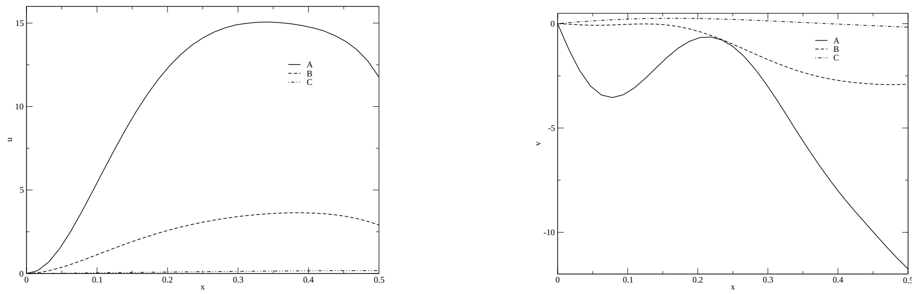


Figure 5. Boundary control (u-component on the left and v-component on the right) on the horizontal part of Γ_c for $\beta = 1 \times 10^{-4}$ (A), 1×10^{-3} (B) and 1×10^{-2} (C).

3.3. Boundary control test 2. In the second numerical experiment we would like to illustrate an example where boundary controls can be efficiently applied to more real situations. Suppose we have a velocity regulator where the inflow over Γ_1 is assigned and the fluid motion near to the output Γ_2 must be controlled by injection or suction along a portion of the boundary Γ_c. In order to model the problem we introduce, as shown in Figure 8 on the left, a L-shape domain with eight small cavities. The cavities are part of the real design and represent the area where the fluid may be controlled. If a control is active in that area then we model such a control as a boundary control, remove the cavity from the domain Ω and use that cavity as a part of the extended domain $\widehat{\Omega}$. A very accurate study of this regulator can be done by taking into account all the seven cavities and the

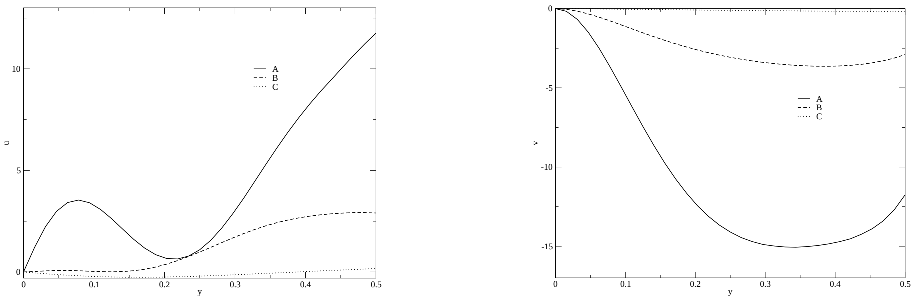


FIGURE 6. Boundary control (u-component on the left and v-component on the right) on the vertical part of Γ_c for $\beta = 1\times 10^{-4}$ (A), 1×10^{-3} (B) and 1×10^{-2} (C).

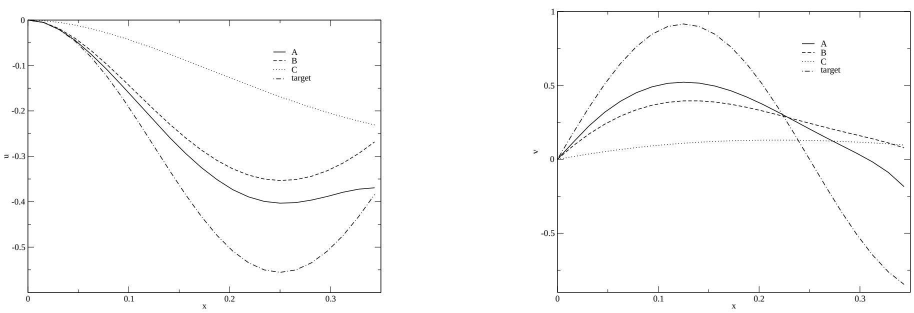


FIGURE 7. Controlled velocity (u-component on the left and v-component on the right) along the horizontal line for $y = 0.2$ and $0 \le x \le .35$ with $\beta = 1\times 10^{-4}$ (A), 1×10^{-3} (B) and 1×10^{-2} (C).

