INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 4, Number 3-4, Pages 489–518

REMARKS ON CONTROLLABILITY OF THE ANYSOTROPIC LAMÉ SYSTEM

S. GUERRERO AND O. YU. IMANUVILOV

Abstract. In this paper we established a Carleman estimate for the elasticity system with the residual stresses. As an application of this estimate we obtain exact controllability results for the same system with locally distributed control.

Key Words. Controllability, anysotropic Lamé system

1. Introduction

Let us denote $x = (x_0, x')$, where x_0 (resp. x') stands for the time (resp. spatial) variable. This paper is concerned with global Carleman estimates for the Lamé system

$$\rho\partial_{x_0}^2 u_i - \sum_{j=1}^{3} \partial_{x_i}(\sigma_{ij}) = f_i \quad \text{in } Q,\ 1 \le i \le 3, \tag{1}$$

where Ω is a bounded domain with boundary $\partial\Omega \in C^3$, $Q = (0,T) \times \Omega$, $\mathbf{u}(x) = (u_1, u_2, u_3)$ is the displacement, $\mathbf{f} = (f_1, f_2, f_3)$ is the density of external forces and σ_{ij} is the stress tensor:

$$\sigma_{ij} = a_{ijhk}(x)\partial_{x_h} u_k.$$

On the boundary, we equip the Lamé system with zero Dirichlet boundary conditions:

$$\mathbf{u} = 0 \quad \text{on } \Sigma,$$

where we have denoted $\Sigma = (0,T) \times \partial\Omega$.

We introduce the following standard assumptions on the coefficients a_{ijhk}

$$\begin{cases} a_{ijhk} = a_{jikh} = a_{hkij}, \\ a_{ijhk}X_{ij}X_{kh} \ge \alpha X_{ij}X_{ij} \quad \forall X \in \mathbb{R}^9 \ \text{ with } \ X_{ij} = X_{ji}, \end{cases} \tag{2}$$

where α is some positive number.

In this paper we will strict to the case of the anisotropic Lamé system with residual free stresses:

$$\begin{aligned} \sigma = \mathbf{R} + (\nabla\mathbf{u})\mathbf{R} + \lambda(tr\epsilon)E_3 + 2\mu\epsilon + \beta_1(tr\epsilon)(tr\mathbf{R})E_3 + \beta_2(tr\mathbf{R})\epsilon \\ + \beta_3((tr\epsilon)\mathbf{R} + tr(\epsilon\mathbf{R})E_3) + \beta_4(\epsilon\mathbf{R} + \mathbf{R}\epsilon), \end{aligned} \tag{3}$$

where E_3 ia a unit matrix,

$$\epsilon = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$$

and

$$\nabla \cdot \mathbf{R} = 0 \quad \text{in } Q. \tag{4}$$

Received by the editors February 24, 2006, and, in revised form, March 8, 2006.
2000 *Mathematics Subject Classification.* 35N.
This work is partially supported by grant BFM2003–06446 of the D.G.E.S. (Spain), and supported by NSF Grant DMS 0205148.

We will assume for simplicity that $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ and therefore

$$\sigma = \mathbf{R} + (\nabla \mathbf{u})\mathbf{R} + \lambda(tr\epsilon)E_3 + 2\mu\epsilon, \tag{5}$$

for some λ, μ and $\mathbf{R}$. We will impose the following regularity assumptions on the Lamé coefficients

(6)

$$\rho,\ \lambda,\ \mu,\ R_{ij} \in C^2(\overline{\Omega})\ i,j \in \{1,2,3\}, \quad \rho > 0,\ \ \mu E_3 - \mathbf{R} \text{ and } (\lambda + 2\mu)E_3 - \mathbf{R} > 0$$

positive definite in Ω.

The first goal of this paper is to establish appropriate global Carleman estimates for the Lamé system with residual free stress. For displacements $\mathbf{u}$ with compact support, such estimates were obtained in the previous works [27], [25], [18]. More results are available for the isotropic Lamé system. Thus, in the stationary case we refer to Dehman-Robbiano [8] and Weck [31] for displacements with compact support and Imanuvilov-Yamamoto [17] for displacements satisfying Dirichlet boundary conditions. For the nonstationary isotropic Lamé system, see [10] for displacements with compact support and [14]–[16] in the other case. In this paper we have extended the techniques in [15] to consider the anisotropic Lamé system (1) with σ given by (5). Our Carleman estimates will hold for displacements $\mathbf{u}$ satisfying zero Dirichlet coditions on Σ.

The last section of this paper is devoted to the exact controllability of the Lamé system. To our best knowledge, the first observability result for the Lamé system was proved in [24] using multipliers of the form $(x_i - x_i^0)\frac{\partial u}{\partial x_i}$, which led to the observability inequality

$$E(x_0) = \int_\Omega (\mu|\nabla \mathbf{u}|^2 + (\lambda + \mu)|\nabla \cdot \mathbf{u}|^2)dx' \leq C \int_{(0,T)\times\Gamma_0} \left|\frac{\partial \mathbf{u}}{\partial n}\right|^2 d\Sigma$$

when the Lamé coefficients μ and λ are constants. Here, $E(t)$ is the energy. The control is exerted at $(0,T)\times\Gamma_0$ where $\Gamma_0 \subset \partial\Omega$ and homogeneous Dirichlet boundary conditions are assumed for u. Further results have been deduced in [1] by Alabau and Komornik for the anisotropic Lamé system by essentially the same multipliers method. Several questions concerning the approximate controllability/uniqueness of the Lamé system were studied in [10] by Eller, Isakov, Nakamura and Tataru by means of Carleman estimates. They obtained approximate controllability with a control distributed over any open subset of the boundary for a sufficiently large time. A series of important results have been obtained quite recently in works of Bellassoued [5]–[7]. In particular, he has proved a "logarithmic type" energy decay estimate in the case where the geometrical control condition of Bardos, Lebeau, Rauch is not fulfilled. An interesting result was also proved by Zuazua in [32], for the isotropic Lamé system with the locally distributed control $\mathbf{v}$ of the form $\chi_\omega \mathbf{v}$, where $\mathbf{v} = (v_1, \dots, v_n)$ and $v_n \equiv 0$. Under some geometric assumptions on the domain D he established the approximate controllability for the isotropic Lamé system.

Several works are devoted to the construction of dissipative "feedback" boundary conditions for the Lamé system. In [2], Alabau and Komornik introduced dissipative boundary conditions of the form $\sigma(\mathbf{u})n + A\mathbf{u} + B\partial_{x_0}\mathbf{u} = 0$ on the controlled part of the boundary Γ_0. Under some geometric conditions on Γ_0, they established the exponential decay of the energy

$$E(x_0) = \frac{1}{2}\int_\Omega (|\partial_{x_0}\mathbf{u}|^2 + \sigma_{ij}\epsilon_{ij}(\mathbf{u}))dx' + \frac{1}{2}\int_{\Gamma_0} A|\mathbf{u}|^2 dS \leq e^{-\omega x_0}$$

for the anisotropic Lamé system. In the isotropic case, Horn [13] and Martinez [26] also constructed stabilizing boundary feedbacks but under less restrictive geometrical assumptions.

A closely related question is the control and stabilization of layered plate models. Concerning the control of thermoelastic systems, Lagnese in [21] proved uniform stabilization of thermoelastic Reissner plates using thermal and mechanical boundary feedbacks. In [20], he also proved the exact controllability of the mechanical component of a thermoelastic Kirchhoff plate using mechanical boundary controls. In one-dimensional cases, this was improved to exact null controllability (of thermal and mechanical components) by the mechanical variable on the boundary in [12]. Lebeau and Zuazua [22] extended this result to the case of a three-dimensional thermoelastic Lamé system with a mechanical distributed control on a neighborhood of the boundary. Avalos and Lasiecka [4] proved a related result for the boundary control of a thermoelastic Kirchhoff plate, but with no restriction on the size of the coupling constant.

In the last part of this paper, we will present a controllability result for the anisotropic Lamé system with distributed controls supported by $Q_\omega = \omega \times (0,T)$, where $\omega \subset \Omega$ is a nonempty open set.

Notation: Let $x = (x_0, x')$, where $x_0 = t$ stands for the time variable. We denote by $H^{2,s}(Q)$ and $H^{1,s}(Q)$ the Hilbert spaces $H^2(Q)$ and $H^1(Q)$ with the norms

$$\|u\|^2_{H^{2,s}(Q)} = \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|D^\alpha u\|^2_{L^2(Q)}$$

and

$$\|u\|^2_{H^{1,s}(Q)} = s^2 \int_Q |u|^2\,dx + \int_Q |\nabla u|^2\,dx,$$

respectively.

We also introduce the following norms:

$$\|\mathbf{u}\|^2_{\mathcal{B}_\phi(Q)} = \int_Q e^{2s\phi}\Big(\textstyle\sum_{|\alpha|=0}^2 s^{4-2|\alpha|}|D^\alpha \mathbf{u}|^2 + s|\nabla(\nabla\times\mathbf{u})|^2 + s^3|\nabla\times\mathbf{u}|^2 + s|\nabla(\nabla\cdot\mathbf{u})|^2 + s^3|\nabla\cdot\mathbf{u}|^2\Big)\,dx \tag{7}$$

and

$$\|\mathbf{u}\|^2_{\mathcal{Y}_\phi(Q)} = \|\mathbf{u}\|^2_{\mathcal{B}_\phi(Q)} + s\left\|e^{s\phi}\frac{\partial\mathbf{u}}{\partial n}\right\|^2_{H^{1,s}(\Sigma)} + s\left\|e^{s\phi}\frac{\partial^2\mathbf{u}}{\partial n^2}\right\|^2_{L^2(\Sigma)}, \tag{8}$$

where ϕ is a given function.

In the sequel, we will denote by $\varepsilon(\delta)$ a positive function such that

$$\lim_{\delta\to 0^+}\varepsilon(\delta) = 0.$$

2. A Carleman estimate for the anisotropic Lamé system

Let us consider the following anisotropic Lamé system completed with Dirichlet boundary conditions:

$$\begin{cases} P(x,D)\mathbf{u} \equiv \rho(x')\partial^2_{x_0}\mathbf{u} - L(x,D)\mathbf{u} = \mathbf{f} & \text{in } Q,\\ \mathbf{u} = 0 & \text{on } \Sigma,\\ \mathbf{u}(\cdot,T) = \partial_{x_0}\mathbf{u}(\cdot,T) = \mathbf{u}(\cdot,0) = \partial_{x_0}\mathbf{u}(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{9}$$

where
$$L(x,D)\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - [(\mathbf{R},\nabla)\nabla^T]\mathbf{u}.$$
Here, $\mathbf{f}$ and $\mathbf{u}$ are three-dimensional vector fields functions and $\mathbf{R}$ is a symmetric matrix-valued function, whose components will be denoted by
$$\mathbf{u} = (u_j)_{j=1}^3, \quad \mathbf{f} = (f_j)_{j=1}^3 \quad \text{and} \quad \mathbf{R} = (R_{jk})_{j,k=1}^3.$$
Let $\xi = (\xi_0,\xi_1,\xi_2,\xi_3) = (\xi_0,\xi')$. We introduce the symbols
$$\begin{cases} p_1(x,\xi) = \rho(x')\xi_0^2 - \mu(x')(\xi_1^2+\xi_2^2+\xi_3^2) + (\mathbf{R}(x')\xi,\xi), \\ p_2(x,\xi) = \rho(x')\xi_0^2 - (\lambda(x')+2\mu(x'))(\xi_1^2+\xi_2^2+\xi_3^2) + (\mathbf{R}(x')\xi,\xi). \end{cases}$$
For any two smooth functions $w(x,\xi)$ and $z(x,\xi)$, let us introduce the formula for the Poisson bracket
$$\{w,z\} = \sum_{j=0}^{3}(\partial_{\xi_j}w\,\partial_{x_j}z - \partial_{\xi_j}z\,\partial_{x_j}w).$$

Through this paper we will assume the existence of a function ψ satisfying the following condition:

Condition A *There exists a function $\psi \in C^3(\overline{Q})$ such that*

- $|\nabla_{x'}\psi|_{\overline{Q\setminus Q_\omega}} \neq 0$
- $\{p_j,\{p_j,\psi\}\}(x,\xi) > 0$ *for all* $(x,\xi) \in (\overline{Q\setminus Q_\omega}) \times \mathbb{R}^4$ *with* $\xi \neq 0$ *such that*
$$p_j(x,\xi) = \langle \nabla_\xi p_j, \nabla\psi\rangle = 0 \quad \text{for } j=1,2$$

and

- $\{p_j(x,\xi - is\nabla\psi(x)), p_j(x,\xi+is\nabla\psi(x))\}/(2is) > 0$ *for all* $\xi \in \mathbb{R}^4\setminus\{0\}$ *and* $s \in \mathbb{R}\setminus\{0\}$ *satisfying*
$$p_j(x,\xi+is\nabla\psi(x)) = \langle\nabla_\xi p_j(x,\xi+is\nabla\psi(x)),\nabla\psi(x)\rangle = 0,\ x \in \overline{Q\setminus Q_\omega},\ j=1,2.$$

On the boundary, the following is required:
$$p_1(x,\nabla\psi(x)) < 0 \quad \forall x \in \overline{\partial\Omega\times(0,T)}$$
and
$$\begin{cases} (\lambda+2\mu)\dfrac{\partial\psi}{\partial n} - \displaystyle\sum_{i,j=1}^{3} R_{ij}\frac{\partial\psi}{\partial x_i}n_j < 0 \quad \textit{and} \quad \mu\frac{\partial\psi}{\partial n} - \sum_{i,j=1}^{3}R_{i3}\frac{\partial\psi}{\partial x_i}n_j < 0 \\ \forall x \in (\overline{\partial\Omega\setminus\partial\omega})\times(0,T). \end{cases}$$

Let $x^* \in [0,T]\times\overline{(\partial\Omega\setminus\partial\omega)}$ be an arbitrary point. Let $\mathcal{O}$ the orthogonal matrix of rotation of the domain Ω around the point $x^{*\prime}$. We assume that normal vector to $\mathcal{O}\Omega$ is $\vec{e}_3$. After the rotation, we translate x'^* into zero and denote the new coordinates as y. In new coordinate system to the point x^* corresponds $y^* = (x_0^*,0,0,0)$. We need to introduce several functions
$$\mathcal{A}(\xi_0,\xi_1,\xi_2) = (\mathrm{Im}((\frac{\tilde{r}_\mu}{s})(y^*,0,\xi')))\mathcal{F}(\xi_1,\xi_2),$$
where $\tilde{\mathbf{R}} = \mathcal{O}\mathbf{R}(x^{*\prime})\mathcal{O}^{-1}$,
$$\begin{aligned}\tilde{r}_\beta(y^*,s,\xi') = (\sum_{j=1}\tilde{R}_{j3}(y^*)(\xi_j+is\varphi_{y_j}(y^*)))^2 - (\beta-\tilde{R}_{33})(y^*)(-\rho(y^*)(\xi_0+is\varphi_{y_0}(y^*))^2 \\ +(\beta-\tilde{R}_{11})(y^*)(\xi_1+is\varphi_{y_1}(y^*))^2 + (\beta-\tilde{R}_{22})(y^*)(\xi_2+is\varphi_{y_2}(y^*))^2 \\ -\tilde{R}_{12}(y^*)((\xi_1+is\varphi_{y_1}(y^*))(\xi_2+is\varphi_{y_2}(y^*))),\end{aligned}$$

$$\mathcal{F}(\xi_1,\xi_2)=\frac{\tilde R_{13}(y^*)\xi_1+\tilde R_{23}(y^*)\xi_2}{(\mu-\tilde R_{33})(y^*)}\times$$

$$(10)\qquad \times\left\{-\frac{(\lambda+2\mu-\tilde R_{33})^2(y^*)((\xi_1)^2+(\xi_2)^2)}{(\mu-\tilde R_{33})(y^*)}-\frac{(\lambda+2\mu-\tilde R_{33})(y^*)(\tilde R_{13}(y^*)\xi_1+\tilde R_{23}(y^*)\xi_2)^2}{(\mu-\tilde R_{33})^2(y^*)}\right\},$$

$$\Omega_{11}=(\mu-\tilde R_{11})(y^*)\xi_1^2+(\mu-\tilde R_{22})(y^*)\xi_2^2-2\tilde R_{12}(y^*)\xi_1\xi_2,$$

$$\mathcal{Q}_\mu(\xi_1,\xi_2)=\left(\Omega_{11}-\frac{(\tilde R_{13}(y^*)\xi_1+\tilde R_{23}(y^*)\xi_2)^2}{(\mu-\tilde R_{33})(y^*)}\right)^{1/2},$$

$$\mathcal{A}_1(\xi_1,\xi_2)=\varphi_{\tilde y_3}(y^*)(|(\lambda+2\mu-\tilde R_{33})(y^*)|(\xi_1,\xi_2)|^2+\frac{(\tilde R_{13}(y^*)\xi_1+\tilde R_{23}(y^*)\tilde\xi_2))^2}{(\mu-\tilde R_{33})(y^*)}|^2$$
$$+\frac{(\tilde R_{13}(y^*)\tilde\xi_1+\tilde R_{23}(y^*)\xi_2))^2}{(\mu-\tilde R_{33})^2(y^*)}\tilde r_{\lambda+2\mu}(y^*,0,\pm\mathcal{Q}_\mu(\xi_1,\xi_2),\xi_1,\xi_2)),$$

$$(11)\qquad \partial_{\tilde y_3,\beta}=-\frac{R_{13}}{\beta-R_{33}}(y^*)\partial_{y_1}-\frac{R_{23}}{\beta-R_{33}}(y^*)\partial_{y_2}+\partial_{y_3}.$$

