INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 4, Number 2, Pages 280–290

STABILITY-PRESERVING FINITE-DIFFERENCE METHODS FOR GENERAL MULTI-DIMENSIONAL AUTONOMOUS DYNAMICAL SYSTEMS

DOBROMIR T. DIMITROV AND HRISTO V. KOJOUHAROV

(Communicated by Pavel Bochev)

Abstract. General multi-dimensional autonomous dynamical systems and their numerical discretizations are considered. Nonstandard stability-preserving finite-difference schemes based on the θ-methods and the second-order Runge-Kutta methods are designed and analyzed. Their elementary stability is established theoretically and is also supported by a set of numerical examples.

Key Words. finite-difference, nonstandard scheme, elementary stability, dynamical systems.

1. Introduction

The increasing study of realistic mathematical models in biology, ecology and medicine is a reflection of their use in helping to understand the dynamic processes involved in such areas as predator-prey and competition interactions, infectious diseases control and multi-species marine societies. Mathematical models usually consist of systems of differential equations that represent the rates of change of the size of each interacting component. In most of the interactions modeled all rates of change are assumed to be time independent, which makes the corresponding systems autonomous.

Numerical methods that approximate continuous dynamical systems are expected to be consistent with the original differential system, to be zero-stable and convergent. Nonstandard finite difference techniques, developed by Mickens [12, 14], have laid the foundation for designing methods that preserve the physical properties, especially the stability properties of equilibria, of the approximated differential system. Anguelov and Lubuma [1] have used Mickens' techniques to design nonstandard versions of the explicit and implicit Euler and the second order Runge-Kutta methods, under the limiting condition that all eigenvalues of the Jacobian at each equilibrium of the original differential system (for simplicity, we name those eigenvalues "equilibria"-eigenvalues) are single and real. However, a wide range of mathematical models do not satisfy the aforementioned limitation. Among them are most of the non-conservative predator-prey systems such as the Lotka-Volterra models [9, 17, 13], most models with Michaelis-Menten functional responses [11], the ratio-dependent models [8, 5], some SI, SIS and SIR epidemiology models [7, 15, 4] and most phytoplankton-nutrient systems [16, 6]. Therefore developing stability-preserving numerical methods for general autonomous dynamical systems

Received by the editors March 1, 2005 and revised May 20, 2005.
2000 *Mathematics Subject Classification.* 37M05, 39A11, 65L12, 65L20.

that have not only single and real but also multiple real and complex "equilibria"-eigenvalues is of critical importance. Dimitrov and Kojouharov [3] have designed a variety of such nonstandard finite-difference schemes for general two-dimensional systems, based on the explicit Euler, the implicit Euler and the second-order Runge-Kutta methods. Lubuma and Roux [10] have constructed nonstandard numerical schemes, based on the θ-methods, that preserve the stability of equilibria for multi-dimensional systems having all of their "equilibria"-eigenvalues in a subregion of the complex plane. In this paper we extend the above nonstandard θ-methods and also develop a new class of stability-preserving nonstandard finite-difference schemes, based on the second-order Runge-Kutta methods, for multi-dimensional autonomous dynamical systems with arbitrary complex "equilibria"-eigenvalues. The proposed new elementary stable nonstandard (ESN) numerical schemes work very well with conservative as well as with non-conservative dynamical systems.

The paper is organized as follows. In Section 2 we provide some definitions and preliminary results. We state our main results in Section 3 and prove them in Section 4. In the last two sections we illustrate our theoretical results by numerical examples and outline some future research directions.

2. Definitions and Preliminaries

A general n-dimensional autonomous system has the following form:

$$\frac{dy}{dt} = f(y); \quad y(t_0) = y_0, \tag{1}$$

where $y = [y^1, y^2, \ldots, y^n]^T : [t_0, T) \to \mathbb{R}^n$, the function $f = [f^1, f^2, \ldots, f^n]^T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is differentiable and $y_0 \in \mathbb{R}^n$. The equilibrium points of System (1) are defined as the solutions of $f(y) = 0$.

Definition 1. *Let y^* be an equilibrium of System (1), $J(y^*)$ be the Jacobian of System (1) at y^* and $\sigma(J(y^*))$ denotes the spectrum of $J(y^*)$. An equilibrium y^* of System (1) is called linearly stable if $Re(\lambda) < 0$ for all $\lambda \in \sigma(J(y^*))$ and linearly unstable if $Re(\lambda) > 0$ for at least one $\lambda \in \sigma(J(y^*))$.*

A one-step numerical scheme with a step size h, that approximates the solution $y(t_k)$ of System (1) can be written in the form:

$$D_h(y_k) = F_h(f; y_k), \tag{2}$$

where $D_h(y_k) \approx \frac{dy}{dt}$, $F_h(f; y_k) \approx f(y)$ and $t_k = t_0 + kh$.

Definition 2. *Let y^* be a fixed point of the scheme (2) and the equation of the perturbed solution $y_k = y^* + \epsilon_k$ be linearly approximated by*

$$D_h \epsilon_k = J_h \epsilon_k, \tag{3}$$

where the right-hand side is the linear term in ϵ_k of the Taylor expansion of $F_h(f; y^* + \epsilon_k)$ around y^*. The fixed point y^* is called stable if $\|\epsilon_k\| \to 0$ as $k \to \infty$, and unstable otherwise, where ϵ_k is the solution of Equation (3).