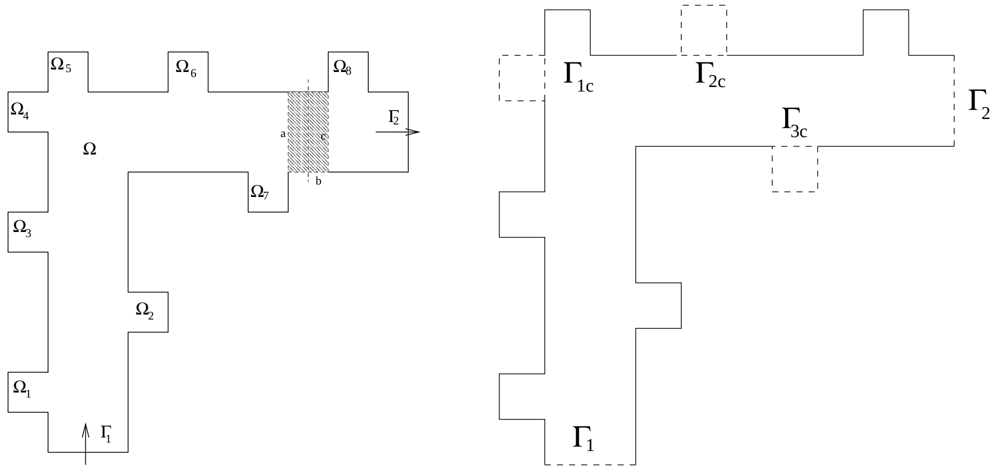


FIGURE 8. The L-shape domain Ω (on the left) and the extended domain $\widehat{\Omega}$ with the controlled boundaries Γ_{1c}, Γ_{2c} and Γ_{3c} (on the right).

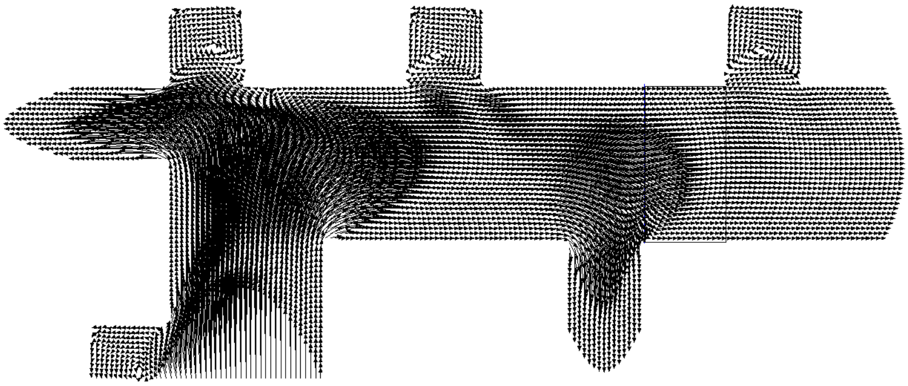


FIGURE 9. The optimal velocity field for $\alpha = 1 \times 10^{-3}$ and $\beta = 1. \times 10^{-3}$ (case A).

