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ANALYSIS OF THE $[L^2, L^2, L^2]$ LEAST-SQUARES FINITE ELEMENT METHOD FOR INCOMPRESSIBLE OSEEN-TYPE PROBLEMS

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Dedicated to Professor Max D. Gunzburger on the occasion of his 60th birthday

Abstract. In this paper we analyze several first-order systems of Oseen-type equations that are obtained from the time-dependent incompressible Navier-Stokes equations after introducing the additional vorticity and possibly total pressure variables, time-discretizing the time derivative and linearizing the non-linear terms. We apply the $[L^2, L^2, L^2]$ least-squares finite element scheme to approximate the solutions of these Oseen-type equations assuming homogeneous velocity boundary conditions. All of the associated least-squares energy functionals are defined to be the sum of squared L^2 norms of the residual equations over an appropriate product space. We first prove that the homogeneous least-squares functionals are coercive in the $H^1 \times L^2 \times L^2$ norm for the velocity, vorticity, and pressure, but only continuous in the $H^1 \times H^1 \times H^1$ norm for these variables. Although equivalence between the homogeneous least-squares functionals and one of the above two product norms is not achieved, by using these *a priori* estimates and additional finite element analysis we are nevertheless able to prove that the least-squares method produces an optimal rate of convergence in the H^1 norm for velocity and suboptimal rate of convergence in the L^2 norm for vorticity and pressure. Numerical experiments with various Reynolds numbers that support the theoretical error estimates are presented. In addition, numerical solutions to the time-dependent incompressible Navier-Stokes problem are given to demonstrate the accuracy of the semi-discrete $[L^2, L^2, L^2]$ least-squares finite element approach.

Key Words. Navier-Stokes equations, Oseen-type equations, finite element methods, least squares.

1. Problem formulation

As a first step towards the finite element solution of the time-dependent incompressible Navier-Stokes problem by using the least-squares principles, in this paper we analyze the $[L^2, L^2, L^2]$ least-squares finite element approximations to several first-order systems of Oseen-type equations all equipped with the homogeneous velocity boundary conditions. These systems are obtained from the time-dependent incompressible Navier-Stokes problem after introducing the additional vorticity and possibly total pressure variables, time-discretizing the time derivative and linearizing the non-linear terms.

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We start with the derivation of these first-order Oseen-type problems and introduce some background and notations. Let Ω be an open bounded and connected domain in $\mathbb{R}^N$ ($N = 2$ or 3) with Lipschitz boundary $\partial\Omega$. The time-dependent incompressible Navier-Stokes problem on the bounded domain Ω can be posed as the following initial-boundary value problem (cf. [13, 14, 15]):

Find $\mathbf{u}(\boldsymbol{x}, t) : \overline{\Omega} \times [0, T] \to \mathbb{R}^N$ *and* $p(\boldsymbol{x}, t) : \overline{\Omega} \times [0, T] \to \mathbb{R}$ *such that*

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \frac{1}{\lambda}\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times (0, T), \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \times [0, T], \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0(\cdot) && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where the symbols Δ, ∇ and $\nabla\cdot$ stand for the Laplacian, gradient and divergence operators with respect to the spatial variable $\boldsymbol{x}$, respectively; $\mathbf{u} = (u_1, \cdots, u_N)^\top$ is the velocity vector; p is the pressure; $\lambda \geq 1$ is the Reynolds number and may be identified with the inverse viscosity constant $1/\nu$; $[0, T]$ is the time interval under consideration; $\mathbf{f} = (f_1, \cdots, f_N)^\top : \Omega \times (0, T) \to \mathbb{R}^N$ is a given vector function representing the density of body force; the initial velocity $\mathbf{u}_0 : \overline{\Omega} \to \mathbb{R}^N$ with $\mathbf{u}_0 = \mathbf{0}$ on $\partial\Omega$ is prescribed. All of them are assumed to be non-dimensionalized.

We now introduce some notations that are used throughout the article. When $N = 2$, we define the curl operator, $\nabla\times$, with respect to the spatial variable $\boldsymbol{x}$ for a smooth scalar function v by

$$\nabla \times v = \Big(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\Big)^\top,$$

and for a smooth 2-component vector function $\mathbf{v} = (v_1, v_2)^\top$ by

$$\nabla \times \mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.$$

When $N = 3$, we define the curl of a smooth 3-component vector function $\mathbf{v} = (v_1, v_2, v_3)^\top$ by

$$\nabla \times \mathbf{v} = \Big(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\Big)^\top.$$

We also define the following cross products. If w is a scalar function and $\mathbf{v} = (v_1, v_2)^\top$, then

$$w \times \mathbf{v} = -\mathbf{v} \times w = (-wv_2, wv_1)^\top.$$

If $\mathbf{w} = (w_1, w_2, w_3)^\top$ and $\mathbf{v} = (v_1, v_2, v_3)^\top$, then

$$\mathbf{w} \times \mathbf{v} = (w_2v_3 - w_3v_2, w_3v_1 - w_1v_3, w_1v_2 - w_2v_1)^\top.$$

With these notations, it can be easily checked that the following identities hold: for a smooth vector function $\mathbf{u} = (u_1, \cdots, u_N)^\top$,

$$\nabla \times (\nabla \times \mathbf{u}) = -\Delta \mathbf{u} + \nabla(\nabla \cdot \mathbf{u}) \tag{1.2}$$

and

$$(\mathbf{w} \times \mathbf{v}) \cdot \mathbf{v} = 0 \tag{1.3}$$

for $\mathbf{w} = (w_1, \cdots, w_{2N-3})^\top$ and $\mathbf{v} = (v_1, \cdots, v_N)^\top$.

Introducing the additional vorticity variable $\boldsymbol{\omega}$ (cf. [2, 7, 10]),

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \qquad \text{on } \overline{\Omega} \times [0, T],$$

and combining the divergence free equation, $\nabla \cdot \mathbf{u} = 0$ in $\Omega \times (0,T)$, with identity (1.2), we can transform the time-dependent incompressible Navier-Stokes problem (1.1) into the following quasi-linear velocity-vorticity-pressure first-order system:

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\lambda}\nabla \times \boldsymbol{\omega} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0,T),\\
\nabla \times \mathbf{u} - \boldsymbol{\omega} &= \mathbf{0} && \text{in } \Omega \times (0,T),\\
\nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times (0,T),\\
\mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \times [0,T],\\
\mathbf{u}(\cdot,0) &= \mathbf{u}_0(\cdot) && \text{in } \Omega.
\end{aligned} \tag{1.4}
$$

If, in addition, introducing the the total pressure b (cf. [1, 15]),

$$b = p + \frac{1}{2}|\mathbf{u}|^2 \qquad \text{on } \overline{\Omega} \times [0,T],$$

as a dependent variable instead of the original pressure p, then one can verify that

$$\nabla b + \boldsymbol{\omega} \times \mathbf{u} = \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} \qquad \text{in } \Omega \times (0,T),$$

and we have the following semi-linear velocity-vorticity-total pressure first-order system:

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\lambda}\nabla \times \boldsymbol{\omega} + \boldsymbol{\omega} \times \mathbf{u} + \nabla b &= \mathbf{f} && \text{in } \Omega \times (0,T),\\
\nabla \times \mathbf{u} - \boldsymbol{\omega} &= \mathbf{0} && \text{in } \Omega \times (0,T),\\
\nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times (0,T),\\
\mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \times [0,T],\\
\mathbf{u}(\cdot,0) &= \mathbf{u}_0(\cdot) && \text{in } \Omega.
\end{aligned} \tag{1.5}
$$

Next, the time discretization can be readily realized by using the finite difference approach such as the backward-Euler scheme [15, 16]. Let $\Delta t = t^{n+1} - t^n \le 1$ be the time step. Given $\mathbf{u}^n$ for the previous time step, the solutions $(\mathbf{u}^{n+1}, \boldsymbol{\omega}^{n+1}, p^{n+1})$ and $(\mathbf{u}^{n+1}, \boldsymbol{\omega}^{n+1}, b^{n+1})$ of the current time step of problems (1.4) and (1.5) are respectively determined from the following problems:

$$
\begin{aligned}
\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \frac{1}{\lambda}\nabla \times \boldsymbol{\omega}^{n+1} + (\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} &= \mathbf{f}^{n+1} && \text{in } \Omega,\\
\nabla \times \mathbf{u}^{n+1} - \boldsymbol{\omega}^{n+1} &= \mathbf{0} && \text{in } \Omega,\\
\nabla \cdot \mathbf{u}^{n+1} &= 0 && \text{in } \Omega,\\
\mathbf{u}^{n+1} &= \mathbf{0} && \text{on } \partial\Omega;
\end{aligned} \tag{1.6}
$$

$$
\begin{aligned}
\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \frac{1}{\lambda}\nabla \times \boldsymbol{\omega}^{n+1} + \boldsymbol{\omega}^{n+1} \times \mathbf{u}^{n+1} + \nabla b^{n+1} &= \mathbf{f}^{n+1} && \text{in } \Omega,\\
\nabla \times \mathbf{u}^{n+1} - \boldsymbol{\omega}^{n+1} &= \mathbf{0} && \text{in } \Omega,\\
\nabla \cdot \mathbf{u}^{n+1} &= 0 && \text{in } \Omega,\\
\mathbf{u}^{n+1} &= \mathbf{0} && \text{on } \partial\Omega.
\end{aligned} \tag{1.7}
$$

The convection term $(\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1}$ in (1.6) and reaction term $\boldsymbol{\omega}^{n+1} \times \mathbf{u}^{n+1}$ in (1.7) can be further linearized by using the simple substitution or Newton's method [15]. More specifically, $(\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1}$ can be approximated as

$$(\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1} \simeq (\mathbf{u}^{n}\cdot\nabla)\mathbf{u}^{n+1}$$

or

$$(\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1} \simeq (\mathbf{u}^{n}\cdot\nabla)\mathbf{u}^{n+1} + (\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n} - (\mathbf{u}^{n}\cdot\nabla)\mathbf{u}^{n},$$

and $\boldsymbol{\omega}^{n+1} \times \mathbf{u}^{n+1}$ can be approximated as

$$\boldsymbol{\omega}^{n+1} \times \mathbf{u}^{n+1} \simeq \boldsymbol{\omega}^{n+1} \times \mathbf{u}^{n}$$

or

$$\boldsymbol{\omega}^{n+1} \times \mathbf{u}^{n+1} \simeq \boldsymbol{\omega}^{n} \times \mathbf{u}^{n+1} + \boldsymbol{\omega}^{n+1} \times \mathbf{u}^{n} - \boldsymbol{\omega}^{n} \times \mathbf{u}^{n},$$

provided $\mathbf{u}(\boldsymbol{x},\cdot), \nabla\mathbf{u}(\boldsymbol{x},\cdot), \boldsymbol{\omega}(\boldsymbol{x},\cdot)$ are continuous on $[0,T]$ for all $\boldsymbol{x} \in \Omega$, and Δt is small enough.