Definition 3. *The finite-difference method (2) is called elementary stable if, for any value of the step size h, the linear stability of each equilibrium y^* of System (1) is the same as the stability of y^* as a fixed point of the discrete method (2).*

We introduce the nonstandard one-step finite-difference method based on a definition given by Anguelov and Lubuma [1].

Definition 4. *The one-step method (2) is called a nonstandard finite-difference method if at least one of the following conditions is satisfied:*

- $D_h(y_k) = \dfrac{y_{k+1} - y_k}{\varphi(h)}$, *where* $\varphi(h) = h + \mathcal{O}(h^2)$ *is a nonnegative function;*
- $F_h(f; y_k) = g(y_k, y_{k+1}, h)$, *where* $g(y_k, y_{k+1}, h)$ *is a nonlocal approximation of the right-hand side function* $f(y)$.

3. Main Results

Assume that System (1) has a finite number of equilibria and $Re(\lambda) \neq 0$, for $\lambda \in \Omega$, where $\Omega = \bigcup_{y^* \in \Gamma} \sigma(J(y^*))$ and Γ represents the set of all equilibria of System (1). The nonstandard stability-preserving finite-difference schemes for solving multi-dimensional autonomous dynamical systems are given in the following theorems:

Theorem 1. *Let* ϕ *be a real-valued function on* $\mathbb{R}$ *that satisfies the property:*

$$\phi(h) = h + O(h^2) \ \textit{and} \ 0 < \phi(h) < 1 \ \textit{for all} \ h > 0. \tag{4}$$

Let $q > \max_{\Omega} \left(\dfrac{|2\theta - 1||\lambda|^2}{2|Re(\lambda)|} \right)$, *where* $0 \leq \theta \leq 1, \theta \neq \dfrac{1}{2}$. *Then the following numerical scheme, based on the standard* θ*-method, represents an elementary stable nonstandard (ESN) method:*

$$\frac{y_{k+1} - y_k}{\phi(hq)/q} = \theta f(y_{k+1}) + (1 - \theta) f(y_k). \tag{5}$$

Remark 1 *In the case of* $\theta = \dfrac{1}{2}$ *the standard* θ*-method:*

$$\frac{y_{k+1} - y_k}{h} = \frac{f(y_{k+1}) + f(y_k)}{2},$$

is elementary stable.

The results of Theorem 1, applied to the forward and backward Euler methods as special cases of the θ-method, are given in the following corollary:

Corollary 1. *Let* ϕ *be a real-valued function on* $\mathbb{R}$ *that satisfies the property (4). Let* $q > \max_{\Omega} \left(\dfrac{|\lambda|^2}{2|Re(\lambda)|} \right)$. *Then the following numerical schemes are ESN methods:*

(a) *the explicit Euler ENS method given by*

$$\frac{y_{k+1} - y_k}{\phi(hq)/q} = f(y_k); \ \textit{and} \tag{6}$$

(b) *the implicit Euler ESN method given by*

$$\frac{y_{k+1} - y_k}{\phi(hq)/q} = f(y_{k+1}). \tag{7}$$

The ESN version of the second-order Runge-Kutta method is given in the following theorem:

Theorem 2. *Let* ϕ *be a real-valued function on* $\mathbb{R}$ *that satisfies the property (4). Let* $q > \max_{\Omega} \left(\dfrac{|\lambda|^2}{2|Re(\lambda)|} \right)$. *Then the following numerical scheme, based on the standard second-order Runge-Kutta method, represents an elementary stable nonstandard (ESN) method:*

$$\frac{y_{k+1} - y_k}{\phi(hq)/q} = \frac{f(y_k) + f(y_k + (\phi(hq)/q) f(y_k))}{2}. \tag{8}$$

4. Proofs of the Main Results

In this section we prove that the designed new ESN methods, based on the standard θ-methods and the second-order Runge-Kutta methods, preserve the stability of all equilibria of System (1).

Proof. **(Theorem 1)** Let us denote $h_1 = \varphi(h) = \frac{\phi(hq)}{q}$. Since $0 < h_1 < \frac{1}{q}$ then

$$h_1 < \frac{2|Re(\lambda)|}{|2\theta - 1||\lambda|^2} \tag{9}$$

for all $\lambda \in \Omega$. Let y^* be an equilibrium of System (1) and $J = J(y^*)$ denote the Jacobian of System (1) at the equilibrium y^*. Equation (3) for the perturbed solution of Scheme (5) has the form

$$\frac{\epsilon_{k+1} - \epsilon_k}{h_1} = J(\theta\epsilon_{k+1} + (1-\theta)\epsilon_k). \tag{10}$$

If Λ is the Jordan form of J, then $J = S^{-1}\Lambda S$, where S is a non-singular complex $n \times n$-matrix. In general, Λ has the following bidiagonal form:

$$\begin{pmatrix} \lambda_1 & \alpha_1 & & & \\ & \lambda_2 & \alpha_2 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{n-1} & \alpha_{n-1} \\ & & & & \lambda_n \end{pmatrix},$$

where $\lambda_i \in \sigma(J)$ and $\alpha_i \in \{0, 1\}, i = 1 \dots n$. After the change of variables $\epsilon_k = S\delta_k$, Equation (10) becomes

$$\frac{\delta_{k+1} - \delta_k}{h_1} = \Lambda(\theta\delta_{k+1} + (1-\theta)\delta_k).$$