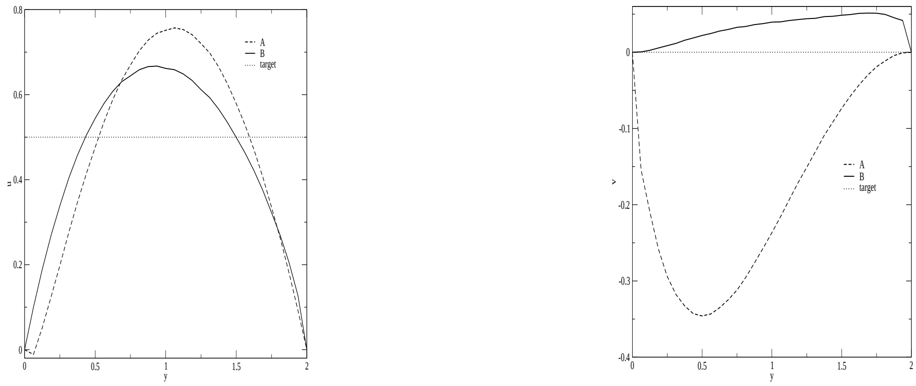


FIGURE 10. Controlled u (left) and v-component (right) of the velocity (case A) along the line $x = a$ (A), and $x = c$ (B) of the controlled area Ω_1 with the desired velocity $\vec{U}$ (target).

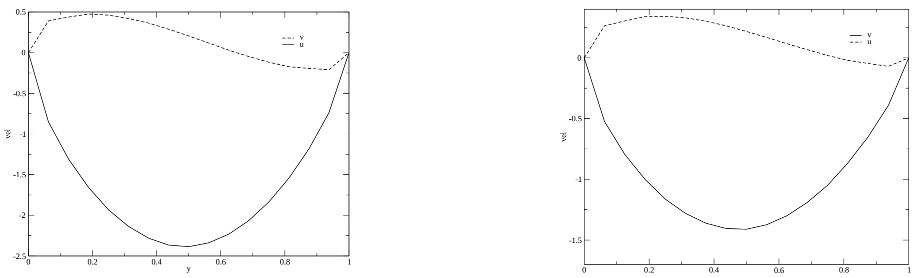
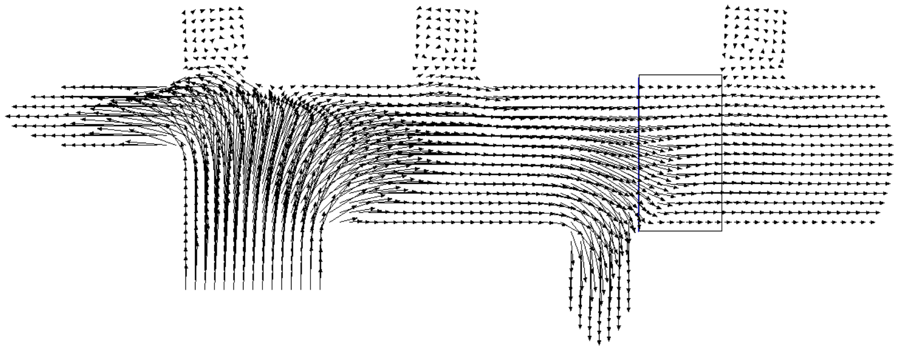


FIGURE 11. Boundary controls (case A). Boundary control over Γ_{1c} on the left and boundary control over Γ_{2c} on the right.

corresponding boundary controls but in this paper we investigate a simulation in which only two parts of the boundary Γ_c are controlled. Γ_c consists of the three parts Γ_{1c}, Γ_{2c} and Γ_{3c} as shown in Figure 8 on the right. The desired velocity is a constant velocity field defined over the controlled area Ω_1. The inlet profile



