INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 4, Number 3-4, Pages 425–440

FINITE ELEMENT APPROXIMATION OF THE NON-ISOTHERMAL STOKES-OLDROYD EQUATIONS

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Dedicated to Max Gunzburger on the occasion of his 60th birthday.

Abstract. We consider the Stokes-Oldroyd equations, defined here as the Stokes equations with the Newtonian constitutive equation explicitly included. Thus a polymer-like stress tensor is included so that the dependent variable structure of a viscoelastic model is in place. The energy equation is coupled with the mass, momentum, and constitutive equations through the use of temperature-dependent viscosity terms in both the constitutive model and the momentum equation. Earlier works assumed temperature-dependent constitutive (polymer) and Newtonian (solvent) viscosities when describing the model equations, but made the simplifying assumption of a constant solvent viscosity when carrying out analysis and computations; we assume no such simplification. Our analysis coupled with numerical solution of the problem with both temperature-dependent viscosities distinguishes this work from earlier efforts.

Key Words. viscous fluid, non-isothermal, finite elements, Stokes-Oldroyd.

1. Introduction

Viscoelastic flows occur in a variety of applications, including polymer processing. The complexity of the governing equations and the physical domains makes analysis of the mathematical models and the associated numerical methods especially difficult. Current efforts to model viscoelastic flows often revolve around the solution of a (modified) Stokes problem, [5]. The isothermal linear elasticity equations, modified in form to have the same dependent variable structure as the equations governing viscoelastic flows, is analyzed along with a numerical solution in [2]. The Stokes problem is a special case (the incompressible limit) of the equations considered in that work.

The purpose of this paper is to analyze the finite element solution of the non-isothermal Stokes problem, modified similarly as in [2]. Thermodynamics play a prominent role in many viscoelastic flow scenarios, especially in polymer processing. Realistic models must ultimately include temperature dependence, since flow characteristics such as viscosity vary widely as temperature varies within normal operating constraints, [1].

The rest of this paper is outlined as follows. The governing equations are presented in the next section, with particular attention given to the manner in which temperature dependence is expressed. Details regarding the weak formulation and

Received by the editors February 21, 2006 and, in revised form, February 23, 2006.
2000 *Mathematics Subject Classification.* 65N30.
This research was supported in part by the ERC program of the National Science Foundation under Award Number EEC-9731680 through the Center for Advanced Engineering Fibers and Films at Clemson University.

corresponding function spaces are provided in Section 3. In Section 4, the finite element formulation is developed along with an existence result for the finite element solution. Convergence results for the finite element solution are derived in Section 5, and numerical confirmation of these results are presented in Section 6. The paper concludes with a summary and a discussion of continuing work.

2. Governing Equations

We consider fluid flowing through a bounded, connected domain $\Omega \subset R^{\acute{d}}$, whose boundary we denote as Γ. Let the velocity be denoted by $\mathbf{u}$, pressure p, extra stress $\underline{\sigma}$, temperature T, and unit outward normal to the boundary $\mathbf{n}$. For viscoelastic fluid flow, the extra stress tensor is often split into a solvent and polymer part,

$$\underline{\sigma} = \underline{\sigma}_s + \underline{\sigma}_p.$$

Normally the solvent part of the extra stress is assumed to be Newtonian, i.e.

$$\underline{\sigma}_s = 2\frac{\eta_s(T)}{\eta_0(T_R)} d(\mathbf{u}),$$

where the rate-of-deformation tensor $d(\mathbf{u})$ is defined as

$$d(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right),$$

η_S is the solvent viscosity which depends at most on the temperature, and $\eta_0(T_R)$ is the zero-shear viscosity at a reference temperature T_R. A nonlinear differential or integral constitutive model is imposed for the polymer part $\underline{\sigma}_p$, [1]. As in [2], we simplify the constitutive model to a Newtonian relationship, and include this equation explicitly to preserve the dependent variable structure associated with viscoelastic constitutive models, such as Giesekus or Oldroyd-B. Whereas only the isothermal case is considered in [2], we analyze the case where both $\underline{\sigma}_p$ and $\underline{\sigma}_s$ depend on temperature. Specifically, we assume that

$$\underline{\sigma}_p - 2\alpha_1(T) d(\mathbf{u}) = \underline{0}, \tag{2.1}$$

where an Arrhenius equation characterizes the dependence of polymer viscosity (also scaled to $\eta_0(T_R)$) upon temperature, i.e.

$$\alpha_1(T) = A_1 \exp\left(\frac{B_1}{T}\right),$$

and $B_1 \neq 0$. The coefficients A_1 and B_1 are defined so that $0 < \alpha_1(T) \leq 1$. We assume the existence of maximum and minimum values for the viscosity

$$\alpha_{1,min} \leq \alpha_1(T) \leq \alpha_{1,max}. \tag{2.2}$$

The (scaled) solvent viscosity is defined in a similar manner so that

$$\underline{\sigma}_s - 2\epsilon\alpha_2(T) d(\mathbf{u}) = \underline{0}, \tag{2.3}$$

with

$$\alpha_2(T) = A_2 \exp\left(\frac{B_2}{T}\right).$$

Once again we choose A_2 and B_2 so that $0 < \alpha_2(T) \leq 1$ and

$$\alpha_{2,min} \leq \alpha_2(T) \leq \alpha_{2,max}, \tag{2.4}$$

but here we may have $B_2 = 0$. The definition of $\underline{\sigma}_s$ includes ϵ because the solvent part of the viscosity is assumed to be much smaller than the polymer part. Furthermore, the term $2\epsilon\alpha_2(T)d(\mathbf{u})$ has special significance in that it increases stability. Hence the parameter ϵ is considered a penalty parameter, and is assumed small.

We define the non-isothermal Stokes-Oldroyd equations as the system consisting of (2.1) along with the conservation of momentum, mass, and energy equations in following form:

$$-\nabla \cdot \left[\underline{\sigma}_p + 2\epsilon\alpha_2(T)d(\mathbf{u})\right] + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.5}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.6}$$

$$-\nabla \cdot (\kappa \nabla T) + \mathbf{u} \cdot \nabla T = Q \quad \text{in } \Omega, \tag{2.7}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.8}$$

$$T = 0 \quad \text{on } \Gamma_1, \tag{2.9}$$

$$\kappa \nabla T \cdot \mathbf{n} = g \quad \text{on } \Gamma_2, \tag{2.10}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$. The thermal conductivity coefficient is denoted by κ, body force $\mathbf{f}$, heat source Q, heat flux g. All variables are dimensionless except for T. Note that if we make the substitution $\underline{\sigma}_p = 2\alpha_1(T)d(\mathbf{u})$, then the momentum equation (2.5) becomes

$$-\nabla \cdot \left[(2\alpha_1(T) + 2\epsilon\alpha_2(T))\, d(\mathbf{u})\right] + \nabla p = \mathbf{f}\,,$$

where $2\alpha_1(T) + 2\epsilon\alpha_2(T)$ is the dimensionless effective viscosity. In the remainder of this paper, the polymer-like part of the extra stress tensor will be referred to as $\underline{\sigma}$, i.e. the subscript will be dropped.

Here we relate the values for A_i and B_i to physical quantities and establish conditions to assure that $0 < \alpha_i(T) \leq 1$. The form for the Arrhenius-type temperature shift factor is

$$a_T = \exp\left[\frac{\Delta E}{R}\left(\frac{1}{T} - \frac{1}{T_R}\right)\right],$$

where ΔE is the activation energy, R is the ideal gas constant, and T_R is a reference temperature, [1]. Assuming that the polymer and solvent viscosities add up to the zero-shear-rate viscosity, (as in [10]), it follows that $\alpha_1(T) = (1-\epsilon)a_T$ and $\alpha_2(T) = a_T$. As a result,

$$B_1 = B_2 = \frac{\Delta E}{R}, \quad A_2 = \exp\left[\frac{-\Delta E}{R T_R}\right], \quad \text{and} \quad A_1 = (1-\epsilon)A_2.$$

The constraint $0 < \alpha_i(T) \leq 1$ will be satisfied as long as the temperature of the system stays above T_R. This condition is simple to satisfy for the application considered in this paper, i.e. flow through a fiber or film-forming die.