Applying the above approximations, we arrive at the following four boundary value problems of Oseen-type equations at each time step:

$$\begin{aligned} \frac{1}{\Delta t}\mathbf{u}^{n+1} + \frac{1}{\lambda}\nabla\times\boldsymbol{\omega}^{n+1} + (\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} &= \mathbf{f}^{n+1} + \frac{1}{\Delta t}\mathbf{u}^n \quad \text{in } \Omega, \\ \nabla\times\mathbf{u}^{n+1} - \boldsymbol{\omega}^{n+1} &= \mathbf{0} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \mathbf{u}^{n+1} &= \mathbf{0} \quad \text{on } \partial\Omega; \end{aligned} \tag{1.8}$$

$$\begin{aligned} \frac{1}{\Delta t}\mathbf{u}^{n+1} + \frac{1}{\lambda}\nabla\times\boldsymbol{\omega}^{n+1} + \qquad & \\ (\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1} + (\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^n + \nabla p^{n+1} &= \mathbf{f}^{n+1} + \frac{1}{\Delta t}\mathbf{u}^n + (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n \quad \text{in } \Omega, \\ \nabla\times\mathbf{u}^{n+1} - \boldsymbol{\omega}^{n+1} &= \mathbf{0} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \mathbf{u}^{n+1} &= \mathbf{0} \quad \text{on } \partial\Omega; \end{aligned} \tag{1.9}$$

$$\begin{aligned} \frac{1}{\Delta t}\mathbf{u}^{n+1} + \frac{1}{\lambda}\nabla\times\boldsymbol{\omega}^{n+1} + \boldsymbol{\omega}^{n+1}\times\mathbf{u}^n + \nabla b^{n+1} &= \mathbf{f}^{n+1} + \frac{1}{\Delta t}\mathbf{u}^n \quad \text{in } \Omega, \\ \nabla\times\mathbf{u}^{n+1} - \boldsymbol{\omega}^{n+1} &= \mathbf{0} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \mathbf{u}^{n+1} &= \mathbf{0} \quad \text{on } \partial\Omega; \end{aligned} \tag{1.10}$$

$$\begin{aligned} \frac{1}{\Delta t}\mathbf{u}^{n+1} + \frac{1}{\lambda}\nabla\times\boldsymbol{\omega}^{n+1} + \boldsymbol{\omega}^{n}\times\mathbf{u}^{n+1} \qquad & \\ +\boldsymbol{\omega}^{n+1}\times\mathbf{u}^n + \nabla b^{n+1} &= \mathbf{f}^{n+1} + \frac{1}{\Delta t}\mathbf{u}^n + \boldsymbol{\omega}^n\times\mathbf{u}^n \quad \text{in } \Omega, \\ \nabla\times\mathbf{u}^{n+1} - \boldsymbol{\omega}^{n+1} &= \mathbf{0} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \mathbf{u}^{n+1} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{1.11}$$

where the superscript "n" denotes the previous time step and "$n+1$" the current time step, and in problem (1.11), $\boldsymbol{\omega}^0$ can be obtained by simply taking

$$\boldsymbol{\omega}^0 = \nabla\times\mathbf{u}^0 = \nabla\times\mathbf{u}_0 \quad \text{in } \Omega.$$

The purpose of this paper is to analyze the $[L^2, L^2, L^2]$ least-squares finite element approximations to the above four Oseen-type problems at each time step. To simplify the notation, we re-write boundary value problems (1.8)–(1.11) in the following generic form:

$$\begin{aligned} k\mathbf{u} + \frac{1}{\lambda}\nabla\times\boldsymbol{\omega} + H(\mathbf{u},\boldsymbol{\omega}) + \nabla d &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla\times\mathbf{u} - \boldsymbol{\omega} &= \mathbf{0} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{1.12}$$

where $k \geq 1$ denotes the constant $1/\Delta t$,

$$H(\mathbf{u}, \boldsymbol{\omega}) = \begin{cases} (\boldsymbol{\alpha} \cdot \nabla)\mathbf{u} & \text{for (1.8)}, \\ (\boldsymbol{\alpha} \cdot \nabla)\mathbf{u} + A\mathbf{u} & \text{for (1.9)}, \\ -\boldsymbol{\alpha} \times \boldsymbol{\omega} & \text{for (1.10)}, \\ \boldsymbol{\beta} \times \mathbf{u} - \boldsymbol{\alpha} \times \boldsymbol{\omega} & \text{for (1.11)}, \end{cases} \tag{1.13}$$

and

$$d = \begin{cases} p & \text{for} \quad \text{(1.8) and (1.9)}, \\ b & \text{for} \quad \text{(1.10) and (1.11)}, \end{cases} \tag{1.14}$$

in which $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_N)^\top$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_{2N-3})^\top$ stem from the previous time step and square matrix A is defined as $A := [(\nabla\alpha_1)^\top, \cdots, (\nabla\alpha_N)^\top]^\top$. Notice that, for simplicity, we still denote the right hand side in the momentum equations by $\mathbf{f}$. We assume that the given functions $\boldsymbol{\alpha} \in [H^1(\Omega)]^N$ and $\boldsymbol{\beta} \in [L^2(\Omega)]^{2N-3}$ are bounded on Ω. For the uniqueness of solution, we also assume that d satisfies the zero mean condition, i.e., $\int_\Omega d = 0$. Notice that if $d = b$ then it does not mean that $\int_\Omega p = 0$.

In the past decade, least-squares finite element methods have become more and more frequently used for computing approximate solution of first-order systems of partial differential equations in fluid dynamics. In [3], the authors summarized most of the former literature. The specific features of the least-squares finite element approach which give it advantages relative to, for instance, the mixed finite element approach [4, 5], are as follows: it leads to a minimization problem; it is not subject to the Ladyzhenskaya-Babuska-Brezzi condition (cf. [4, 5]); simple equal low-order finite elements, such as the continuous linear elements, can be used for the approximation of all unknowns; the resulting linear system is symmetric and positive definite with the condition number of order $O(h^{-2})$, where h denotes some measure of the mesh size; the value of the homogeneous least-squares functional of the approximate solution provides a practical and sharp *a posteriori* error estimator at no additional cost [15], etc.

Up to now, in the mathematics and engineering communities, most existing least-squares methods for the stationary viscous incompressible flow problems are based on the velocity-vorticity-pressure (or total pressure) formulation, see [1, 2, 3, 7, 8, 10, 11, 12, 15, 19, 20] for examples. Some other first-order system formulations which involve velocity-flux or stress variables can also be found in [6, 9, 11, 15, 17, 19, 21]. However, the latter formulations lead to the introduction of extra variables and more equations, and hence these approaches also involve a higher number of degrees of freedom in the solution procedure. Therefore, we adopt the former formulation in the present paper.

The objective of this paper is to investigate the $[L^2, L^2, L^2]$ least-squares finite element approximations to the Oseen-type problem (1.12) obtained from the time-dependent Navier-Stokes problem after time-discretization and linearization. The least-squares functional is defined to be the sum of squared L^2 norms of the residuals of the partial differential equations over an appropriate product space. We first prove that the homogeneous least-squares functional is coercive in the $H^1 \times L^2 \times L^2$ norm for the velocity, vorticity, and pressure (or total pressure), but only continuous in the $H^1 \times H^1 \times H^1$ norm for these variables. Although equivalence between the homogeneous least-squares functional and one of the above two product norms is not achieved, with the use of these *a priori* estimates and additional finite element analysis, we still prove that with respect to the order of approximation for smooth exact solutions the least-squares method produces an optimal rate of convergence in the H^1 norm for velocity and suboptimal rate of convergence in the L^2 norm for vorticity and pressure (or total pressure). Numerical examples in two dimensions with various Reynolds numbers that support the theoretical error estimates are presented.

On the other hand, though it seems that the analysis for the Oseen-type and Navier-Stokes problems follows directly from the corresponding results for the Stokes problem, measuring and trying to reduce the dependence of the coercivity and continuity estimates of the homogeneous least-squares functional on the Reynolds number λ and other parameters should be a major concern, when passing from the Stokes problem to the Oseen-type and Navier-Stokes problems by using the least-squares finite element approach. Unfortunately, in the present paper, we are not able to identify the dependence of the these coercivity and continuity estimates on the Reynolds number λ and time step Δt. But numerical results reported in Sections 5 and 6 indicate that the accuracy does not degrade and the error estimates still hold for larger λ.

Let us briefly discuss the recent work [18]. The $[H^{-1}, L^2, L^2]$ least-squares finite element method for Oseen-type problems associated with (1.8) and (1.10) is analyzed. In their formulation, the least-squares energy functional is defined to be the sum of squared H^{-1} and L^2 norms of the residual equations over a suitable product space. The homogeneous least-squares functional is proved to be equivalent to the $H^1 \times L^2 \times L^2$ product norm. The authors then analyzed the case where the H^{-1} norm in the least-squares functional is replaced by a discrete functional to make the computation feasible. With the help of coercivity and continuity estimates of the homogeneous least-squares functional, optimal error estimates in order of approximation as well as the required regularity of the exact solution can be derived. However, this approach is rather tricky to program in practice. Thus, the L^2-type method has become the most popular least-squares approach in the engineering community (see many references contained in [15]).

Finally, based on the analysis for the Oseen-type problem (1.12), numerical approximations produced from schemes (1.8)–(1.11) for the time-dependent incompressible Navier-Stokes problem are also reported in this paper, which demonstrate the accuracy of the semi-discrete $[L^2, L^2, L^2]$ least-squares finite element approach.

The remainder of this paper is organized as follows. In Section 2, we introduce some notations and present some preliminaries. In Section 3, under suitable assumptions, we derive the continuity and coercivity estimates for the homogeneous least-squares functional associated with the Oseen-type problem (1.12). With the aid of such *a priori* estimates, in Section 4, we provide the error analysis of the $[L^2, L^2, L^2]$ least-squares finite element approximations. In Section 5, we illustrate our analysis by some numerical examples. Finally, in Section 6, numerical results for the time-dependent incompressible Navier-Stokes problem are given.

2. Preliminaries

We shall use standard notation and definition for the Sobolev space $H^m(\Omega)$ for non-negative integer m. The standard associated inner product and norm are denoted by $(\cdot,\cdot)_m$ and $\|\cdot\|_m$, respectively. As usual, $L^2(\Omega) = H^0(\Omega)$. We define the following two subspaces of $L^2(\Omega)$ and $H^1(\Omega)$, respectively,

$$\begin{aligned} L_0^2(\Omega) &= \{q \in L^2(\Omega) : (q,1)_0 = 0\}, \\ H_0^1(\Omega) &= \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}. \end{aligned}$$

Let $[H^m(\Omega)]^N$ denote the corresponding product space of $H^m(\Omega)$, and the inner product and norm will be still denoted by $(\cdot,\cdot)_m$ and $\|\cdot\|_m$, respectively, when there is no risk of confusion. Furthermore, we define the Banach space,

$$L^\infty(\Omega) = \{v : v \text{ is a measurable function on } \Omega \text{ and } \|v\|_\infty := \operatorname*{ess\,sup}_{\Omega} |v| < +\infty\},$$

and let $[L^\infty(\Omega)]^m$ denote the corresponding product space and the norm will be still denoted by $\|\cdot\|_\infty$.