Consider the polynomial

$$(12)\qquad \mathcal{P}(\xi_1,\xi_2)=(\mathcal{A}(0,\xi_1,\xi_2)+\mathcal{A}_1(\xi_1,\xi_2))^2-(\varphi_{y_0}(y^*)\mathcal{Q}_\mu(\xi_1,\xi_2)\mathcal{F}(\xi_1,\xi_2))^2.$$

Note that this polynomial is homogeneous of order four in ξ_1 and ξ_2. Moreover, this is a homogeneous function of order four. Introducing the new variable $z=\xi_1/\xi_2$ the polynomial $\mathcal{P}_1$:

$$\mathcal{P}_1(z)=\frac{1}{\xi_2^4}\mathcal{P}(\xi_1,\xi_2)$$

Condition B. *Assume that for any* $x\in[0,T]\times\overline{(\partial\Omega\setminus\partial\omega)}$

$$\mathcal{A}(\pm\mathcal{Q}_\mu(0,1),0,1)+\mathcal{A}_1(0,1)>0$$

and polynomial $\mathcal{P}_1(z)$ *does not have real roots.*

Denote

$$\mathcal{Q}_{\lambda+2\mu}(\xi_1,\xi_2)=\left(\Omega_{22}-2\tilde R_{12}(y^*)\xi_1\xi_2\frac{(\tilde R_{13}(y^*)\xi_1+\tilde R_{23}(y^*)\xi_2)^2}{(\lambda+2\mu-\tilde R_{33})(y^*)}\right)^{1/2}$$

where

$$\Omega_{22}=(\lambda+2\mu-\tilde R_{11})(y^*)(\xi_1)^2+(\lambda+2\mu-\tilde R_{22})(y^*)(\xi_2)^2$$

and

$$q(\xi_1,\xi_2)=(\tilde R_{13}(y^*)\xi_1+\tilde R_{23}(y^*)\xi_2\ \ \Sigma^{1,3}_{\lambda+2\mu}(z_{4,\nu})-\frac{(\mu-\tilde R_{33})(\lambda+2\mu-\tilde R_{33})(y^*)(\xi_1^2+\xi_2^2)}{\sqrt{r_\mu}^+(y^*,0,\xi')-\tilde R_{13}(y^*)\xi_1-\tilde R_{23}(y^*)\xi_2})$$

Condition C. *Assume that for any* $x\in[0,T]\times\overline{(\partial\Omega\setminus\partial\omega)}$

$$(13)\qquad (\tilde R^2_{13}(y^*)+\tilde R^2_{23}(x))<(\lambda+\mu)(x)(\lambda+2\mu-\tilde R_{33})(y^*)$$

and $\forall(\xi_1,\xi_2)\in\{(\xi_1,\xi_2)||(\xi_1,\xi_2)|=1,$
$\varphi_{\tilde y_3,\mu}(y^*)\le(Im(r_\mu(y^*,s,\mathcal{Q}_{\lambda+2\mu}(\xi_1,\xi_2)\xi_1,\xi_2)/s))/(2\sqrt{Re(r_\mu(y^*,0,\pm\mathcal{Q}_{\lambda+2\mu}(\xi_1,\xi_2),\xi_1,\xi_2))})\}$

$$(14)\qquad q(\xi_1,\xi_2)^2-q(\xi_1,\xi_2)(Imr_{\lambda+2\mu}/s)(y^*,s,\pm\mathcal{Q}_{\lambda+2\mu}(\xi_1,\xi_2),\xi_1,\xi_2)>0\quad.$$

Let thus fix a function ψ satisfying the previous conditions and let us set

$$\phi(x) = e^{\tau\psi(x)} \quad \forall x \in \overline{Q}$$

for some positive parameter τ which will be chosen later on.

Theorem 1. *Let $\mathbf{f} \in \mathbf{H}^1(Q)$ and let (2), (4) and (6) hold. Suppose there exists a function ψ satisfying conditions A-C. Then, there exists $\tau^* > 0$ such that, for any $\tau > \tau^*$, there exists $s^* > 0$ such that*

$$\|\mathbf{u}\|_{\mathcal{Y}_\phi(Q)} \leq C(\|e^{s\phi}\mathbf{f}\|_{\mathbf{H}^{1,s}(Q)} + \|\mathbf{u}\|_{\mathcal{B}_\phi(Q_\omega)}) \quad \forall s > s^* \tag{15}$$

for some $C > 0$ independent of s and for any solution $u \in L^2(0,T;\mathbf{H}^2(\Omega)) \cap \mathbf{H}^1(Q)$ of system (9).

Proof: Let us first write down the equations verified by $\nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u}$:

$$\begin{aligned} P_\mu(x,D)(\nabla \times \mathbf{u}) &\equiv \partial^2_{x_0}(\nabla \times \mathbf{u}) - \mu\Delta(\nabla \times \mathbf{u}) + (R,\nabla)\nabla^T(\nabla \times \mathbf{u}) \\ &= \nabla \times \mathbf{f} + \tilde{P}_1(x,D)\mathbf{u} \end{aligned} \tag{16}$$

and

$$\begin{aligned} P_{\lambda+2\mu}(x,D)(\nabla \cdot \mathbf{u}) &\equiv \partial^2_{x_0}(\nabla \cdot \mathbf{u}) - (\lambda+2\mu)\Delta(\nabla \cdot \mathbf{u}) + (R,\nabla)\nabla^T(\nabla \cdot \mathbf{u}) \\ &= \nabla \cdot \mathbf{f} + \tilde{P}_2(x,D)\mathbf{u}, \end{aligned} \tag{17}$$

where the $\tilde{P}_j(x,D)$ are second order differential operators.

In this situation, we are able to apply the Carleman estimates obtained in [28] and [9] to the equations (16) and (17). Combining them, we obtain

$$\begin{aligned} & s^3\tau^3(\|\phi^{3/2}e^{s\phi}(\nabla \times \mathbf{u})\|^2_{\mathbf{L}^2(Q)} + \|\phi^{3/2}e^{s\phi}(\nabla \cdot \mathbf{u})\|^2_{\mathbf{L}^2(Q)}) \\ & + s\tau(\|\phi^{1/2}e^{s\phi}(\nabla \cdot \mathbf{u})\|^2_{\mathbf{H}^1(Q)} + \|\phi^{1/2}e^{s\phi}(\nabla \times \mathbf{u})\|^2_{\mathbf{H}^1(Q)}) \\ & \leq C \Bigg(\|e^{s\phi}\mathbf{f}\|^2_{\mathbf{H}^1(Q)} + \sum_{|\alpha|\leq 2} \|e^{s\phi}D^\alpha \mathbf{u}\|^2_{\mathbf{L}^2(Q)} \\ & + s^3\tau^3 \left\|\phi^{3/2}e^{s\phi}\frac{\partial \mathbf{u}}{\partial n}\right\|^2_{\mathbf{L}^2(\Sigma)} + s\tau \left\|\phi^{1/2}e^{s\phi}\frac{\partial(\partial_{tg}\mathbf{u})}{\partial n}\right\|^2_{\mathbf{L}^2(\Sigma)} \\ & + s\tau\|\phi^{1/2}e^{s\phi}\nabla(\nabla \times \mathbf{u})\|^2_{\mathbf{L}^2(Q_\omega)} + s\tau\|\phi^{1/2}e^{s\phi}\nabla(\nabla \cdot \mathbf{u})\|^2_{\mathbf{L}^2(Q_\omega)} \\ & + s^3\tau^3\|\phi^{3/2}e^{s\phi}(\nabla \times \mathbf{u})\|^2_{\mathbf{L}^2(Q_\omega)} + s^3\tau^3\|\phi^{3/2}e^{s\phi}(\nabla \cdot \mathbf{u})\|^2_{\mathbf{L}^2(Q_\omega)} \\ & + \sum_{|\alpha|=0}^{2} (s\tau)^{5-2|\alpha|}\|\phi^{\frac{5}{2}-|\alpha|}e^{s\varphi}D^\alpha \mathbf{u}\|^2_{\mathbf{L}^2(Q_\omega)} \Bigg) \quad \forall \tau \geq \tau_0,\ \forall s \geq s_0(\tau), \end{aligned} \tag{18}$$

where C is independent of s and τ. We recall that the definition of $\|\cdot\|_{\mathcal{B}_\phi(Q)}$ was given in (7).

Now, from the well known identity

$$\Delta \mathbf{u} = -\nabla \times (\nabla \times \mathbf{u}) + \nabla(\nabla \cdot \mathbf{u}),$$

we can use the Carleman inequality proved in [11] for the case of a (simpler) elliptic equation with homogeneous Dirichlet conditions and combine this with (18), so we deduce in a standard way that

$$
\begin{aligned}
& s^4\tau^5\|\phi^2 e^{s\phi}\mathbf{u}\|^2_{\mathbf{L}^2(Q)} + s^2\tau^3\|\phi e^{s\phi}\nabla\mathbf{u}\|^2_{\mathbf{L}^2(Q)} + \tau\|e^{s\phi}D^2\mathbf{u}\|^2_{\mathbf{L}^2(Q)} \\
& \leq C\left(\|e^{s\phi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(Q)} + \sum_{|\alpha|\leq 2}\|e^{s\phi}D^\alpha\mathbf{u}\|^2_{\mathbf{L}^2(Q)}\right. \\
& +s^3\tau^3\left\|\phi^{3/2}e^{s\phi}\frac{\partial\mathbf{u}}{\partial n}\right\|^2_{\mathbf{L}^2(\Sigma)} + s\tau\left\|\phi^{1/2}e^{s\phi}\frac{\partial(\partial_{tg}\mathbf{u})}{\partial n}\right\|^2_{\mathbf{L}^2(\Sigma)} \\
& +s\tau\|\phi^{1/2}e^{s\phi}\nabla(\nabla\times\mathbf{u})\|^2_{\mathbf{L}^2(Q_\omega)} + s\tau\|\phi^{1/2}e^{s\phi}\nabla(\nabla\cdot\mathbf{u})\|^2_{\mathbf{L}^2(Q_\omega)} \\
& +s^3\tau^3\|\phi^{3/2}e^{s\phi}(\nabla\times\mathbf{u})\|^2_{\mathbf{L}^2(Q_\omega)} + s^3\tau^3\|\phi^{3/2}e^{s\phi}(\nabla\cdot\mathbf{u})\|^2_{\mathbf{L}^2(Q_\omega)} \\
& \left. + \sum_{|\alpha|=0}^{2}(s\tau)^{5-2|\alpha|}\|\phi^{\frac{5}{2}-|\alpha|}e^{s\varphi}D^\alpha\mathbf{u}\|^2_{\mathbf{L}^2(Q_\omega)}\right) \quad \forall\tau\geq\tau_1,\ s\geq s_1(\tau).
\end{aligned} \tag{19}
$$

Taking τ large enough the global term in the right hand side concerning $\mathbf{u}$ can be absorbed.

On the other hand, at this point of the proof we forget about the dependence on τ and possible powers of ϕ (which is a regular function) in our inequalities, since it will not be crucial. Consequently, for the moment, we have

$$
\begin{aligned}
\|\mathbf{u}\|^2_{\mathcal{Y}_\phi(Q)} \leq C\Big(&\|e^{s\phi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(Q)} + s\left\|e^{s\phi}\frac{\partial\mathbf{u}}{\partial\vec{n}}\right\|^2_{\mathbf{H}^{1,s}(\Sigma)} + s\left\|e^{s\phi}\frac{\partial^2\mathbf{u}}{\partial\vec{n}^2}\right\|^2_{\mathbf{L}^2(\Sigma)} \\
&+\|\mathbf{u}\|^2_{\mathcal{B}_\phi(Q_\omega)}\Big) \quad \forall\tau\geq\tau_1,\ s\geq s_1(\tau),
\end{aligned} \tag{20}
$$

where constant C is independent of s.

The norm $\|\cdot\|_{\mathcal{Y}_\phi(Q)}$ was introduced in (8).

The goal will be now to estimate the boundary terms appearing in the previous inequality. For this purpose, we consider another weight function φ such that $\varphi = \phi$ on Σ. For instance, let us take

$$
\varphi = e^{\tau\tilde{\psi}} \quad \text{with} \quad \tilde{\psi} = \psi - \frac{1}{Z^2}\ell_1 + Z\ell_1^2, \tag{21}
$$

where Z is a large positive number and ℓ_1 is a regular function verifying

$$
\ell_1 = 0 \ \text{ on } \partial\Omega, \quad \ell_1 > 0 \ \text{ in } \Omega \quad \text{and} \quad \nabla\ell_1 \neq 0 \ \text{ on } \partial\Omega.
$$

Moreover, if we set

$$
\Omega_{1/Z^2} = \{x' = (x_1, x_2, x_3) \in \Omega : dist(x', \partial\Omega) < 1/Z^2\},
$$

we can suppose that the function ℓ_1 is chosen such that

$$
\varphi(x) < \phi(x) \quad \forall x \in \Omega_{1/Z^2} \times (0, T)
$$

provided Z is large enough.

For this new weight function, we will be able to prove the following lemma:

Lemma 1. *Under the previous conditions, the following inequality holds*

$$
\|\mathbf{u}\|_{\mathcal{Y}_\varphi(Q)} \leq C\left(\|e^{s\varphi}\mathbf{f}\|_{\mathbf{H}^{1,s}(Q)} + \|\mathbf{u}\|_{\mathcal{B}_\varphi(Q_\omega)}\right) \quad \forall s \geq s_0, \tag{22}
$$

for functions $\mathbf{u}$ *verifying*

$$
supp\, \mathbf{u} \subset \overline{\Omega}_{1/Z^2} \times [0, T].
$$

Let us suppose that lemma 1 holds and let us deduce theorem 1 from it. Suppose that Z is already fixed such that (22) holds and take $\varepsilon \in (0, 1/Z^2)$. Then we have

$$\varphi(x) < \phi(x) \quad \forall x \in (\overline{\Omega_\varepsilon \setminus \Omega_{\varepsilon/2}}) \times [0,T]. \tag{23}$$

Let us now introduce a cut-off function $\theta \in C_c^2(\Omega_\varepsilon)$ such that $\theta \equiv 1$ in $\Omega_{\varepsilon/2}$. Then, it readily follows that the function $\theta\mathbf{u}$ fulfills the following system

$$\begin{cases} P(x,D)(\theta\mathbf{u}) \equiv \theta\mathbf{f} + [P,\theta]\mathbf{u} & \text{in } Q, \\ \theta\mathbf{u} = 0 & \text{on } \Sigma, \\ (\theta\mathbf{u})(\cdot,T) = \partial_{x_0}(\theta\mathbf{u})(\cdot,T) = (\theta\mathbf{u})(\cdot,0) = \partial_{x_0}(\theta\mathbf{u})(\cdot,0) = 0 & \text{in } \Omega. \end{cases}$$

Consequently, we can apply estimate (22) to $\theta\mathbf{u}$ and deduce that

$$\begin{aligned} & s\left\|e^{s\phi}\frac{\partial \mathbf{u}}{\partial n}\right\|^2_{\mathbf{H}^{1,s}(\Sigma)} + s\left\|e^{s\phi}\frac{\partial^2 \mathbf{u}}{\partial n^2}\right\|^2_{\mathbf{L}^2(\Sigma)} \\ & \leq C/s\left(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(Q)} + \|e^{s\varphi}[P,\theta]\|^2_{\mathbf{H}^{1,s}(Q)} + \|\mathbf{u}\|^2_{\mathcal{B}_\varphi(Q_\omega)}\right) \quad \forall s \geq s_0, \end{aligned} \tag{24}$$

since $\varphi \equiv \phi$ on the boundary. Next, we observe that the support of the function $[P,\theta]\mathbf{u}$ is contained in $\overline{\Omega_\varepsilon \setminus \Omega_{\varepsilon/2}} \times [0,T]$, while $\varphi < \phi$ in that set (see (23)). Thus,

$$\|e^{s\varphi}[P,\theta]\mathbf{u}\|_{\mathbf{H}^{1,s}(Q)} \leq C\sum_{|\alpha|=0}^{2}\|e^{s\phi}D^\alpha\mathbf{u}\|_{\mathbf{L}^2(Q)} \quad \forall s > 0.$$

Finally, we put this together with (24) and (19) and we obtain the desired inequality (15).

This ends the proof of theorem 1.

Proof of Lemma 1: Assume that $supp\,\mathbf{u} \subset B_\delta \cap (\overline{\Omega}_{1/Z^2} \times [0,T])$. By means of a translation and a rotation, one can always suppose that the small part of the boundary where we are working on is given by the equation

$$x_3 = \ell(x_1, x_2).$$

Without lack of generality, the function $\ell \in C^3$ can be taken to satisfy

$$\nabla'\ell(0,0) = (\ell_{x_1}, \ell_{x_2})(0,0) = 0.$$

Observe that if we denote by $\mathcal{O}$ the orthogonal matrix which defines the rotation and transform our original domain Ω into the new one $\tilde{\Omega}$ ($x' \in \Omega \Rightarrow \tilde{x}' = \mathcal{O}x' \in \tilde{\Omega}$), the equation satisfied by our function $\tilde{\mathbf{u}}(\tilde{x}) := \mathcal{O}\mathbf{u}(x_0, \mathcal{O}^{-1}\tilde{x})$ is now

$$\partial_t\tilde{\mathbf{u}} - \mu\Delta\tilde{\mathbf{u}} - (\lambda+\mu)\nabla(\nabla\cdot\tilde{\mathbf{u}}) + [(\tilde{\mathbf{R}},\nabla)\nabla^T]\tilde{\mathbf{u}} = \mathcal{O}f(\tilde{x}_0, \mathcal{O}^{-1}\tilde{x}'), \tag{25}$$

where $\tilde{\mathbf{R}}(\tilde{x}') = \mathcal{O}\mathbf{R}(\mathcal{O}^{-1}\tilde{x}')\mathcal{O}^{-1}$.