3. Weak Formulation

In this section we develop the variational formulation of the modified non-isothermal Stokes-Oldroyd equations. We use the Sobolev spaces $W^{m,p}(D)$ with norms $\|\cdot\|_{m,p,D}$ if $p < \infty$, $\|\cdot\|_{m,\infty,D}$ if $p = \infty$. We denote the Sobolev space $W^{m,2}$ by H^m with the norm $\|\cdot\|_m$. The corresponding spaces of vector-valued and tensor-valued functions are denoted by $\mathbf{H}^m$. If $D = \Omega$, D is omitted, i.e., $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ and $\|\cdot\| = \|\cdot\|_\Omega$. We define the following subspaces

$$\begin{aligned} \mathbf{H}_0^1(\Omega) &= \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_\Gamma = \mathbf{0}\}, \\ L_0^2(\Omega) &= \{q \in L^2(\Omega) : \int_\Omega q\, d\Omega = 0\}, \end{aligned}$$

and the solution spaces:

$$\begin{aligned}
\text{Velocity Space} &: \mathbf{X} := \mathbf{H}_0^1(\Omega),\\
\text{Pressure Space} &: P := L_0^2(\Omega),\\
\text{Stress Space} &: \mathbf{\Sigma} := \left(L^2(\Omega)\right)^{\acute{d}\times\acute{d}} \cap \{\underline{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji}; \tau_{ij} \in L^2(\Omega)\},\\
\text{Temperature Space} &: E := \{S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_1\}
\end{aligned}$$

Notice that the velocity and pressure spaces, $\mathbf{X}$ and P respectively, satisfy the $inf - sup$ condition [4, 6]

$$\inf_{q\in P} \sup_{\mathbf{v}\in\mathbf{X}} \frac{(q, \nabla\cdot\mathbf{v})}{||q||_0||\mathbf{v}||_1} \geq \beta > 0.$$

We also define the weak divergence free space as

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : \int_\Omega q\nabla\cdot\mathbf{v}d\Omega = 0, \forall q \in P\}.$$

For weak formulation, we will consider the scaled constitutive equation

$$\frac{\underline{\sigma}}{2\alpha_1(T)} - d(\mathbf{u}) = \underline{0}, \tag{3.11}$$

in order to simplify analysis. The weak formulation of (2.5)-(2.10) and (3.11) is then to find $(\mathbf{u}, p, \underline{\sigma}, T) \in \mathbf{X} \times P \times \mathbf{\Sigma} \times E$ such that

$$\left(\frac{1}{2\alpha_1(T)}\underline{\sigma}, \underline{\tau}\right) - (d(\mathbf{u}), \underline{\tau}) = 0 \quad \forall\underline{\tau} \in \mathbf{\Sigma}, \tag{3.12}$$

$$(\underline{\sigma}, d(\mathbf{v})) + 2\epsilon(\alpha_2(T)d(\mathbf{u}), d(\mathbf{v})) - (p, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \forall\mathbf{v} \in \mathbf{X}, \tag{3.13}$$

$$(q, \nabla\cdot\mathbf{u}) = 0 \quad \forall q \in P, \tag{3.14}$$

$$\kappa(\nabla T, \nabla S) + (\mathbf{u}\cdot\nabla T, S) = (Q, S) + (g, S)_{\Gamma_2} \quad \forall S \in E. \tag{3.15}$$

Recall that $\mathbf{u}$ satisfies conservation of mass ($\nabla\cdot\mathbf{u} = 0$) and is zero on the boundary Γ, and hence we have that

$$(\mathbf{u}\cdot\nabla T, T) = 0. \tag{3.16}$$

Also, note that the nonlinear term in (3.15) is bounded as follows. See, for instance, [7].

$$(\mathbf{u}\cdot\nabla T, S) \leq C\|\mathbf{u}\|_1\|T\|_1\|S\|_1. \tag{3.17}$$

Using the weak-divergence free space, (3.12)-(3.15) is written in the equivalent form: finding $(\mathbf{u}, \underline{\sigma}, T) \in \mathbf{V} \times \mathbf{\Sigma} \times E$ such that

$$\left(\frac{1}{2\alpha_1(T)}\underline{\sigma}, \underline{\tau}\right) - (d(\mathbf{u}), \underline{\tau}) = 0 \qquad \forall\underline{\tau} \in \mathbf{\Sigma}, \tag{3.18}$$

$$(\underline{\sigma}, d(\mathbf{v})) + 2\epsilon(\alpha_2(T)d(\mathbf{u}), d(\mathbf{v})) = (\mathbf{f}, \mathbf{v}) \qquad \forall\mathbf{v} \in \mathbf{V}, \tag{3.19}$$

$$\kappa(\nabla T, \nabla S) + (\mathbf{u}\cdot\nabla T, S) = (Q, S) + (g, S)_{\Gamma_2} \qquad \forall S \in E. \tag{3.20}$$

4. Finite Element Approximation

Suppose that we have a triangulation T_h of our domain Ω such that $\bar{\Omega} = \{\cup K : K \in T_h\}$, i.e., K is an element of the triangulation. Further suppose that there exist positive constants c_1 and c_2 such that

$$c_1 h \leq h_K \leq c_2\rho_K,$$

where h_K is the diameter of K, ρ_K is the diameter of the greatest ball included in K, and $h = \max_{K\in T_h} h_K$. Denote the space of polynomials of degree less than or equal to k on $K \in T_h$ by $P_k(K)$. To approximate the solution $(\mathbf{u}, p, \underline{\sigma}, T)$, we define the following finite-element spaces.

$$\begin{aligned}
\mathbf{X}^h &:= \{\mathbf{v} \in \mathbf{X} \cap (C^0(\bar{\Omega}))^{\acute{d}} : \mathbf{v}|_K \in P_2(K), \forall K \in T_h\},\\
P^h &:= \{q \in P \cap C^0(\bar{\Omega}) : q|_K \in P_1(K), \forall K \in T_h\},\\
\mathbf{\Sigma}^h &:= \{\underline{\tau} \in \mathbf{\Sigma} : \underline{\tau}|_K \in P_1(K), \forall K \in T_h\},\\
E^h &:= \{T \in E \cap C^0(\bar{\Omega}) : T|_K \in P_2(K), \forall K \in T_h\}.
\end{aligned}$$

We use continuous piecewise quadratic elements for velocity and temperature, continuous piecewise linear elements for pressure, and discontinuous piecewise linear elements for stress. The discontinuous finite element space is adopted in anticipation of applying this method to complex constitutive models which requires a form of upwinding in the numerical approximation.

Analogous to the continuous function spaces, the discrete spaces $\mathbf{X}^h$ and P^h satisfy the discrete $inf - sup$ condition [4, 6]

$$\inf_{q^h \in P^h} \sup_{\mathbf{v}^h \in \mathbf{X}^h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{||q^h||_0 ||\mathbf{v}^h||_1} \geq \beta > 0. \tag{4.21}$$

We also define the discrete weak divergence free space.

$$\mathbf{V}^h := \{\mathbf{v}^h \in \mathbf{X}^h : (q^h, \nabla \cdot \mathbf{v}^h) = 0, \forall q^h \in P^h\}.$$

We now consider the following problem: find $(\mathbf{u}^h, p^h, \underline{\sigma}^h, T^h) \in (\mathbf{X}^h, P^h, \mathbf{\Sigma}^h, E^h)$ such that

$$\left(\frac{1}{2\alpha_1(T^h)}\underline{\sigma}^h, \underline{\tau}^h\right) - (d(\mathbf{u}^h), \underline{\tau}^h) = 0 \quad \forall \underline{\tau}^h \in \mathbf{\Sigma}^h, \tag{4.22}$$

$$(\underline{\sigma}^h, d(\mathbf{v}^h)) + 2\epsilon(\alpha_2(T^h) d(\mathbf{u}^h), d(\mathbf{v}^h)) - (p^h, \nabla \cdot \mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathbf{X}^h, \tag{4.23}$$

$$(q^h, \nabla \cdot \mathbf{u}^h) = 0 \quad \forall q^h \in P^h, \tag{4.24}$$

$$\kappa(\nabla T^h, \nabla S^h) + (\mathbf{u}^h \cdot \nabla T^h, S^h) = (Q, S^h) + (g, S^h)_{\Gamma_2} \quad \forall S^h \in E^h. \tag{4.25}$$

Using the discrete weak divergence free space $\mathbf{V}^h$, (4.22)-(4.25) can be written as

$$\left(\frac{1}{2\alpha_1(T^h)}\underline{\sigma}^h, \underline{\tau}^h\right) - (d(\mathbf{u}^h), \underline{\tau}^h) = 0 \qquad \forall \underline{\tau}^h \in \mathbf{\Sigma}^h, \tag{4.26}$$

$$(\underline{\sigma}^h, d(\mathbf{v}^h)) + 2\epsilon(\alpha_2(T^h) d(\mathbf{u}^h), d(\mathbf{v}^h)) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall \mathbf{v}^h \in \mathbf{v}^h, \tag{4.27}$$

$$\kappa(\nabla T^h, \nabla S^h) + (\mathbf{u}^h \cdot \nabla T^h, S^h) = (Q, S^h) + (g, S^h)_{\Gamma_2} \qquad \forall S^h \in E^h. \tag{4.28}$$

In the next theorem we will now show the existence of a solution to the system (4.22)-(4.25).