We also introduce the following Hilbert spaces with natural norms (cf. [14]):

$$\begin{aligned} \mathbf{H}(\nabla\cdot;\Omega) &= \{\mathbf{v} \in [L^2(\Omega)]^N : \nabla\cdot\mathbf{v} \in L^2(\Omega)\}, \\ \mathbf{H}_0(\nabla\cdot;\Omega) &= \{\mathbf{v} \in \mathbf{H}(\nabla\cdot;\Omega) : \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}, \\ \mathbf{H}(\nabla\times;\Omega) &= \{\boldsymbol{\varphi} \in [L^2(\Omega)]^N : \nabla\times\boldsymbol{\varphi} \in [L^2(\Omega)]^{2N-3}\}, \\ \mathbf{H}_0(\nabla\times;\Omega) &= \{\boldsymbol{\varphi} \in \mathbf{H}(\nabla\times;\Omega) : \gamma\boldsymbol{\varphi}|_{\partial\Omega} = \mathbf{0}\}, \end{aligned}$$

where $\gamma\boldsymbol{\varphi} = \boldsymbol{\varphi}\cdot\mathbf{t}$ if $N = 2$, $\gamma\boldsymbol{\varphi} = \boldsymbol{\varphi}\times\mathbf{n}$ if $N = 3$; $\mathbf{n}$ and $\mathbf{t}$ denote the unit outward normal vector and tangent vector to $\partial\Omega$, respectively. The dual space of $H_0^1(\Omega)$ is denoted as $H^{-1}(\Omega)$ with norm defined by

$$\|v\|_{-1} = \sup_{0\neq\varphi\in H_0^1(\Omega)} \frac{(v,\varphi)}{\|\varphi\|_1},$$

where (v,φ) denotes the value of the functional v at φ. The corresponding product space is denoted by $[H^{-1}(\Omega)]^N$ with norm still denoted by $\|\cdot\|_{-1}$.

With these notations, we have the following Green-type formulas which are applications of the usual Green's formula.

Lemma 2.1. *The following three Green-type formulas hold:*

$$(\mathbf{v},\nabla q)_0 + (\nabla\cdot\mathbf{v}, q)_0 = (\mathbf{v}\cdot\mathbf{n}, q)_{0,\partial\Omega} \tag{2.1}$$

for all $\mathbf{v} \in \mathbf{H}(\nabla\cdot;\Omega)$ *and* $q \in H^1(\Omega)$*;*

$$(\nabla\times\boldsymbol{\varphi}, \Phi)_0 - (\boldsymbol{\varphi}, \nabla\times\Phi)_0 = (\gamma\boldsymbol{\varphi}, \Phi)_{0,\partial\Omega} \tag{2.2}$$

for all $\boldsymbol{\varphi} \in \mathbf{H}(\nabla\times;\Omega)$ *and* $\Phi \in [H^1(\Omega)]^{2N-3}$*;*

$$((\boldsymbol{\alpha}\cdot\nabla)\mathbf{v}, \mathbf{w})_0 = -(\mathbf{v}, (\boldsymbol{\alpha}\cdot\nabla)\mathbf{w})_0 - ((\nabla\cdot\boldsymbol{\alpha})\mathbf{v}, \mathbf{w})_0 \tag{2.3}$$

for all $\mathbf{v}$*,* $\mathbf{w} \in [H_0^1(\Omega)]^N$*.*

Proof. See (1.19), (2.17), and (2.22) in [14]. □

Furthermore, we have the following Poincaré-Friedrichs-type inequalities:

Lemma 2.2. *There exists a positive constant C such that for any $\mathbf{v} \in \mathbf{H}_0(\nabla\cdot;\Omega) \cap \mathbf{H}_0(\nabla\times;\Omega)$,*

$$\|\nabla \mathbf{v}\|_0 \le C\{\|\nabla\cdot\mathbf{v}\|_0 + \|\nabla\times\mathbf{v}\|_0\}. \tag{2.4}$$

Combining (2.4) *with the usual Poincaré-Friedrichs inequality, we have*

$$\|\mathbf{v}\|_0 \le C\{\|\nabla\cdot\mathbf{v}\|_0 + \|\nabla\times\mathbf{v}\|_0\}, \tag{2.5}$$

for all $\mathbf{v} \in [H_0^1(\Omega)]^N$.

Proof. See page 36 in [14]. □

We remark that, throughout this paper, in any estimate or inequality, the quantity C with or without subscripts will denote a generic positive constant always independent of the mesh size h that will be introduced later, and need not be the same constant in different occurrences.

The next lemma is an immediate consequence of a general result of functional analysis due to Nečas:

Lemma 2.3. *Let Ω be an open, connected, and bounded domain with Lipschitz boundary. There exists a constant C, depending only on Ω, such that*

$$\|\dot{q}\|_{L^2(\Omega)/\mathbb{R}} \le C\|\nabla\dot{q}\|_{-1}, \tag{2.6}$$

for all $\dot{q} \in L^2(\Omega)/\mathbb{R}$, where $L^2(\Omega)/\mathbb{R}$ denotes the quotient space of $L^2(\Omega)$ by $\mathbb{R}$.

Proof. See [14], page 20, Corollary 2.1, part 2. □

Note the fact that

$$\|\dot{q}\|_{L^2(\Omega)/\mathbb{R}} := \inf_{c\in\mathbb{R}} \|q+c\|_0 = \inf_{c\in\mathbb{R}} \left(\int_\Omega q^2 + 2cq + c^2\right)^{\frac{1}{2}} = \|q\|_0,$$

for any $q \in L_0^2(\Omega)$. Thus, we have

$$\|q\|_0 \le C\|\nabla q\|_{-1} \qquad \text{for all } q \in L_0^2(\Omega). \tag{2.7}$$

In the proof of Theorem 3.1 below we will often use the following ε-inequality:

$$2ab \le \varepsilon a^2 + \varepsilon^{-1} b^2, \tag{2.8}$$

for any a, b, $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$.

We are interested in the following three function spaces with respect to the three unknown functions: velocity $\mathbf{u}$, vorticity $\boldsymbol{\omega}$, and pressure (or total pressure) d,

$$\mathcal{V} = [H_0^1(\Omega)]^N, \quad \mathcal{W} = [H^1(\Omega)]^{2N-3}, \quad \text{and} \quad \mathcal{Q} = H^1(\Omega) \cap L_0^2(\Omega). \tag{2.9}$$

It is now in the position to introduce the following $[L^2, L^2, L^2]$ least-squares energy functional $\mathcal{F}(\cdot;\mathbf{f})$ over the product space $\mathcal{V}\times\mathcal{W}\times\mathcal{Q}$:

$$\begin{aligned} \mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{f}) \quad &= \quad \|k\mathbf{v} + \frac{1}{\lambda}\nabla\times\boldsymbol{\varphi} + H(\mathbf{v},\boldsymbol{\varphi}) + \nabla q - \mathbf{f}\|_0^2 \\ &\quad + \|\nabla\times\mathbf{v} - \boldsymbol{\varphi}\|_0^2 \\ &\quad + \|\nabla\cdot\mathbf{v}\|_0^2. \end{aligned} \tag{2.10}$$

Here the least-squares energy functional $\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{f})$ is defined to be the sum of squared L^2 norms of the residuals of the partial differential equations in (1.12) over the product space $\mathcal{V}\times\mathcal{W}\times\mathcal{Q}$.

3. *A priori* estimates

We now derive the continuity and coercivity estimates of the homogeneous least-squares functional $\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0})$ for the first-order system problem (1.12). Such *a priori* estimates play crucial roles in the error estimates for the $[L^2,L^2,L^2]$ least-squares finite element scheme that will be introduced in next section.

Theorem 3.1. *Consider the homogeneous $[L^2,L^2,L^2]$ least-squares energy functional $\mathcal{F}(\cdot;\mathbf{0})$ over the product space $\mathcal{V}\times\mathcal{W}\times\mathcal{Q}$.*

(1) *There exists a positive constant C_1 such that for any $(\mathbf{v},\boldsymbol{\varphi},q)\in\mathcal{V}\times\mathcal{W}\times\mathcal{Q}$,*

$$\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0})\le C_1\Big(\|\mathbf{v}\|_1^2+\|\boldsymbol{\varphi}\|_1^2+\|q\|_1^2\Big). \tag{3.1}$$

(2) *Under the following assumptions:*

$$\begin{cases}\nabla\cdot\boldsymbol{\alpha}\le 2k \quad \text{in } \Omega & \text{for (1.8) and (1.9)},\\ 4\|\boldsymbol{\alpha}\|_\infty^2\le \dfrac{k}{\lambda} & \text{for (1.10) and (1.11)},\end{cases} \tag{3.2}$$

there exists a positive constant C_2 such that for any $(\mathbf{v},\boldsymbol{\varphi},q)\in\mathcal{V}\times\mathcal{W}\times\mathcal{Q}$,

$$\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0})\ge C_2\Big(\|\mathbf{v}\|_1^2+\|\boldsymbol{\varphi}\|_0^2+\|q\|_0^2\Big). \tag{3.3}$$

Proof. The upper bound (3.1) is straightforward from the triangle inequality and ε-inequality (2.8). We proceed to show the validity of (3.3). For any $\Phi\in[H_0^1(\Omega)]^N$, by Green-type formula (2.2), we have

$$\begin{aligned}(\nabla q,\Phi)_0 &= (k\mathbf{v}+\frac{1}{\lambda}\nabla\times\boldsymbol{\varphi}+H(\mathbf{v},\boldsymbol{\varphi})+\nabla q,\Phi)_0+\frac{1}{\lambda}(\nabla\times\mathbf{v}-\boldsymbol{\varphi},\nabla\times\Phi)_0\\ &\quad-\frac{1}{\lambda}(\nabla\times\mathbf{v},\nabla\times\Phi)_0-k(\mathbf{v},\Phi)_0-(H(\mathbf{v},\boldsymbol{\varphi}),\Phi)_0\\ &\le \|k\mathbf{v}+\frac{1}{\lambda}\nabla\times\boldsymbol{\varphi}+H(\mathbf{v},\boldsymbol{\varphi})+\nabla q\|_0\|\Phi\|_0\\ &\quad+\frac{1}{\lambda}\|\nabla\times\mathbf{v}-\boldsymbol{\varphi}\|_0\|\nabla\times\Phi\|_0+\frac{1}{\lambda}\|\nabla\times\mathbf{v}\|_0\|\nabla\times\Phi\|_0\\ &\quad+k\|\mathbf{v}\|_0\|\Phi\|_0+\|H(\mathbf{v},\boldsymbol{\varphi})\|_0\|\Phi\|_0.\end{aligned} \tag{3.4}$$

We estimate the term $\|H(\mathbf{v},\boldsymbol{\varphi})\|_0$ in (3.4) by considering the following cases:

(1) $H(\mathbf{v},\boldsymbol{\varphi})=(\boldsymbol{\alpha}\cdot\nabla)\mathbf{v}:$

$$\begin{aligned}\|H(\mathbf{v},\boldsymbol{\varphi})\|_0 &\le C\|\boldsymbol{\alpha}\|_\infty\|\mathbf{v}\|_1\\ &\le C(\|\nabla\cdot\mathbf{v}\|_0+\|\nabla\times\mathbf{v}\|_0)\\ &\le C\mathcal{F}^{\frac12}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0})+C\|\nabla\times\mathbf{v}\|_0.\end{aligned} \tag{3.5}$$

(2) $H(\mathbf{v},\boldsymbol{\varphi})=(\boldsymbol{\alpha}\cdot\nabla)\mathbf{v}+A\mathbf{v}:$

$$\begin{aligned}\|H(\mathbf{v},\boldsymbol{\varphi})\|_0 &\le C\|\boldsymbol{\alpha}\|_\infty\|\mathbf{v}\|_1+C\|\boldsymbol{\alpha}\|_1\|\mathbf{v}\|_0\\ &\le C(\|\nabla\cdot\mathbf{v}\|_0+\|\nabla\times\mathbf{v}\|_0)\\ &\le C\mathcal{F}^{\frac12}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0})+C\|\nabla\times\mathbf{v}\|_0.\end{aligned} \tag{3.6}$$