The above equation is equivalent to

$$\delta_{k+1} = (I - h_1\theta\Lambda)^{-1}(I + h_1(1-\theta)\Lambda)\delta_k,$$

where I denotes the identity $n \times n$-matrix. Since the matrix Λ is upper triangular, then the matrix $(I - h_1\theta\Lambda)^{-1}(I + h_1(1-\theta)\Lambda)$ is upper triangular also and its eigenvalues are given by $\mu_i = \frac{1 + h_1(1-\theta)\lambda_i}{1 - h_1\theta\lambda_i}$, where $\lambda_i \in \sigma(J), i = 1 \dots n$. Since $\|\epsilon_k\| \to 0$ is equivalent to $\|\delta_k\| \to 0$, then y^* is a stable fixed point of (5) if $|\mu_i| < 1$ for all $i = 1 \dots n$ and an unstable fixed point if $|\mu_i| > 1$ for at least one $i \in \{1 \dots n\}$. For $\lambda \in \mathbb{C}$ the following is true:

$$\begin{aligned} \left|\frac{1 + h_1(1-\theta)\lambda}{1 - h_1\theta\lambda}\right| < 1 \quad &\Longleftrightarrow \quad |1 + h_1(1-\theta)\lambda| < |1 - h_1\theta\lambda| \\ &\Longleftrightarrow \quad h_1(1-2\theta)|\lambda|^2 < -2Re\lambda. \end{aligned}$$

Therefore, y^* is a stable fixed point of Scheme (5) if the inequality

$$h_1(1-2\theta) < \frac{-2Re\lambda}{|\lambda|^2} \tag{11}$$

holds for all $\lambda \in \sigma(J)$ and y^* is an unstable fixed point if the opposite inequality

$$h_1(1-2\theta) > \frac{-2Re\lambda}{|\lambda|^2} \tag{12}$$

holds for at least one $\lambda \in \sigma(J)$.

Let us consider the following two cases for the parameter θ:

(1) $\theta < \frac{1}{2}$. If y^* is a stable equilibrium of System (1) and $\lambda \in \sigma(J)$ then $Re\lambda < 0$ and the inequality (11) is satisfied because of Inequality (9). Therefore y^* is a stable fixed point of Scheme (5). If y^* is unstable then $Re\lambda > 0$ for some $\lambda \in \sigma(J)$. Thus the inequality (12) is satisfied for that λ and therefore y^* is an unstable fixed point of (5).

(2) $\theta > \frac{1}{2}$. If y^* is a stable equilibrium of System (1) and $\lambda \in \sigma(J)$ then $Re\lambda < 0$. The inequality (11) is satisfied, because the left-hand side is negative while the right-hand side is positive. Therefore y^* is a stable fixed point of Scheme (5). If y^* is unstable then $Re\lambda > 0$ for some $\lambda \in \sigma(J)$. Thus the inequality (12) is satisfied for that λ because of Inequality (9) and therefore y^* is an unstable fixed point of (5). □

Proof. **(Corollary 1)** The forward and backward Euler schemes are special cases of the θ-method with $\theta = 0$ and $\theta = 1$, respectively. In both cases $|2\theta - 1| = 1$ and the proof of the corollary follows directly from the results of Theorem 1. □

Proof. **(Theorem 2)** Let us denote $h_1 = \varphi(h) = \dfrac{\phi(hq)}{q}$. Since $0 < h_1 < \dfrac{1}{q}$ then

$$h_1 < \frac{2|Re\lambda|}{|\lambda|^2} \tag{13}$$

for all $\lambda \in \Omega$. Let y^* be an equilibrium of System (1) and $J = J(y^*)$ denotes the Jacobian of System (1) at the equilibrium y^*. For the method (8) the linearized equation of the perturbed solution is given by:

$$\epsilon_{k+1} = \left(I + h_1 J + \frac{h_1^2 J^2}{2}\right)\epsilon_k, \tag{14}$$

After the change of variables $\epsilon_k = S\delta_k$, Equation (14) becomes $\delta_{k+1} = (I + h_1\Lambda + \frac{h_1^2\Lambda^2}{2})\delta_k$. The eigenvalues of $I + h_1\Lambda + \dfrac{h_1^2\Lambda^2}{2}$ are given by $\mu_i = 1 + h_1\lambda_i + \dfrac{h_1^2\lambda_i^2}{2}$, where $\lambda_i \in \sigma(J), i = 1\ldots n$. For $\lambda \in \mathbb{C}$ the following is true:

$$\begin{aligned}\left|1 + h_1\lambda + \tfrac{h_1^2\lambda^2}{2}\right| < 1 \quad &\Longleftrightarrow \quad (1 + h_1 Re\lambda + \tfrac{h_1^2}{2}((Re\lambda)^2 - (Im\lambda)^2))^2 \\ &\qquad\quad + h_1^2(Im\lambda)^2(1 + (Re\lambda)h_1)^2 < 1 \\ &\Longleftrightarrow \quad 2Re\lambda + 2(Re\lambda)^2 h_1 + Re\lambda|\lambda|^2 h_1^2 + \tfrac{|\lambda|^4}{4}h_1^3 < 0.\end{aligned}$$