Theorem 4.1. *There exists a solution* $(\mathbf{u}^h, p^h, \underline{\sigma}^h, T^h) \in \mathbf{X}^h \times P^h \times \mathbf{\Sigma}^h \times E^h$ *of the equations (4.22)-(4.25) satisfying*

$$\|\mathbf{u}^h\|_1 + \|\underline{\sigma}^h\|_0 + \|T^h\|_1 \leq C(\|\mathbf{f}\|_0 + \|Q\|_0 + \|g\|_{0,\Gamma_2}). \tag{4.29}$$

Proof: We will first show existence of a solution $(\mathbf{u}^h, \underline{\sigma}^h, T^h) \in \mathbf{V}^h \times \mathbf{\Sigma}^h \times E^h$ to the problem (4.26)-(4.28). Then, existence of $(\mathbf{u}^h, p^h, \underline{\sigma}^h, T^h) \in \mathbf{X}^h \times P^h \times \mathbf{\Sigma}^h \times E^h$ satisfying (4.22)-(4.25) will be obtained using the $inf - sup$ condition (4.21).

For a given $\mathbf{u}^h \in \mathbf{V}^h$, consider the bilinear form $A(\cdot, \cdot) : E^h \times E^h \to R$ defined as

$$A(T^h, S^h) = \kappa(\nabla T^h, \nabla S^h) + (\mathbf{u}^h \cdot \nabla T^h, S^h).$$

It can be easily shown using (3.16)-(3.17) that A is continuous and coercive. Thus, by the Lax-Milgram theorem, for given $g \in L^2(\Gamma)$ and $Q \in L^2(\Omega)$, we can find a unique solution $T^h \in E^h$ satisfying (4.28) and the estimate

$$\|T^h\|_1 \leq C(\|Q\|_0 + \|g\|_{0,\Gamma_2})\,. \tag{4.30}$$

Define the mapping $F : \mathbf{V}^h \to E^h$ by

$$F(\mathbf{u}^h) = T^h, \tag{4.31}$$

where $(\mathbf{u},^h, T^h)$ satisfies (4.28). The problem (4.26)-(4.28) can be reduced to finding $(\mathbf{u}^h, \underline{\sigma}^h) \in \mathbf{V}^h \times \mathbf{\Sigma}^h$ such that

$$\left(\frac{1}{2\alpha_1(F(\mathbf{u}^h))}\underline{\sigma}^h, \underline{\tau}^h\right) - (d(\mathbf{u}^h), \underline{\tau}^h) = 0, \tag{4.32}$$

$$(\underline{\sigma}^h, d(\mathbf{v}^h)) + 2\epsilon(\alpha_2(F(\mathbf{u}^h))d(\mathbf{u}^h), d(\mathbf{v}^h)) = (\mathbf{f}, \mathbf{v}^h), \tag{4.33}$$

for all $(\mathbf{v}^h, \underline{\tau}^h) \in \mathbf{V}^h \times \mathbf{\Sigma}^h$. Define the mapping

$$\mathcal{D} : \mathbf{V}^h \times \mathbf{\Sigma}^h \to \mathbf{V}^h \times \mathbf{\Sigma}^h,$$

implicitly by $\mathcal{D}(\mathbf{z}^h, \underline{\lambda}^h) = (\mathbf{u}^h, \underline{\sigma}^h)$, if and only if

$$\left(\frac{1}{2\alpha_1(F(\mathbf{z}^h))}\underline{\sigma}^h, \underline{\tau}^h\right) - (d(\mathbf{u}^h), \underline{\tau}^h) = 0, \tag{4.34}$$

$$(\underline{\sigma}^h, d(\mathbf{v}^h)) + 2\epsilon(\alpha_2(F(\mathbf{z}^h))d(\mathbf{u}^h), d(\mathbf{v}^h)) = (\mathbf{f}, \mathbf{v}^h)_\Omega, \tag{4.35}$$

for all $(\mathbf{v}^h, \underline{\tau}^h) \in \mathbf{V}^h \times \mathbf{\Sigma}^h$. Now, $(\mathbf{u}^h, \underline{\sigma}^h)$ will be a solution of (4.32)-(4.33) if it is a fixed point of $\mathcal{D}(\cdot)$. By the Leray-Schauder Principle [3, 8], $\mathcal{D}(\cdot)$ has at least one fixed point if

(i) $\mathcal{D}(\cdot)$ is absolutely continuous.
(ii) There exists a $M > 0$ such that, for all $\Lambda \in [0, 1]$ and $(\mathbf{v}^h, \underline{\tau}^h) \in \mathbf{V}^h \times \mathbf{\Sigma}^h$, if
$$(\mathbf{v}^h, \underline{\tau}^h) = \Lambda\mathcal{D}((\mathbf{v}^h, \underline{\tau}^h)),$$
then $(\mathbf{v}^h, \underline{\tau}^h)$ satisfies $\|(\mathbf{v}^h, \underline{\tau}^h)\|_{\mathbf{V}^h\times\mathbf{\Sigma}^h} \leq M$.

We will show (i) and (ii) to prove $\mathcal{D}(\cdot)$ has a fixed point.

Absolute continuity

Suppose

$$\mathcal{D}(\mathbf{z}_i^h, \underline{\lambda}_i^h) = (\mathbf{u}_i^h, \underline{\sigma}_i^h),$$

where $(\mathbf{z}_i^h, \underline{\lambda}_i^h), (\mathbf{u}_i^h, \underline{\sigma}_i^h) \in \mathbf{V}^h \times \mathbf{\Sigma}^h$ for $i = 1, 2$, i.e.,

$$\left(\frac{1}{2\alpha_1(F(\mathbf{z}_i^h))}\underline{\sigma}_i^h, \underline{\tau}^h\right) - (d(\mathbf{u}_i^h), \underline{\tau}^h) = 0, \tag{4.36}$$

$$(\underline{\sigma}_i^h, d(\mathbf{v}^h)) + 2\epsilon(\alpha_2(F(\mathbf{z}_i^h))d(\mathbf{u}_i^h), d(\mathbf{v}^h)) = (\mathbf{f}, \mathbf{v}^h), \tag{4.37}$$

for all $(\mathbf{v}^h, \underline{\tau}^h) \in \mathbf{V}^h \times \mathbf{\Sigma}^h$. We will show there exists a constant C_h such that

$$\|(\mathbf{u}_2^h, \underline{\sigma}_2^h) - (\mathbf{u}_1^h, \underline{\sigma}_1^h)\|_{\mathbf{V}^h\times\mathbf{\Sigma}^h} \leq C_h \|(\mathbf{z}_2^h, \underline{\lambda}_2^h) - (\mathbf{z}_1^h, \underline{\lambda}_1^h)\|_{\mathbf{V}^h\times\mathbf{\Sigma}^h}. \tag{4.38}$$

From (4.36)-(4.37) we have

$$\left(\frac{1}{2\alpha_1(F(\mathbf{z}_2^h))}\underline{\sigma}_2^h - \frac{1}{2\alpha_1(F(\mathbf{z}_1^h))}\underline{\sigma}_1^h, \underline{\tau}^h\right) - (d(\mathbf{u}_2^h - \mathbf{u}_1^h), \underline{\tau}^h)$$
$$(\underline{\sigma}_2^h - \underline{\sigma}_1^h, d(\mathbf{v}^h)) + 2\epsilon(\alpha_2(F(\mathbf{z}_2^h))d(\mathbf{u}_2^h) - \alpha_2(F(\mathbf{z}_1^h))d(\mathbf{u}_1^h), d(\mathbf{v}^h)) = 0,$$

which can be written as

$$\left(\frac{1}{2\alpha_1(F(\mathbf{z}_2^h))}(\underline{\sigma}_2^h-\underline{\sigma}_1^h),\underline{\tau}^h\right)-(d(\mathbf{u}_2^h-\mathbf{u}_1^h),\underline{\tau}^h)+(\underline{\sigma}_2^h-\underline{\sigma}_1^h,d(\mathbf{v}^h))$$
$$+2\epsilon(\alpha_2(F(\mathbf{z}_2^h))d(\mathbf{u}_2^h-\mathbf{u}_1^h),d(\mathbf{v}^h))$$
$$=-\left(\left(\frac{1}{2\alpha_1(F(\mathbf{z}_2^h))}-\frac{1}{2\alpha_1(F(\mathbf{z}_1^h))}\right)\underline{\sigma}_1^h,\underline{\tau}^h\right)$$
$$-2\epsilon((\alpha_2(F(\mathbf{z}_2^h))-\alpha_2(F(\mathbf{z}_1^h)))d(\mathbf{u}_1^h),d(\mathbf{v}^h)).$$