(3) $H(\mathbf{v},\boldsymbol{\varphi})=-\boldsymbol{\alpha}\times\boldsymbol{\varphi}:$

$$\begin{aligned}\|H(\mathbf{v},\boldsymbol{\varphi})\|_0 &\le C\|\boldsymbol{\alpha}\|_\infty\|\boldsymbol{\varphi}\|_0\\ &\le C\|\boldsymbol{\alpha}\|_\infty(\|\nabla\times\mathbf{v}-\boldsymbol{\varphi}\|_0+\|\nabla\times\mathbf{v}\|_0)\\ &\le C\mathcal{F}^{\frac12}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0})+C\|\nabla\times\mathbf{v}\|_0.\end{aligned} \tag{3.7}$$

(4) $H(\mathbf{v},\boldsymbol{\varphi}) = \boldsymbol{\beta}\times\mathbf{v} - \boldsymbol{\alpha}\times\boldsymbol{\varphi}$:

$$\begin{aligned}
\|H(\mathbf{v},\boldsymbol{\varphi})\|_0 &\le C\|\boldsymbol{\beta}\|_\infty\|\mathbf{v}\|_0 + C\|\boldsymbol{\alpha}\|_\infty\|\boldsymbol{\varphi}\|_0\\
&\le C\|\boldsymbol{\beta}\|_\infty\|\mathbf{v}\|_0 + C\|\boldsymbol{\alpha}\|_\infty(\|\nabla\times\mathbf{v}-\boldsymbol{\varphi}\|_0 + \|\nabla\times\mathbf{v}\|_0)\\
&\le C(\|\nabla\cdot\mathbf{v}\|_0 + \|\nabla\times\mathbf{v}\|_0) + C\|\nabla\times\mathbf{v}-\boldsymbol{\varphi}\|_0\\
&\le C\mathcal{F}^{\frac12}((\mathbf{v},\boldsymbol{\varphi},q);\,\mathbf{0}) + C\|\nabla\times\mathbf{v}\|_0. \qquad (3.8)
\end{aligned}$$

Combining above estimates (3.5)–(3.8) with (2.5), (2.7), and (3.4), we obtain

$$\begin{aligned}
\|q\|_0 &\le C\|\nabla q\|_{-1}\\
&\le C(\|k\mathbf{v} + \frac{1}{\lambda}\nabla\times\boldsymbol{\varphi} + H(\mathbf{v},\boldsymbol{\varphi}) + \nabla q\|_0 + \|\nabla\times\mathbf{v}-\boldsymbol{\varphi}\|_0\\
&\qquad + \|\mathbf{v}\|_0 + \mathcal{F}^{\frac12}((\mathbf{v},\boldsymbol{\varphi},q);\,\mathbf{0}) + \|\nabla\times\mathbf{v}\|_0)\\
&\le C(\mathcal{F}^{\frac12}((\mathbf{v},\boldsymbol{\varphi},q);\,\mathbf{0}) + \|\nabla\times\mathbf{v}\|_0). \qquad (3.9)
\end{aligned}$$

Next, we are going to estimate $\|\nabla\times\mathbf{v}\|_0$ in terms of $\mathcal{F}^{\frac12}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0})$. We first expand $\|\nabla\times\mathbf{v}\|_0^2$ by adding and subtracting some related terms. Then by the Cauchy-Schwarz inequality, ε-inequality (2.8), Green-type formulas (2.1)–(2.2), and estimate (3.9), we have

$$\begin{aligned}
\|\nabla\times\mathbf{v}\|_0^2 &= (\nabla\times\mathbf{v}-\boldsymbol{\varphi},\nabla\times\mathbf{v})_0 + (k\mathbf{v} + \frac{1}{\lambda}\nabla\times\boldsymbol{\varphi} + H(\mathbf{v},\boldsymbol{\varphi}) + \nabla q, \lambda\mathbf{v})_0\\
&\qquad -(k\mathbf{v} + H(\mathbf{v},\boldsymbol{\varphi}),\lambda\mathbf{v})_0 + \lambda(q,\nabla\cdot\mathbf{v})_0\\
&\le 4\|\nabla\times\mathbf{v}-\boldsymbol{\varphi}\|_0^2 + \frac14\|\nabla\times\mathbf{v}\|_0^2\\
&\qquad + C\|k\mathbf{v} + \frac{1}{\lambda}\nabla\times\boldsymbol{\varphi} + H(\mathbf{v},\boldsymbol{\varphi}) + \nabla q\|_0\|\nabla\cdot\mathbf{v}\|_0\\
&\qquad + C\|k\mathbf{v} + \frac{1}{\lambda}\nabla\times\boldsymbol{\varphi} + H(\mathbf{v},\boldsymbol{\varphi}) + \nabla q\|_0^2 + \frac14\|\nabla\times\mathbf{v}\|_0^2\\
&\qquad -(k\mathbf{v} + H(\mathbf{v},\boldsymbol{\varphi}),\lambda\mathbf{v})_0 + C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\,\mathbf{0})\\
&\qquad + C\|\nabla\cdot\mathbf{v}\|_0^2 + \frac14\|\nabla\times\mathbf{v}\|_0^2,
\end{aligned}$$

which implies

$$\frac14\|\nabla\times\mathbf{v}\|_0^2 \le C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\,\mathbf{0}) - (k\mathbf{v} + H(\mathbf{v},\boldsymbol{\varphi}),\lambda\mathbf{v})_0. \qquad (3.10)$$

We now estimate the term $-(k\mathbf{v}+H(\mathbf{v},\boldsymbol{\varphi}),\lambda\mathbf{v})_0$ by considering the following cases:

(1) $H(\mathbf{v},\boldsymbol{\varphi}) = (\boldsymbol{\alpha}\cdot\nabla)\mathbf{v}$: by Green-type formula (2.3) with assumption (3.2),

$$-(k\mathbf{v} + H(\mathbf{v},\boldsymbol{\varphi}),\lambda\mathbf{v})_0 = \lambda\Big((-k + \frac12(\nabla\cdot\boldsymbol{\alpha}))\mathbf{v},\mathbf{v}\Big)_0 \le 0. \qquad (3.11)$$

(2) $H(\mathbf{v},\boldsymbol{\varphi}) = (\boldsymbol{\alpha}\cdot\nabla)\mathbf{v} + A\mathbf{v}$: by Green-type formulas (2.1), (2.3), Poincaré-Friedrichs-type inequality (2.5), ε-inequality (2.8), and assumption (3.2),

$$\begin{aligned}
-(k\mathbf{v} + H(\mathbf{v},\boldsymbol{\varphi}),\lambda\mathbf{v})_0 &\le -\lambda(A\mathbf{v},\mathbf{v})_0\\
&\le \lambda\sum_{i=1}^{N}\Big\{(\nabla\cdot\mathbf{v},\alpha_i v_i)_0 + (\mathbf{v}\cdot\nabla v_i,\alpha_i)_0\Big\}\\
&\le 4\lambda\|\boldsymbol{\alpha}\|_\infty\|\nabla\cdot\mathbf{v}\|_0\|\mathbf{v}\|_0\\
&\le C\lambda\|\boldsymbol{\alpha}\|_\infty\|\nabla\cdot\mathbf{v}\|_0(\|\nabla\cdot\mathbf{v}\|_0 + \|\nabla\times\mathbf{v}\|_0)\\
&\le C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\,\mathbf{0}) + \frac18\|\nabla\times\mathbf{v}\|_0^2. \qquad (3.12)
\end{aligned}$$

(3) $H(\mathbf{v},\boldsymbol{\varphi}) = -\boldsymbol{\alpha}\times\boldsymbol{\varphi}$: owing to assumption (3.2),

$$\begin{aligned} -(k\mathbf{v}+H(\mathbf{v},\boldsymbol{\varphi}),\lambda\mathbf{v})_0 &= -(k\mathbf{v}-\boldsymbol{\alpha}\times\boldsymbol{\varphi},\lambda\mathbf{v})_0 \\ &\le -k\lambda\|\mathbf{v}\|_0^2 + \frac{1}{16}\|\boldsymbol{\varphi}\|_0^2 + 4\lambda^2\|\boldsymbol{\alpha}\|_\infty^2\|\mathbf{v}\|_0^2 \\ &\le \frac{1}{16}\Big(\|\nabla\times\mathbf{v}-\boldsymbol{\varphi}\|_0 + \|\nabla\times\mathbf{v}\|_0\Big)^2 \\ &\le \frac{1}{8}\|\nabla\times\mathbf{v}-\boldsymbol{\varphi}\|_0^2 + \frac{1}{8}\|\nabla\times\mathbf{v}\|_0^2 \\ &\le C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0}) + \frac{1}{8}\|\nabla\times\mathbf{v}\|_0^2. \end{aligned} \tag{3.13}$$

(4) $H(\mathbf{v},\boldsymbol{\varphi}) = \boldsymbol{\beta}\times\mathbf{v}-\boldsymbol{\alpha}\times\boldsymbol{\varphi}$: since $(\boldsymbol{\beta}\times\mathbf{v})\cdot\mathbf{v}=0$, by (3.13) we have

$$-(k\mathbf{v}+H(\mathbf{v},\boldsymbol{\varphi}),\lambda\mathbf{v})_0 \le C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0}) + \frac{1}{8}\|\nabla\times\mathbf{v}\|_0^2. \tag{3.14}$$

Combining (3.10) with (3.11)–(3.14), we obtain

$$\|\nabla\times\mathbf{v}\|_0^2 \le C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0}). \tag{3.15}$$

Now, by (3.9) and (3.15), we have

$$\|q\|_0^2 \le C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0}). \tag{3.16}$$

Since

$$\|\nabla\cdot\mathbf{v}\|_0^2 \le C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0}), \tag{3.17}$$

by (2.4), (2.5), and (3.15) we have

$$\|\mathbf{v}\|_1^2 \le C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0}). \tag{3.18}$$

Again, by (3.15) with the triangle inequality, we have

$$\begin{aligned} \|\boldsymbol{\varphi}\|_0^2 &\le 2\|\nabla\times\mathbf{v}-\boldsymbol{\varphi}\|_0^2 + 2\|\nabla\times\mathbf{v}\|_0^2 \\ &\le C\mathcal{F}((\mathbf{v},\boldsymbol{\varphi},q);\mathbf{0}). \end{aligned} \tag{3.19}$$

Finally, combining (3.16), (3.18), and (3.19), we obtain (3.3). This completes the proof of the theorem. □

We conclude this section with the following remark concerning (3.2).