In order to work in an appropriate frame, we perform the change of variables

$$\begin{cases} y_1 = x_1, \\ y_2 = x_2, \\ y_3 = x_3 - \ell(x_1, x_2) \end{cases}$$

and we set $y^* = (y_0, 0, 0, 0)$. Let us denote by $\mathbf{w} = (\mathbf{w}', w_4)$ the functions $\nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u}$ in the new coordinates. Then, some simple computations show that, in the new variables, the main symbols of equations (16) and (17) are

$$\begin{aligned} p_\beta(y,\xi) = & -\rho\xi_0^2 + (\beta - R_{11})\xi_1^2 + (\beta - R_{22})\xi_2^2 + \{[(\beta E_3 - R)G^T]G\}\xi_3^2 \\ & -2R_{12}\xi_1\xi_2 - 2\textstyle\sum_{j=1}^2(R_{j3} + \ell_{y_j}(\beta - R_{jj}) - \ell_{y_{3-j}}R_{12})\xi_j\xi_3, \end{aligned}$$

where

$$G = (-\ell_{y_1}, -\ell_{y_2}, 1)^t. \tag{26}$$

Let us set some other standard ingredients in the microlocal analysis frame. Thus, we consider the unit sphere in $\mathbb{R}^4$, say

$$S^3 = \{\zeta = (s, \xi') : s^2 + \xi_0^2 + \xi_1^2 + \xi_2^2 = 1\}$$

and the following associated finite covering:

$$\{\zeta \in S^3 : |\zeta - \zeta_\nu^*| < \delta_1\}_{1 \le \nu \le M(\delta_1)},$$

with $\zeta_\nu^* \in S^3$. To this covering we associate the partition of unity $\{\chi_\nu\}_{1 \le \nu \le M}$, extending χ_ν out of S^3 like a homogenous function of order 0 with support contained in the conic neighborhood

$$\mathcal{O}(\delta_1) = \left\{ \zeta : \left| \frac{\zeta}{|\zeta|} - \zeta_\nu^* \right| < \delta_1 \right\}.$$

In order to finish the proof, we need another lemma:

Lemma 2. *Let $\gamma^* = (y^*, \zeta^*) \in \partial\mathcal{G} \times S^3$ be fixed and $\operatorname{supp} \chi_\nu \subset \mathcal{O}(\delta_1)$. Then, for sufficiently small δ and δ_1, we have*

$$\begin{aligned} &s\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}) \\ &\quad \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \end{aligned} \tag{27}$$

Here, we have denoted $\mathcal{G} = \mathbb{R}^3 \times [0, 1/Z^2]$ and $\mathbf{z}_\nu = \chi_\nu(s, D')\mathbf{z}$, with $\mathbf{z} = e^{s\varphi}\mathbf{w}$.

Let us suppose that lemma 2 holds. Then we have

$$\begin{aligned} &s\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s(\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}\|^2_{\mathbf{L}^2(\partial\mathcal{G})}) \\ &\quad \le Cs\sum_{\nu=1}^{M}(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}) \\ &\quad \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|e^{s\varphi}\mathbf{u}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}). \end{aligned} \tag{28}$$

Let us now use the result stated in proposition 4.2 of [14]:

$$Z\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|e^{s\varphi}D^\alpha_{y'}\mathbf{u}\|^2_{\mathbf{L}^2(\mathcal{G})} \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|e^{s\varphi}\mathbf{u}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}),$$

where C is independent of s and Z.

From the differential equation in (9), we deduce that $\|e^{s\varphi}\partial^2_{y_0}\mathbf{u}\|^2_{\mathbf{L}^2(\mathcal{G})}$ can be added to the left hand side of the previous inequality. The same can be said of $\|e^{s\varphi}\partial^2_{y_0y_1}\mathbf{u}\|^2_{\mathbf{L}^2(\mathcal{G})}$ in view of well known interpolation arguments. Consequently, we have

$$Z\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|e^{s\varphi}D^\alpha\mathbf{u}\|^2_{\mathbf{L}^2(\mathcal{G})} \le C\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}.$$

Combining this with (28), we obtain:

$$\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|e^{s\varphi}D^\alpha\mathbf{u}\|^2_{\mathbf{L}^2(\mathcal{G})} + s(\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}\|^2_{\mathbf{L}^2(\partial\mathcal{G})}) \le C\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}. \tag{29}$$

Let us finally see that we can deduce inequality (22) from (29). To do this, we just have to provide 'good' estimates for the terms

$$s\,\|e^{s\varphi}\partial_{y_3}\mathbf{u}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \quad \text{and} \quad s\,\left\|e^{s\varphi}\partial^2_{y_3}\mathbf{u}\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})}.$$

From the definition of the y variables in terms of x and the Dirichlet boundary condition on u, we see that the following estimates hold:

$$|\partial_{y_3}\mathbf{u}_j| \le |(\nabla\times\mathbf{u})_{3-j}| + \varepsilon(\delta)|\partial_{y_3}\mathbf{u}| \quad \text{on } \partial\mathcal{G} \quad \text{for } j=1,2,$$

$$|\partial_{y_3}u_3| \le |\nabla\cdot\mathbf{u}| + \varepsilon(\delta)|\partial_{y_3}\mathbf{u}| \quad \text{on } \partial\mathcal{G}.$$

These two estimates tell that

$$|e^{s\varphi}\partial_{y_3}\mathbf{u}| \le |z_1| + |z_2| + |z_4| \quad \text{on } \partial\mathcal{G}. \tag{30}$$

Additionally,

$$|\partial^2_{y_jy_3}u_k| \le |\partial_{y_j}(\nabla\times\mathbf{u})_{3-k}| + \varepsilon(\delta)|\partial_{y_jy_3}\mathbf{u}| \quad \text{on } \partial\mathcal{G} \quad \text{for } j=0,1,2,\ k=1,2$$

and

$$|\partial^2_{y_jy_3}u_3| \le |\partial_{y_j}(\nabla\cdot\mathbf{u})| + \varepsilon(\delta)|\partial_{y_jy_3}\mathbf{u}| \quad \text{on } \partial\mathcal{G} \quad \text{for } j=0,1,2,$$

whence we deduce that

$$|e^{s\varphi}\nabla^{tg}_y\partial_{y_3}\mathbf{u}| \le |\nabla^{tg}_y z_1| + |\nabla^{tg}_y z_2| + |\nabla^{tg}_y z_4| \quad \text{on } \partial\mathcal{G}. \tag{31}$$

Finally, we have

$$|\partial^2_{y_3}u_j| \le |\partial_{y_3}(\nabla\times\mathbf{u})_{3-j}| + |\partial_{y_j}(\nabla\cdot\mathbf{u})| + \varepsilon(\delta)|\partial_{y_3}\nabla_y\mathbf{u}| \quad \text{on } \partial\mathcal{G} \quad \text{for } j=1,2$$

and

$$|\partial^2_{y_3}u_3| \le |\partial_{y_1}(\nabla\times\mathbf{u})_2| + |\partial_{y_2}(\nabla\times\mathbf{u})_1| + |\partial_{y_3}(\nabla\cdot\mathbf{u})| + \varepsilon(\delta)|\partial_{y_3}\nabla_y\mathbf{u}| \quad \text{on } \partial\mathcal{G},$$

which lead to the estimate

$$|e^{s\varphi}\partial^2_{y_3}\mathbf{u}| \le |\nabla_y z_1| + |\nabla_y z_2| + |\nabla_y z_4| + \varepsilon(\delta)|e^{s\varphi}\partial_{y_3}\nabla^{tg}_y\mathbf{u}| \quad \text{on } \partial\mathcal{G}. \tag{32}$$

One can readily see that (30)–(32) imply that

$$\|e^{s\varphi}\partial_{y_3}\mathbf{u}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \left\|e^{s\varphi}\partial^2_{y_3}\mathbf{u}\right\|^2_{\mathbf{L}^2(\partial\mathcal{G})} \le C(\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}\|^2_{\mathbf{L}^2(\partial\mathcal{G})}),$$

as we wanted to prove.

As a conclusion it suffices to prove lemma 2, so the rest of this section will be dedicated to it.

Proof of lemma 2: Let us introduce the notation

$$I\!\!D = D + is\nabla\varphi.$$

Then the main symbol of the differential operator $P_\beta(y, I\!\!D)$ is

$$\begin{aligned} p_{\beta,s}(y,s,\xi) &= -\rho(\xi_0+is\varphi_{y_0})^2 + (\beta-R_{11})(\xi_1+is\varphi_{y_1})^2 \\ &\quad +(\beta-R_{22})(\xi_2+is\varphi_{y_2})^2 + \{[(\beta E_3-R)G^T]G\}(\xi_3+is\varphi_{y_3})^2 \\ &\quad -2R_{12}(\xi_1+is\varphi_{y_1})(\xi_2+is\varphi_{y_2}) \\ &\quad -2\textstyle\sum_{j=1}^2(R_{j3}+\ell_{y_j}(\beta-R_{jj})-\ell_{y_{3-j}}R_{12})(\xi_j+is\varphi_{y_j})(\xi_3+is\varphi_{y_3}). \end{aligned} \tag{33}$$

We recall that $G=(-\ell_{y_1},-\ell_{y_2},1)^t$. The roots of this polynomial with respect to the ξ_3 variable are

$$\Gamma^\pm_\beta(y,s,\xi') = -is\varphi_{y_3} + \alpha^\pm_\beta(y,s,\xi'), \tag{34}$$

where

$$\alpha_\beta^\pm(y,s,\xi') = \frac{\sum_{j=1}^2 (R_{j3} + \ell_{y_j}(\beta - R_{jj}) - \ell_{y_{3-j}} R_{12})(\xi_j + is\varphi_{y_j}) \pm \sqrt{r_\beta(y,s,\xi')}}{[(\beta E_3 - R)G^t]G}$$

and

$$\begin{aligned} r_\beta(y,s,\xi') = {} & \Big(\sum_{j=1}^2 (R_{j3} + \ell_{y_j}(\beta - R_{jj}) - \ell_{y_{3-j}} R_{12})(\xi_j + is\varphi_{y_j})\Big)^2 \\ & -[(\beta E_3 - R)G^T]G(-\rho(\xi_0 + is\varphi_{y_0})^2 + (\beta - R_{11})(\xi_1 + is\varphi_{y_1})^2 \\ & +(\beta - R_{22})(\xi_2 + is\varphi_{y_2})^2 - 2R_{12}(\xi_1 + is\varphi_{y_1})(\xi_2 + is\varphi_{y_2})). \end{aligned} \tag{35}$$

It will be useful for the sequel to factorize $P_\beta(y, \mathbb{D})$ as the product of two first order operators. This is made in the following proposition:

Proposition 1. *Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(\gamma)| \geq C > 0$ for all $\gamma \in (B_\delta \cap \mathcal{G}) \times \mathcal{O}(2\delta_1)$. Then, for any function v such that $\operatorname{supp} v \subset B_\delta \cap \mathcal{G}$, we have*

$$\begin{aligned} & P_{\beta,s}(y,D)v_\nu \\ & \quad = [(\beta E_3 - R)G^T]G(D_{y_3} - \Gamma_\beta^-(y,s,D'))(D_{y_3} - \Gamma_\beta^+(y,s,D'))v_\nu + T_{\beta,s}v_\nu, \end{aligned}$$

where $T_{\beta,s}$ is a continuous operator

$$T_{\beta,s} : \mathbf{H}^{1,s}(\mathcal{G}) \longmapsto \mathbf{L}^2(\mathcal{G}).$$

Once this decomposition can be done, one would desire to obtain appropriate estimates of some norms of v_ν. More precisely, let $\tilde{v}$ satisfy

$$(D_{y_3} - \Gamma_\beta^-(y,s,D'))\tilde{v}_\nu = q, \quad \tilde{v}|_{y_3 = 1/Z^2} = 0, \quad \operatorname{supp} \tilde{v} \subset B_\delta \cap \mathcal{G}.$$

We can then prove the following:

Proposition 2. *Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(\gamma)| \geq C > 0$ for all $\gamma \in B_\delta \times \mathcal{O}(2\delta_1)$. Then,*

$$s\|\tilde{v}_\nu\|^2_{L^2(\partial\mathcal{G})} \leq C\|q\|^2_{L^2(\partial\mathcal{G})},$$

for some positive constant C independent of s and Z.

Proposition 1 and proposition 2 can both be proved as in [14].

The next step will be to obtain a Carleman inequality for a function satisfying our (second order hyperbolic) differential equation but with no imposed boundary conditions. This will be crucial to obtain the desired estimate (27). Indeed, let w satisfy

$$P_\beta(y, \mathbb{D})w = g \text{ in } \mathcal{G}, \quad \operatorname{supp} w \subset B_\delta \times [0, 1/Z^2).$$

Let us denote by $P_\beta^*(y, \mathbb{D})$ the adjoint operator of $P_\beta(y, \mathbb{D})$ ($\beta \in \{\mu, \lambda + 2\mu\}$) and let us set

$$L_{+,\beta}(y,s,D) = \frac{P_\beta(y,\mathbb{D}) + P_\beta^*(y,\mathbb{D})}{2}, \quad L_{-,\beta}(y,s,D) = \frac{P_\beta(y,\mathbb{D}) - P_\beta^*(y,\mathbb{D})}{2}.$$

We have

$$L_{+,\beta}(y,s,D)w + L_{-,\beta}(y,s,D)w = g.$$

After several computations involving integration by parts, we get

$$\begin{aligned} & \|L_{+,\beta}w\|^2_{L^2(\mathcal{G})} + \|L_{-,\beta}w\|^2_{L^2(\mathcal{G})} + \operatorname{Re} \int_{\mathcal{G}} [L_{+,\beta}, L_{-,\beta}]w \, dy + \Sigma_\beta(w) \\ & \qquad = \|g\|^2_{L^2(\mathcal{G})}, \end{aligned} \tag{36}$$

with $\Sigma_\beta(w) = (\Sigma_\beta^1 + \Sigma_\beta^2)(w)$, where Σ_β^1 can be written as $\Sigma_\beta^{1,1} + \Sigma_\beta^{1,2} + \Sigma_\beta^{1,3}$ with

$$\Sigma_\beta^{1,1}(w) = s \int_{\partial\mathcal{G}} (\beta - R_{33})^2(y^*)\varphi_{\tilde{y}_3}(y^*)(|\partial_{\tilde{y}_3} w|^2 + s^2\varphi_{\tilde{y}_3}^2|w|^2)\,dy', \tag{37}$$

$$\begin{aligned}\Sigma_\beta^{1,2}(w) = s \int_{\partial\mathcal{G}} (\beta - R_{33})(y^*)\varphi_{\tilde{y}_3}(y^*) \Big[&\rho(y^*)|\partial_{y_0} w|^2 - s^2\rho(y^*)\varphi_{y_0}^2(y^*)|w|^2 \\ &- \left(\beta - R_{11} - \frac{R_{13}^2}{\beta - R_{33}}\right)(y^*)(|\partial_{y_1} w|^2 - s^2\varphi_{y_1}^2(y^*)|w|^2) \\ &- \left(\beta - R_{22} - \frac{R_{23}^2}{\beta - R_{33}}\right)(y^*)(|\partial_{y_2} w|^2 - s^2\varphi_{y_2}^2(y^*)|w|^2) \\ &+2\left(R_{12} + \frac{R_{13}R_{23}}{\beta - R_{33}}\right)(y^*)(\partial_{y_1} w\,\partial_{y_2} w - s^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*)|w|^2)\Big]\,dy'\end{aligned} \tag{38}$$

and

$$\begin{aligned}\Sigma_\beta^{1,3}(w) = -2s\,\mathrm{Re} \int_{\partial\mathcal{G}} &((\beta - R_{33})(y^*)\partial_{y_3} w - R_{13}\partial_{y_1} w - R_{23}\partial_{y_2} w) \\ &\overline{\left(\varphi_{y_0}(y^*)\rho(y^*)\partial_{y_0} w - \left(\beta - R_{11} - \frac{R_{13}^2}{\beta - R_{33}}\right)(y^*)\varphi_{y_1}(y^*)\partial_{y_1} w\right.} \\ &\overline{- \left(\beta - R_{22} - \frac{R_{23}^2}{\beta - R_{33}}\right)(y^*)\varphi_{y_2}(y^*)\partial_{y_2} w} \\ &\overline{\left.+ \left(R_{12} + \frac{R_{13}R_{23}}{\beta - R_{33}}\right)(y^*)(\varphi_{y_2}(y^*)\partial_{y_1} w + \varphi_{y_1}(y^*)\partial_{y_2} w)\right)}\,dy'\end{aligned} \tag{39}$$

and

$$\Sigma_\beta^2(w) \le \varepsilon(\delta)s\left(\|w\|_{H^{1,s}(\partial\mathcal{G})}^2 + \|\partial_{y_3} w\|_{L^2(\partial\mathcal{G})}^2\right), \tag{40}$$

with $\varepsilon(\delta) \to 0$ when $\delta \to 0^+$. We remind that the differential operator $\partial_{\tilde{y}_3}$ was introduced in (11).