Letting $\mathbf{v}^h=\mathbf{u}_2^h-\mathbf{u}_1^h$, $\underline{\tau}^h=\underline{\sigma}_2^h-\underline{\sigma}_1^h$ and using (2.2), (2.4), we have

$$\frac{1}{2\alpha_{1,max}}\|\underline{\sigma}_2^h-\underline{\sigma}_1^h\|_0^2+2\epsilon\alpha_{2,min}\|d(\mathbf{u}_2^h-\mathbf{u}_1^h)\|_0^2$$
$$\leq C\left[\frac{1}{2}\left\|\frac{1}{\alpha_1(F(\mathbf{z}_2^h))}-\frac{1}{\alpha_1(F(\mathbf{z}_1^h))}\right\|_0||\underline{\sigma}_1^h||_\infty\|\underline{\sigma}_2^h-\underline{\sigma}_1^h\|_0\right.$$
$$\left.+2\epsilon\|\alpha_2(F(\mathbf{z}_2^h))-\alpha_2(F(\mathbf{z}_1^h))\|_0||d(\mathbf{u}_1^h)||_\infty\|d(\mathbf{u}_2^h-\mathbf{u}_1^h)\|_0\right].$$

Note that $\frac{1}{\alpha_1(\cdot)}$ and $\alpha_2(\cdot)$ are Lipschitz continuous as they are absolutely bounded exponential functions. Hence, there exists a constant $\hat{C}$ such that

$$\left\|\frac{1}{\alpha_1(T_2)}-\frac{1}{\alpha_1(T_1)}\right\|_0 \leq \hat{C}\|T_2-T_1\|_0, \tag{4.39}$$

$$\|\alpha_2(T_2)-\alpha_2(T_1)\|_0 \leq \hat{C}\|T_2-T_1|\|_0. \tag{4.40}$$

Thus, using the inverse inequalities [4]

$$\|\underline{\sigma}^h\|_\infty \leq C\,h^{-1}\,\|\underline{\sigma}^h\|_0,$$
$$\|d(\mathbf{u}^h)\|_\infty \leq C\,h^{-1}\,\|d(\mathbf{u}^h)\|_0,$$

the Poincaré-Friedrichs inequality and (4.39)-(4.40),

$$\frac{1}{2\alpha_{1,max}}\|\underline{\sigma}_2^h-\underline{\sigma}_1^h\|_0^2+2\epsilon\alpha_{2,min}\|\mathbf{u}_2^h-\mathbf{u}_1^h\|_1^2$$
$$\leq C\,h^{-1}\|F(\mathbf{z}_2^h)-F(\mathbf{z}_1^h)\|_1(\|\underline{\sigma}_1^h\|_0\|\underline{\sigma}_2^h-\underline{\sigma}_1^h\|_0+\|\mathbf{u}_1^h\|_1\|\mathbf{u}_2^h-\mathbf{u}_1^h\|_1)\,. \tag{4.41}$$

Now, we estimate $\|F(\mathbf{z}_2^h)-F(\mathbf{z}_1^h)\|_1$. From (4.28) and (4.31),

$$\kappa(\nabla F(\mathbf{z}_i^h),\nabla S^h)+(\mathbf{z}_i^h\cdot\nabla F(\mathbf{z}_i^h),S^h)=(Q,S^h)-(\tilde{g},S^h)_\Gamma \qquad \forall S^h\in E^h$$

for $i=1,2$. Hence, we have

$$\kappa(\nabla(F(\mathbf{z}_2^h)-F(\mathbf{z}_1^h)),\nabla S^h)=-(\mathbf{z}_2^h\cdot\nabla F(\mathbf{z}_2^h)-\mathbf{z}_1^h\cdot\nabla F(\mathbf{z}_1^h),S^h)\,.$$

Let $S^h=F(\mathbf{z}_2^h)-F(\mathbf{z}_1^h)$, add and subtract terms to obtain

$$\kappa\|\nabla(F(\mathbf{z}_2^h)-F(\mathbf{z}_1^h))\|_0^2=-((\mathbf{z}_2^h-\mathbf{z}_1^h)\cdot\nabla F(\mathbf{z}_2^h),F(\mathbf{z}_2^h)-F(\mathbf{z}_1^h))$$
$$-((\mathbf{z}_1^h)\cdot\nabla(F(\mathbf{z}_2^h)-F(\mathbf{z}_1^h)),F(\mathbf{z}_2^h)-F(\mathbf{z}_1^h))\,. \tag{4.42}$$

By (3.16), (3.17), (4.30) and the Poincaré-Friedrichs inequality,

$$\|F(\mathbf{z}_2^h)-F(\mathbf{z}_1^h)\|_1\leq\|\mathbf{z}_2^h-\mathbf{z}_1^h\|_1(\|Q\|_0+\|\tilde{g}\|_{0,\Gamma})\,. \tag{4.43}$$

Finally, we estimate $\|\underline{\sigma}_i^h\|_0$ and $\|\mathbf{u}_i^h\|_1$. Letting $\underline{\tau}_i^h=\underline{\sigma}_i^h$, $\mathbf{v}_i^h=\mathbf{u}_i^h$ in (4.36), (4.37) and using (2.2), (2.4),

$$\|\underline{\sigma}_i^h\|_0^2+\|d(\mathbf{u}_i^h)\|_0^2\leq C\|\mathbf{f}\|_0\|\mathbf{u}_i^h\|_0,$$

which implies

$$\|\mathbf{u}_i^h\|_1\leq C\,\|\mathbf{f}\|_0. \tag{4.44}$$

Also

$$\|\underline{\sigma}_i^h\|_0 \le C\,\|\mathbf{f}\|_0 \tag{4.45}$$

is obtained by (2.2), (4.36) and (4.44). Now the estimate (4.38) follows from (4.41), (4.43), (4.44) and (4.45).

Contraction mapping

Assume that $\Lambda \in [0,1]$, and let $(\mathbf{z}^h, \underline{\lambda}^h) \in \mathbf{V}^h \times \mathbf{\Sigma}^h$ be such that

$$\Lambda\mathcal{D}((\mathbf{z}^h, \underline{\lambda}^h)) = (\mathbf{z}^h, \underline{\lambda}^h).$$

If $\Lambda = 0$, then

$$(\mathbf{0}, \underline{0}) = \Lambda\mathcal{D}((\mathbf{z}^h, \underline{\lambda}^h)) = (\mathbf{z}^h, \underline{\lambda}^h),$$

which implies that

$$\mathbf{z}^h = \mathbf{0}, \qquad \underline{\lambda}^h = \underline{0},$$

hence $\|(\mathbf{z}^h, \underline{\lambda}^h)\|_{\mathbf{V}^h\times\mathbf{\Sigma}^h} = 0$ and any M will work. Now assume that $\Lambda \in (0,1]$. $\Lambda\mathcal{D}((\mathbf{z}^h, \underline{\lambda}^h)) = (\mathbf{z}^h, \underline{\lambda}^h)$ implies $\mathcal{D}((\mathbf{z}^h, \underline{\lambda}^h)) = \left(\frac{1}{\Lambda}\mathbf{z}^h, \frac{1}{\Lambda}\underline{\lambda}^h\right)$, which is equivalent to

$$\begin{aligned}\left(\frac{1}{2\alpha_1(F(\mathbf{z}^h))}\frac{\underline{\lambda}^h}{\Lambda}, \underline{\tau}^h\right) - \left(d\left(\frac{\mathbf{z}^h}{\Lambda}\right), \underline{\tau}^h\right) &= 0,\\ \left(\frac{\underline{\lambda}^h}{\Lambda}, d(\mathbf{v}^h)\right) + 2\epsilon\left(\alpha_2(F(\mathbf{z}^h))d\left(\frac{\mathbf{z}^h}{\Lambda}\right), d(\mathbf{v}^h)\right) &= (\mathbf{f}, \mathbf{v}^h).\end{aligned}$$