Remark 3.1. *The first assumption in* (3.2) *is similar to the usual assumption,*

$$k - \frac{1}{2}(\nabla\cdot\boldsymbol{\alpha}) \ge \sigma > 0 \quad \text{in } \Omega, \tag{3.20}$$

in the stationary convection-diffusion problem that ensures the solvability and uniqueness by using the standard finite element method [16]*. In our case, $\nabla\cdot\boldsymbol{\alpha}$ is small enough because $\boldsymbol{\alpha}$ stems from the previous approximate velocity vector $\mathbf{u}_h$. Hence, it is a quite reasonable assumption. In contrast to the first assumption, the second assumption in* (3.2) *is a little bit strict constraint. However, it provides a cue for choosing the suitable time step for various Reynolds numbers.*

4. The least-squares finite element scheme: stability, convergence, and error estimates

In this section, we will derive the $[L^2, L^2, L^2]$ least-squares finite element scheme for the Oseen-type problem (1.12) and then give the error analysis of the scheme. First, we note that the exact solution $(\mathbf{u}, \boldsymbol{\omega}, d) \in \mathcal{V} \times \mathcal{W} \times \mathcal{Q}$ of problem (1.12) must be the zero minimizer of the functional $\mathcal{F}$ on $\mathcal{V} \times \mathcal{W} \times \mathcal{Q}$, i.e.,

$$\mathcal{F}((\mathbf{u}, \boldsymbol{\omega}, d); \mathbf{f}) = 0 = \min\{\mathcal{F}((\mathbf{v}, \boldsymbol{\varphi}, q); \mathbf{f}) : (\mathbf{v}, \boldsymbol{\varphi}, q) \in \mathcal{V} \times \mathcal{W} \times \mathcal{Q}\}.$$

Since $\mathcal{F}((\mathbf{u}, \boldsymbol{\omega}, d) + \delta(\mathbf{v}, \boldsymbol{\varphi}, q); \mathbf{f})$ is a nonnegative quadratic functional in the variable $\delta \in \mathbb{R}$, for any given $(\mathbf{v}, \boldsymbol{\varphi}, q) \in \mathcal{V} \times \mathcal{W} \times \mathcal{Q}$, we have

$$\frac{d}{d\delta}\mathcal{F}((\mathbf{u}, \boldsymbol{\omega}, d) + \delta(\mathbf{v}, \boldsymbol{\varphi}, q); \mathbf{f})\Big|_{\delta=0} = 0$$

which is equivalent to

$$\mathcal{B}((\mathbf{u}, \boldsymbol{\omega}, d), (\mathbf{v}, \boldsymbol{\varphi}, q)) = \mathcal{L}((\mathbf{v}, \boldsymbol{\varphi}, q)), \quad \forall\ (\mathbf{v}, \boldsymbol{\varphi}, q) \in \mathcal{V} \times \mathcal{W} \times \mathcal{Q}, \tag{4.1}$$

where the continuous bilinear form $\mathcal{B}(\cdot,\cdot)$ and the continuous linear form $\mathcal{L}(\cdot)$ are respectively defined as follows:

$$\begin{aligned}
\mathcal{B}((\mathbf{u}, \boldsymbol{\omega}, d), (\mathbf{v}, \boldsymbol{\varphi}, q)) &= \int_\Omega (k\mathbf{u} + \frac{1}{\lambda}\nabla \times \boldsymbol{\omega} + H(\mathbf{u}, \boldsymbol{\omega}) + \nabla d) \\
&\qquad \cdot(k\mathbf{v} + \frac{1}{\lambda}\nabla \times \boldsymbol{\varphi} + H(\mathbf{v}, \boldsymbol{\varphi}) + \nabla q) \\
&\quad + (\nabla \times \mathbf{u} - \boldsymbol{\omega}) \cdot (\nabla \times \mathbf{v} - \boldsymbol{\varphi}) \\
&\quad + (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})\ d\Omega, \\
\mathcal{L}((\mathbf{v}, \boldsymbol{\varphi}, q)) &= \int_\Omega \mathbf{f} \cdot (k\mathbf{v} + \frac{1}{\lambda}\nabla \times \boldsymbol{\varphi} + H(\mathbf{v}, \boldsymbol{\varphi}) + \nabla q)\ d\Omega.
\end{aligned}$$

Observe that we have the following identity:

$$\mathcal{B}((\mathbf{v}, \boldsymbol{\varphi}, q), (\mathbf{v}, \boldsymbol{\varphi}, q)) = \mathcal{F}((\mathbf{v}, \boldsymbol{\varphi}, q); \mathbf{0}), \quad \forall\ (\mathbf{v}, \boldsymbol{\varphi}, q) \in \mathcal{V} \times \mathcal{W} \times \mathcal{Q}. \tag{4.2}$$

Therefore, the *a priori* estimates (3.1) and (3.3) give continuity and coercivity estimates for the bilinear form $\mathcal{B}(\cdot,\cdot)$, respectively. As a consequence, one can further verify that the bilinear form $\mathcal{B}(\cdot,\cdot)$ defines an inner product on the product space $\mathcal{V} \times \mathcal{W} \times \mathcal{Q}$. We denote its associated norm by

$$|||(\mathbf{v}, \boldsymbol{\varphi}, q)|||^2 = \mathcal{B}((\mathbf{v}, \boldsymbol{\varphi}, q), (\mathbf{v}, \boldsymbol{\varphi}, q)), \quad \forall\ (\mathbf{v}, \boldsymbol{\varphi}, q) \in \mathcal{V} \times \mathcal{W} \times \mathcal{Q}. \tag{4.3}$$

Now, we define the finite element spaces. Let $\{\mathcal{T}_h\}$ be a family of regular triangulations [4, 16] of the domain Ω, where

$$\mathcal{T}_h = \{\Omega_i^h : i = 1, 2, \cdots, T(h)\},$$

$h = \max\{\mathrm{diam}(\Omega_i^h) : \Omega_i^h \in \mathcal{T}_h\}$ denotes the grid size, and $T(h)$ denotes the number of triangles. Let $P_r(\Omega_i^h)$ denote the space of polynomials of degree less than or equal to r defined over Ω_i^h. Define the following three continuous approximating function spaces,

$$\mathcal{V}_h^r = \{\mathbf{v}_h \in \mathcal{V} : \mathbf{v}_h|_{\Omega_i^h} \in [P_r(\Omega_i^h)]^N,\ i = 1, 2, \cdots, T(h)\}, \tag{4.4}$$

$$\mathcal{W}_h^r = \{\boldsymbol{\varphi}_h \in \mathcal{W} : \boldsymbol{\varphi}_h|_{\Omega_i^h} \in [P_r(\Omega_i^h)]^{2N-3},\ i = 1, 2, \cdots, T(h)\}, \tag{4.5}$$

$$\mathcal{Q}_h^r = \{q_h \in \mathcal{Q} : q_h|_{\Omega_i^h} \in P_r(\Omega_i^h), i = 1, 2, \cdots, T(h)\}. \tag{4.6}$$

It is well-known that the finite element spaces $\mathcal{V}_h^r$, $\mathcal{W}_h^r$, and $\mathcal{Q}_h^r$ satisfy the following approximation properties: for any $\mathbf{v} \in \mathcal{V} \cap [H^{r+1}(\Omega)]^N$, $\boldsymbol{\varphi} \in \mathcal{W} \cap [H^{r+1}(\Omega)]^{2N-3}$, and $q \in \mathcal{Q} \cap H^{r+1}(\Omega)$, there exist $\mathbf{v}_h \in \mathcal{V}_h^r$, $\boldsymbol{\varphi}_h \in \mathcal{W}_h^r$, and $q_h \in \mathcal{Q}_h^r$ such that

$$\|\mathbf{v} - \mathbf{v}_h\|_0 + h\|\mathbf{v} - \mathbf{v}_h\|_1 \leq Ch^{r+1}\|\mathbf{v}\|_{r+1}, \tag{4.7}$$

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_0 + h\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_1 \leq Ch^{r+1}\|\boldsymbol{\varphi}\|_{r+1}, \tag{4.8}$$

$$\|q - q_h\|_0 + h\|q - q_h\|_1 \leq Ch^{r+1}\|q\|_{r+1}, \tag{4.9}$$

where C is a positive constant independent of $\mathbf{v}$, $\boldsymbol{\varphi}$, q, and h.

With above notations, the $[L^2, L^2, L^2]$ least-squares finite element scheme for problem (1.12) is then defined to be the following problem:

Find $(\mathbf{u}_h, \boldsymbol{\omega}_h, d_h) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$ *such that*

$$\mathcal{B}((\mathbf{u}_h, \boldsymbol{\omega}_h, d_h), (\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h)) = \mathcal{L}((\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h)), \tag{4.10}$$

for all $(\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$.

This finite-dimensional problem (4.10) can be shown to be uniquely solvable.

Theorem 4.1. (existence and uniqueness) *Suppose assumption* (3.2) *holds. Then problem* (4.10) *has a unique solution* $(\mathbf{u}_h, \boldsymbol{\omega}_h, d_h) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$.

Proof. Since $\mathcal{B}(\cdot,\cdot)$ is an inner product on $\mathcal{V} \times \mathcal{W} \times \mathcal{Q}$ and $\mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$ is a finite-dimensional subspace of $\mathcal{V} \times \mathcal{W} \times \mathcal{Q}$, by the Fredholm alternative, we know that problem (4.10) possesses a unique solution $(\mathbf{u}_h, \boldsymbol{\omega}_h, d_h) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$. □

We next give the stability and convergence analysis with error estimates for the $[L^2, L^2, L^2]$ least-squares finite element approximations to problem (1.12).

Theorem 4.2. (stability) *Suppose assumption* (3.2) *holds. Then the least-squares finite element scheme* (4.10) *is stable in the following sense: there exists a positive constant* C *independent of* h *such that*

$$\|\mathbf{u}_h\|_1 + \|\boldsymbol{\omega}_h\|_0 + \|d_h\|_0 \leq C\|\mathbf{f}\|_0. \tag{4.11}$$

Proof. By (4.3) and (4.10), we have

$$\begin{aligned} |||(\mathbf{u}_h, \boldsymbol{\omega}_h, d_h)|||^2 &= \mathcal{L}((\mathbf{u}_h, \boldsymbol{\omega}_h, d_h)) \\ &\leq \|\mathbf{f}\|_0 \|k\mathbf{u}_h + \frac{1}{\lambda}\nabla \times \boldsymbol{\omega}_h + H(\mathbf{u}_h, \boldsymbol{\omega}_h) + \nabla d_h\|_0 \\ &\leq \|\mathbf{f}\|_0 |||(\mathbf{u}_h, \boldsymbol{\omega}_h, d_h)||| \end{aligned}$$

which together with (3.3) yields the conclusion. □

Let $(\mathbf{u}, \boldsymbol{\omega}, d) \in \mathcal{V} \times \mathcal{W} \times \mathcal{Q}$ and $(\mathbf{u}_h, \boldsymbol{\omega}_h, d_h) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$ denote the solutions of problems (1.12) and (4.10), respectively. Since $\mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r \subset \mathcal{V} \times \mathcal{W} \times \mathcal{Q}$, by using (4.1) and (4.10), we have the following orthogonality relation:

$$\mathcal{B}((\mathbf{u}, \boldsymbol{\omega}, d) - (\mathbf{u}_h, \boldsymbol{\omega}_h, d_h), (\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h)) = 0, \tag{4.12}$$

for all $(\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$. Combining (4.12) with the usual Cauchy-Schwarz inequality, we easily obtain

$$|||(\mathbf{u}, \boldsymbol{\omega}, d) - (\mathbf{u}_h, \boldsymbol{\omega}_h, d_h)||| \leq |||(\mathbf{u}, \boldsymbol{\omega}, d) - (\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h)|||, \tag{4.13}$$

for all $(\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$. Inequality (4.13) indicates that the $[L^2, L^2, L^2]$ least-squares finite element solution $(\mathbf{u}_h, \boldsymbol{\omega}_h, d_h)$ is a best approximation to the exact solution $(\mathbf{u}, \boldsymbol{\omega}, d)$ in the finite-dimensional space $\mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$ with respect to the energy norm $|||\cdot|||$.