In fact, the expressions of $L_{+,\beta}(y,s,D)w$ and $L_{-,\beta}(y,s,D)w$ are

$$\begin{aligned}L_{+,\beta}(y,s,D)w &= \rho\partial_{y_0}^2 w + s^2\rho\varphi_{y_0}^2 w - (\beta - R_{11})(\partial_{y_1}^2 w + s^2\varphi_{y_1}^2 w) \\ &-(\beta - R_{22})(\partial_{y_2}^2 w + s^2\varphi_{y_2}^2 w) - (\beta - R_{33})(\partial_{y_3}^2 w + s^2\varphi_{y_3}^2 w) \\ &+2R_{12}(\partial_{y_1y_2}^2 w + s^2\varphi_{y_1}\varphi_{y_2} w) + 2R_{13}(\partial_{y_1y_3}^2 w + s^2\varphi_{y_1}\varphi_{y_3} w) \\ &+2R_{23}(\partial_{y_2y_3}^2 w + s^2\varphi_{y_2}\varphi_{y_3} w)\end{aligned}$$

and

$$\begin{aligned}L_{-,\beta}(y,s,D)w = s\,(&-\partial_{y_0}(\rho\varphi_{y_0} w) - rho\varphi_{y_0}\partial_{y_0} w + (\beta - R_{11})(\partial_{y_1}(\varphi_{y_1} w) + \varphi_{y_1}\partial_{y_1} w) \\ &+(\beta - R_{22})(\partial_{y_2}(\varphi_{y_2} w) + \varphi_{y_2}\partial_{y_2} w) + (\beta - R_{33})(\partial_{y_3}(\varphi_{y_3} w) + \varphi_{y_3}\partial_{y_3} w) \\ &-R_{12}(\partial_{y_2}(\varphi_{y_1} w) + \varphi_{y_1}\partial_{y_2} w + \partial_{y_1}(\varphi_{y_2} w) + \varphi_{y_2}\partial_{y_1} w) \\ &-R_{13}(\partial_{y_3}(\varphi_{y_1} w) + \varphi_{y_1}\partial_{y_3} w + \partial_{y_1}(\varphi_{y_3} w) + \varphi_{y_3}\partial_{y_1} w) \\ &-R_{23}(\partial_{y_2}(\varphi_{y_3} w) + \varphi_{y_3}\partial_{y_2} w + \partial_{y_3}(\varphi_{y_2} w) + \varphi_{y_2}\partial_{y_3} w)).\end{aligned}$$

One just has to integrate by parts, keeping the boundary terms in order to conclude that

$$\Sigma_\beta(w) = \Sigma_\beta^1(w) + \Sigma_\beta^2(w),$$

where $\Sigma_\beta^1(w)$ is given by (37)–(39) and Σ_β^2 verifies the estimate (40).

Using (36), one can prove in the same way as in Appendix II of [14] that there exists $c_0 > 0$ such that the following inequality holds:

$$sc_0\|w\|^2_{H^{1,s}(\mathcal{G})} \le \|L_{+,\beta}w\|^2_{L^2(\mathcal{G})} + \|L_{-,\beta}w\|^2_{L^2(\mathcal{G})} + \operatorname{Re}([L_{+,\beta}, L_{-,\beta}]w, w)_{L^2(\mathcal{G})} \\ +sC\|w\|_{L^2(\partial\mathcal{G})}\|\partial_{y_3}w\|_{L^2(\partial\mathcal{G})} \quad \forall s \ge s_0.$$

Combining this with (36), we get

(41)
$$C_1 s\|w\|^2_{H^{1,s}(\mathcal{G})} + \Sigma_\beta(w) \le C_2(\|g\|^2_{L^2(\mathcal{G})} + s\|w\|_{L^2(\partial\mathcal{G})}\|\partial_{y_3}w\|_{L^2(\partial\mathcal{G})}) \quad \forall s \ge s_0.$$

Having this inequality in mind, we will prove lemma 2 distinguishing several cases according to the values of $r_\beta(\gamma^*)$ (recall that $\gamma^* = (y^*, \zeta^*)$):

- **First Case:** $r_\mu(\gamma^*) = 0$, $r_{\lambda+2\mu}(\gamma^*) \neq 0$.
- **Second case:** $r_{\lambda+2\mu}(\gamma^*) = 0$, $r_\mu(\gamma^*) \neq 0$.
- **Third Case:** $r_\mu(\gamma^*) \neq 0$, $r_{\lambda+2\mu}(\gamma^*) \neq 0$ or $r_\mu(\gamma^*) = r_{\lambda+2\mu}(\gamma^*) = 0$.

2.1. First Case: $r_\mu(\gamma^*) = 0$, $r_{\lambda+2\mu}(\gamma^*) \neq 0$.. In this situation, taking δ and δ_1 small enough, one can suppose that

$$|r_{\lambda+2\mu}(\gamma)| \ge C > 0 \quad \forall \gamma = (y, \zeta) \in B_\delta \times (\mathcal{O}(\delta_1) \cap \{|\zeta| \ge 1\}). \tag{42}$$

Let us start applying estimate (41) to $\mathbf{z}'_\nu = \chi_\nu(s, D')e^{s\varphi}(\nabla \times \mathbf{u})$. Recall that $\mathbf{w} = (\mathbf{w}', w_4) = (\nabla \times \mathbf{u}, \nabla \cdot \mathbf{u})$, where the differential operators are taken in the y variables. This yields

$$s\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \Sigma^1_\mu(\mathbf{z}'_\nu) \le \varepsilon(\delta)s(\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}) \\ +C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}), \tag{43}$$

with $\Sigma^1_\mu(\mathbf{z}'_\nu)$ given by (37)–(39). Let us rewrite the boundary terms in the form

$$\Sigma^1_\mu(\mathbf{z}'_\nu) = \Sigma^{1,1}_\mu(\mathbf{z}'_\nu) + \Sigma^{1,2}_\mu(\mathbf{z}'_\nu) + \Sigma^{1,3}_\mu(\mathbf{z}'_\nu), \tag{44}$$

with

$$\Sigma^{1,1}_\mu(\mathbf{z}'_\nu) = s\int_{\partial\mathcal{G}} (\mu - R_{33})^2(y^*)\varphi_{\tilde{y}_3,\mu}(y^*)(|\partial_{\tilde{y}_3}\mathbf{z}'_\nu|^2 + s^2\varphi^2_{\tilde{y}_3}|\mathbf{z}'_\nu|^2)\,dy',$$

$$\Sigma^{1,2}_\mu(\mathbf{z}'_\nu) = s\int_{\partial\mathcal{G}} (\mu - R_{33})(y^*)\varphi_{\tilde{y}_3,\mu}(y^*)\Bigg[\rho(y^*)|\partial_{y_0}\mathbf{z}'_\nu|^2 - s^2\rho(y^*)\varphi^2_{y_0}(y^*)|\mathbf{z}'_\nu|^2 \\ -\left(\mu - R_{11} - \frac{R^2_{13}}{\mu - R_{33}}\right)(y^*)(|\partial_{y_1}\mathbf{z}'_\nu|^2 - s^2\varphi^2_{y_1}(y^*)|\mathbf{z}'_\nu|^2) \\ -\left(\mu - R_{22} - \frac{R^2_{23}}{\mu - R_{33}}\right)(y^*)(|\partial_{y_2}\mathbf{z}'_\nu|^2 - s^2\varphi^2_{y_2}(y^*)|\mathbf{z}'_\nu|^2) \\ +2\left(R_{12} + \frac{R_{13}R_{23}}{\mu - R_{33}}\right)(y^*)(\partial_{y_1}\mathbf{z}'_\nu\,\partial_{y_2}\mathbf{z}'_\nu - s^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*))|\mathbf{z}'_\nu|^2\Bigg]\,dy'$$

and
(45)

$$\Sigma^{1,3}_\mu(\mathbf{z}'_\nu) = -2s\,\mathrm{Re}\int_{\partial\mathcal{G}} ((\mu - R_{33})(y^*)\partial_{y_3}\mathbf{z}'_\nu - R_{13}(y^*)\partial_{y_1}\mathbf{z}'_\nu - R_{23}(y^*)\partial_{y_2}\mathbf{z}'_\nu)$$
$$\times\Bigg(\overline{\varphi_{y_0}(y^*)\rho(y^*)\partial_{y_0}\mathbf{z}'_\nu - \left(\mu - R_{11} - \frac{R_{13}^2}{\mu - R_{33}}\right)(y^*)\varphi_{y_1}(y^*)\partial_{y_1}\mathbf{z}'_\nu}$$
$$\overline{-\left(\mu - R_{22} - \frac{R_{23}^2}{\mu - R_{33}}\right)(y^*)\varphi_{y_2}(y^*)\partial_{y_2}\mathbf{z}'_\nu}$$
$$\overline{+\left(R_{12} + \frac{R_{13}R_{23}}{\mu - R_{33}}\right)(y^*)(\varphi_{y_2}(y^*)\partial_{y_1}\mathbf{z}'_\nu + \varphi_{y_1}(y^*)\partial_{y_2}\mathbf{z}'_\nu)}\Bigg)\,dy'.$$

Taking into account (35), we observe that

$$\begin{aligned}
&0 = \mathrm{Re}\,r_\mu(\gamma^*) = R_{13}^2[(\xi_1^*)^2 - (s^*\varphi_{y_1}(y^*))^2] + R_{23}^2[(\xi_2^*)^2 - (s^*\varphi_{y_2}(y^*))^2]\\
&+2R_{13}(y^*)R_{23}(y^*)[\xi_1^*\xi_2^* - (s^*)^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*)]\\
&-(\mu - R_{33})(y^*)\{[-\rho(y^*)(\xi_0^*)^2 + \rho(y^*)(s^*\varphi_{y_0}(y^*))^2] + (\mu - R_{11})(y^*)[(\xi_1^*)^2 - (s^*\varphi_{y_1}(y^*))^2]\\
&+(\mu - R_{22})(y^*)[(\xi_2^*)^2 - (s^*\varphi_{y_2}(y^*))^2] - 2R_{12}(y^*)[\xi_1^*\,\xi_2^* - (s^*)^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*)]\}.
\end{aligned}$$

Consequently,

$$|\mathrm{Re}\,r_\mu(\gamma)| \le \varepsilon(\delta_1)(s^2 + |\xi_0|^2 + |\xi_1|^2 + |\xi_2|^2) \quad \forall\gamma = (y^*, s, \xi') \in \mathcal{O}(\delta_1).$$

Since

$$\Sigma^{1,2}_\mu(\mathbf{z}'_\nu) = s\int_{\mathbf{R}^3} (\mu - R_{33})(y^*)\varphi_{\tilde{y}_3,\mu}(y^*)\,\mathrm{Re}\,r_\mu(y^*, s, \xi')|\widehat{\mathbf{z}'_\nu}|^2\,d\xi',$$

we readily deduce that

$$\Sigma^{1,2}_\mu(\mathbf{z}'_\mu) \le \varepsilon(\delta_1)s\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}. \tag{46}$$

Also, from $r_\mu(\gamma^*) = 0$ we find that

$$|\xi_0|^2 \le C(s^2 + |\xi_1|^2 + |\xi_2|^2) \quad \forall(s, \xi_0, \xi_1, \xi_2) \in \mathcal{O}(\delta_1) \tag{47}$$

for a positive constant C, provided δ_1 is small enough.

In order to estimate $\Sigma^{1,3}_\mu(\mathbf{z}'_\nu)$, we will have to distinguish again whether s^* is equal to zero or not.

Taking into account (42), an application of proposition 1 provides the identity

$$P_{\lambda+2\mu,s}(y, D)z_{4,\nu} = [(\lambda + 2\mu)E_3 - R)G^T]G(D_{y_3} - \Gamma^-_{\lambda+2\mu}(y, s, D'))z^+_{4,\nu} + Tz_{4,\nu},$$

for some $T \in \mathcal{L}(H^{1,s}(\mathcal{G}); L^2(\mathcal{G}))$, where we have set

$$z^+_{4,\nu} = (D_{y_3} - \Gamma^+_{\lambda+2\mu}(y, s, D'))z_{4,\nu}.$$

Then, proposition 2 applied to $z^+_{4,\nu}$ yields

$$s\|(\not{D}_{y_3} - \alpha^+_{\lambda+2\mu})z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})} \le (\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{48}$$

Case 2.1.1: $s^* \neq 0$

We recall here the expression of $\operatorname{Im} r_\mu(\gamma^*)$:

$$\begin{aligned}0 = \operatorname{Im} r_\mu(\gamma^*) = 2s^*\{&R_{13}^2(y^*)\varphi_{y_1}(y^*)\xi_1^* + R_{23}^2(y^*)\varphi_{y_2}(y^*)\xi_2^*\\ &+R_{13}(y^*)R_{23}(y^*)(\varphi_{y_2}(y^*)\xi_1^* + \varphi_{y_1}(y^*)\xi_2^*)\\ &-(\mu - R_{33})(y^*)[-\rho(y^*)\varphi_{y_0}(y^*)\xi_0^* + (\mu - R_{11})(y^*)(\varphi_{y_1}(y^*)\xi_1^*)\\ &+(\mu - R_{22})(y^*)(\varphi_{y_2}(y^*)\xi_2^*) - R_{12}(y^*)(\varphi_{y_2}(y^*)\xi_1^* + \varphi_{y_1}(y^*)\xi_2^*)]\},\end{aligned}$$

so we have

$$|\operatorname{Im} r_\mu(\gamma)| \leq \varepsilon(\delta_1)(s^2 + |\xi_0|^2 + |\xi_1|^2 + |\xi_2|^2) \quad \forall\gamma = (y^*, s, \xi') \in \mathcal{O}(\delta_1).$$

Then, a similar argument to the one above leads us to estimate the term $\Sigma_\mu^{1,3}(\mathbf{z}'_\nu)$:

$$\Sigma_\mu^{1,3}(\mathbf{z}'_\nu) \leq \varepsilon(\delta_1)s(\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}). \tag{49}$$

Therefore, from the expression of $\Sigma_\mu^1(\mathbf{z}'_\nu)$ (see (37)–(39)), (46), (49) and the positiveness of $\Sigma_\mu^{1,1}(\mathbf{z}'_\nu)$, we deduce that

$$\Sigma_\nu^1(\mathbf{z}'_\mu) \geq Cs(\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})})$$

for some positive constant C. Here, we have used Condition A (at the beginning of section 2) on ψ, $\mu - R_{33} > 0$ (see (6)) and the fact that $s^2 \geq C(\xi_0^2 + \xi_1^2 + \xi_2^2)$ in $\mathcal{O}(\delta_1)$. Combining this and (43), we get

$$\begin{aligned}s(\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})})&\\ \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}),&\end{aligned} \tag{50}$$

In order to estimate the boundary norms of $z_{4,\nu}$, we will use the boundary Dirichlet conditions and the equations of the Lamé system written on $\partial\mathcal{G}$. Indeed, from the two first ones, it is not difficult to deduce that

$$\begin{cases} |\mathbb{D}_{y_j} z_{4,\nu}| \leq C(|e^{s\varphi}\mathbf{f}| + s|\mathbf{z}'_\nu| + |\nabla_y^{tg}\mathbf{z}'_\nu|) + \varepsilon(\delta)(s|\mathbf{z}_\nu| + |\nabla_y^{tg}\mathbf{z}_\nu| + |\partial_{y_3}\mathbf{z}_\nu|)\\ \text{on } \partial\mathcal{G}, \quad \text{for } j = 1, 2\end{cases}$$

and from the third one

$$\begin{aligned}|\mathbb{D}_{y_3} z_{4,\nu}| \leq C(&|e^{s\varphi}\mathbf{f}| + s|\mathbf{z}'_\nu| + |\nabla_y^{tg}\mathbf{z}'_\nu| + |\mathbb{D}_{y_1} z_{4,\nu}| + |\mathbb{D}_{y_2} z_{4,\nu}|)\\ &+\varepsilon(\delta)(s|\mathbf{z}_\nu| + |\nabla_y^{tg}\mathbf{z}_\nu| + |\partial_{y_3}\mathbf{z}_\nu|) \quad \text{on } \partial\mathcal{G}.\end{aligned}$$

Now, using that $\lambda + 2\mu - R_{33} > 0$ along with the Dirichlet boundary conditions, we have

$$\begin{aligned}\|\mathbb{D}_{y_3} z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})} \leq C(&\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{L}^2(\partial\mathcal{G})} + \|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})})\\ &+\varepsilon(\delta)(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}).\end{aligned} \tag{51}$$

In addition, combining (51) and (48), we find an estimate of the L^2 norm of z_4 and its tangential derivatives on $\partial\mathcal{G}$:

$$\begin{aligned}s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} \leq C(&\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})})\\ &+\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}).\end{aligned} \tag{52}$$

Indeed, in view of (42), we can apply Gärding's inequality after (48) and obtain (52).

Finally, (52) and (51) give an estimate of the normal derivative of $z_{4,\nu}$ in the L^2 norm:

$$\begin{aligned}s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + s\|\partial_{y_3} z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})} \leq C(&\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}\\ &+s\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}) + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}).\end{aligned} \tag{53}$$

This and (50) provide the desired inequality (27). This ends the proof of lemma 2 in this case.