Notice that there is no Λ associated with the argument of the terms $\alpha_i(\cdot)$. This is because their argument is determined by the argument of $\mathcal{D}(\cdot)$, which is unaffected by Λ. Sum the above equations and multiply through by Λ to obtain

$$\begin{aligned}&\left(\frac{1}{2\alpha_1(F(\mathbf{z}^h))}\underline{\lambda}^h, \underline{\tau}^h\right) - (d(\mathbf{z}^h), \underline{\tau}^h)\\ &\qquad +(\underline{\lambda}^h, d(\mathbf{v}^h)) + 2\epsilon(\alpha_2(F(\mathbf{z}^h))d(\mathbf{z}^h), d(\mathbf{v}^h)) = \Lambda(\mathbf{f}, \mathbf{v}^h).\end{aligned}$$

Now letting $\underline{\tau}^h = \underline{\lambda}^h$, and $\mathbf{v}^h = \mathbf{z}^h$, we obtain

$$\left(\frac{1}{2\alpha_1(F(\mathbf{z}^h))}\underline{\lambda}^h, \underline{\lambda}^h\right) + 2\epsilon(\alpha_2(F(\mathbf{z}^h))d(\mathbf{z}^h), d(\mathbf{z}^h)) = \Lambda(\mathbf{f}, \mathbf{z}^h),$$

which implies

$$||\underline{\lambda}^h||_0^2 + ||\mathbf{z}^h||_1^2 \le \Lambda\, C\,\|\mathbf{f}\|_0\|\mathbf{z}^h\|_1 \tag{4.46}$$

by (2.2) and (2.4). Therefore, we obtain

$$\|(\mathbf{z}^h, \underline{\lambda}^h)\|_{\mathbf{V}^h\times\mathbf{\Sigma}^h} \le C\|\mathbf{f}\|_0\,.$$

As we have shown the hypotheses of the Leray-Schauder Principle, we have the existence of a solution $(\mathbf{u}^h, \underline{\sigma}^h, F(\mathbf{u}^h)) = (\mathbf{u}^h, \underline{\sigma}^h, T^h) \in \mathbf{V}^h \times \mathbf{\Sigma}^h \times E^h$. And since $(\mathbf{X}^h, P^h)$ satisfies the inf-sup condition (4.21), there exists $p^h \in P^h$ such that

$$(\underline{\sigma}^h, d(\mathbf{v}^h)) + 2\epsilon(\alpha_2(T^h)d(\mathbf{u}^h), d(\mathbf{v}^h)) - (p^h, \nabla\cdot\mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \qquad \forall\mathbf{v}^h \in \mathbf{X}^h.$$

Finally, the estimate (4.29) follows directly from (4.30), (4.44) and (4.45). □

Remark 4.2. *Theorem 4.1 establishes existence of the solution* $(\mathbf{u}^h, p^h, \underline{\sigma}^h, T^h)$ *in the finite element space* $\mathbf{V}^h \times P^h \times \mathbf{\Sigma}^h \times E^h$. *Note the dependence of the absolute continuity constant on* h *in (4.38).*

5. Error Estimate

We begin with introducing the standard approximation results, which will be used in error estimation. Let $\underline{\tilde{\sigma}}^h \in \mathbf{\Sigma}^h$ be the orthogonal projection of $\underline{\sigma}$ on T_h in $\mathbf{\Sigma}$, and $\tilde{\mathbf{u}}^h \in \mathbf{V}^h$, $\tilde{T}^h \in E^h$ be the interpolants of $\mathbf{u}$ in $\mathbf{V}$ and T in E, respectively, and $\tilde{p}^h \in P^h$ the orthogonal projection of p on T_h in P. Then, we have the following standard estimates [6]:

$$\|\mathbf{u} - \tilde{\mathbf{u}}^h\|_1 \leq \overline{C} h^m \|\mathbf{u}\|_{m+1} \quad \forall \mathbf{u} \in \mathbf{H}^{m+1}(\Omega), \tag{5.47}$$

$$\|\underline{\sigma} - \underline{\tilde{\sigma}}^h\|_0 \leq \overline{C} h^m \|\underline{\sigma}\|_m \quad \forall \underline{\sigma} \in \mathbf{H}^m(\Omega), \tag{5.48}$$

$$\|T - \tilde{T}^h\|_1 \leq \overline{C} h^m \|T\|_{m+1} \quad \forall T \in H^{m+1}(\Omega), \tag{5.49}$$

$$\|p - \tilde{p}^h\|_0 \leq \overline{C} h^m \|p\|_m \quad \forall p \in H^m(\Omega). \tag{5.50}$$

Theorem 5.1. *If (2.1), (2.5)-(2.10) admit a solution* $(\mathbf{u}, p, \underline{\sigma}, T) \in \mathbf{H}^3(\Omega) \times H^2(\Omega) \times \mathbf{H}^2(\Omega) \times H^3(\Omega)$ *such that* $max\{\|\mathbf{u}\|_3, \|p\|_2, \|\underline{\sigma}\|_2, \|T\|_3\} \leq M$ *and the bound* M *is sufficiently small, then we have the following error estimate:*

$$||\underline{\sigma} - \underline{\sigma}^h||_0 + ||\mathbf{u} - \mathbf{u}^h||_1 + ||T - T^h||_1 \leq Ch^2 \,. \tag{5.51}$$

Proof: If $(\mathbf{u}, p, \underline{\sigma}, T)$ is a solution of (2.1) and (2.5)-(2.10), it also solves the scaled constitutive equation (3.11). Hence $(\mathbf{u}, \underline{\sigma}, T)$ satisfies the weak problem (3.18)-(3.20). Now subtracting the discrete system (4.26)-(4.28) from the continuous system (3.18)-(3.20) yields

$$\begin{aligned}
\left(\frac{1}{2\alpha_1(T)}\underline{\sigma} - \frac{1}{2\alpha_1(T^h)}\underline{\sigma}^h, \underline{\tau}^h\right)_\Omega - (d(\mathbf{u}) - d(\mathbf{u}^h), \underline{\sigma}^h) &= 0, \\
(\underline{\sigma} - \underline{\sigma}^h, d(\mathbf{v}^h))_\Omega + 2\epsilon(\alpha_2(T)d(\mathbf{u}) - \alpha_2(T^h)d(\mathbf{u}^h), d(\mathbf{v}^h))_\Omega - (p, \nabla \cdot \mathbf{v}^h) &= 0, \\
\kappa(\nabla(T - T^h), S^h) + (\mathbf{u} \cdot \nabla T - \mathbf{u}^h \cdot \nabla T^h, S^h)_\Omega &= 0,
\end{aligned}$$

for $(\underline{\tau}^h, \mathbf{v}^h, S^h) \in \mathbf{\Sigma}^h \times \mathbf{V}^h \times E^h$. Notice that $(p, \nabla \cdot \mathbf{v}^h) \neq 0$ for $p \in P$, $\mathbf{v}^h \in \mathbf{V}^h$. The above system is equivalent to

$$\begin{aligned}
&\left(\frac{1}{2\alpha_1(T)}\underline{\sigma} - \frac{1}{2\alpha_1(T^h)}\underline{\sigma}^h, \underline{\tau}^h\right) \\
&\quad + 2\epsilon(\alpha_2(T)d(\mathbf{u}) - \alpha_2(T^h)d(\mathbf{u}^h), d(\mathbf{v}^h)) + (\mathbf{u} \cdot \nabla T - \mathbf{u}^h \cdot \nabla T^h, S^h) \\
&\quad - (d(\mathbf{u} - \mathbf{u}^h), \underline{\tau}^h) + (\underline{\sigma} - \underline{\sigma}^h, d(\mathbf{v}^h)) - (p, \nabla \cdot \mathbf{v}^h) + \kappa(\nabla(T - T^h), \nabla S^h) \\
&\quad = 0 \,.
\end{aligned} \tag{5.52}$$