We need the following results in the proof of Theorem 4.3. Let $\mathcal{D}(\Omega)$ denote the linear space of infinitely differentiable functions with compact support in Ω, and let $\mathcal{D}(\overline{\Omega})$ denote the space of restrictions of the functions in $\mathcal{D}(\mathbb{R}^N)$ to $\overline{\Omega}$. Let $\widetilde{\mathcal{D}}(\overline{\Omega}) = \mathcal{D}(\overline{\Omega}) \cap L_0^2(\Omega)$. Since a bounded, Lipschitz continuous open set is the union of a finite number of star-shaped, Lipschitz continuous open sets (see page 22 in [14]), we can prove that the space $\widetilde{\mathcal{D}}(\overline{\Omega})$ is dense in $H^1(\Omega) \cap L_0^2(\Omega)$ (see Lemma 4.2 in [10]).

Theorem 4.3. (convergence) *Suppose assumption* (3.2) *holds. Then the least-squares finite element scheme* (4.10) *is convergent in the following sense:*

$$\lim_{h\to 0}(\|\mathbf{u}-\mathbf{u}_h\|_1 + \|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_0 + \|d-d_h\|_0) = 0. \tag{4.14}$$

Proof. We will prove (4.14) by using the density argument. Since the spaces $[\mathcal{D}(\Omega)]^N$, $[\mathcal{D}(\overline{\Omega})]^{2N-3}$, and $\widetilde{\mathcal{D}}(\overline{\Omega})$ are dense in $[H_0^1(\Omega)]^N$, $[H^1(\Omega)]^{2N-3}$, and $H^1(\Omega)$ $\cap L_0^2(\Omega)$ with respect to the $\|\cdot\|_1$ norm, respectively, for any $\epsilon > 0$ there exist $\mathbf{u}^* \in [\mathcal{D}(\Omega)]^N$, $\boldsymbol{\omega}^* \in [\mathcal{D}(\overline{\Omega})]^{2N-3}$, and $d^* \in \widetilde{\mathcal{D}}(\overline{\Omega})$ all independent of h such that

$$\|\mathbf{u}-\mathbf{u}^*\|_1 + \|\boldsymbol{\omega}-\boldsymbol{\omega}^*\|_1 + \|d-d^*\|_1 < \frac{\epsilon}{2\sqrt{C_1}},$$

that together with (3.1) implies

$$|||(\mathbf{u},\boldsymbol{\omega},d)-(\mathbf{u}^*,\boldsymbol{\omega}^*,d^*)||| < \frac{\epsilon}{2}.$$

For this fixed sufficiently smooth function $(\mathbf{u}^*, \boldsymbol{\omega}^*, d^*)$, by approximation properties (4.7)–(4.9), we can find $(\mathbf{u}_h^*, \boldsymbol{\omega}_h^*, d_h^*) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$ so that

$$\|\mathbf{u}^*-\mathbf{u}_h^*\|_1 + \|\boldsymbol{\omega}^*-\boldsymbol{\omega}_h^*\|_1 + \|d^*-d_h^*\|_1 \le Ch^r(\|\mathbf{u}^*\|_{r+1} + \|\boldsymbol{\omega}^*\|_{r+1} + \|d^*\|_{r+1})$$

which implies, for sufficiently small h,

$$|||(\mathbf{u}^*,\boldsymbol{\omega}^*,d^*)-(\mathbf{u}_h^*,\boldsymbol{\omega}_h^*,d_h^*)||| < \frac{\epsilon}{2}.$$

Utilizing (4.13) and the above inequalities, we immediately obtain

$$\begin{aligned} 0 &\le |||(\mathbf{u},\boldsymbol{\omega},d)-(\mathbf{u}_h,\boldsymbol{\omega}_h,d_h)||| \\ &\le |||(\mathbf{u},\boldsymbol{\omega},d)-(\mathbf{u}_h^*,\boldsymbol{\omega}_h^*,d_h^*)||| \\ &\le |||(\mathbf{u},\boldsymbol{\omega},d)-(\mathbf{u}^*,\boldsymbol{\omega}^*,d^*)||| + |||(\mathbf{u}^*,\boldsymbol{\omega}^*,d^*)-(\mathbf{u}_h^*,\boldsymbol{\omega}_h^*,d_h^*)||| \\ &< \epsilon \end{aligned}$$

which implies

$$\lim_{h\to 0}|||(\mathbf{u},\boldsymbol{\omega},d)-(\mathbf{u}_h,\boldsymbol{\omega}_h,d_h)||| = 0. \tag{4.15}$$

Combining (4.15) with (3.3), we obtain (4.14). This completes the proof. □

Since the bilinear form $\mathcal{B}(\cdot,\cdot)$ is symmetric and positive definite on the product space $\mathcal{V} \times \mathcal{W} \times \mathcal{Q}$, once a basis for $\mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$ is chosen, we can immediately conclude that the matrix $\mathcal{M}$ of the linear system associated with problem (4.10) is symmetric and positive definite. Furthermore, using (3.1), (3.3) and inverse estimates for finite element spaces defined over quasi-uniform triangulations [4], we have the following results:

Theorem 4.4. (condition number) *Suppose assumption* (3.2) *holds. Then the matrix* $\mathcal{M}$ *of the linear system associated with the least-squares finite element scheme* (4.10) *is symmetric and positive definite. If the family* $\{\mathcal{T}_h\}$ *of regular triangulations of the domain* Ω *is quasi-uniform, then the condition number of* $\mathcal{M}$ *is of order* $O(h^{-2})$.

Proof. Details of the proof can be found in [10]. □

We are now in the position to give the error estimates of the least-squares finite element solution $(\mathbf{u}_h, \boldsymbol{\omega}_h, d_h)$.

Theorem 4.5. (error estimates) *Suppose assumption* (3.2) *holds. If the exact solution* $(\mathbf{u}, \boldsymbol{\omega}, d) \in (\mathcal{V} \times \mathcal{W} \times \mathcal{Q}) \cap [H^{r+1}(\Omega)]^{3N-2}$, *then we have the following error estimates:*

$$\|\mathbf{u}-\mathbf{u}_h\|_1 + \|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_0 + \|d-d_h\|_0 \le Ch^r(\|\mathbf{u}\|_{r+1} + \|\boldsymbol{\omega}\|_{r+1} + \|d\|_{r+1}), \tag{4.16}$$

where C *is a positive constant independent of* h.

Proof. By (3.3), (4.13), and (3.1), we have

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_1 + \|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_0 + \|d-d_h\|_0 &\le C|||(\mathbf{u}, \boldsymbol{\omega}, d) - (\mathbf{u}_h, \boldsymbol{\omega}_h, d_h)||| \\
&\le C|||(\mathbf{u}, \boldsymbol{\omega}, d) - (\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h)||| \\
&\le C(\|\mathbf{u}-\mathbf{v}_h\|_1 + \|\boldsymbol{\omega}-\boldsymbol{\varphi}_h\|_1 + \|d-q_h\|_1),
\end{aligned}$$

for any $(\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$. Choosing $(\mathbf{v}_h, \boldsymbol{\varphi}_h, q_h) \in \mathcal{V}_h^r \times \mathcal{W}_h^r \times \mathcal{Q}_h^r$ such that the approximation properties (4.7)–(4.9) are satisfied when $(\mathbf{v}, \boldsymbol{\varphi}, q)$ is replaced by $(\mathbf{u}, \boldsymbol{\omega}, d)$, we obtain (4.16). This completes the proof. □

Although the constant C in error estimates (4.16) might depend on k and the Reynolds number λ, it indicates that, with respect to the order r of approximation for smooth exact solutions, the $[L^2, L^2, L^2]$ least-squares method produces an optimal rate of convergence in the H^1 norm for velocity $\mathbf{u}$ and suboptimal rate of convergence in the L^2 norm for vorticity $\boldsymbol{\omega}$ and pressure p (or total pressure b).

5. Numerical examples for the Oseen-type problem

In this section, we consider numerical experiments for the Oseen-type problem (1.12) in two-dimensional domain Ω with various Reynolds numbers. We first take for our domain the unit square $\Omega = (0,1)^2$ and construct a problem with the following smooth exact solution which is also discussed for the Stokes problem in [10]:

$$\begin{aligned}
u_1(x,y) &= x^2(1-x)^2(2y-6y^2+4y^3), \\
u_2(x,y) &= y^2(1-y)^2(-2x+6x^2-4x^3), \\
\omega(x,y) &= x^2(1-x)^2(-2+12y-12y^2) + y^2(1-y)^2(-2+12x-12x^2), \\
d(x,y) &= x^2+y^2-\frac{20}{3}xy+x+y.
\end{aligned}$$

Substituting the above exact solution with $k = 10$, $\boldsymbol{\alpha} = (1,1)^\top$, $\boldsymbol{\beta} = \beta_1 = 1$ into the Oseen-type problem (1.12), we can readily get the right-hand side data functions

$\mathbf{f} = (f_1, f_2)^\top$, and we have

$$(5.1)\qquad H(\mathbf{u}, \boldsymbol{\omega}) = \begin{cases} \text{case I} : (\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y}, \frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial y})^\top & \text{for (1.8) and (1.9)}, \\ \text{case II} : (-\omega, \omega)^\top & \text{for (1.10)}, \\ \text{case III} : (-(u_2 + \omega), (u_1 + \omega))^\top & \text{for (1.11)}. \end{cases}$$

To simplify the numerical implementation, we shall assume that the square domain Ω is uniformly partitioned into a set of $1/h^2$ square subdomains Ω_i^h with side-length h. For simplicity, we also set $d_h(0,0) = 0$, instead of $(d_h, 1)_0 = 0$, in the approximations to ensure the uniqueness of solution. Piecewise bilinear finite elements are used to approximate all components of the exact solution. In other words, we should have the error estimates (4.16) with $r = 1$. Notice that the first assumption in (3.2) is always fulfilled because $\nabla \cdot \boldsymbol{\alpha} = 0$ in Ω. The second assumption in (3.2) is satisfied if the Reynolds number $\lambda \leq 10/4$.

Now we consider case I, case II, and case III all with various Reynolds numbers: $\lambda = 1, 10, 100, 1000$. A double precision conjugate gradient solver is applied to solve the linear system associated with the $[L^2, L^2, L^2]$ least-squares finite element scheme (4.10). The numerical results are collected in Figure 1 – Figure 12. The asymptotic rates of convergence are also indicated in each figure. We estimate the asymptotic rate of convergence for the approximations in the following intuitive way: for any two consecutive sets of data with respect to the mesh sizes $h_1 = (1/64) > h_2 = (1/128)$,

$$(5.2)\qquad \text{asymptotic rate of convergence } \theta \approx \ln\left(\frac{\|e_1\|_*}{\|e_2\|_*}\right) / \ln 2,$$

where $\|e_i\|_*$ denotes the error in the $\|\cdot\|_*$ norm with respect to the mesh size h_i for $i = 1, 2$.

Numerical results in Figure 1 – Figure 12 confirm the theoretical error estimates (4.16) with $r = 1$ for velocity. That is, the $[L^2, L^2, L^2]$ least-squares finite element approximations to the velocity field are optimal in the H^1 norm. However, numerical evidences also show that our theoretical estimates (4.16) for vorticity and pressure (or total pressure) appears to be non-sharp because, in some cases, both the L^2 rates are greater than one as $h \to 0$. Furthermore, an examination of the numerical results shows that the asymptotic convergence rate for pressure (or total pressure) in the H^1 norm seems to be optimal, although we have not provided the H^1 error analysis for pressure (or total pressure).