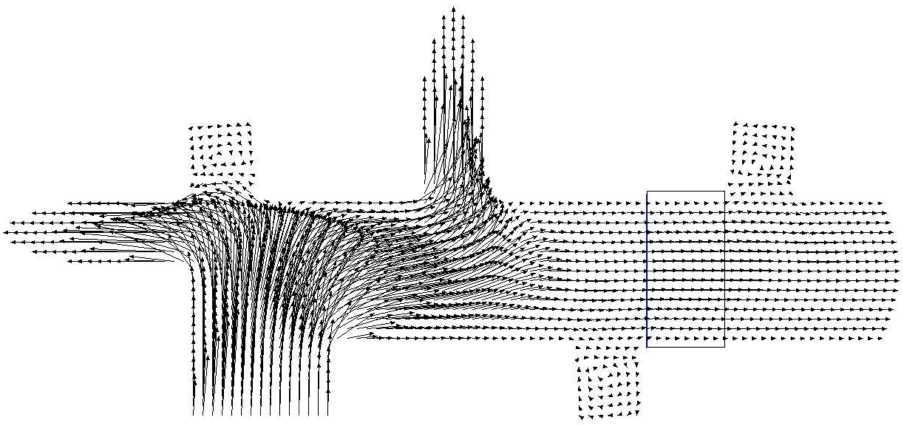


FIGURE 12. The velocity field for the case A (top) and B (bottom) for $\beta = 10^{-4}$ and $\alpha = 1 \times 10^{-3}$.

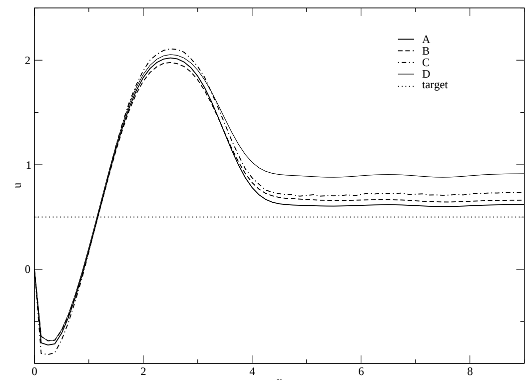


FIGURE 13. Controlled u-component of the velocity (case B) along the center line ofthe top branch of the L-shape channel for $\beta = 1 \times 10^{-3}$ (A), $\beta = 1 \times 10^{-2}$ (B), $\beta = 1 \times 10^{-1}$ (C), and $\beta = 1$ (D) with the desired velocity $\vec{U}$ (target) .

of velocity is a parabolic profile over Γ_1 with maximal velocity of $2.5m/s$. The

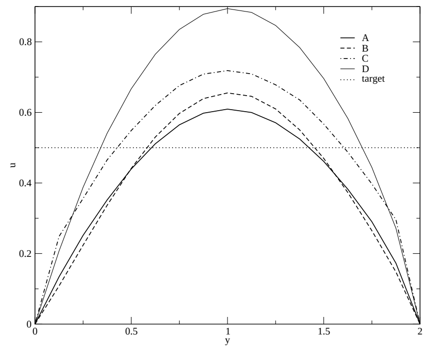


FIGURE 14. Controlled u-component of the velocity (case B) along the line $x = a$ for $\beta = 1\times10^{-3}$ (A), $\beta = 1\times10^{-2}$ (B), $\beta = 1\times10^{-1}$ (C), and $\beta = 1$ (D) with the desired velocity $\vec{U}$ (target).

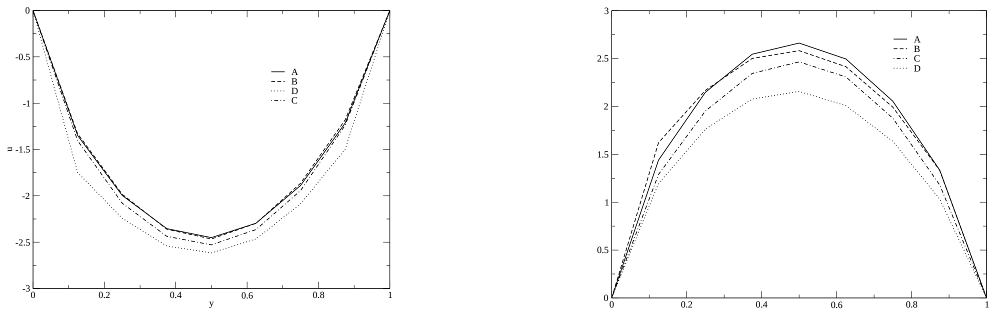


FIGURE 15. Boundary controls (case B). Boundary control over Γ_{1c} on the left and boundary control over Γ_{3c} on the right for $\beta = 1$ (D), $\beta = 1 \times 10^{-1}$ (C), $\beta = 1 \times 10^{-2}$ (B) and $\beta = 1 \times 10^{-3}$ (A) with the desired velocity $\vec{U}$ (target) .