Adding and subtracting terms, we obtain

$$\begin{aligned}
\left(\frac{1}{2\alpha_1(T)}\underline{\sigma} - \frac{1}{2\alpha_1(T^h)}\underline{\sigma}^h, \underline{\tau}^h\right) &= \left(\left[\frac{1}{2\alpha_1(T)} - \frac{1}{2\alpha_1(\tilde{T}^h)}\right]\underline{\sigma}, \underline{\tau}^h\right) \\
&+ \left(\left[\frac{1}{2\alpha_1(\tilde{T}^h)} - \frac{1}{2\alpha_1(T^h)}\right]\underline{\sigma}, \underline{\tau}^h\right) + \left(\frac{1}{2\alpha_1(T^h)}(\underline{\sigma} - \underline{\tilde{\sigma}}^h), \underline{\tau}^h\right) \\
&+ \left(\frac{1}{2\alpha_1(T^h)}(\underline{\tilde{\sigma}}^h - \underline{\sigma}^h), \underline{\tau}^h\right) \,.
\end{aligned} \tag{5.53}$$

Similarly,

$$
\begin{aligned}
&2\epsilon(\alpha_2(T)d(\mathbf{u}) - \alpha_2(T^h)d(\mathbf{u}^h), d(\mathbf{v}^h)) + (\mathbf{u}\cdot\nabla T - \mathbf{u}^h\cdot\nabla T^h, S^h)\\
&\quad = 2\epsilon([\alpha_2(T) - \alpha_2(\tilde{T}^h)]d(\mathbf{u}), d(\mathbf{v}^h)) + 2\epsilon([\alpha_2(\tilde{T}^h) - \alpha_2(T^h)]d(\mathbf{u}), d(\mathbf{v}^h))\\
&\qquad +2\epsilon(\alpha_2(T^h)(d(\mathbf{u} - \tilde{\mathbf{u}}^h)), d(\mathbf{v}^h)) + 2\epsilon(\alpha_2(T^h)(d(\tilde{\mathbf{u}}^h - \mathbf{u}^h), d(\mathbf{v}^h))\\
&\qquad +((\mathbf{u} - \tilde{\mathbf{u}}^h)\cdot\nabla T, S^h) + ((\tilde{\mathbf{u}}^h - \mathbf{u}^h)\cdot\nabla T, S^h)\\
&\qquad +(\mathbf{u}^h\cdot\nabla(T - \tilde{T}^h), S^h) + (\mathbf{u}^h\cdot\nabla(\tilde{T}^h - T^h), S^h)\,, \qquad (5.54)
\end{aligned}
$$

and

$$
\begin{aligned}
&-(d(\mathbf{u} - \mathbf{u}^h), \underline{\tau}^h) + (\underline{\sigma} - \underline{\sigma}^h, d(\mathbf{v}^h)) - (p, \nabla\cdot\mathbf{v}^h) + \kappa(\nabla(T - T^h), \nabla S^h)\\
&\quad = -(d(\mathbf{u} - \tilde{\mathbf{u}}^h), \underline{\tau}^h) - (d(\tilde{\mathbf{u}}^h - \mathbf{u}^h), \underline{\tau}^h)\\
&\qquad +(\underline{\sigma} - \tilde{\underline{\sigma}}^h, d(\mathbf{v}^h)) + (\tilde{\underline{\sigma}}^h - \underline{\sigma}^h, d(\mathbf{v}^h))\\
&\qquad -(p - \tilde{p}^h, \nabla\cdot\mathbf{v}^h) + \kappa(\nabla(T - \tilde{T}^h), \nabla S^h) + \kappa(\nabla(\tilde{T}^h - T^h), \nabla S^h)\,, \qquad (5.55)
\end{aligned}
$$

where we used $(\tilde{p}^h, \nabla\cdot\mathbf{v}^h) = 0$ for $\mathbf{v}^h \in V^h$. Therefore, rearranging terms, we can rewrite (5.52) as

$$
\begin{aligned}
&\left(\frac{1}{2\alpha_1(T^h)}(\tilde{\underline{\sigma}}^h - \underline{\sigma}^h), \underline{\tau}^h\right) + 2\epsilon\;(\alpha_2(T^h)\; d(\tilde{\mathbf{u}}^h - \mathbf{u}^h), d(\mathbf{v}^h))\\
&+(\mathbf{u}^h\cdot\nabla(\tilde{T}^h - T^h), S^h)\\
&-(d(\tilde{\mathbf{u}}^h - \mathbf{u}^h), \underline{\tau}^h) + (\tilde{\underline{\sigma}}^h - \underline{\sigma}^h, d(\mathbf{v}^h)) + \kappa(\nabla(\tilde{T}^h - T^h), \nabla S^h)\\
&\quad = -\left(\left[\frac{1}{2\alpha_1(T)} - \frac{1}{2\alpha_1(\tilde{T}^h)}\right]\underline{\sigma}, \underline{\tau}^h\right) - \left(\left[\frac{1}{2\alpha_1(\tilde{T}^h)} - \frac{1}{2\alpha_1(T^h)})\right]\underline{\sigma}, \underline{\tau}^h\right)\\
&\qquad\qquad - \left(\frac{1}{2\alpha_1(T^h)}(\underline{\sigma} - \tilde{\underline{\sigma}}^h), \underline{\tau}^h\right)\\
&\qquad -2\epsilon([\alpha_2(T) - \alpha_2(\tilde{T}^h)]d(\mathbf{u}), d(\mathbf{v}^h)) - 2\epsilon([\alpha_2(\tilde{T}^h) - \alpha_2(T^h)]d(\mathbf{u}), d(\mathbf{v}^h))\\
&\qquad\qquad -2\epsilon(\alpha_2(T^h)\; d(\mathbf{u} - \tilde{\mathbf{u}}^h), d(\mathbf{v}^h))\\
&\qquad -((\mathbf{u} - \tilde{\mathbf{u}}^h)\cdot\nabla T, S^h) - ((\tilde{\mathbf{u}}^h - \mathbf{u}^h)\cdot\nabla T, S^h) - (\mathbf{u}^h\cdot\nabla(T - \tilde{T}^h), S^h)\\
&\qquad +(d(\mathbf{u} - \tilde{\mathbf{u}}^h), \underline{\tau}^h) - (\underline{\sigma} - \tilde{\underline{\sigma}}^h, d(\mathbf{v}^h))\\
&\qquad +(p - \tilde{p}^h, \nabla\cdot\mathbf{v}^h) - \kappa(\nabla(T - \tilde{T}^h), \nabla S^h)\,. \qquad (5.56)
\end{aligned}
$$

Letting $\mathbf{v}^h = \tilde{\mathbf{u}}^h - \mathbf{u}^h$, $\underline{\tau}^h = \tilde{\underline{\sigma}}^h - \underline{\sigma}^h$, $S^h = \tilde{T}^h - T^h$ and using (2.2), (2.4), (3.16),

$$
\begin{aligned}
&\text{LHS of (5.56)}\\
&\quad \geq \frac{1}{2\alpha_{1,max}}\|\tilde{\underline{\sigma}}^h - \underline{\sigma}^h\|_0^2 + 2\epsilon\alpha_{2,min}\|d(\tilde{\mathbf{u}}^h - \mathbf{u}^h)\|_0^2 + \kappa\|\nabla(\tilde{T}^h - T^h)\|_0^2\,. \qquad (5.57)
\end{aligned}
$$

Now, we will get an estimate for the right hand side of (5.56). Using the imbedding theorem of H^2 in L^∞, (2.2), (4.39), (5.48), (5.49) and the Young's inequality,

$$
\begin{aligned}
&-\left(\left[\frac{1}{2\alpha_1(T)}-\frac{1}{2\alpha_1(\tilde{T}^h)}\right]\underline{\sigma},\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\right)-\left(\left[\frac{1}{2\alpha_1(\tilde{T}^h)}-\frac{1}{2\alpha_1(T^h)}\right]\underline{\sigma},\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\right)\\
&\qquad\qquad\qquad\qquad-\left(\frac{1}{2\alpha_1(T^h)}(\underline{\sigma}-\underline{\tilde{\sigma}}^h),\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\right)\\
&\le C\left[\left\|\frac{1}{\alpha_1(T)}-\frac{1}{\alpha_1(\tilde{T}^h)}\right\|_0\|\underline{\sigma}\|_2\|\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\|_0\right.\\
&\left.+\left\|\frac{1}{\alpha_1(\tilde{T}^h)}-\frac{1}{\alpha_1(T^h)}\right\|_0\|\underline{\sigma}\|_2\|\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\|+\frac{1}{2\alpha_{1,min}}\|\underline{\sigma}-\underline{\tilde{\sigma}}^h\|_0\|\underline{\tilde{\sigma}}^h-\sigma^h\|_0\right]\\
&\le C\left[\hat{C}\|T-\tilde{T}^h\|_1M\|\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\|_0+\hat{C}\|\tilde{T}^h-T^h\|_1M\|\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\|_0\right.\\
&\qquad\qquad\qquad\qquad\left.+\frac{1}{2\alpha_{1,min}}\|\underline{\sigma}-\underline{\tilde{\sigma}}^h\|_0\|\underline{\tilde{\sigma}}^h-\sigma^h\|_0\right]\\
&\le C\left[\frac{\hat{C}^2\overline{C}^2M^4h^4}{4\epsilon_1}+\frac{\hat{C}^2M^2}{4\epsilon_2}\|\tilde{T}^h-T^h\|_1^2+\frac{\overline{C}^2M^2h^4}{16\alpha_{1,min}^2\epsilon_3}\right.\\
&\qquad\qquad\qquad\qquad\left.+(\epsilon_1+\epsilon_2+\epsilon_3)\|\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\|_0^2\right]. \qquad (5.58)
\end{aligned}
$$