We conclude this section with the following remark. Note that the positive constants C_1 and C_2 in the *a priori* estimates (3.1) and (3.3) obviously depend on the parameters k and λ and, as mentioned in Section 1, k is supposedly equal to $1/\Delta t$. Hence, it is possible that the error may be large if the Reynolds number λ is large or the time step Δt is small. However, numerical results reported in this section identify that the accuracy does not degrade and the error estimates still hold for larger λ.

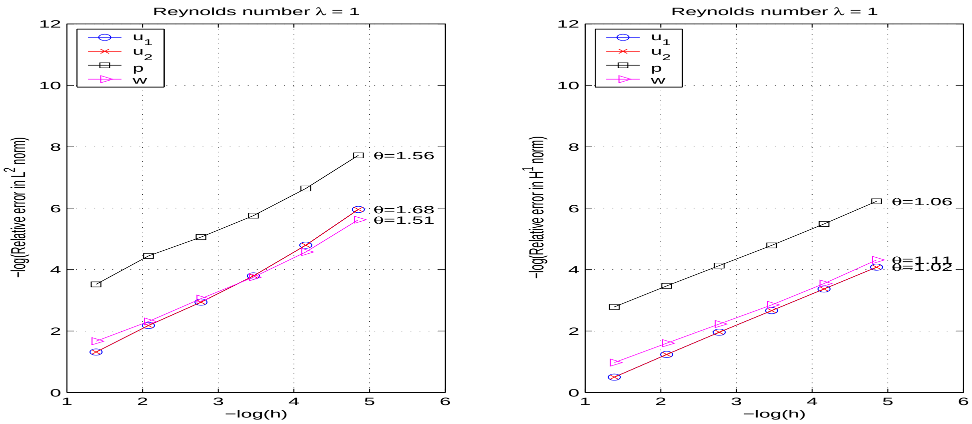

Figure 1. Numerical results of case I with Reynolds number $\lambda = 1$

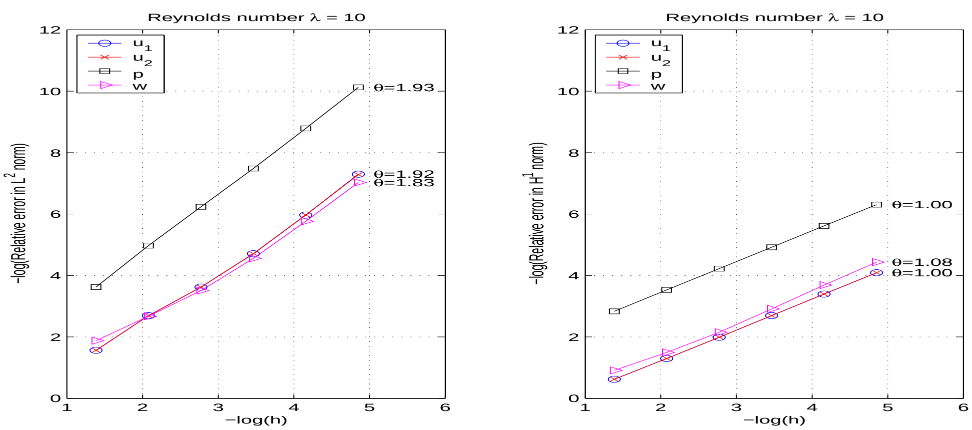

Figure 2. Numerical results of case I with Reynolds number $\lambda = 10$

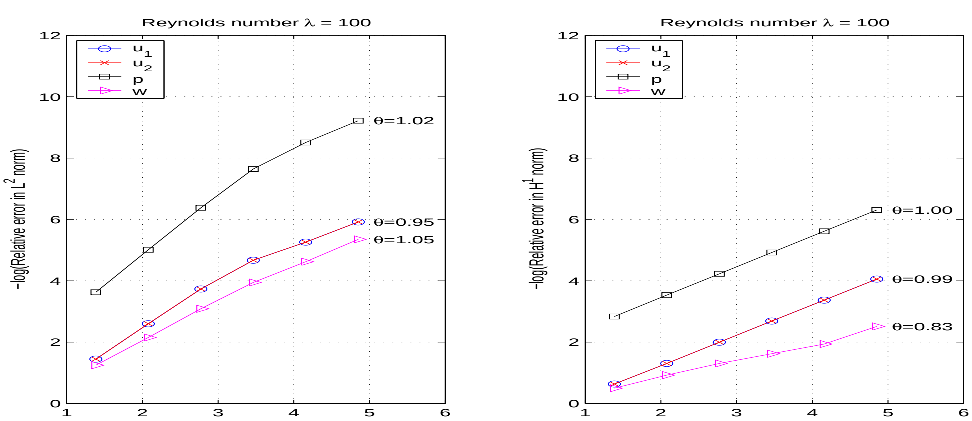

Figure 3. Numerical results of case I with Reynolds number $\lambda = 100$

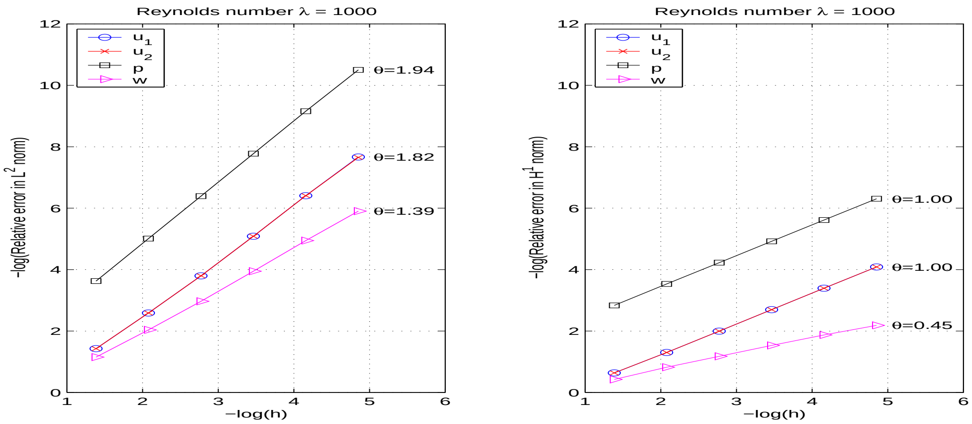

Figure 4. Numerical results of case I with Reynolds number $\lambda = 1000$

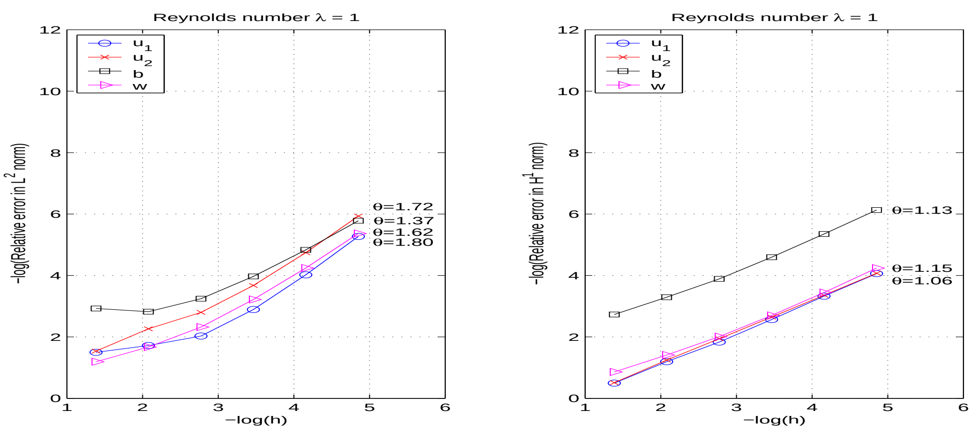

Figure 5. Numerical results of case II with Reynolds number $\lambda = 1$

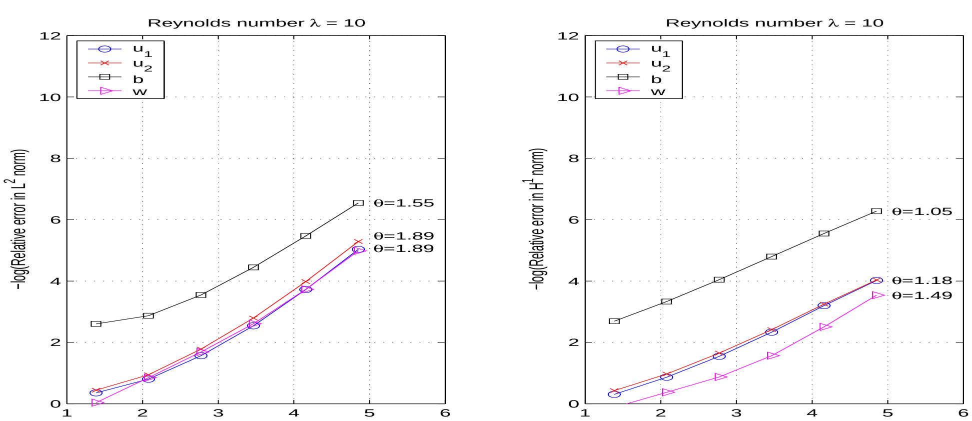

Figure 6. Numerical results of case II with Reynolds number $\lambda = 10$

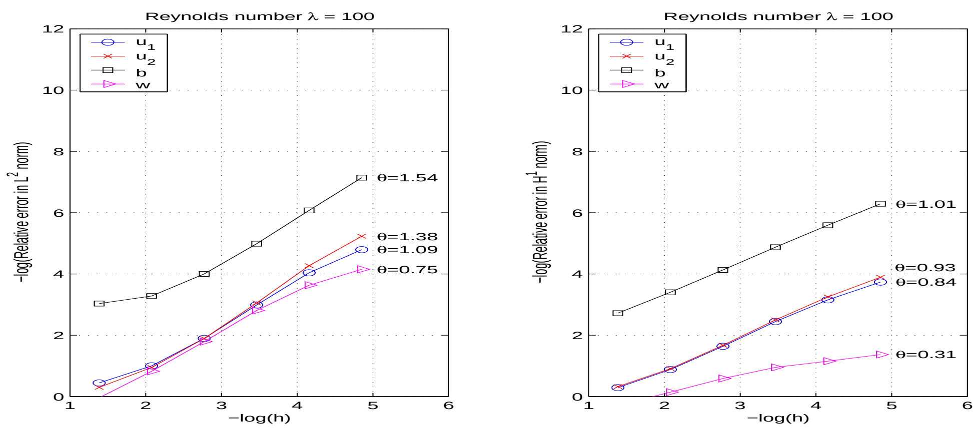

Figure 7. Numerical results of case II with Reynolds number $\lambda = 100$

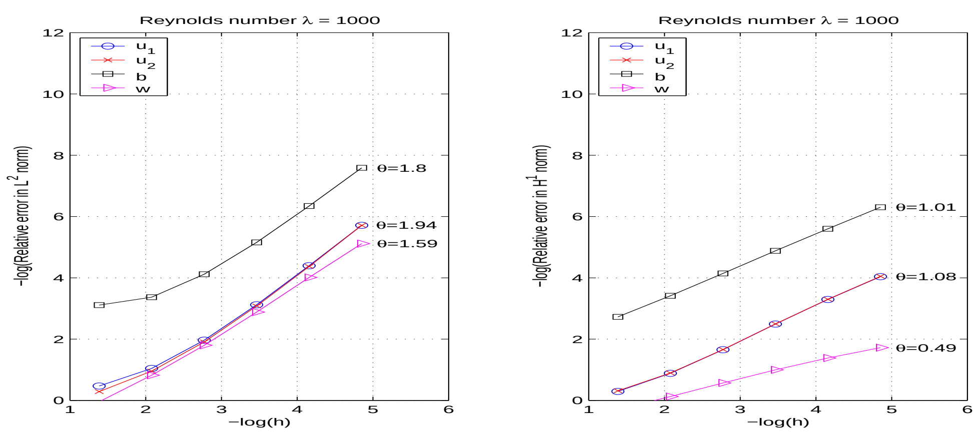

Figure 8. Numerical results of case II with Reynolds number $\lambda = 1000$

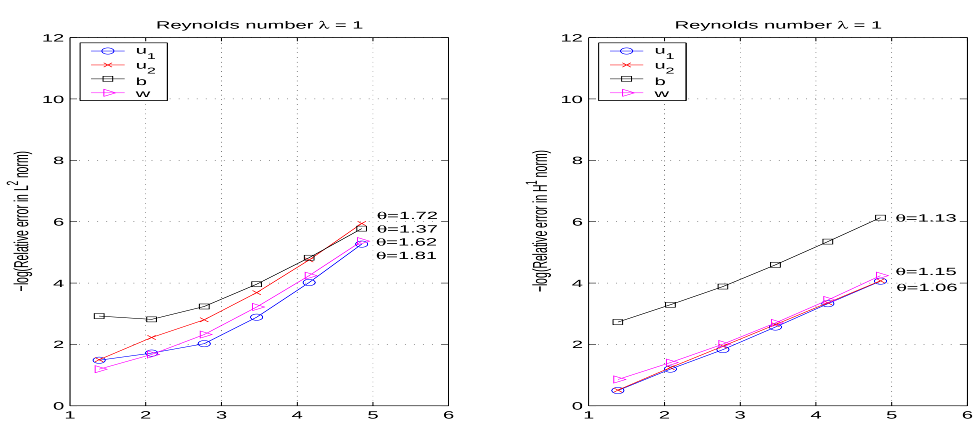

Figure 9. Numerical results of case III with Reynolds number $\lambda = 1$

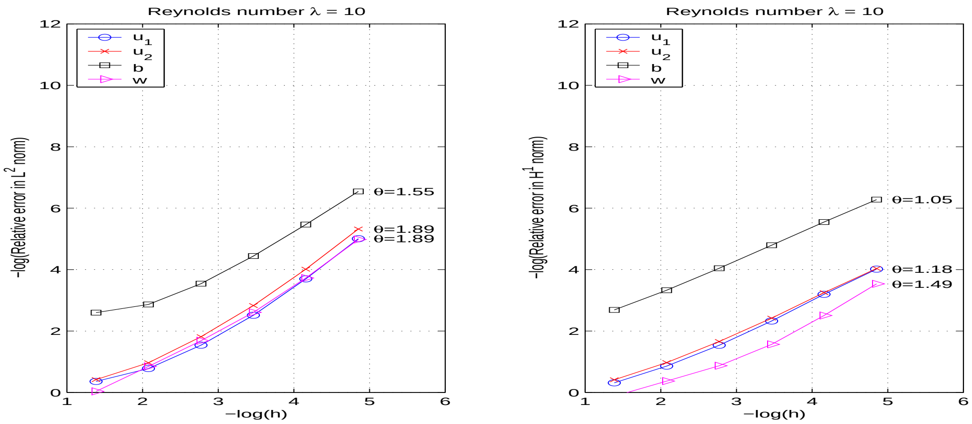

Figure 10. Numerical results of case III with Reynolds number $\lambda = 10$

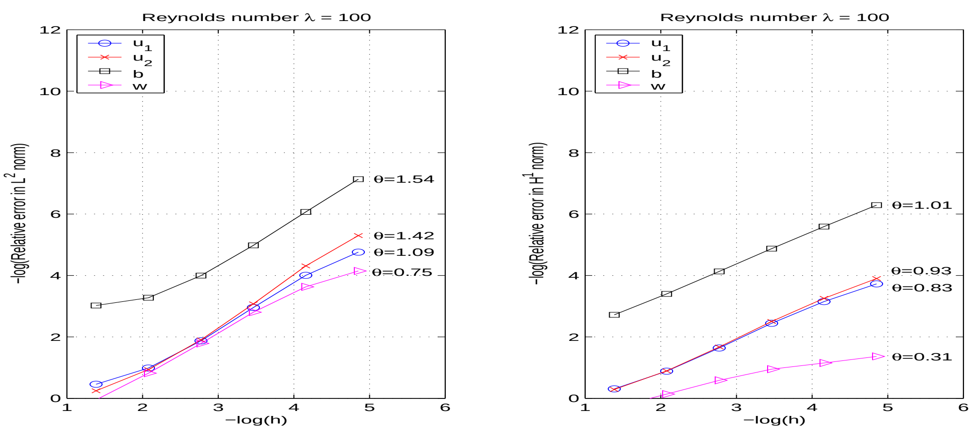

Figure 11. Numerical results of case III with Reynolds number $\lambda = 100$

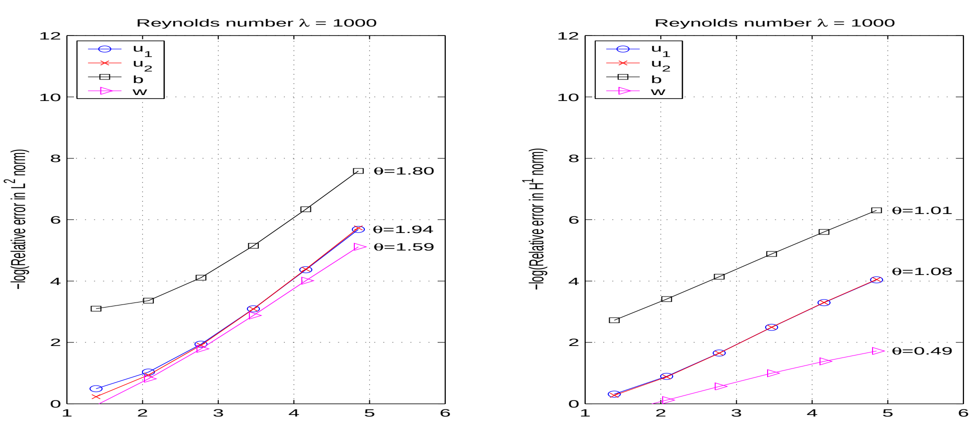

Figure 12. Numerical results of case III with Reynolds number $\lambda = 1000$

6. Semi-discrete least-squares finite elements for the time-dependent incompressible Navier-Stokes problem

In this section, we study numerically the semi-discrete $[L^2, L^2, L^2]$ least-squares finite element method for the time-dependent incompressible Naiver-Stokes problem in the velocity-vorticity-pressure formulation (1.4) and velocity-vorticity-total pressure formulation (1.5). We will apply the $[L^2, L^2, L^2]$ least-squares scheme to problems (1.8) – (1.11) at each time step.

We take the spatial domain $\Omega = (0,1)^2$, the time interval $[0,T] = [0,5]$, and then construct a problem with the following smooth exact solution:

$$u_1^{\mathrm{NS}}(x,y,t) = (t+5)u_1(x,y), \quad u_2^{\mathrm{NS}}(x,y,t) = (t+5)u_2(x,y),$$
$$\omega^{\mathrm{NS}}(x,y,t) = (t+5)\omega(x,y), \quad d^{\mathrm{NS}}(x,y,t) = (t+5)d(x,y),$$