Let us denote $\alpha(t) = 2Re\lambda + 2(Re\lambda)^2 t + Re\lambda|\lambda|^2 t^2 + \dfrac{|\lambda|^4}{4}t^3$. Therefore, y^* is a stable fixed point of (8) if the inequality $\alpha(h_1) < 0$ is satisfied for all $\lambda \in \sigma(J)$ and y^* is unstable if $\alpha(h_1) > 0$ for at least one $\lambda \in \sigma(J)$. The derivative $\alpha'(t) = 2(Re\lambda)^2 + 2Re\lambda|\lambda|^2 t + \dfrac{3|\lambda|^4}{4}t^2 = \dfrac{(Re\lambda)^2}{4}\beta\left(\dfrac{|\lambda|^2}{Re\lambda}t\right)$, where $\beta(t) = 8 + 8t + 3t^2$. Since $\beta(t)$ is positive for all t, the derivative $\alpha'(t)$ is positive for all t and $\alpha(t)$ is an increasing function. Thus the inequality (13) implies that $\alpha(0) < \alpha(h_1) \le \alpha\left(\dfrac{2|Re\lambda|}{|\lambda|^2}\right)$.

If y^* is a stable equilibrium of System (1) and $\lambda \in \sigma(J)$ then $Re\lambda < 0$ and $\alpha(h_1) \le \alpha\left(\dfrac{-2Re\lambda}{|\lambda|^2}\right) = \dfrac{2Re\lambda(Im\lambda)^2}{|\lambda|^2} < 0$. Therefore, y^* is a stable fixed point of (8). If y^* is an unstable equilibrium of System (1), then there exists $\lambda \in \sigma(J)$ with $Re\lambda > 0$. Since $\alpha(h_1) > \alpha(0) = 2Re\lambda > 0$, then y^* is an unstable fixed point of (8). □

Remark 2 *The definition of the scheme (5) guarantees that all of the scheme's fixed points are equilibria of System (1) and vice versa.*

Remark 3 *There exists a variety of functions* ϕ *that satisfy condition (4), e.g.,* $\phi(h) = 1 - e^{-h}$, *i.e.,* $\varphi(h) = \phi(hq)/q = (1 - e^{-hq})/q$.

5. Numerical Examples

To illustrate the efficiency of the designed new ESN methods, we first consider the following predator-prey system with Beddington-DeAngelis functional response [2]:

$$\frac{dx}{dt} = x - \frac{Axy}{1+x+y},$$
$$\frac{dy}{dt} = \frac{Exy}{1+x+y} - Dy, \tag{15}$$

where x and y represent the prey and predator population sizes, respectively, and the values of the constants are $A = 6.0$, $D = 5.0$ and $E = 7.5$.

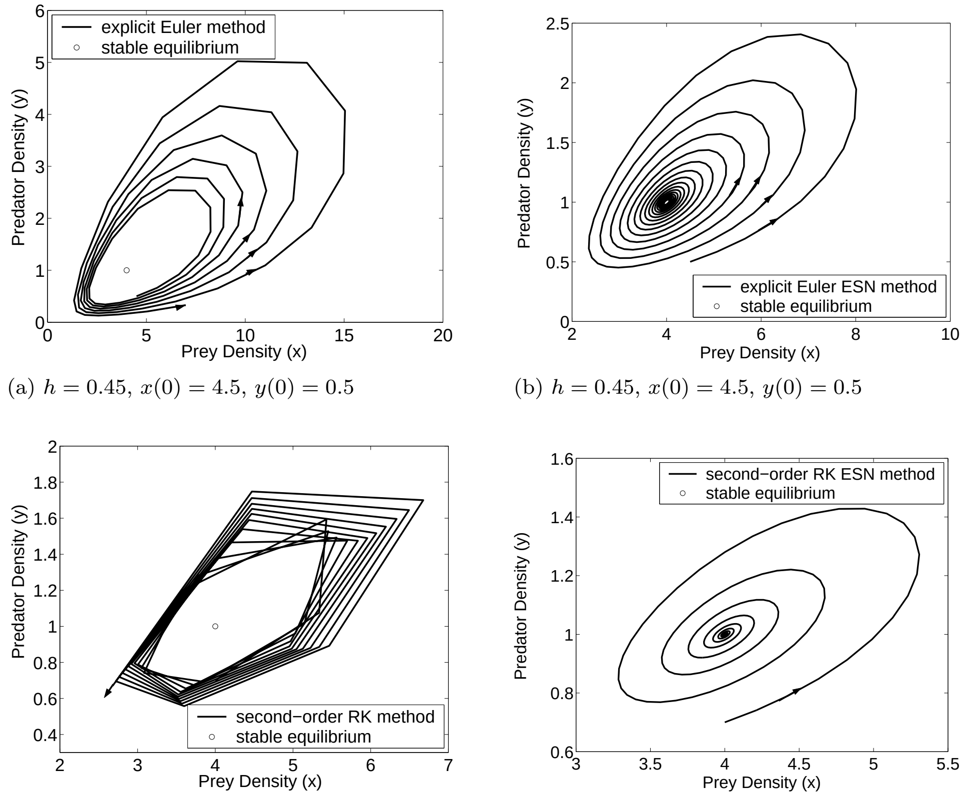
Mathematical analysis of System (15) shows that there exist two equilibria $(0,0)$ and $\left(\frac{AD}{AE-E-AD}, \frac{E}{AE-E-AD}\right) = (4,1)$, with the equilibrium $(4,1)$ being globally stable in the interior of the first quadrant [2]. The eigenvalues of $J(0,0)$ are given by $\lambda_1 = 1$ and $\lambda_2 = 5$, and the eigenvalues of $J(4,1)$ are given by $\lambda_{3,4} = -\frac{1}{12} \pm i\frac{\sqrt{119}}{12}$. Numerical approximations of the solution of System (15) with initial values $x(0) = 4.5$ and $y(0) = 0.5$ and step-sizes $h = 0.45$ and $h = 1.18$ using the θ-methods and the second-order Runge-Kutta methods, respectively, (see Fig.1 (a)-(d)) support the results of Theorem 1 and Theorem 2. The ESN methods, using a denominator function $\varphi(h) = \phi(hq)/q = (1 - e^{-hq})/q$ with $q = 5.1 > q^*$, where $q^* = 5$ is the threshold value, preserve the equilibrium $(4,1)$, while approximations obtained by the standard methods diverge. The nonstandard θ-method is a stability-preserving method despite of the existence of complex "equilibrium"-eigenvalues, which lie outside of the complex region considered by Lubuma and Roux [10]. We also examine the ESN method using different values of q below the the threshold value $q^* = 5$. The experiments show that the second-order Runge-Kutta ESN method is still elementary stable even for $q = 1$ (see Fig.3(a),(b)). For $q = 0.8$ the method is no longer elementary stable, however it still preserves the correct stability of the equilibria for a wide range of step-sizes, e.g. $h = 2.5$ (Fig.3(c)), compared to the standard second-order Runge-Kutta method which diverges for $h = 1.18$ (Fig.1(c)).