Case 2.1.2: $s^* = 0$

We first remark that, thanks to the Dirichlet boundary conditions, we have

$$s\|z_{3,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} \le \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}). \tag{54}$$

Once the tangential derivatives of $z_{3,\nu}$ are bounded, an application of (41) for $w = z_{3,\nu}$ also gives an estimate for its normal derivative :

$$\begin{aligned} s(\|z_{3,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{3,\nu}\|^2_{L^2(\partial\mathcal{G})}) &\le \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \\ &+\|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}) + C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \end{aligned} \tag{55}$$

We will next estimate the terms $\Sigma^{1,3}_\mu(z_{1,\nu})$ and $\Sigma^{1,3}_\mu(z_{2,\nu})$. To this end, let us introduce the following differential operator:

$$\begin{aligned} &M(y,s,D')(z_{1,\nu},z_{2,\nu}) \\ &= (\mathbb{D}_{y_1}z_{1,\nu} + \mathbb{D}_{y_2}z_{2,\nu}, (\mu - R_{33})(\mathbb{D}_{y_1}z_{2,\nu} - \mathbb{D}_{y_2}z_{1,\nu})) = (F_1, F_2). \end{aligned} \tag{56}$$

From the third equation of our Lamé system, we have (observe that $s^* = 0$)

$$\begin{aligned} F_2 &= (\lambda + 2\mu - R_{33})\mathbb{D}_{y_3}z_{4,\nu} - 2R_{13}\mathbb{D}_{y_1}z_{4,\nu} - 2R_{23}\mathbb{D}_{y_2}z_{4,\nu} + F_3 \\ &= \sqrt{r_{\lambda+2\mu}}^{+}(y,s,D')z_{4,\nu} + V^+_\mu - R_{13}\mathbb{D}_{y_1}z_{4,\nu} - R_{23}\mathbb{D}_{y_2}z_{4,\nu} + F_3 \end{aligned}$$

where F_3 verifies

$$\begin{aligned} s\|F_3\|^2_{\mathbf{L}^2(\partial\mathcal{G})} &\le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) \\ &+\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}). \end{aligned} \tag{57}$$

Here, we have denoted

$$V^+_\mu = ((\lambda + 2\mu - R_{33})\mathbb{D}_{y_3} - R_{13}\mathbb{D}_{y_1} - R_{23}\mathbb{D}_{y_2})z_{4,\nu} - \sqrt{r_{\lambda+2\mu}}^{+}(y,s,D')z_{4,\nu}.$$

Taking into account (48), we deduce that

$$\begin{aligned} s\|V^+_\mu\|^2_{L^2(\partial\mathcal{G})} &\le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) \\ &+\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}). \end{aligned}$$

Then, since the divergence of a curl is identically zero, we find that

$$s\|F_1\|^2_{L^2(\partial\mathcal{G})} \le \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}).$$

Next, using (55), we rewrite the two first equations of our Lamé system (9) as

$$\left\{ \begin{aligned} &-(\mu - R_{33})\mathbb{D}_{y_3}z_{2,\nu} + 2R_{13}\mathbb{D}_{y_1}z_{2,\nu} + 2R_{23}\mathbb{D}_{y_2}z_{2,\nu} \\ &\qquad\qquad = F_4 + (\lambda + 2\mu - R_{33})\mathbb{D}_{y_1}z_{4,\nu}, \\ &(\mu - R_{33})\mathbb{D}_{y_3}z_{1,\nu} - 2R_{13}\mathbb{D}_{y_1}z_{1,\nu} - 2R_{23}\mathbb{D}_{y_2}z_{1,\nu} \\ &\qquad\qquad = F_5 + (\lambda + 2\mu - R_{33})\mathbb{D}_{y_2}z_{4,\nu}, \end{aligned} \right. \tag{58}$$

with F_4 and F_5 satisfying estimate (57).

The principal symbol of the operator M is

$$\begin{pmatrix} \xi_1 + is\varphi_{y_1} & \xi_2 + is\varphi_{y_2} \\ -(\mu - R_{33})(\xi_2 + is\varphi_{y_2}) & (\mu - R_{33})(\xi_1 + is\varphi_{y_1}) \end{pmatrix},$$

which clearly has a nonzero determinant at the point γ^*.

Therefore, from the definition of M given in (56), we deduce that there exists a parametrix of this operator such that

(59)
$$\begin{aligned}(z_{1,\nu}, z_{2,\nu}) &= M^{-1}(y,s,D')(0, (\sqrt{r_{\lambda+2\mu}}^{+}(y,s,D') - R_{13}\mathbb{D}_{y_1} - R_{23}\mathbb{D}_{y_2})z_{4,\nu}) \\ &\quad + M^{-1}(y,s,D')(F_1, F_6) + T(z_{1,\nu}, z_{2,\nu}).\end{aligned}$$

In terms of the ξ variables, the principal part of (59) reads

(60)
$$\begin{aligned}\widehat{z}_{k,\nu} &= (-1)^k \frac{(\xi_{3-k} + is\varphi_{y_{3-k}}(y^*))(\sqrt{r_{\lambda+2\mu}}^{+}(y^*,s,\xi') - R_{13}(y^*)(\xi_1 + is\varphi_{y_1}(y^*))}{(\mu - R_{33})(y^*)((\xi_1 + is\varphi_{y_1}(y^*))^2 + (\xi_2 + is\varphi_{y_2}(y^*))^2)} \\ &\quad \frac{-R_{23}(y^*)(\xi_2 + is\varphi_{y_2}(y^*)))}{} \widehat{z}_{4,\nu} + \widehat{F}_7,\end{aligned}$$

for $k = 1, 2$, with F_7 satisfying estimate (53). From the fact that $\zeta^* = (s^*, \xi_0^*, \xi_1^*, \xi_2^*) \in S^3$ and $r_\mu(\gamma^*) = 0$, we deduce there exists $\alpha, \alpha' \in \mathbf{R}$ such that $\alpha\xi_1^* + \alpha'\xi_2^* = 1$ and $\sqrt{\alpha^2 + (\alpha')^2} = 1$.

Let us then introduce the function

$$z_{5,\nu} = -\alpha' z_{1,\nu} + \alpha z_{2,\nu},$$

which, by virtue of (60), satisfies

$$\begin{aligned}\widehat{z}_{5,\nu} &= \frac{(\alpha(\xi_1 + is\varphi_{y_1}(y^*)) + \alpha'(\xi_2 + is\varphi_{y_2}(y^*)))\ \Omega_{33}}{(\mu - R_{33})(y^*)((\xi_1 + is\varphi_{y_1}(y^*))^2 + (\xi_2 + is\varphi_{y_2}(y^*))^2)} \\ &\quad \frac{-R_{23}(y^*)(\xi_2 + is\varphi_{y_2}(y^*)))}{} \widehat{z}_{4,\nu} + \widehat{F}_8,\end{aligned} \tag{61}$$

for all $\zeta \in \mathcal{O}(\delta_1)$, where

$$\Omega_{33} = \sqrt{r_{\lambda+2\mu}}^{+}(y^*,s,\xi') - R_{13}(y^*)(\xi_1 + is\varphi_{y_1}(y^*)$$

and F_8 satisfies estimate (53).

With all these ingredients, we will be able to estimate the term $\Sigma_\mu^{1,3}(z_{5,\nu})$. Indeed, we can plug identities (58) and (61) into the expression of $\Sigma_\mu^{1,3}$ given in (45) in order to express it in terms of the $z_{4,\nu}$ variable. This yields

$$\begin{aligned}&\Sigma_\mu^{1,3}(z_{5,\nu}) = \\ &-2s\,\mathrm{Re}\int_{\mathbf{R}^3} \tfrac{\alpha\xi_1 + \alpha'\xi_2}{(\mu - R_{33})(y^*)(\xi_1^2 + \xi_2^2)} \Bigg[-(\lambda + 2\mu - R_{33})(y^*)(\alpha\xi_1 + \alpha'\xi_2) \\ &\qquad + \frac{(\alpha\xi_1 + \alpha'\xi_2)(R_{13}(y^*)\xi_1 + R_{23}(y^*)\xi_2)\ \Omega_{44}}{(\mu - R_{33})(y^*)(\xi_1^2 + \xi_2^2)} \Bigg] \times \\ &\quad \Sigma_\mu^{1,3} \times \overline{\left(\varphi_{y_0}(y^*)\rho(y^*)\xi_0 - \left(\mu - R_{11} - \frac{R_{13}^2}{\mu - R_{33}}\right)(y^*)\varphi_{y_1}(y^*)\xi_1 - \Omega_{55}\right.} \\ &\qquad \overline{\left. + \left(R_{12} + \frac{R_{13}R_{23}}{\mu - R_{33}}\right)(y^*)(\varphi_{y_2}(y^*)\xi_1 + \varphi_{y_1}(y^*)\xi_2)\right)} \\ &\qquad \overline{(\sqrt{r_{\lambda+2\mu}}^{+}(y^*,s^*,\xi') - R_{13}(y^*)\xi_1 - R_{23}(y^*)\xi_2)}\,|\widehat{z}_{4,\nu}|^2\,d\xi' + I_1(z_\nu),\end{aligned} \tag{62}$$

where

$$\Omega_{44} = \sqrt{r_{\lambda+2\mu}}^{+}(y^*,s^*,\xi') - R_{13}(y^*)\xi_1 - R_{23}(y^*)\xi_2,$$

$$\Omega_{55} = \left(\mu - R_{22} - \frac{R_{23}^2}{\mu - R_{33}}\right)(y^*)\varphi_{y_2}(y^*)\xi_2.$$

Here, the term $I_1(z_\nu)$ is bounded by

(63) $$I_1(z_\nu) \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}).$$

We set $\tilde{\xi}' = C\xi^{*\prime}$ where C is a positive constant. The real part of the main symbol of the pseudodifferential operator appearing in (62) at point $(y^*, \tilde{\xi}')$ equals (observe that $\sqrt{r_{\lambda+2\mu}}^+(\gamma^*)$ is a pure imaginary number, since $r_{\lambda+2\mu}(\gamma^*)$ is real and negative)

(64)
$$\frac{\mathcal{A}(\tilde{\xi}_0, \tilde{\xi}_1, \tilde{\xi}_2)}{((\tilde{\xi}_1)^2 + (\tilde{\xi}_2)^2)} = (\mathrm{Im}((\frac{r_\mu}{s})(y^*, s, \tilde{\xi}')))\frac{(R_{13}(y^*)\tilde{\xi}_1 + R_{23}(y^*)\tilde{\xi}_2)}{(\mu - R_{33})(y^*)((\tilde{\xi}_1)^2 + (\tilde{\xi}_2)^2)} \times$$
$$\times \left\{ -(\lambda + 2\mu - R_{33})(y^*)((\tilde{\xi}_1)^2 + (\tilde{\xi}_2)^2) - \frac{(R_{13}(y^*)\xi_1 + R_{23}(y^*)\tilde{\xi}_2)^2 + |r_{\lambda+2\mu}(y^*, \tilde{\xi}')|}{(\mu - R_{33})(y^*)} \right\}.$$

Here, we have taken into account that $\alpha\xi_1^* + \alpha'\xi_2^* = |(\tilde{\xi}_1, \tilde{\xi}_2)|$.

Plugging here the expression of $r_{\lambda+2\mu}(y^*, \tilde{\xi}')$ and taking into account that $r_\mu(\gamma^*) = 0$, we get

$$\mathcal{A}(\tilde{\xi}_0, \tilde{\xi}_1, \tilde{\xi}_2) = (\mathrm{Im}((\frac{r_\mu}{s})(y^*, 0, \tilde{\xi}')))\mathcal{F}(\tilde{\xi}_1, \tilde{\xi}_2)$$

where

(65)
$$\mathcal{F}(\xi_1, \xi_2) = \frac{R_{13}(y^*)\xi_1 + R_{23}(y^*)\xi_2}{(\mu - R_{33})(y^*)} \times$$
$$\times \left\{ -\frac{(\lambda + 2\mu - R_{33})^2(y^*)}{(\mu - R_{33})(y^*)}((\xi_1)^2 + (\xi_2)^2) - \Omega_{66} \right\}.$$

where

$$\Omega_{66} = \frac{(\lambda + 2\mu - R_{33})(y^*)(R_{13}(y^*)\xi_1 + R_{23}(y^*)\xi_2)^2}{(\mu - R_{33})^2(y^*)}.$$

Again from $r_\mu(\gamma^*) = 0$, we can put $\tilde{\xi}_0$ in terms of $\tilde{\xi}_1$ and $\tilde{\xi}_2$ in the following way: $\tilde{\xi}_0 = \pm\mathcal{Q}(\tilde{\xi}_1, \tilde{\xi}_2)$ where

$$\mathcal{Q}(\xi_1, \xi_2) = \big((\mu - R_{11})(y^*)(\xi_1)^2 + (\mu - R_{22})(y^*)(\xi_2)^2 - 2R_{12}(y^*)\xi_1\xi_2 - \Omega 77\big)^{1/2}$$

where

$$\Omega_{77} = \frac{(R_{13}(y^*)\xi_1 + R_{23}(y^*)\xi_2)^2}{(\mu - R_{33})(y^*)}.$$

Plugging this into the expression (65), we obtain

$$\mathcal{A}(\pm\mathcal{Q}(\tilde{\xi}_1, \tilde{\xi}_2), \tilde{\xi}_1, \tilde{\xi}_2) = (\mathrm{Im}((\frac{r_\mu}{s})(y^*, 0, \pm\mathcal{Q}(\tilde{\xi}_1, \tilde{\xi}_2), \tilde{\xi}_1, \tilde{\xi}_2))/2)\mathcal{F}(\tilde{\xi}_1, \tilde{\xi}_2)$$

(66)
$$= \Big\{ \pm\varphi_{y_0}\rho(y^*)(y^*) \Big((\mu - R_{11})(y^*)(\tilde{\xi}_1)^2 + (\mu - R_{22})(y^*)(\tilde{\xi}_2)^2 - 2R_{12}(y^*)\tilde{\xi}_1\tilde{\xi}_2$$
$$- \frac{(R_{13}(y^*)\xi_1^* + R_{23}(y^*)\tilde{\xi}_2)^2}{(\mu - R_{33})(y^*)} \Big)^{1/2} - \left(\mu - R_{11} - \frac{R_{13}^2}{\mu - R_{33}}\right)(y^*)\varphi_{y_1}(y^*)\tilde{\xi}_1$$
$$- \left(\mu - R_{22} - \frac{R_{23}^2}{\mu - R_{33}}\right)(y^*)\varphi_{y_2}(y^*)\tilde{\xi}_2 + \Omega_{88} \times (y^*)(\varphi_{y_2}(y^*)\tilde{\xi}_1 + \varphi_{y_1}(y^*)\tilde{\xi}_2) \Big\}$$
$$\frac{R_{13}(y^*)\tilde{\xi}_1 + R_{23}(y^*)\tilde{\xi}_2}{(\mu - R_{33})(y^*)((\tilde{\xi}_1)^2 + (\tilde{\xi}_2)^2)} \left\{ - \frac{(\lambda + 2\mu - R_{33})(y^*)(R_{13}(y^*)\xi_1^* + R_{23}(y^*)\tilde{\xi}_2)^2}{(\mu - R_{33})^2(y^*)((\tilde{\xi}_1)^2 + (\tilde{\xi}_2)^2)} \right.$$
$$\left. - \frac{(\lambda + 2\mu - R_{33})^2(y^*)}{(\mu - R_{33})(y^*)} \right\}.$$

where

$$\Omega_{88} = \left(R_{12} + \frac{R_{13}R_{23}}{\mu - R_{33}}\right).$$

Now, from the expression of $\Sigma^{1,1}_\mu(z_{5,\nu})$ together with the first and second equations of the Lamé system, we have

$$\Sigma^{1,1}_\mu(z_{5,\nu}) = s\int_{\mathbf{R}^3} \varphi_{\tilde y_3,\mu}(y^*)\left\{|-(\lambda+2\mu-R_{33})(y^*)(\alpha\xi_1+\alpha'\xi_2)+\Omega_{99}\times\right.$$
$$\left.(\sqrt{r_{\lambda+2\mu}}^+(y^*,s^*,\xi) - R_{13}(y^*)\xi_1 - R_{23}(y^*)\xi_2)\right|^2\Big\}\,|\widehat z_{4,\nu}|^2\,d\xi' + I_2(z_{4,\nu}),$$

where

$$\Omega_{99} = \frac{(R_{13}(y^*)\xi_1 + R_{23}(y^*)\xi_2)}{(\mu - R_{33})(y^*)(\xi_1^2+\xi_2^2)}(\alpha\xi_1+\alpha'\xi_2)$$

and $I_2(z_{4,\nu})$ verifies estimate (63). Consequently, the real part of the principal symbol of the operator which is now acting on $z_{4,\nu}$ at point $(y^*, \tilde\xi')$ is

$$\frac{\mathcal{A}_1(\tilde\xi_1,\tilde\xi_2)}{|(\tilde\xi_1,\tilde\xi_2)|} = \varphi_{\tilde y_3}(y^*)(|(\lambda+2\mu-R_{33})(y^*)|(\tilde\xi_1,\tilde\xi_2)|^2 + \frac{(R_{13}(y^*)\tilde\xi_1+R_{23}(y^*)\tilde\xi_2))^2}{(\mu-R_{33})(y^*)}|^2$$
$$+\frac{(R_{13}(y^*)\tilde\xi_1+R_{23}(y^*)\tilde\xi_2))^2}{(\mu-R_{33})^2(y^*)|(\tilde\xi_1,\tilde\xi_2)|^2} r_{\lambda+2\mu}(y^*,0,\pm\mathcal{Q}_\mu(\tilde\xi_1,\tilde\xi_2),\tilde\xi_1,\tilde\xi_2)).$$

Let us show that Condition B implies

$$\mathcal{A}(\pm\mathcal{Q}_\mu(\xi_1,\xi_2),\xi_1,\xi_2) + \mathcal{A}_1(\xi_1,\xi_2) > 0 \quad \forall(\xi_1,\xi_2)\ne 0. \tag{67}$$

First we note that, according to our assumption, we have

$$\mathcal{A}(\pm\mathcal{Q}_\mu(0,1),0,1) + \mathcal{A}_1(1,0) > 0. \tag{68}$$

Therefore since the function $\mathcal{A}+\mathcal{A}_1$ is homogeneous and continuous on S^2, in order to prove (67) one need to show that equation

$$\mathcal{A}(\pm\mathcal{Q}_\mu(\xi_1,\xi_2),\xi_1,\xi_2) + \mathcal{A}_1(\xi_1,\xi_2) = 0 \tag{69}$$

does not have any solutions on S^2. Let us fix the sign of $\pm\mathcal{Q}_\mu(\xi_1,\xi_2)$ in such a way, that $\varphi_{y_0}(y^*)(\pm\mathcal{Q}_\mu(\xi_1^*,\xi_2^*)\mathcal{F}(\xi_1^*,\xi_2^*)) \le 0$ and after that we move this term into the right hand side of equation (69). In this new equation, we take the square of both sides. As a result, we have

$$\begin{aligned}&(\mathcal{A}(0,\xi_1,\xi_2)+\mathcal{A}_1(\xi_1,\xi_2))^2\\ &= (\varphi_{y_0}(y^*)(\mathcal{Q}_\mu(\xi_1,\xi_2)\mathcal{F}(\xi_1,\xi_2))^2.\end{aligned} \tag{70}$$

In equation (70) we will move all terms from the right hand side into the left hand side and, as a result, we have a polynomial of order four of ξ_1 and ξ_2. As a result we obtain the polynomial $\mathcal{P}(\xi_1,\xi_2)$ introduced in (12). Since polynomial $\mathcal{P}_1$ does not have a real roots the equation $\mathcal{P}(\xi_1,\xi_2)$ does not have any solutions of the form (ξ_1,ξ_2), $\xi_2\ne 0$. On the other hand by (87) the point $(0,1)$ is not solution to this equation also. The proof of (67) is finished.