where u_1, u_2, ω, and d are given in Section 5. Substituting the above exact solution into (1.4) and (1.5), we have the right-hand side functions $\mathbf{f}$ and initial velocities $\mathbf{u}_0$. We choose the time step $\Delta t = 0.01$ (i.e., $k = 100$), and partition the unit domain Ω into 64×64 elements (i.e., $h = 1/64$). For simplicity, we set $d_h^{\mathrm{NS}}(0,0,t) = 0$ instead of $\int_\Omega d_h^{\mathrm{NS}}\, d\Omega = 0$ in the approximations to ensure the uniqueness of solution for all time $t \in [0,T]$. We still adopt piecewise bilinear finite elements for all unknown functions in our numerical simulation.

Numerical results for $\lambda = 100$ and $\lambda = 1000$ at time $T = 5$ are collected in Table 1 – Table 4, that demonstrate the accuracy of the semi-discrete $[L^2, L^2, L^2]$ least-squares approach for the time-dependent incompressible Navier-Stokes problem.

Table 1: Numerical results of scheme (1.8) at time $T = 5$

λ	Relative Error	L^2 norm	H^1 norm
100	u_{1h}^{NS}	0.002648484416	0.033654821309
	u_{2h}^{NS}	0.002648647931	0.033654883599
	ω_h^{NS}	0.003681394714	0.041579547550
	p_h^{NS}	0.000104675230	0.003645812130
1000	u_{1h}^{NS}	0.004267549310	0.036353102020
	u_{2h}^{NS}	0.004268892416	0.036353631679
	ω_h^{NS}	0.017193424058	0.218767286634
	p_h^{NS}	0.000107672790	0.003645804555

Table 2: Numerical results of scheme (1.9) at time $T = 5$

λ	Relative Error	L^2 norm	H^1 norm
100	u_{1h}^{NS}	0.002606361771	0.033646406778
	u_{2h}^{NS}	0.002606550618	0.033646490678
	ω_h^{NS}	0.003637785983	0.041567667579
	p_h^{NS}	0.000104063726	0.003645810869
1000	u_{1h}^{NS}	0.004239296048	0.036354432214
	u_{2h}^{NS}	0.004240933238	0.036355042784
	ω_h^{NS}	0.017200648572	0.218816124439
	p_h^{NS}	0.000107019540	0.003645803116

Table 3: Numerical results of scheme (1.10) at time $T = 5$

λ	Relative Error	L^2 norm	H^1 norm
100	$u_{1h}^{\rm NS}$	0.002675622356	0.033661380469
	$u_{2h}^{\rm NS}$	0.002675842040	0.033661446217
	$\omega_h^{\rm NS}$	0.003719692262	0.041564040874
	$b_h^{\rm NS}$	0.000106233856	0.003645814296
1000	$u_{1h}^{\rm NS}$	0.004303768802	0.036328001517
	$u_{2h}^{\rm NS}$	0.004305491772	0.036328606902
	$\omega_h^{\rm NS}$	0.017131995428	0.218566078025
	$b_h^{\rm NS}$	0.000106528886	0.003645809213

Table 4: Numerical results of scheme (1.11) at time $T = 5$

λ	Relative Error	L^2 norm	H^1 norm
100	$u_{1h}^{\rm NS}$	0.002676540675	0.033661364458
	$u_{2h}^{\rm NS}$	0.002676821554	0.033661446347
	$\omega_h^{\rm NS}$	0.003720545716	0.041563365351
	$b_h^{\rm NS}$	0.000105948751	0.003645813358
1000	$u_{1h}^{\rm NS}$	0.004324244094	0.036328207301
	$u_{2h}^{\rm NS}$	0.004325841280	0.036328686277
	$\omega_h^{\rm NS}$	0.017132346049	0.218527829299
	$b_h^{\rm NS}$	0.000106410433	0.003645808185

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