In the second example, we consider the following vaccination model with multiple endemic states [7]:

$$\frac{dS}{dt} = \mu N - \beta SI/N - (\mu + \phi)S + cI + \delta V,$$
$$\frac{dI}{dt} = \beta SI/N - (\mu + c)I, \tag{16}$$
$$\frac{dV}{dt} = \phi S - (\mu + \delta)V,$$

where the constants $\beta = 0.7$, $c = 0.1$, $\mu = 0.8$, $\delta = 0.8$ and $\phi = 0.8$. In the above model the total (constant) population size $N = 100$ is divided into three classes - susceptibles (S), infectives (I) and vaccinated (V) and it is assumed that the vaccine is completely effective in preventing infection.

Mathematical analysis of System (16) shows that the disease free equilibrium $(S^*, I^*, V^*) = \left(\frac{(\mu+\delta)N}{\mu+\delta+\phi}, 0, \frac{\phi N}{\mu+\delta+\phi}\right) = (\frac{200}{3}, 0, \frac{100}{3})$ is globally asymptotically stable



(a) $h = 0.45$, $x(0) = 4.5$, $y(0) = 0.5$ (b) $h = 0.45$, $x(0) = 4.5$, $y(0) = 0.5$

(c) $h = 1.18$, $x(0) = 4.5$, $y(0) = 0.5$ (d) $h = 1.18$, $x(0) = 4.5$, $y(0) = 0.5$