Reynolds number of this initial configuration is 150 Reynolds with laminar motion everywhere. We are interested in the investigation of two cases:
1) constant horizontal target velocity of $.5m/s$ over the controlled area Ω_1 with controlled boundary $\Gamma_{1c} \cup \Gamma_{2c}$ (case A);
2) constant horizontal target velocity of $.5m/s$ over the controlled area Ω_1 with controlled boundary $\Gamma_{1c} \cup \Gamma_{3c}$ (case B);
The controlled area Ω_1, shown in Figure 8 on the left, is bounded by the line a and c. The vertical centerline of the controlled area is label by b. By using the solution algorithm introduced in the numerical section it is possible to solve the complete optimality system over the extended domain and recover the boundary control $\vec{g}$ as the trace of the extended velocity. For the case A the velocity solution is computed and plotted in Figure 9. The cavities Ω_1, Ω_2, Ω_3, Ω_5, Ω_6, Ω_8 are assumed to be part of the regular domain Ω and the cavities Ω_4 , Ω_7 are assumed to be part of the extended domain $\widehat{\Omega}$. In Figure 9 the stationary computations are performed with the penalty parameter β equal to 1×10^{-4} and $\alpha = 1 \times 10^{-3}$. If α is zero the boundary controls are in $\mathbf{H}^{3/2}(\Gamma)$ which implies continuity along the boundary. For high values of α and small values of β it may be possible to have control in $\mathbf{H}^{1/2}(\Gamma)$ and the boundary discretization with continuous polynomials

may not be an appropriate discretization. In order to avoid this situation we search a rather smooth control and keep α small. In this test we use Neumann boundary conditions over the extended domain remarking that in this case it is necessary in order to obtain a significant control. The desired target velocity cannot be reached if the inlet flux is not modified by the boundary control. First the boundary control must tune the average flux through the section of the L-shape channel and then the geometrical distribution along the controlled section. The effectiveness of the boundary control can be measured by the capacity to drain or add fluid to the system to obtain reasonable velocity profile around the desired one.

In Figure 10 the controlled and the desired u-component of the velocity for case A are shown along the line $x = a$ (A), and the line $x = c$ (B) of the controlled area Ω_1 together with the desired constant profile (target) on the left. The v-component in the same section of the channel is shown on the right. We note that the boundary suction and injection can control efficiently the average velocity to the target. There is a strong suction in both boundary controls Γ_{1c} and Γ_{2c} in order to reduce the velocity over the controlled area. In Fig.11 we see the velocity field for the u and v component over the boundary Γ_{1c} on the left and the boundary Γ_{2c} on the right.

The case B can be treated in the similar way. Again the target velocity is a constant velocity $\vec{u} = (.5, 0)$ over the subset Ω_1. The control is distributed over Γ_{1c} and Γ_{3c}. The inlet velocity is high and some fluid must be drained through the boundary control to match the desired profile. By using Neumann boundary conditions the control easily can tune the average mass of fluid as we can see in Figure 12. In Figure 12 we see the solution of the optimality system for the velocity field in the case A and B for $\alpha = 1 \times 10^{-3}$ and $\beta = 1 \times 10^{-3}$. In the case B the cavities Ω_1, Ω_2, Ω_3, Ω_5, Ω_7, Ω_8 are assumed to be part of the regular domain Ω and the cavities Ω_4 , Ω_6 are assumed to be part of the extended domain $\widehat{\Omega}$.

In Figure 13- 14 we see the u-component of the velocity field for the case B along the central line of the top branch of the L-shape channel and along the line $x = a$ of the controlled area Ω_1 respectively. Figure 13- 14 show the different profiles for the different values of the parameter $\beta = 1 \times 10^{-3}$ (A), 1×10^{-2} (B), 1×10^{-1} (C), 1 (D) with the desired velocity $\vec{U}$ (target). The profile quickly matches the average required value as β decreases.