The next three terms in (5.56) are bounded using (2.4), (4.40), (5.47), (5.49), the imbedding theorem of H^2 in L^∞ and the Poincaré-Friedrichs inequality:

$$
\begin{aligned}
&-2\epsilon([\alpha_2(T)-\alpha_2(\tilde{T}^h)]d(\mathbf{u}),d(\tilde{\mathbf{u}}^h-\mathbf{u}^h))\\
&\qquad\qquad-2\epsilon([\alpha_2(\tilde{T}^h)-\alpha_2(T^h)]d(\mathbf{u}),d(\tilde{\mathbf{u}}^h-\mathbf{u}^h))\\
&\qquad\qquad-2\epsilon(\alpha_2(T^h)\,d(\mathbf{u}-\tilde{\mathbf{u}}^h),d(\tilde{\mathbf{u}}^h-\mathbf{u}^h))\\
&\le C\left[\|\alpha_2(T)-\alpha_2(\tilde{T}^h)\|_0\|d(\mathbf{u})\|_\infty\|d(\tilde{\mathbf{u}}^h-\mathbf{u}^h)\|_0\right.\\
&\qquad\qquad+\|\alpha_2(\tilde{T}^h)-\alpha_2(T^h)\|_0\|d(\mathbf{u})\|_\infty\|d(\tilde{\mathbf{u}}^h-\mathbf{u}^h)\|_0\\
&\qquad\qquad\left.+\alpha_{2,max}\|d(\mathbf{u}-\tilde{\mathbf{u}}^h)\|_0\|d(\tilde{\mathbf{u}}^h-\mathbf{u}^h)\|_0\right]\\
&\le C\left[\hat{C}\|T-\tilde{T}^h\|_1M\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1+\hat{C}\|\tilde{T}^h-T^h\|_1M\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1\right.\\
&\qquad\qquad\qquad\left.+\alpha_{2,max}\|\mathbf{u}-\tilde{\mathbf{u}}^h\|_1\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1\right]\\
&\le C\left[\frac{\hat{C}^2\overline{C}^2M^4h^4}{4\epsilon_4}+\frac{\hat{C}^2M^2}{4\epsilon_5}\|\tilde{T}^h-T^h\|_1^2+\frac{\alpha_{2,max}^2\overline{C}^2M^2h^4}{4\epsilon_6}\right.\\
&\qquad\qquad\qquad\left.+(\epsilon_4+\epsilon_5+\epsilon_6)\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1^2\right]. \qquad (5.59)
\end{aligned}
$$

Also, by (3.17), (4.29), (5.47), (5.49) and the Young's inequality,

$$
\begin{aligned}
&-((\mathbf{u}-\tilde{\mathbf{u}}^h)\cdot\nabla T,\tilde{T}^h-T^h)-((\tilde{\mathbf{u}}^h-\mathbf{u}^h)\cdot\nabla T,\tilde{T}^h-T^h)\\
&\qquad\qquad -(\mathbf{u}^h\cdot\nabla(T-\tilde{T}^h),\tilde{T}^h-T^h)\\
&\le C\Big[\|\mathbf{u}-\tilde{\mathbf{u}}^h\|_1\|T\|_1\|\tilde{T}^h-T^h\|_1+\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1\|T\|_1\|\tilde{T}^h-T^h\|_1\\
&\qquad\qquad +\|\mathbf{u}^h\|_1\|T-\tilde{T}^h\|_1\|\tilde{T}^h-T^h\|_1\Big]\\
&\le C'\Big[\overline{C}h^2M^2\|\tilde{T}^h-T^h\|_1+M\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1\|\tilde{T}^h-T^h\|_1\\
&\qquad\qquad +\overline{C}h^2M\|\tilde{T}^h-T^h\|_1\Big]\\
&\le C'\Bigg[\frac{\overline{C}^2M^4h^4}{4\epsilon_7}+\frac{M^2}{4\epsilon_8}\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1^2+\frac{\overline{C}^2M^2h^4}{4\epsilon_9}\\
&\qquad\qquad +(\epsilon_7+\epsilon_8+\epsilon_9)\|\tilde{T}^h-T^h\|_1^2\Bigg]. \qquad (5.60)
\end{aligned}
$$

The last four terms in (5.56) are bounded using (5.47)-(5.50):

$$
\begin{aligned}
&(d(\mathbf{u}-\tilde{\mathbf{u}}^h),\underline{\tilde{\sigma}}^h-\underline{\sigma}^h)-(\underline{\sigma}-\underline{\tilde{\sigma}}^h,d(\tilde{\mathbf{u}}^h-\mathbf{u}^h))+(p-\tilde{p}^h,\nabla\cdot(\tilde{\mathbf{u}}^h-\mathbf{u}^h))\\
&\qquad\qquad -\kappa(\nabla(T-\tilde{T}^h),\nabla(\tilde{T}^h-T^h))\\
&\le C\big[\|\mathbf{u}-\tilde{\mathbf{u}}^h\|_1\|\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\|_0+\|\sigma-\underline{\tilde{\sigma}}^h\|_0\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1\\
&\qquad\qquad +\|p-\tilde{p}^h\|_0\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1+\kappa\|T-\tilde{T}^h\|_1\|\tilde{T}^h-T^h\|_1\big]\\
&\le C\Bigg[\frac{\overline{C}^2M^2h^4}{4\epsilon_{10}}+\frac{\overline{C}^2M^2h^4}{4\epsilon_{11}}+\frac{\overline{C}^2M^2h^4}{4\epsilon_{12}}+\frac{\overline{C}^2M^2h^4}{4\epsilon_{13}}+\epsilon_{10}\|\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\|_0^2\\
&\qquad\qquad +(\epsilon_{11}+\epsilon_{12})\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_1+\epsilon_{13}\|\tilde{T}^h-T^h\|_1^2\Bigg]. \qquad (5.61)
\end{aligned}
$$

Combining all estimates (5.58)-(5.61) and the lower bound (5.57), we have

$$
\begin{aligned}
&\left[\frac{1}{2\alpha_{1,max}}-C(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_{10})\right]\|\underline{\tilde{\sigma}}^h-\underline{\sigma}^h\|_0^2\\
&\qquad +\left[2\epsilon\alpha_{2,min}-C(\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_{11}+\epsilon_{12}+\frac{M^2}{4\epsilon_8})\right]\|\tilde{\mathbf{u}}^h-\mathbf{u}^h\|_0^2\\
&\qquad +\left[\kappa-C(\epsilon_7+\epsilon_8+\epsilon_9+\epsilon_{13}+\frac{\hat{C}^2M^2}{4\epsilon_2}+\frac{\hat{C}^2M^2}{4\epsilon_5})\right]\|\tilde{T}^h-T^h\|_1^2\\
&\le C\Bigg[\frac{\hat{C}^2\overline{C}^2M^4h^4}{4\epsilon_1}+\frac{\overline{C}^2M^2h^4}{16\alpha_{1,min}^2\epsilon_3}+\frac{\hat{C}^2\overline{C}^2M^4h^4}{4\epsilon_4}+\frac{\alpha_{2,max}^2\overline{C}^2M^2h^4}{4\epsilon_6}\\
&\quad +\frac{\overline{C}^2M^4h^4}{4\epsilon_7}+\frac{\overline{C}^2M^2h^4}{4\epsilon_9}+\frac{\overline{C}^2M^2h^4}{4\epsilon_{10}}+\frac{\overline{C}^2M^2h^4}{4\epsilon_{11}}+\frac{\overline{C}^2M^2h^4}{4\epsilon_{12}}+\frac{\overline{C}^2M^2h^4}{4\epsilon_{13}}\Bigg]. \qquad (5.62)
\end{aligned}
$$

Finally, if we choose ϵ_i appropriately for $i=1,2,\ldots,13$ and M is sufficiently small, the error bound (5.51) is obtained by the triangle inequality, (5.47)-(5.49) and (5.62). □

6. Numerical Examples

We now present numerical results of two examples. The first is a test problem with the domain $\Omega = \{(x, y)|0 \le x \le 1, 0 \le y \le 1\}$ and a specified solution. The second is a four-to-one contraction channel flow problem with mixed boundary conditions. The following viscosity parameters used in both examples correspond to polystyrene [10]:

$$\begin{aligned} \frac{\Delta E}{R} &= 14500, \\ \epsilon &= 0.001. \end{aligned}$$

We also set $T_R = 500$ and $\kappa = 1$.