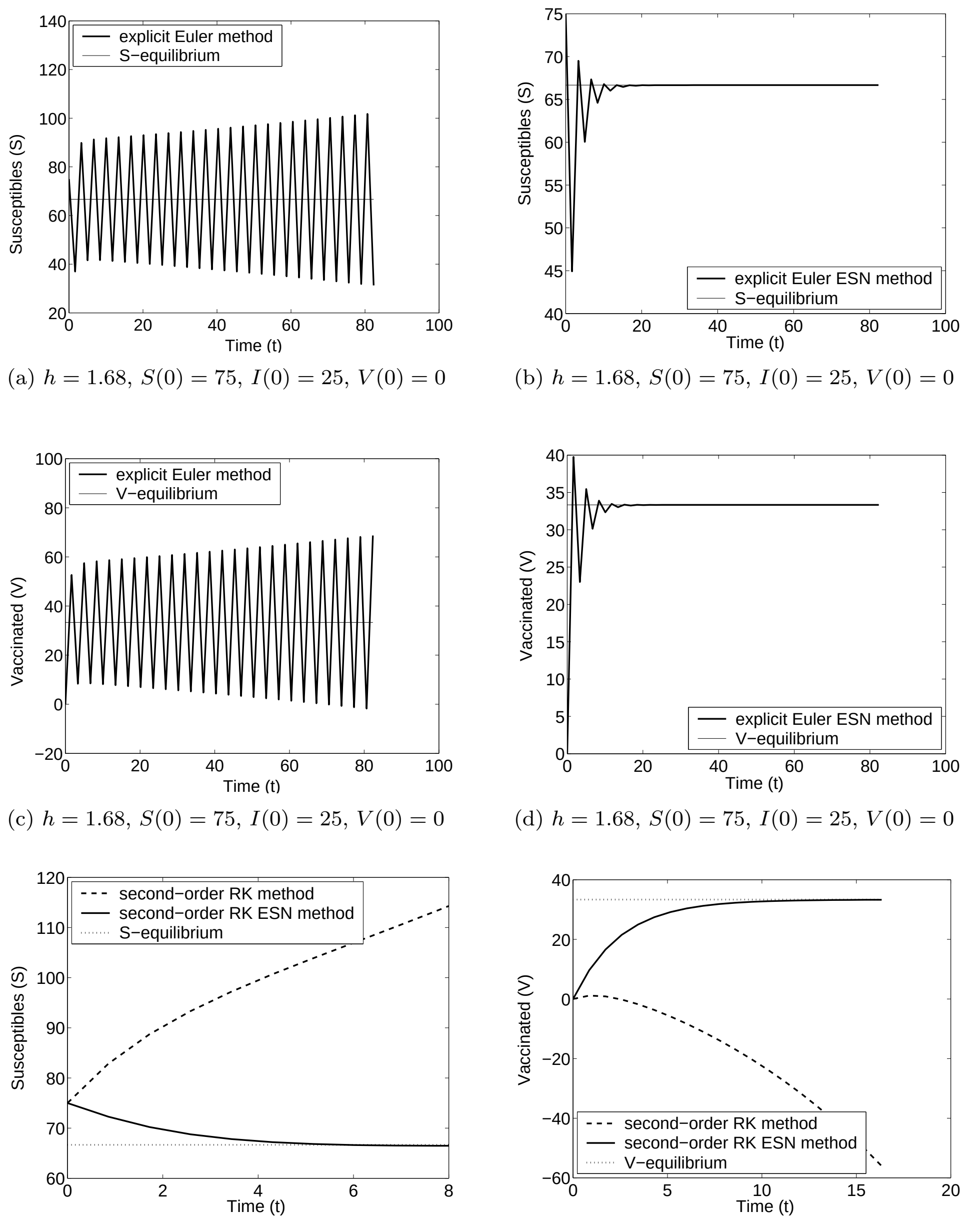
FIGURE 1. Numerical approximations of the solution of Equation (15).

[7]. The eigenvalues of $J(S^*, I^*, V^*)$ are given by $\lambda_1 = -0.8$, $\lambda_2 = -2.4$ and $\lambda_3 = -\frac{13}{30}$. Numerical approximations of the solution of System (16) with initial values $S(0) = 75$, $I(0) = 25$ and $V(0) = 0$ for step-sizes $h = 1.68$ and $h = 0.86$ using the θ-methods and the second-order Runge-Kutta methods, respectively, (see Fig.2 (a)-(f)) support the results of Theorem 1 and Theorem 2. The ESN methods, using a denominator function $\varphi(h) = \phi(hq)/q = (1 - e^{-hq})/q$ with $q = 1.3 > q^*$, where $q^* = 1.2$ is the threshold value, preserve the stability of the equilibrium (S^*, I^*, V^*), while approximations obtained by the standard methods diverge.

In the last sets of simulations (Fig.3(d)-(f)) we investigate the behavior of the new ESN numerical methods when applied to dynamical systems with nonhyperbolic equilibria. Since the threshold value for the parameter q can not be defined as in Theorems 1 and 2, the parameter q is selected in the following way:

$$q > \max_{\Omega_0} \left(\frac{|\lambda|^2}{2|Re(\lambda)|} \right), \qquad (17)$$

where $\Omega_0 = \bigcup_{y^* \in \Gamma_0} \sigma(J(y^*))$ and Γ_0 is the set of all nonhyperbolic equilibria of the dynamical system. First, we analyze the vaccination model (16) with $\beta = 1.35$, $c = 0.1$, $\mu = 0.8$, $\delta = 0.8$ and $\phi = 0.8$. The mathematical analysis shows that the system undergoes the transcritical bifurcation for the above set of parameter values and therefore the existing disease-free equilibrium (S^*, I^*, V^*) is still asymptotically



(a) $h = 1.68$, $S(0) = 75$, $I(0) = 25$, $V(0) = 0$ (b) $h = 1.68$, $S(0) = 75$, $I(0) = 25$, $V(0) = 0$

(c) $h = 1.68$, $S(0) = 75$, $I(0) = 25$, $V(0) = 0$ (d) $h = 1.68$, $S(0) = 75$, $I(0) = 25$, $V(0) = 0$

(e) $h = 0.86$, $S(0) = 75$, $I(0) = 25$, $V(0) = 0$ (f) $h = 0.86$, $S(0) = 75$, $I(0) = 25$, $V(0) = 0$

FIGURE 2. Numerical approximations of the solution of Equation (16).

stable. We compare the approximations of the system by the second-order Runge-Kutta method and second-order Runge-Kutta ESN method for a step-size $h = 0.86$ using $\varphi(h) = \phi(hq)/q = (1 - e^{-hq})/q$ with $q = 1.3 > q^*$, where $q^* = 1.2$ as defined by (17). The simulations show that the ESN method preserves the stability of the nonhyperbolic equilibria (S^*, I^*, V^*), while the solution of the standard method diverges (Fig.3(e),(f)). Second, we analyze the Beddington-DeAngelis system (15) with $A = 7.5$, $D = 5.0$ and $E = 7.5$. All trajectories of the system for this set of parameter values are periodic and the interior equilibrium is stable, but not asymptotically stable. We compare the approximations of the system by the second-order Runge-Kutta method and second-order Runge-Kutta ESN method for a step-size $h = 1.18$ using $\varphi(h) = \phi(hq)/q = (1 - e^{-hq})/q$ with $q = 3 > q^*$, where $q^* = 2.5$ as defined by (17). The solution obtained by the standard numerical method blows up after several time-steps, while the solution of the ESN numerical method expresses the periodic behavior of the exact solution. However, the ESN method does not preserve the exact periodic orbit and presents the trajectory as moving away from the equilibrium (Fig.3(d)).