By (67) there exists $\hat C > 0$ such that

$$\Sigma^1_\mu(z_{5,\nu}) \ge \hat C s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3} z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}) - C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})})$$
$$-\epsilon(\delta)s(\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}\|^2_{\mathbf{L}^2(\partial\mathcal{G})})$$

This estimate and (55),(58),(60) imply

$$\Sigma^1_\mu(z_{5,\nu}) \ge \hat C_1 s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})}) - C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})})$$

On the other hand this inequality and the estimate (41) applied to $z_{5,\nu}$ imply

$$s(\|\mathbf{z}_\nu\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{L^2(\partial\mathcal{G})}) \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})})$$

This estimate and (41) implies (27).

2.2. Second Case: $r_\mu(\gamma^*) \neq 0$, $r_{\lambda+2\mu}(\gamma^*) = 0$.**.** In a way similar to above, one can take δ and δ_1 to be small enough so that

$$|r_\mu(\gamma)| \ge C > 0 \quad \forall\gamma = (y,\zeta) \in B_\delta \times \mathcal{O}(\delta_1). \tag{71}$$

In this situation, condition $r_{\lambda+2\mu}(\gamma^*) = 0$ also provides the estimate (47).

We distinguish now three different situations depending on s^* and $\varphi_{\tilde{y}_3,\mu}(y^*)$.

Case 2.2.1: $s^* = 0$ and $\varphi_{\tilde{y}_3,\mu}(y^*) > (\mathrm{Im}(r_\mu(\gamma^*)/s))/(2\sqrt{\mathrm{Re}(r_\mu(\gamma^*))})$

Under these hypotheses, we can take δ and δ_1 small enough and suppose that

$$-\mathrm{Im}\,\Gamma^\pm_\mu(y,\zeta) \ge Cs \quad \forall\gamma = (y,\zeta) \in B_\delta \times \mathcal{O}(\delta_1),$$

since $\varphi_{\tilde{y}_3,\mu} > 0$ (see the expression of $\Gamma^\pm_\mu$ in (34)).

Consequently, we can apply proposition 1 in two different ways and deduce that

$$\begin{aligned} P_{\mu,s}(y,D)\mathbf{z}'_\nu &= [(\mu E_3 - R)G^T]G(D_{y_3} - \Gamma^-_\mu(y,s,D'))\mathbf{z}'^+_\nu + T^+_{\mu,s}w'^+_\nu \\ &= [(\mu E_3 - R)G^T]G(D_{y_3} - \Gamma^+_\mu(y,s,D'))\mathbf{z}'^-_\nu + T^-_{\mu,s}\mathbf{w}'^-_\nu, \end{aligned}$$

with $T^\pm_{\mu,s} \in \mathcal{L}(\mathbf{H}^{1,s}(\mathcal{G}), \mathbf{L}^2(\mathcal{G}))$. Here, we have set

$$\mathbf{w}'^\pm_\nu = (D_{y_3} - \Gamma^\pm_\mu(y,s,D'))\mathbf{z}'_\nu.$$

Next, we apply proposition 2 and we get the following estimates:

$$s\|\mathbf{w}'^\pm_\nu\|^2_{L^2(\partial\mathcal{G})} \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{72}$$

On the other hand, since

$$\mathbf{w}'^-_\nu - \mathbf{w}'^+_\nu = 2\sqrt{r_\mu}(y,s,D')\mathbf{z}'_\nu \quad \text{on } \partial\mathcal{G}$$

and $r_\mu(\gamma^*) \neq 0$, we can apply Gärding's inequality and obtain from (72) that

$$s\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Again, (72) indicates that the normal derivative of $\mathbf{z}'_\nu$ is also bounded:

$$s(\|\mathbf{z}'_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}'_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}) \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{73}$$

Next, we can estimate $\partial_{y_1}z_{4,\nu}$ and $\partial_{y_2}z_{4,\nu}$ by means of (58). This, together with (47), provides the estimate

$$\begin{aligned} s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} &\le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) \\ &\quad + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}). \end{aligned} \tag{74}$$

Finally, the third equation of the Lamé system (9) gives an estimate of the normal derivative of $z_{4,\nu}$. Combining this and (74), (73) and (43), we deduce the desired inequality (27).

Case 2.2.2: $s^* = 0$ and $\varphi_{\tilde{y}_3,\mu}(y^*) \le (\mathrm{Im}(r_\mu(\gamma^*)/s))/(2\sqrt{\mathrm{Re}(r_\mu(\gamma^*))})$

We first apply estimate (41) to $z_{4,\nu}$ and we obtain

$$\begin{aligned} s\|z_{4,\nu}\|^2_{H^{1,s}(\mathcal{G})} + \Sigma^1_{\lambda+2\mu}(z_{4,\nu}) &\le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) \\ &\quad + \varepsilon(\delta)s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}), \end{aligned} \tag{75}$$

where $\Sigma^1_{\lambda+2\mu}(z_{4,\nu})$ has the form

$$\Sigma^1_{\lambda+2\mu}(z_{4,\nu}) = \Sigma^{1,1}_{\lambda+2\mu}(z_{4,\nu}) + \Sigma^{1,2}_{\lambda+2\mu}(z_{4,\nu}) + \Sigma^{1,3}_{\lambda+2\mu}(z_{4,\nu}) \tag{76}$$

and $\Sigma^{1,k}_{\lambda+2\mu}(z_{4,\nu})$ $(k = 1, 2, 3)$ can be obtained from the expressions of $\Sigma^{1,k}_{\mu}$ given in (44) by just changing μ by $\lambda + 2\mu$.

From the fact that $r_{\lambda+2\mu}(\gamma^*) = 0$, one can obtain an estimate of $\Sigma^{1,2}_{\lambda+2\mu}(z_{4,\nu})$ just in the same way as we did in the previous paragraph:

$$|\Sigma^{1,2}_{\lambda+2\mu}(z_{4,\nu})| \le \varepsilon(\delta) s \|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}. \tag{77}$$

On the other hand, let us recall the expression of $\Sigma^{1,3}_{\lambda+2\mu}(z_{4,\nu})$:

(78)

$$\begin{aligned}
&\Sigma^{1,3}_{\lambda+2\mu}(z_{4,\nu})\\
&= -2s \operatorname{Re} \int_{\partial\mathcal{G}} \left((\lambda + 2\mu - R_{33})(y^*)D_{y_3} - R_{13}(y^*)D_{y_1} - R_{23}(y^*)D_{y_2}\right) z_{4,\nu}\\
&\times \Bigg(\overline{\varphi_{y_0}(y^*)\rho(y^*)D_{y_0} z_{4,\nu} - \left(\lambda + 2\mu - R_{11} - \frac{R_{13}^2}{\lambda + 2\mu - R_{33}}\right)(y^*)\varphi_{y_1}(y^*)D_{y_1} z_{4,\nu}}\\
&\quad \overline{- \left(\lambda + 2\mu - R_{22} - \frac{R_{23}^2}{\lambda + 2\mu - R_{33}}\right)(y^*)\varphi_{y_2}(y^*)D_{y_2} z_{4,\nu}}\\
&\quad \overline{+ \left(R_{12} + \frac{R_{13}R_{23}}{\lambda + 2\mu - R_{33}}\right)(y^*)(\varphi_{y_2}(y^*)D_{y_1} z_{4,\nu} + \varphi_{y_1}(y^*)D_{y_2} z_{4,\nu})}\Bigg)\, dy'.
\end{aligned}$$

The next step will be to get an expression of $\Sigma^{1,3}_{\lambda+2\mu}(z_{4,\nu})$ just in terms of tangential derivatives of $z_{4,\nu}$.

First, from proposition 2 applied to $\mathbf{z}'_\nu$, estimate (72) holds for $\mathbf{w}'^{+}_\nu = (D_{y_3} - \Gamma^+_\mu(y, s, D'))\mathbf{z}'_\nu$, so we have

$$\partial_{\tilde{y}_3,\mu}\mathbf{z}'_\nu = (\sqrt{r_\mu}^{+}(y, s, D')/(\mu - R_{33}))\mathbf{z}'_\nu + \mathbf{w}'^{+}_\nu,$$

with

$$s\|\mathbf{w}'^{+}_\nu\|^2_{L^2(\partial\mathcal{G})} \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Now, combining this with the two first equations of the Lamé system (see (58)), we have

$$\begin{aligned}
&(\sqrt{r_\mu}^{+}(y^*, s^*, D') - R_{13}(y^*)D_{y_1} - R_{23}(y^*)D_{y_2})z_{k,\nu}\\
&\qquad\qquad = (-1)^{k+1}(\lambda + 2\mu - R_{33})(y^*)D_{y_{3-k}} z_{4,\nu} + G_k
\end{aligned} \tag{79}$$

for $k = 1, 2$, with G_k satisfying estimate

(80)

$$s\|G_k\|^2_{L^2(\partial\mathcal{G})} \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) + \varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}).$$

Let us see that

$$|r_\mu(\gamma^*)| > (R_{13}(y^*)\xi_1^* + R_{23}(y^*)\xi_2^*)^2. \tag{81}$$

Looking at the expression of r_μ and taking into account that $r_{\lambda+2\mu}(\gamma^*) = 0$, we deduce that $r_\mu(\gamma^*)$ is a positive real number, which coincides with

$$(\lambda + \mu)(y^*)\left(\frac{(R_{13}(y^*)\xi_1^* + R_{23}(y^*)\xi_2^*)^2}{(\lambda + 2\mu - R_{33})(y^*)} + (\mu - R_{33})(y^*)((\xi_1^*)^2 + (\xi_2^*)^2)\right). \tag{82}$$

Consequently,

$$\begin{aligned}
&r_\mu(\gamma^*) - (R_{13}\xi_1^* + R_{23}\xi_2^*)^2\\
&\qquad = (\mu - R_{33})(y^*)\left(\frac{-(R_{13}(y^*)\xi_1^* + R_{23}(y^*)\xi_2^*)^2}{(\lambda + 2\mu - R_{33})(y^*)} + (\lambda + \mu)(y^*)((\xi_1^*)^2 + (\xi_2^*)^2)\right),
\end{aligned}$$

which is a positive number as long as condition (13) is satisfied, so (81) holds.

This yields that from (79) we can express $z_{1,\nu}$ and $z_{2,\nu}$ in terms of tangential derivatives of $z_{4,\nu}$ in the following way:

(83)
$$\widehat{z}_{k,\nu} = (-1)^{k+1}\frac{(\lambda+2\mu-R_{33})(y^*)\xi_{3-k}}{\sqrt{r_\mu}^+(y^*,s^*,\xi')-R_{13}(y^*)\xi_1-R_{23}(y^*)\xi_2}\widehat{z}_{4,\nu}+T(\widehat{z}_{k,\nu})+\widehat{G}_{k+2},$$

$k=1,2$, with G_3 and G_4 satisfying estimate (80).

Now, we recall that from the last equation of the our Lamé we can deduce that

(84)
$$\begin{aligned}&(\lambda+2\mu-R_{33})(y^*)\partial_{\tilde{y}_3,\lambda+2\mu}z_{4,\nu}\\ &=R_{13}(y^*)\partial_{y_1}z_{4,\nu}+R_{23}(y^*)\partial_{y_2}z_{4,\nu}+(\mu-R_{33})(y^*)(\partial_{y_2}z_{1,\nu}-\partial_{y_1}z_{2,\nu})+G_5.\end{aligned}$$

with G_5 satisfying estimate (80). Denote

$$\begin{aligned}q(\xi_1,\xi_2)=(&R_{13}(y^*)\xi_1+R_{23}(y^*)\xi_2\\ &-\frac{(\mu-R_{33})(\lambda+2\mu-R_{33})(y^*)(\xi_1^2+\xi_2^2)}{\sqrt{r_\mu}^+(y^*,0,\xi')-R_{13}(y^*)\xi_1-R_{23}(y^*)\xi_2})\end{aligned}$$

Plugging (83) into (78) and taking into account (84), we obtain

$$\Sigma^{1,3}_{\lambda+2\mu}(z_{4,\nu})=I_3(z_\nu)-\operatorname{Re}\int_{\mathbf{R}^3}q(\xi_1,\xi_2)\operatorname{Im}r_{\lambda+2\mu}(y^*,s,\xi')|\widehat{z}_{4,\nu}|^2\,d\xi', \tag{85}$$

where $I_3(z_\nu)$ is bounded by the expression in (63). In this situation, since $r_{\lambda+2\mu}(\gamma^*)=0$, we have

$$\begin{aligned}\xi_0^*&=\pm\mathcal{Q}_{\lambda+2\mu}(\xi_1^*,\xi_2^*)\\ &=\pm\Big((\lambda+2\mu-R_{11})(y^*)(\xi_1^*)^2+(\lambda+2\mu-R_{22})(y^*)(\xi_2^*)^2-2R_{12}(y^*)\xi_1^*\xi_2^*\\ &\qquad-\frac{(R_{13}(y^*)\xi_1^*+R_{23}(y^*)\xi_2^*)^2}{(\lambda+2\mu-R_{33})(y^*)}\Big)^{1/2}.\end{aligned}$$

Let us now compute the term $\Sigma^{1,1}_{\lambda+2\mu}(z_{4,\nu})$. We first recall that it is given by

$$\begin{aligned}\Sigma^{1,1}_{\lambda+2\mu}(z_{4,\nu})=s\int_{\partial\mathcal{G}}&(\lambda+2\mu-R_{33})^2(y^*)\varphi_{\tilde{y}_3,\lambda+2\mu}(y^*)\\ &\times(|\partial_{\tilde{y}_3,\lambda+2\mu}z_{4,\nu}|^2+s^2\varphi^2_{\tilde{y}_3,\lambda+2\mu}|z_{4,\nu}|^2)\,dy',\end{aligned}$$

so we can write it like follows:

$$\Sigma^{1,1}_{\lambda+2\mu}(z_{4,\nu})=s\int_{\partial\mathcal{G}}(\lambda+2\mu-R_{33})^2(y^*)\varphi_{\tilde{y}_3,\lambda+2\mu}(y^*)|\partial_{\tilde{y}_3,\lambda+2\mu}z_{4,\nu}|^2\,dy'+I_4(z_\nu),$$

where $I_4(z_\nu)$ satisfies estimate (63).

Using the same argument as before (that is to say, using the third equation of the Lamé system together with (83)), we obtain

$$\Sigma^{1,1}_{\lambda+2\mu}(z_{4,\nu})=s\int_{\mathbf{R}^3}\varphi_{\tilde{y}_3,\lambda+2\mu}(y^*)|q(\xi_1,\xi_2)|^2|\hat{z}_{4,\nu}|^2\,d\xi'+I_5(\mathbf{z}_\nu),$$

(recall that the principal symbol of the operator $\sqrt{r_\mu}^+(y,s,D')$ at point γ^* is real), where $I_5(\mathbf{z}_\nu)$ satisfies estimate (63).

Now let us show that Condition **C** implies that

$$q(\xi_1^*,\xi_2^*)^2-q(\xi_1^*,\xi_2^*)(\operatorname{Im}r_{\lambda+2\mu}/s)(y^*,s,(\xi')^*)>0 \tag{86}$$

Really this is equivalent to

(87)
$$q(\xi_1,\xi_2)^2-q(\xi_1,\xi_2)(\operatorname{Im}r_{\lambda+2\mu}/s)(y^*,s,\pm\mathcal{Q}_{\lambda+2\mu}(\xi_1,\xi_2),\xi_1,\xi_2)>0\quad\forall|(\xi_1,\xi_2)|=1.$$

Using then (86) to apply Garding's inequality and taking into account (76)-(77), we obtain estimates of the tangential derivatives of $z_{4,\nu}$:

$$\Sigma^1_{\lambda+2\mu}(z_{4,\nu}) \geq Cs\|z_{4,\nu}\|^2_{H^{1,s}(\mathcal{G})} - C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Putting this together with (75) and using (54) and (72), we have

(88)

$$s\sum_{j=3}^4(\|z_{j,\nu}\|^2_{H^{1,s}(\mathcal{G})} + \|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}) + s\|\partial_{y_3}z_{3,\nu}\|^2_{L^2(\partial\mathcal{G})} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) + \varepsilon(\delta)s\|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}.$$

We use now the representation formulas of $\widehat{z}_{1,\nu}$ and $\widehat{z}_{2,\nu}$ in terms of $\widehat{z}_{4,\nu}$ (see (83)). Then, (88) also provides estimates for the tangential derivatives of $z_{1,\nu}$ and $z_{2,\nu}$:

(89)

$$s\sum_{j=1}^4(\|z_{j,\nu}\|^2_{H^{1,s}(\mathcal{G})} + \|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}) + s\|\partial_{y_3}z_{3,\nu}\|^2_{L^2(\partial\mathcal{G})} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) + \varepsilon(\delta)s\|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}.$$

Finally, the two first equations of the Lamé system (see (58)) provides estimates of the normal derivatives of $z_{1,\nu}$ and $z_{2,\nu}$, while the third equation (see (84)) gives the estimate for the normal derivative of $z_{4,\nu}$. Consequently, from (89) we deduce the desired estimate (27) in this case.