Finally in Figure 15 the u-component and the v-component of the boundary control are plotted as a function of the edge coordinate of the cavity. Again the different profiles of the control over Γ_{1c} and Γ_{3c} are shown for $\beta = 1$ (D), 1×10^{-1} (C), 1×10^{-2} (B), 1×10^{-3} (A). In this case there is suction in Γ_{1c} and Γ_{3c}.

4. Conclusions

We have introduced an extended method for boundary controls which allows tracking and matching velocity field very efficiently. It is accurate and avoids the cumberstone coupling of the boundary equation with the Navier-Stokes and the adjoint system. This methods allows to solve the problem for normal boundary control which must obey to the compatibility condition and boundary control corners. All this leads to improved computability and reliability for the numerical solution of steady boundary control for Navier-Stokes system.

References

[1] F. Abergel and R. Temam, On some control problems in fluid mechanics, Theor. Comp. Fluid Dyn., 1, pp. 303–326, 1990.

[2] R. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[3] V. Alekseev, V. Tikhomirov, and S. Fomin, Optimal Control, Consultants Bureau, New York, 1987.

[4] G. Armugan and O. Pironneau, On the problem of riblets as a drag reduction device, Optim. Control Appl. Meth., 10, pp. 93–112, 1989.

[5] I. Babuska, The finite element method with Lagrangian multipliers, Numer. Math., 16, pp. 179–192, 1973.

[6] M. Berggren, Numerical solution of a flow control problem: Vorticity reduction by dynamic boundary action, Siam J. Sci. Comput., 19(3), 1998.

[7] D. Bedivan and G. Fix, An extension theorem for the space H_{div}, Appl. Math. Lett., 9(5), pp. 17-20(4) , 1996

[8] D. Begis and R. Glowinski, Application de la m*é*thode des *é*l*é*ments finis *à* l'approximation d'un probleme de domain optimal. M*é*thodes de r*é*solution des probl*è*mes approach*é*s, Applied Mathematics and Optimization 2, pp. 130–169, 1975.

[9] T. Bewley, R. Temam, and M. Ziane, A robust framework for robust control in fluid mechanics, Report, Center for Turbulence Research, Stanford University, Stanford, 1998.

[10] Z. Chen, On the convergence of non-nested multigrid methods with nested spaces on coarse grids, Numer. Methods in Partial Differential Equations, pp. 265-284, 16, 2000

[11] P. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam 1978.

[12] P. Ciarlet, Introduction to Numerical Linear Algebra and Optimization, Cambridge University, Cambridge, 1989.

[13] M. Desai and K. Ito, Optimal control of Navier-Stokes equations, SIAM J. Cont. Optim. **32**, 1994, pp. 1428–1446.

[14] A. Fursikov, M. Gunzburger and L. Hou, Boundary value problems and optimal boundary control for the Navier-Stokes system: the two-dimensional case , SIAM J. Control Optim. **3** 36, 1998.

[15] A. Fursikov, Properties of solutions of some extremal problems connected with the Navier-Stokes system, Math USSR Sb., 43, pp.323-351, 1983

[16] A. Fursikov, On some control problems and results concerning the unique solvability of a mixed boundary value problem for the three dimensional Navier-Stokes and Euler system, Soviet Math Dokl., 3, pp. 889-893, 1980

[17] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer, New York, 1984.

[18] M. Gunzburger, L. Hou and T. Svobodny, Analysis and finite element approximations of optimal control problems for the stationary Navier-Stokes equations with Dirichlet controls, Math. Model. Numer. Anal., 25, pp. 711–748, 1991.

[19] M. Gunzburger, L. Hou and T. Svobodny, Optimal control and optimization of viscous, incompressible flow, in Incompressible Computational Fluid Dynamics, M. Gunzburger and R. Nicolaides, Eds., Cambridge University, New York, 1993, pp. 109–150.

[20] M. Gunzburger, L. Hou and T. Svobodny, Analysis and finite element approximations of optimal control problems for the stationary Navier-Stokes equations with distributed and Neumann controls, Math. Comp, 57, pp. 123–151, 1991.