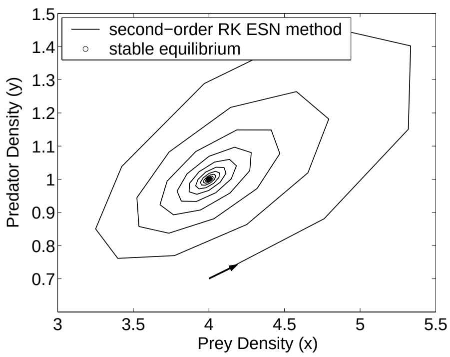
6. Discussion and Conclusions

Stability-preserving finite-difference schemes, based on the standard θ- and the second-order Runge-Kutta methods, were developed and analyzed. The nonstandard (ESN) numerical methods represent generalizations of results obtained earlier by Anguelov and Lubuma [1] and Lubuma and Roux [10] which makes them applicable to solving arbitrary multi-dimensional autonomous dynamical systems.

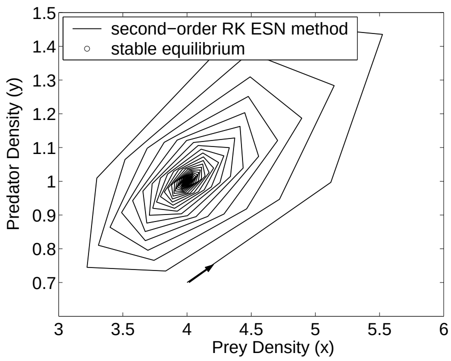
The new ESN numerical methods guarantee that the stability of a given equilibrium E^* of the dynamical system is the same as the stability of E^* as a fixed point of the numerical method for an arbitrary step-size h, when the dynamical system has only hyperbolic equilibria and the denominator parameter q is selected above the threshold value, as defined in Theorems 1 and 2.

We have also investigated how well the ESN numerical methods work when the denominator parameter q is selected below the threshold value q^*. We examined the predator-prey system with Beddington-DeAngelis functional response (15) using the second-order Runge-Kutta ESN method for a variety of different values of q below the threshold value. The experiments showed that the nonstandard method preserves the correct stability of the equilibria for a much wider range of step-sizes h when compared to the corresponding standard second-order Runge-Kutta method. These results are not surprising, given the limiting condition for q in Theorems 1 and 2 is only a sufficient, but not a necessary condition. Therefore, a choice of q below the threshold q^* can produce an elementary stable nonstandard (ESN) method or a nonstandard method with stability properties much better than the corresponding standard method.

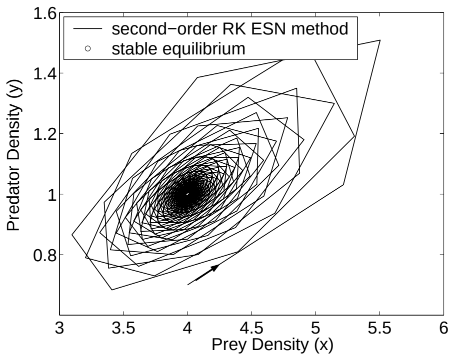
In addition, we have also investigated the behavior of the ESN numerical methods when applied to dynamical systems with nonhyperbolic equilibria. First, we compared the numerical approximations of the vaccination system (16) by the second-order Runge-Kutta method and its ESN version for a set of parameter values when the system undergoes a transcritical bifurcation and therefore the existing disease-free equilibrium is still asymptotically stable. The simulations showed that the nonstandard method preserves the stability of the nonhyperbolic equilibria, while the solution obtained by the standard method diverges. Therefore, in this case the existence of a nonhyperbolic equilibrium did not affect the stability-preserving property of the ESN numerical method. Second, we analyzed the Beddington-DeAngelis system (15) for a set of parameter values when all trajectories of the system are periodic and the interior equilibrium is stable, but not asymptotically stable. We compared again the numerical approximations of the system by the second-order Runge-Kutta method and its ESN version. The solution obtained by the standard numerical method blew up after several steps, while the ESN numerical method expressed the periodic behavior of the exact solution but failed to preserve the stability of the interior equilibrium.