Case 2.2.3: $s^* \neq 0$

We first apply estimate (41) to $z_{4,\nu}$ and get

$$\begin{aligned} &s\|z_{4,\nu}\|^2_{H^{1,s}(\mathcal{G})} + \Sigma^1_{\lambda+2\mu}(z_{4,\nu}) \\ &\leq C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|z\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) + \varepsilon(\delta)s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}), \end{aligned} \tag{90}$$

with $\Sigma^1_{\lambda+2\mu}(z_{4,\nu})$ given by (76).

In order to estimate $\Sigma^{1,2}_{\lambda+2\mu}(z_{4,\nu})$, we use that $r_{\lambda+2\mu}(\gamma^*) = 0$ and we obtain

$$|\Sigma^{1,2}_{\lambda+2\mu}(z_{4,\nu})| \leq \varepsilon(\delta_1)s\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}.$$

Also, from $r_{\lambda+2\mu}(\gamma^*) = 0$ and (47), we obtain for small δ_1 that

$$\begin{aligned} s\Bigg|&\rho(y^*)\varphi_{y_0}(y^*)\xi_0 - \left(\lambda+2\mu - R_{11} - \frac{R_{13}^2}{\lambda+2\mu-R_{33}}\right)(y^*)\varphi_{y_1}(y^*)\xi_1 \\ &- \left(\lambda+2\mu - R_{22} - \frac{R_{23}^2}{\lambda+2\mu-R_{33}}\right)\varphi_{y_2}(y^*)\xi_2 \\ &+ \left(R_{12} + \frac{R_{13}R_{23}}{\lambda+2\mu-R_{33}}\right)(y^*)\left(\varphi_{y_2}(y^*)\xi_1 + \varphi_{y_1}(y^*)\xi_2\right)\Bigg| \\ &\qquad \leq \varepsilon(\delta_1)(s^2+\xi_1^2+\xi_2^2) \quad \forall\zeta\in\mathcal{O}(\delta_1), \end{aligned}$$

which yields

$$|\Sigma^{1,3}_{\lambda+2\mu}(z_{4,\nu})| \leq \varepsilon(\delta_1)s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}).$$

Consequently, we have

$$\begin{aligned} \Sigma^1_{\lambda+2\mu}(z_{4,\nu}) \geq &-\varepsilon(\delta_1)s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}) \\ &+s(s^2\|z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}). \end{aligned}$$

From the fact that $s^* \neq 0$, we also deduce that

$$\Sigma^1_{\lambda+2\mu}(z_{4,\nu}) \geq s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}),$$

which, together with (90), gives

$$s\|z_{4,\nu}\|^2_{H^{1,s}(\mathcal{G})}+s(\|z_{4,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}+\|\partial_{y_3}z_{4,\nu}\|^2_{L^2(\partial\mathcal{G})}) \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{91}$$

In this situation, estimates (54) and (72) also hold and give

$$s(\|z_{3,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}+\|\partial_{y_3}z_{3,\nu}\|^2_{L^2(\partial\mathcal{G})}) \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) +\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}). \tag{92}$$

The next step will be to estimate the functions $z_{1,\nu}$ and $z_{2,\nu}$. To this end, we first observe that the case where

$$(\sqrt{r_\mu}^{+}(\gamma)-R_{13}(\xi_1^*+is^*\varphi_{y_1})-R_{23}(\xi_2^*+is^*\varphi_{y_2}))\neq 0 \tag{93}$$

directly gives good estimates; indeed, the two first equations in (9) can be rewritten as

$$(\sqrt{r_\mu}^{+}(y,s,D')-R_{13}(D_{y_1}+is\varphi_{y_1})-R_{23}(D_{y_2}+is\varphi_{y_2}))z_{2,\nu}=F_8$$

and

$$(\sqrt{r_\mu}^{+}(y,s,D')-R_{13}(D_{y_1}+is\varphi_{y_1})-R_{23}(D_{y_2}+is\varphi_{y_2}))z_{1,\nu}=F_9$$

respectively, with F_8 and F_9 satisfying estimate (57). To prove this, it suffices to take into account (91) and (72). Now, from (93), we can apply Gärding's inequality to the two previous equations and we find out estimates of the $H^{1,s}$ norm of $z_{1,\nu}$ and $z_{2,\nu}$ on $\partial\mathcal{G}$. Finally, (72) provides the estimate of the normal derivative too.

Outside this particular situation, we distinguish another two cases when (93) does not hold:

- If $(\xi_1^*+is\varphi_{y_1}(y^*))^2+(\xi_2^*+is\varphi_{y_2}(y^*))^2\neq 0$

Then, one can also represent $(z_{1,\nu},z_{2,\nu})$ on $\partial\mathcal{G}$ as in (59) by means of a similar argument but using (91) instead of (48). Consequently,

$$s\sum_{j=1}^{2}\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}\le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) +\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}) \tag{94}$$

Finally, the estimate of $\partial_{y_3}z_{j,\nu}$ $(j=1,2)$ comes from (72). As a conclusion, we have

$$s\sum_{j=1}^{2}(\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}+\|\partial_{y_3}z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})})\le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}) +\varepsilon(\delta)s(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}).$$

- If $(\xi_1^*+is^*\varphi_{y_1}(y^*))^2+(\xi_2^*+is^*\varphi_{y_2}(y^*))^2=0$

Let us apply estimate (43) to $z_{1,\nu}$ and $z_{2,\nu}$:

$$s\|z_{j,\nu}\|^2_{H^{1,s}(\mathcal{G})}+\Sigma^1_\mu(z_{j,\nu})\le\varepsilon(\delta)s(\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}+\|\partial_{y_3}z_{j,\nu}\|^2_{L^2(\partial\mathcal{G})}) +C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})})\quad\text{for }\ j=1,2, \tag{95}$$

where $\Sigma^1_\mu(z_{j,\nu})$ are given just after (44).

Let us estimate $\Sigma_\mu^{1,2}(z_{j,\nu})$. First, we rewrite it in the form

$$\Sigma_\mu^{1,2}(z_{j,\nu}) = s\int_{\partial\mathcal{G}}(\mu - R_{33})(y^*)\varphi_{\tilde{y}_3,\lambda+2\mu}(y^*)\left[\rho(y^*)|\partial_{y_0}z_{j,\nu}|^2 - s^2\rho(y^*)\varphi_{y_0}^2(y^*)|z_{j,\nu}|^2\right.$$
$$-\left(\lambda+2\mu-R_{11}-\frac{R_{13}^2}{\lambda+2\mu-R_{33}}\right)(y^*)(|\partial_{y_1}z_{j,\nu}|^2 - s^2\varphi_{y_1}^2(y^*)|z_{j,\nu}|^2)$$
$$-\left(\lambda+2\mu-R_{22}-\frac{R_{23}^2}{\lambda+2\mu-R_{33}}\right)(y^*)(|\partial_{y_2}z_{j,\nu}|^2 - s^2\varphi_{y_2}^2(y^*)|z_{j,\nu}|^2)$$
$$\left.+2\left(R_{12}+\frac{R_{13}R_{23}}{\lambda+2\mu-R_{33}}\right)(y^*)(\partial_{y_1}z_{j,\nu}\,\partial_{y_2}z_{j,\nu} - s^2\varphi_{y_1}(y^*)\varphi_{y_2}(y^*))|z_{j,\nu}|^2\right]dy'$$
$$+s\int_{\partial\mathcal{G}}(\mu - R_{33})(y^*)\varphi_{\tilde{y}_3}\left[(\lambda+\mu)\left(|\partial_{y_1}z_{j,\nu}|^2 + |\partial_{y_2}z_{j,\nu}|^2 - s^2\varphi_{y_1}^2(y^*)|z_{j,\nu}|^2\right.\right.$$
$$\left.-s^2\varphi_{y_2}^2(y^*)|z_{j,\nu}|^2\right) - (\lambda+2\mu-R_{33})^{-1}(y^*)\left(R_{13}^2(y^*)\left(|\partial_{y_1}z_{j,\nu}|^2\right.\right.$$
$$\left.-s^2\varphi_{y_1}^2(y^*)|z_{j,\nu}|^2\right) + R_{23}^2(y^*)\left(|\partial_{y_2}z_{j,\nu}|^2 - s^2\varphi_{y_2}^2(y^*)|z_{j,\nu}|^2\right)$$
$$\left.\left.+2R_{13}(y^*)R_{23}(y^*)\left(\partial_{y_1}z_{j,\nu}\partial_{y_2}z_{j,\nu} - s^2\varphi_{y_1}\varphi_{y_2}|z_{j,\nu}|^2\right)\right)\right]dy'$$
$$= J_4(z_{j,\nu}) + J_5(z_{j,\nu}).$$

From $\mathrm{Re}(r_{\lambda+2\mu}(\gamma^*)) = 0$, we deduce that

$$|J_4(z_{j,\nu})| \le \varepsilon(\delta_1)s\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}. \tag{96}$$

Let us now estimate $J_5(z_{j,\nu})$; we observe that

$$J_5(z_{j,\nu}) = s\int_{\mathbf{R}^3}(\mu - R_{33})(y^*)\varphi_{\tilde{y}_3,\lambda+2\mu}(y^*)\left[(\lambda+\mu)\left(\xi_1^2+\xi_2^2 - s^2\varphi_{y_1}^2(y^*)\right.\right.$$
$$\left.-s^2\varphi_{y_2}^2(y^*)\right) - (\lambda+2\mu-R_{33})^{-1}(y^*)\,\mathrm{Re}\left(R_{13}(y^*)(\xi_1 + is\varphi_{y_1}(y^*))\right.$$
$$\left.\left.+R_{23}(y^*)(\xi_2 + is\varphi_{y_2}(y^*))\right)^2\right]|\hat{z}_{j,\nu}|^2\,d\xi_1\,d\xi_2.$$

We first realize that, since $r_{\lambda+2\mu}(\gamma^*) = 0$ and

$$(\sqrt{r_\mu}^{+}(\gamma^*) - R_{13}(\xi_1^* + is^*\varphi_{y_1}) - R_{23}(\xi_2^* + is^*\varphi_{y_2})) = 0,$$

we have

$$\mathrm{Re}\left(R_{13}(y^*)(\xi_1^* + is\varphi_{y_1}(y^*)) + R_{23}(y^*)(\xi_2^* + is\varphi_{y_2}(y^*))\right)^2$$
$$= (\lambda+2\mu-R_{33})(y^*)(\lambda+\mu)\,\mathrm{Re}\left((\xi_1^* + is\varphi_{y_1}(y^*))^2 + (\xi_2^* + is\varphi_{y_2}(y^*))^2\right).$$

Then, it is readily seen that $J_5(z_{j,\nu})$ can be bounded as in (96). Consequently,

$$|J_5(z_{j,\nu})| \le \varepsilon(\delta_1)s\|z_{j,\nu}\|^2_{H^{1,s}(\partial\mathcal{G})}. \tag{97}$$

For $\Sigma_\mu^{1,3}(z_{j,\nu})$, we use the third equation of the Lamé system (9) (see (58)) in combination with (91) and we apply Young's inequality. We deduce that

$$|\Sigma_\mu^{1,3}(z_{j,\nu})|$$
$$\le \varepsilon(\delta_1)(\|\mathbf{z}_\nu\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\partial_{y_3}\mathbf{z}_\nu\|^2_{\mathbf{L}^2(\partial\mathcal{G})}) + C(\delta_1)(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{z}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}),$$

where the constant $C(\delta_1)$ may tend to infinity when δ_1 tends to zero.

As a conclusion of this, (91), (95) and (97), we deduce the desired inequality (27).

2.3. Third Case: $r_\mu(\gamma^*) \neq 0$ and $r_{\lambda+2\mu}(\gamma^*) \neq 0$ or $r_\mu(\gamma^*) = r_{\lambda+2\mu}(\gamma^*) = 0$. It is in this paragraph where we will use Calderon's method. The ideas we develop here are similar to those in [15].

Let us introduce the variables $\mathbf{U} = (U_j)_{j=1}^6$, given by

$$(U_1, U_2, U_3) = \Lambda(s, D')(e^{s\varphi}\mathbf{u}), \quad (U_4, U_5, U_6) = I\!\!D_{y_3}(e^{s\varphi}\mathbf{u}),$$

where Λ is the pseudodifferential operator with symbol $(1 + s^2 + |\xi'|^2)$. With these notations, we can rewrite the Lamé system (9) as

$$\text{(98)} \qquad \begin{cases} D_{y_3}\mathbf{U} = K(y, s, D')\mathbf{U} + \mathbf{F} & \text{in } \mathbf{R}^3 \times [0, 1/Z^2], \\ (U_1, U_2, U_3) = 0 & \text{on } \{y_3 \equiv 0\}, \\ \mathbf{U} = 0 & \text{on } \{y_3 \equiv 1/Z^2\}, \end{cases}$$

where $\mathbf{F} = (0, e^{s\varphi}\mathbf{f})$ and $K(y, s, D')$ is a pseudodifferential operator whose principal symbol is

$$K_1(\gamma) = \begin{pmatrix} 0 & \Lambda_1 E_3 \\ A^{-1}K_{11}\Lambda^{-1} & A^{-1}K_{12} \end{pmatrix} - is\varphi_{y_3}E_6.$$

Here, we have set

$$\text{(99)} \qquad \begin{aligned} &A = (\lambda + \mu)GG^T + [((\mu E_3 - R)G^T)G]E_3, \\ &K_{11} = -(\lambda + \mu)\theta\theta^T + [(\xi_0 + is\varphi_{y_0})^2 - ((\mu E_3 - R)\theta^T)\theta]E_3, \\ &K_{12} = -(\lambda + \mu)(\theta G^T + G\theta^T) - 2((\mu E_3 - R)\theta^T)GE_3, \end{aligned}$$

with $\theta = (\xi_1 + is\varphi_{y_1}, \xi_2 + is\varphi_{y_2}, 0)^T$. Recall that G was defined (also as a column vector field) in (26).

In this context, one can check that the eigenvalues of the matrix K_1 coincide with $\Gamma_\beta^\pm$ (given in (34)).

Let us consider again several situations:

Case 2.3.1: $r_\mu(\gamma^*) = 0 = r_{\lambda+2\mu}(\gamma^*)$

Then, the expression of r_β indicates directly that $\operatorname{Im}(\Gamma_\beta^\pm((\gamma^*)) < 0$ for $\beta \in \{\mu, \lambda+2\mu\}$; this comes from the fact that $\varphi_{\tilde{y}_3,\beta} > 0$. Consequently, the eigenvalues of the matrix K_1 have negative imaginary parts and we can suppose that

$$\operatorname{Im}(\Gamma_\beta^\pm(\gamma)) < -C|\zeta| \quad \forall \gamma \in B_\delta \times \mathcal{O}(\delta_1).$$

Now, using standard arguments (see, for instance, Chapter 7 in [19]), we obtain that

$$\|\chi_\nu \mathbf{U}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})} \leq C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|e^{s\varphi}\mathbf{u}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

From this inequality, estimate (27) is readily deduced.

Case 2.3.2: $r_\mu(\gamma^*) \neq 0$, $r_{\lambda+2\mu}(\gamma^*) \neq 0$ and $r_\mu(\gamma^*) \neq r_{\lambda+2\mu}(\gamma^*)$

In this situation, the matrix K_1 has four eigenvalues which are given by (34) and the corresponding eigenvectors:

$$\text{(100)} \qquad \begin{aligned} v_1^\pm &= ((\theta^T + \alpha_{\lambda+2\mu}^\pm G^t)\Lambda_1^{-1}, \alpha_{\lambda+2\mu}^\pm(\theta^T + \alpha_{\lambda+2\mu}^\pm G^T)\Lambda_1^{-2})^T, \\ v_2^\pm &= (v_{21}^\pm, \alpha_\mu^\pm v_{21}^\pm \Lambda_1^{-1})^T, \quad v_3^\pm = (v_{31}^\pm, \alpha_\mu^\pm v_{31}^\pm \Lambda_1^{-1})^T, \end{aligned}$$

with

$$v_{21}^\pm = (\ell_{y_2}\alpha_\mu^\pm - \theta_2, -\ell_{y_1}\alpha_\mu^\pm + \theta_1, 0)\Lambda_1^{-1}$$

and

$$v_{31}^\pm = (\alpha_\mu^\pm(\theta_1 - \ell_{y_1}\alpha_\mu^\pm), \alpha_\mu^\pm(\theta_2 - \ell_{y_2}\alpha_\mu^\pm), -\sum_{j=1}^2(\theta_j - \ell_{y_j}\alpha_\mu^\pm)^2)\Lambda_1^{-2}.$$

Observe that $\{v_2^\pm, v_3^\pm\}$ is a basis of the orthogonal space to the vector $\theta + \alpha_\mu^\pm G$.