[21] M. Gunzburger and S. Manservisi, Some numerical computations of optimal shapes for Navier-Stokes flows,Mathematical Modelling and Numerical Analysis, Vol. 34, N 6, pp. 1233-1258 (2000)

[22] M. Gunzburger, L. Hou and T. Svobodny, Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet controls, Math. Mod. Num. Anal. **25**, 1991, pp. 771-748.

[23] M. Gunzburger and S. Manservisi, Analysis and approximation of the velocity tracking problem for Navier-Stokes flows with distributed control SIAM Journal of Numerical Analysis, Vol. 37/5, pp. 1481-1512 (2000)

[24] M. Gunzburger and S. Manservisi, The velocity tracking problem for for Navier-Stokes flows with boundary control, in SIAM J. Contr. Optim., Vol. 39 Number 2 pp. 594-634 (2000)

[25] M. Gunzburger and S. Manservisi, The velocity tracking problem for Navier-Stokes flows with bounded distributed control, SIAM J. Cont. Optim.,37, pp. 1913–1945, 1999.

[26] M. Gunzburger and S. Manservisi, A variational inequality formulation of an inverse elasticity problem, Comp. Meth. Appl. Mech. Engrg., Vol. 34/1, pp.99-126 (2000)

[27] M. Gunzburger and S. Manservisi, On a shape control problem for the stationary Navier-Stokes equations, Mathematical Modelling and Numerical Analysis, Vol. 34, N 6, pp. 1233-1258 (2000)

[28] M. Gunzburger and S. Manservisi, The velocity tracking problem for Navier-Stokes flow with linear feedback control, Journal of Comput. Meth. Appl. Mech. Eng., 189 (3) pp. 803-823 (2000)

[29] V. Girault and P. Raviart, The Finite Element Method for Navier-Stokes Equations: Theory and Algorithms, Springer, New York, 1986.

[30] L. Hou and T. Svobodny, Optimization problems for the Navier-Stokes equations with regular boundary control, J.Math.Anal.Appl. , 177, pp. 342-367-748, 1993.

[31] L. Hou and Y.Yan, Dynamics for controlled Navier-Stokes system with distributed control, SIAM J.Control Optim., 35, pp.654-677, 1997.

[32] M. Hinze and K. Kunish, Control strategies for fluid flows - optimal versus suboptimal control, Preprint n.573/1998, Technische Universitat, Germany 1998

[33] V. John, P. Knobloch, G. Matthies, and L. Tobiska, Non-nested Multi-level Solvers for Finite Element Discretizations of Mixed Problems, MGnet, preprint (2000)

[34] V. John and L. Tobiska, Numerical performance of smoother in coupled multigrid methods for parallel solution of the incompressible Navier-Stokes equations, Int. J. Numer. Meth. Fluids, **33**, pp.453-473 (2000) MGnet, preprint (2000)

[35] P. Koumoutsakos, Active control of vortex-wall interactions, Phys. Fluids, **9**, 1997, pp. 3808–3816.

[36] J.L. Lions and E. Magenes, Problemes aux Limites Non Homogenes et Applications, Vol.1 et 2, Dunod, Paris, 1968.

[37] V. Shaidurov, Multigrid Methods for finite elements, Kluwer Academic Publisher, 1989.

[38] S. Sritharan, Dynamic programming of Navier-Stokes equations, System and Control Letters, 16, 1991, pp. 229-307.

[39] S. Turek, Efficient solvers for incompressible flow problems, Lectures notes in Comput. Sci. Engrg. 6, Spring-Verlag, New York 1999.

[40] R.Temam, Navier-Stokes equation, North-Holland, Amsterdam, 1979.

[41] R.Temam, Navier-Stokes equations and Nonlinear Functional Analysis, SIAM, Philadelphia, 1993.

[42] V. Tikhomirov, Fundamental Principles of the Theory of Extremal Problems, Wiley, Chichester, 1986.

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