Example 1. We adjust the right hand side functions in (2.1), (2.5)-(2.7) so that the system has the exact solution

$$\begin{aligned} \mathbf{u}(x, y) &= \begin{bmatrix} x^2(1-x)y(1-y) \\ -(2x - 3x^2)\left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right) \end{bmatrix}, \\ p(x, y) &= -100x^2 + 100, \\ T(x, y) &= x^2(1-x)y(1-y) + 600. \end{aligned}$$

Although we assumed that $\mathbf{u} = \mathbf{0}$ on Γ for analysis, it turns out that the homogeneous boundary condition is not necessary for the numerical experiments. We calculated errors on successively refined meshes. The convergence rate was then computed, and the results are shown in Table 1. The table shows that our computational results are well matched with the analytical result (5.51).

Mesh Size	$\mathbf{u}$		$\underline{\sigma}$		T	
h	$\mathbf{H}^1$ Error	Rate	L^2 Error	Rate	H^1 Error	Rate
1	0.129×10^0	NA	0.164×10^{-2}	NA	0.663×10^{-1}	NA
$\frac{1}{2}$	0.425×10^{-1}	1.60	0.580×10^{-3}	1.50	0.254×10^{-1}	1.38
$\frac{1}{4}$	0.111×10^{-1}	1.94	0.155×10^{-3}	1.91	0.720×10^{-2}	1.82
$\frac{1}{8}$	0.282×10^{-2}	1.98	0.399×10^{-4}	1.96	0.186×10^{-2}	1.95
$\frac{1}{16}$	0.709×10^{-3}	1.99	0.101×10^{-4}	1.98	0.471×10^{-3}	1.99
$\frac{1}{32}$	0.181×10^{-3}	1.97	0.256×10^{-5}	1.98	0.118×10^{-3}	1.99

Table 1. Example 1: Errors and rates

Example 2. We consider a four-to-one contraction domain, as depicted in Figure 1. Fluid flows from left to right, and this domain is characterized by the sudden width reduction of 75%. The boundary Γ is divided into four parts: the inflow boundary Γ_{in}, the wall boundary Γ_{wall}, the outflow boundary Γ_{out}, and the symmetry boundary Γ_{sym} (so that $\Gamma = \Gamma_{in} \cup \Gamma_{wall} \cup \Gamma_{out} \cup \Gamma_{sym}$). Let $\mathbf{n}$ and $\mathbf{t}$ be the unit outward normal and tangential vectors to Γ, respectively. Let $\underline{\pi}$ be the total stress. Then the boundary conditions are as follows [10].

- Inflow boundary Γ_{in}:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_{in}, \\ T &= T_{in}. \end{aligned}$$

- Wall boundary Γ_{wall}:

$$\begin{gathered} \mathbf{u} = \mathbf{0}, \\ \nabla T \cdot \mathbf{n} = 0. \end{gathered}$$

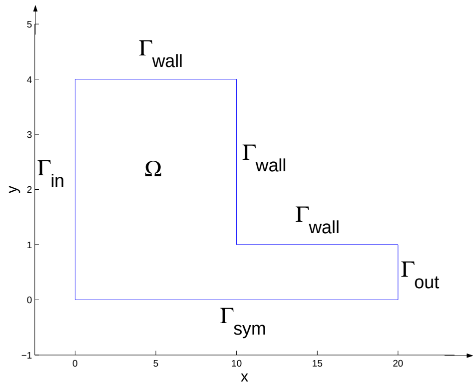


FIGURE 1. Example 2: Computational domain Ω.

- Outflow boundary Γ_{out}:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_{out}, \\ \nabla T \cdot \mathbf{n} &= 0. \end{aligned}$$

- Symmetry boundary Γ_{sym}:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 0, \\ \nabla T \cdot \mathbf{n} &= 0, \\ \underline{\pi} : \mathbf{nt} &= 0. \end{aligned}$$

Notice that we can interpret $\nabla T \cdot \mathbf{n} = 0$ as an insulated boundary condition, $\mathbf{u} = \mathbf{0}$ as the fluid does not move if it touches the wall, and that the condition $\mathbf{u} \cdot \mathbf{n} = 0$ along Γ_{sym} implies that the flow in this half of the domain does not affect the flow in the other 'half' of the domain. The condition $\underline{\pi} : \mathbf{nt} = 0$ along Γ_{sym} is interpreted as a vanishing tangential contact force. We then set

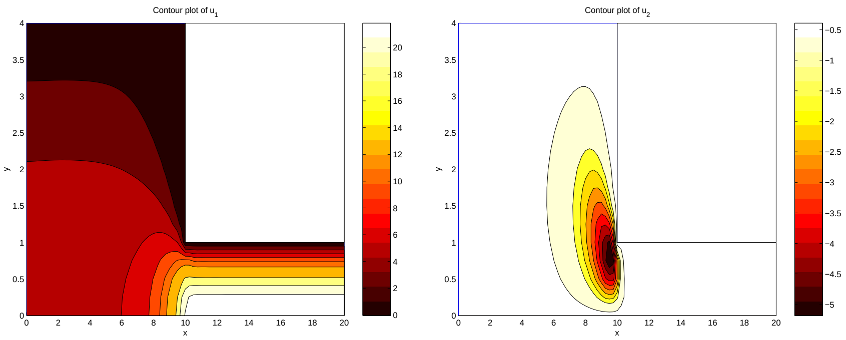
$$\mathbf{u}_{in} = \begin{bmatrix} 6\left(1 - \left(\frac{y}{4}\right)^2\right) \\ 0 \end{bmatrix}, \quad \mathbf{u}_{out} = \begin{bmatrix} 24(1-y^2) \\ 0 \end{bmatrix}$$

$$T_{in} = 540 + \frac{5}{2}y.$$

Note that $\int_\Gamma \mathbf{u} \cdot \mathbf{n} \, d\Gamma = 0$ for the incompressibility condition $\nabla \cdot \mathbf{u} = 0$. The equations are simulated, and typical profiles for velocity, temperature, and pressure are presented in Figures 2 and 3.

7. Concluding remarks

In this paper, the finite element solution of the non-isothermal Stokes-Oldroyd problem was investigated. We proved that a bounded approximate solution exists and also derived an error estimate for the solution. As mentioned earlier, this work is our initial step toward the numerical study of the equations governing non-isothermal viscoelastic flows characterized by, for example, the Oldroyd-B or Giesekus constitutive model. There are numerous engineering publications (for example, [9, 10, 11]) which consider non-isothermal viscoelastic flows and related application problems. However, further mathematical and numerical analysis of the problem is needed due to the complexity of the model equations. Details concerning numerical analysis and other computational issues will be addressed in a later paper.



(a) $u_1(x, y)$ (b) $u_2(x, y)$

FIGURE 2. Solution profiles of the components of velocity.

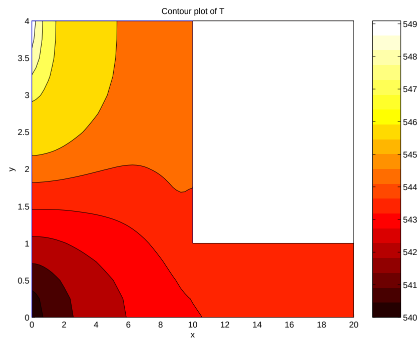


FIGURE 3. Solution profile of temperature.

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