Let us take the symbol $S(\gamma)$ in the form $S = \{v_1^+, v_2^+, v_3^+, v_1^-, v_2^-, v_3^-\}$ and let us extend it as an homogeneous function of order 0 and of class C^3 in the ζ variables. Now, $\mathbf{U}$ is determined by (98). Let us introduce $\mathbf{W} = S^{-1}(y, s, D')\mathbf{U}$. Then, system (98) takes the form

$$D_{y_3}\mathbf{W} = \widetilde{K}(y, s, D')\mathbf{W} + T(y, s, D')\mathbf{W} + \widetilde{\mathbf{F}},$$

where $\widetilde{K}$ is a diagonal matrix and $T \in L^\infty(0, 1; \mathcal{L}(\mathbf{H}^{1,s}(\mathbf{R}^3)); \mathbf{H}^{1,s}(\mathbf{R}^3))$. A standard argument for pseudodifferential systems allows to estimate the last three components of $\mathbf{W}$ as follows (see for instance [19]):

$$s\|(W_4, W_5, W_6)\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|e^{s\varphi}\mathbf{u}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}). \tag{101}$$

Finally, the first three components can be bounded in terms of the last three by means of the first-squared 3×3 matrix inside S, which is

$$\Lambda_1^{-1}\begin{pmatrix} \theta_1 - \alpha^+_{\lambda+2\mu}\ell_{y_1} & -\theta_2 + \ell_{y_2}\alpha^+_\mu & \alpha^+_\mu(\theta_1 - \ell_{y_1}\alpha^+_\mu)\Lambda_1^{-1} \\ \theta_2 - \alpha^+_{\lambda+2\mu}\ell_{y_2} & \theta_1 - \ell_{y_1}\alpha^+_\mu & \alpha^+_\mu(\theta_2 - \ell_{y_2}\alpha^+_\mu)\Lambda_1^{-1} \\ \alpha^+_{\lambda+2\mu} & 0 & -\sum_{j=1}^2(\theta_j - \ell_{y_j}\alpha^+_\mu)^2\Lambda_1^{-1} \end{pmatrix}$$

This yields

$$s\|(W_1, W_2, W_3)\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C(\|e^{s\varphi}\mathbf{f}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|e^{s\varphi}\mathbf{u}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}),$$

which, in combination with (101) and (41), provides (27).

Case 2.3.3: $r_\mu(\gamma^*) = r_{\lambda+2\mu}(\gamma^*) \neq 0$

The argument needed to prove estimate (27) in this case coincides exactly with the one developed in [15].

This ends the proof of Lemma 2.

3. Controllability of the Lamé system.

In this section we obtain some exact controllability results for the Lamé system with controls locally distributed over the cylinder $Q_\omega = \omega \times (0, T)$.

First, we need a Carleman estimate similar to (15) but with the right hand side in the spaces $L^2(Q)$ and $H^{-1}(Q)$. In order to prove such an estimate, we need a pseudoconvex function ψ which satisfies the additional condition:

$$\partial_{x_0}\psi(\cdot, 0) > 0, \quad \partial_{x_0}\psi(\cdot, T) < 0 \quad \forall x \in \overline{\Omega}. \tag{102}$$

We have the following result:

Theorem 2. *Let $\mathbf{f} \in \mathbf{L}^2(Q)$ and assume that* (2)*,* (4)*,* (6) *and* (102) *hold true. Suppose there exists a function ψ which satisfies condition A. Then, there exists $\tau^* > 0$ such that, for any $\tau > \tau^*$, there exists $s^* > 0$ such that*

$$\|\mathbf{u}\|_{\mathbf{H}^{1,s}(Q)} \le C(\|e^{s\phi}\mathbf{f}\|_{\mathbf{L}^2(Q)} + \|\mathbf{u}\|_{\mathbf{H}^{1,s}(Q_\omega)}) \quad \forall s > s^* \tag{103}$$

for some positive constant C independent of s and for any solution $\mathbf{u} \in \mathbf{H}^1(Q)$ of (9).

Next, we consider a situation with the function $\mathbf{f}$ in the space $\mathbf{H}^{-1}(Q)$.

Theorem 3. *Let* $\mathbf{f} = \mathbf{f}_{-1} + \sum_{k=0}^{3} \partial_{x_k} \mathbf{f}_k$, *where* $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 \in \mathbf{L}^2(Q)$ *and assume that* (2), (4), (6) *and* (102) *hold true. Suppose there exists a function* ψ *which satisfies condition A. Then, there exists* $\tau^* > 0$ *such that, for any* $\tau > \tau^*$, *there exists* $s^* > 0$ *such that*

$$(104)\quad \|\mathbf{u}\|_{\mathbf{L}^2(Q)} \le C(\|e^{s\phi}\mathbf{f}_{-1}\|_{\mathbf{H}^{-1}(Q)} + \sum_{k=0}^{3} \|\mathbf{f}_k e^{s\phi}\|_{\mathbf{L}^2(Q)} + \|\mathbf{u}\|_{\mathbf{L}^2(Q_\omega)}) \quad \forall s > s^*$$

for some positive constant C *independent of* s *and for any solution* $\mathbf{u} \in \mathbf{L}^2(Q)$ *of* (9).

The proofs of theorems 2 and 3 rely on theorem 1 and are similar to the proofs of the similar results presented in [14]).

Next, we consider the controllability problem

$$(105)\quad \begin{cases} P(x,D)\mathbf{u} = \mathbf{f} + \chi_\omega \mathbf{v} & \text{in } Q, \quad \mathbf{u} = 0 \quad \text{on } \Sigma, \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0, \quad \partial_{x_0}\mathbf{u}(\cdot,0) = \mathbf{u}_1 & \text{in } \Omega. \end{cases}$$

Here $\mathbf{f}, \mathbf{u}_0, \mathbf{u}_1$ are given functions and v is a control. Recall that the operator P was defined in (9). Suppose that two target functions $\mathbf{u}_2$ and $\mathbf{u}_3$ are given. We have to find a control $\mathbf{v}$ such that

$$(106)\quad \mathbf{u}(\cdot,T) = \mathbf{u}_2, \quad \partial_{x_0}\mathbf{u}(\cdot,T) = \mathbf{u}_3 \quad \text{in } \Omega.$$

Condition B. Suppose that there exists $t^* \in (0,T)$ such that

$$\min_{x' \in \overline{\Omega \setminus \omega}} \psi(x', t^*) > \max\{ \max_{x' \in \overline{\Omega \setminus \omega}} \psi(x', 0), \max_{x' \in \overline{\Omega \setminus \omega}} \psi(x', T)\}.$$

Before presenting the main result of this section, let us remind a well known result on the solvability of the Lamé system

$$(107)\quad \begin{cases} P(x,D)\mathbf{p} = \mathbf{q} & \text{in } Q, \quad \mathbf{p} = 0 \quad \text{on } \Sigma, \\ \mathbf{p}(\cdot,0) = \mathbf{p}_0, \quad \partial_{x_0}\mathbf{p}(\cdot,0) = \mathbf{p}_1 & \text{in } \Omega. \end{cases}$$

We have the following:

Theorem 4. *Assume that* (2) *and* (6) *hold. Then, if* $\mathbf{q} \in \mathbf{L}^2(Q)$, $\mathbf{p}_0 \in \mathbf{H}^1_0(\Omega)$ *and* $\mathbf{p}_1 \in \mathbf{L}^2(\Omega)$, *problem* (107) *possesses exactly one solution* $\mathbf{p} \in \mathbf{H}^1(Q)$.

Furthermore, if $\mathbf{q} \in \mathbf{H}^{-1}(Q)$, $\mathbf{p}_0 \in \mathbf{L}^2(\Omega)$ *and* $\mathbf{p}_1 \in \mathbf{H}^{-1}(\Omega)$ *then there exists a solution to problem* (107) *possesses exactly one solution* $\mathbf{p} \in \mathbf{L}^2(Q)$.

Now, we state the main result of this section:

Theorem 5. *Assume that* (2), (4) *and* (6) *hold. Suppose that there exists a function* ψ *which satisfies condition A, condition B and* (102). *Then if* $\mathbf{f} \in \mathbf{L}^2(Q)$, $\mathbf{u}_0 \in \mathbf{H}^1_0(\Omega)$ *and* $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$, *there exists a solution* $(\mathbf{u}, \mathbf{v}) \in \mathbf{H}^1(Q) \times \mathbf{L}^2(Q_\omega)$ *to the controllability problem* (105)–(106).

Furthermore, if $\mathbf{f} \in \mathbf{H}^{-1}(Q)$, $\mathbf{u}_0 \in L^2(\Omega)$ *and* $\mathbf{u}_1 \in \mathbf{H}^{-1}(\Omega)$, *there exists a solution* $(\mathbf{u}, \mathbf{v}) \in \mathbf{L}^2(Q) \times \mathbf{H}^{-1}(Q_\omega)$ *to* (105)–(106).

Proof. Let ϵ be a sufficiently small positive number such that $t^* \in (\epsilon, T-\epsilon)$ and set

$$M = \min_{x' \in \overline{\Omega \setminus \omega}} \psi(x', t^*) > \max\{ \max_{x' \in \overline{\Omega \setminus \omega}} \psi(x', \epsilon), \max_{x' \in \overline{\Omega \setminus \omega}} \psi(x', T-\epsilon)\}$$

We introduce a cut-off function $\chi(x_0) \in C_0^\infty([0,T])$ such that $\chi|_{[\epsilon,T-\epsilon]} = 1$. Let $\mathbf{p}$ be a solution to the system

$$\begin{cases} P(x,D)\mathbf{p} \equiv \rho(x')\partial_{x_0}^2\mathbf{p} - L(x,D)\mathbf{p} = \mathbf{q} & \text{in } Q, \\ \mathbf{p} = 0 & \text{on } \Sigma. \end{cases} \tag{108}$$

Then the function $\tilde{\mathbf{p}} = \chi(x_0)\mathbf{p}$ satisfies the equation

$$\begin{cases} P(x,D)\tilde{\mathbf{p}}(x) = \chi\mathbf{q} - [\chi, \partial_{x_0}^2]\mathbf{p} & \text{in } Q, \\ \mathbf{p} = 0 & \text{on } \Sigma. \end{cases} \tag{109}$$

Note that $[\chi, \partial_{x_0}^2]$ is a first order operator with coefficients having support in $[0,\epsilon] \cup [T-\epsilon, T] \times \Omega$. Therefore, we have the following a priori estimates:

$$\|[\chi, \partial_{x_0}^2]\mathbf{p}e^{s\phi}\|_{\mathbf{L}^2(Q)} \le e^{\delta M}\|\mathbf{p}\|_{\mathbf{H}^1(Q)}, \quad \|[\chi, \partial_{x_0}^2]\mathbf{p}e^{s\phi}\|_{\mathbf{H}^{-1}(Q)} \le e^{\delta M}\|\mathbf{p}\|_{\mathbf{L}^2(Q)}.$$

Additionally, we apply Carleman estimate (15) to the equation (109) and we obtain that

$$\|\tilde{\mathbf{p}}\|_{\mathbf{H}^{k,s}(Q)} \le C(\|\mathbf{q}e^{s\phi}\|_{\mathbf{H}^{k-1}(Q)} + \|\tilde{\mathbf{p}}\|_{\mathbf{H}^{k,s}(Q_\omega)} + e^{s\delta M}\|\mathbf{p}\|_{\mathbf{H}^{1-k}(Q)}).$$

Note that, for any sufficiently small $\epsilon > 0$ there exists a $\delta_1 > \delta$ such that

$$e^{s\delta_1 M}\|(\mathbf{p}(\cdot,t^*), \partial_{x_0}\mathbf{p}(\cdot,t^*))\|_{\mathbf{H}^k(\Omega)\times\mathbf{H}^{k-1}(\Omega)} \\ \le C(\|\tilde{\mathbf{p}}\|_{\mathbf{H}^{k,s}(Q)} + \|\mathbf{q}e^{s\varphi}\|_{\mathbf{H}^{k-1}(Q)} + \|\tilde{\mathbf{p}}\|_{\mathbf{H}^{k,s}(Q_\omega)} + e^{\delta M}\|\mathbf{p}\|_{\mathbf{H}^{1-k}(Q)}).$$

Since $\delta_1 > \delta$, taking the parameter s sufficiently large in the previous inequality thanks to the theorem 4, we arrive at the observability estimate

$$\|\mathbf{p}\|_{\mathbf{H}^k(Q)} \le C(\|\mathbf{q}\|_{\mathbf{H}^{k-1}(Q)} + \|\mathbf{p}\|_{\mathbf{H}^k(Q_\omega)}).$$

This observability estimate can be readily converted into the controllability results stated in our theorem by the well known HUM method (see [24]).

This ends the proof of Theorem 5. ■

References

[1] F. Alabau, V. Komornik *Boundary observability and controllability of linear elastodynamic systems* in Optimization methods in partial differential equations, AMS, Providence (1997), 1-8.

[2] F. Alabau, V. Komornik, *Boundary observability, controllability, and stabilization of linear elastodynamic systems*, SIAM J. Control Optim., 37 (1998) pp. 521-542.

[3] P. Albano, D. Tataru, *Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system*, Electron. J. Differential Equations (2000), no 53, 13.

[4] G. Avalos, I. Lasiecka, *Exact–Approximate Boundary Reachability of Thermoelastic Plates Under Variable Thermal Coupling*, Inverse Problems **16**, (2000), 978-996.

[5] M. Bellassoued, *Distribution of resonances and decay of the local energy for the elastic wave equations*, Comm. Math. Phys. **215** (2000), 375-408.

[6] M. Bellassoued, *Carleman estimates and decay rate of the local energy for the Neumann problem of elasticity*, Progr. Nonlinear Differential Equations Appl. **46** (2001), 15-36.

[7] M. Bellassoued,*Unicité et contrôle pour le système de Lamé*, ESIAM. **6** (2001), 561-592.

[8] B. Dehman, L. Robbiano, *La propriété du prolongement unique pour un système elliptique. le système de Lamé*, J. Math. Pures Appl. **72** (1993), 475-492.

[9] M. Eller, V. Isakov, *Carleman estimates with two large parameters and applications*, Differential geometric methods in the control of partial differential equations (Boulder, CO, 1999), 117–136, Contemp. Math., 268, Amer. Math. Soc., Providence, RI, 2000.

[10] M. Eller, V. Isakov, G. Nakamura, D.Tataru, *Uniqueness and stability in the Cauchy problem for Maxwell's and the elasticity system*, in "Nonlinear Partial Differential Equations", Vol.14, Collège de France Seminar, Elsevier-Gauthier Villaras "Series in Applied Mathematics", **31** (2002), 329–350.

[11] A. Fursikov, O. Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes #34, Seoul National University, Korea, 1996.

[12] S.W. Hansen, *Boundary control of a one-dimensional linear thermoelastic rod*, SIAM J. Control Optim., **32** (1994), 1052- 1074.

[13] M. A. Horn, *Implications of sharp regularity results on boundary stabilization of the system of linear elastisity* , J. Math. Analysis and Applications **223** (1998), 126-150.

[14] O. Yu. Imanuvilov, M. Yamamoto, *Carleman estimates for the non-stationary Lamé system and the application to an inverse problem*, ESAIM Control Optim. Calc. Var. 11 (2005), no. 1, 1–56.

[15] O. Yu. Imanuvilov, M. Yamamoto, *Carleman estimates for the three-dimensional non-stationary Lamé system and the application to an inverse problem*, In Series of Lecture Notes in Pure and Applied Mathematics Control Theory of Partial Differential Equations, 242, Chapman & Hall/CRC, 2005, 337-374.

[16] O. Yu. Imanuvilov, M. Yamamoto, *Carleman estimates for the Lamé system with stress boundary condition and the application to an inverse problem*, (Preprint: University of Tokyo Graduate School of Mathematical Sciences **UTMS 2003-2**).

[17] O. Yu. Imanuvilov, M. Yamamoto, *Carleman estimate for stationary isotropic Lamé system*, **83**, (2004), 243-270.

[18] V. Isakov, G. Nakamura, J.-N. Wang *Uniqueness and stability in the Cauchy problem for the elasticity system with residual stress*, Contemp. Math. **333** (2003), 99–113.

[19] H. Kumano-go, *Pseudodifferential Operators*, MIT Press, Cambridge (1981).

[20] J. Lagnese, *The reachibility for thermoelastic plates*, Arch. Rat. Mech. Anal. **112** (1990), 223-267.

[21] J. Lagnese, *Boundary Stabilization of Thin Plates*, Vol. 10 of *SIAM studies in applied mathematics*, (Society for Industrial and Applied Mathematics, 1989).

[22] G. Lebeau, E. Zuazua *Null-Controllability of system of linear thermoelasticity* , Arch. Rat. Mech. Anal. **141** (1998), 297-329.

[23] G. Lebeau, E. Zuazua *Decay rates for three dimentional linear system of thermoelasticity* Archives for Rational Mechanics and Analysis **148** (1999), 179-231.

[24] J. L. Lions, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Review **30** (1988) no. 1, 1–68.

[25] C.-L. Lin, J.-N. Wang, *Uniqueness in inverse problems for an elasticity system with residual stress by a single measurement*, Inverse Problems **19** (2003), no. 4, 807–820.

[26] P. Martinez, *Uniform boundary stabilization of elasticity systems of cubic crystals by nonlinear feedbacks*, Nonlinear Analysis **37**, 719-733.

[27] G. Nakamura, J.-N. Wang, *Unique continuation for an elasticity system with the residual stress and its applications*, SIAM J. Math. Anal **35** (2003), no 2, 304-317.

[28] D. Tataru, *Carleman estimates and unique continuation for solutions to boundary value problems*, J. Math. Pures Appl., **75** (1996), 367-408.

[29] M. E. Taylor, *Pseudodifferential operators*, Princeton Mathematical Series, **34**. Princeton University Press, Princeton, N.J. (1981).

[30] K. Yamamoto *Singularities of solutions to the boundary value problems for elastic and Maxwell's equations* Japan J. Math. **14** (1988), no 1, 119-163.

[31] N. Weck, *Unique continuation for systems with Lamé principal part*, Math. Meth. Appl. Sci., **24** (2001), no. 9, 595-605.

[32] E. Zuazua *A uniqueness result for the linear system of elasticity and its control theoretical consequences*, SIAM J. Cont. Optim. **34** (1996), 1473-1495; Erratum **37** 330-331.

Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Boîte Courrier 187, 75252 Paris Cedex 05, France
E-mail: guerrero@ann.jussieu.fr

Department of Mathematics, Iowa State University, 400 Carver Hall, Ames, IA 50011-2064
E-mail: vika@iastate.edu