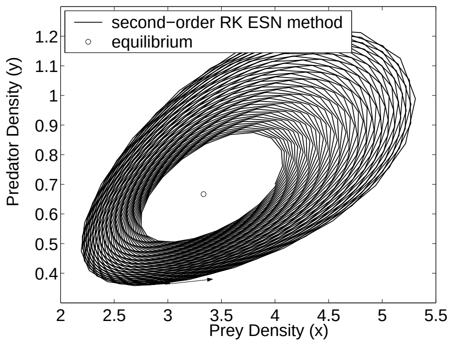
(a) $q = 1$, $h = 1.18$, $x(0) = 4$, $y(0) = 0.7$



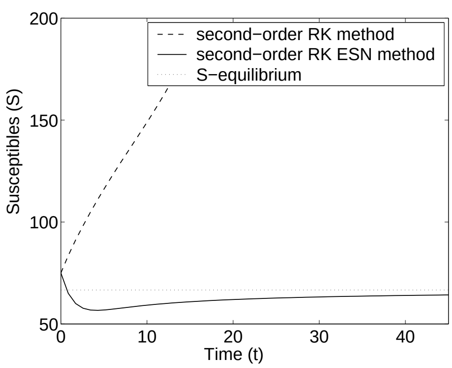
(b) $q = 1$, $h = 500$, $x(0) = 4$, $y(0) = 0.7$



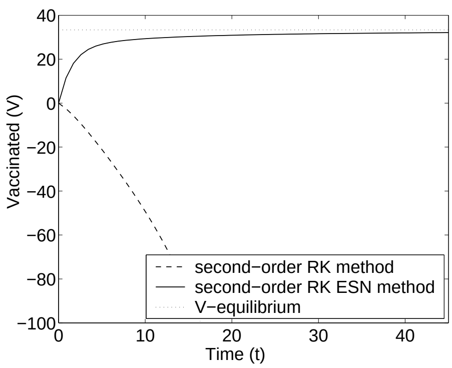
(c) $q = 0.8$, $h = 2.5$, $x(0) = 4$, $y(0) = 0.7$



(d) $q = 3$, $h = 1.18$, $x(0) = 4$, $y(0) = 0.7$



(e) $h = 0.86$, $S(0) = 75$, $I(0) = 25$, $V(0) = 0$



(f) $h = 0.86$, $S(0) = 75$, $I(0) = 25$, $V(0) = 0$

FIGURE 3. Numerical approximations of the solution of Equation (15) (top and middle) and Equation (16) (bottom).

Future research directions include the construction of nonstandard numerical schemes that preserve not only the stability but also most of the other essential qualitative properties of the exact solutions of dynamical systems.

Acknowledgments

The authors thank the anonymous reviewers for their valuable comments and suggestions regarding the presentation of the paper.

References

[1] Anguelov R., Lubuma J.M.-S., Contributions to the mathematics of the nonstandard finite difference method and applications, Numer. Methods Partial Diff. Equations **17:5,** (2001) 518-543.

[2] Dimitrov D.T., Kojouharov H.V., Complete mathematical analysis of predator-prey models with linear prey growth and Beddington-DeAngelis functional response, Appl. Math. Comput. **162:2,** (2005) 523-538.

[3] Dimitrov D.T., Kojouharov H.V., Nonstandard finite-difference schemes for general two-dimensional autonomous dynamical systems, Appl. Math. Lett. **18:7,** (2005) 769-774.

[4] Gumel, A.B., Twizell, E.H., Yu, P. Numerical and bifurcation analysis for a population model of HIV chemotherapy. Math. Comput. Simulation **54:1-3,** (2000) 169-181

[5] Hsu S.-B., Hwang T.-W., Kuang Y., Global analysis of the Michaelis-Menten-type ratio-dependent predator-prey system, J. Math. Biol. **42,** (2001) 489-506.

[6] Jang S.R.-J., Allen L.J.S., A simple food chain with a growth inhibiting nutrient, Appl. Math. Comp. **104,** (1999) 277-298.

[7] Kribs-Zaleta C.M. and Velasco-Hernández J.X., A simple vaccination model with multiple endemic states, Math. Biosci. **164,** (2000) 183-201.

[8] Kuang Y. and Beretta E., Global qualitative analysis of a ratio-dependent predator-prey system, J. Math. Biol. **36,** (1998) 389-406.

[9] Lotka, A. J., *Elements of physical biology* (Williams and Wilkins, Baltimore 1925)

[10] Lubuma, J.M.-S. and Roux A., An improved theta-method for systems of ordinary differential equations, J. Differ. Equations Appl. **9:11,** (2003) 1023-1035.

[11] May, R. M., *Stability and Complexity in Model Ecosystem* (Princeton Univ. Press, Princeton 1974)

[12] Mickens, R. E., *Nonstandard finite difference model of differential equations* (World Scientific, Singapore 1994)

[13] Mickens, R.E., A nonstandard finite-difference scheme for the Lotka-Volterra system, Appl. Numer. Math. **45,** (2003) 309-314.

[14] Mickens, R.E., Nonstandard finite difference schemes for differential equations, J. Differ. Equations Appl. **8:9,** (2002) 823-847.

[15] Piyawong W., Twizell E.H., Gumel A.B., An unconditionally convergent finite-difference scheme for the SIR model, Appl. Math. Comput. **146,** (2003) 611-625.

[16] Taylor A.H., Harris J.R.W. and Aiken J., The interaction of physical and biological process in a model of the vertical distribution of phytoplankton under stratification, Mar. Int. Ecohyrd., Jacques C.Nihoul (editor) **42,** (1986) 313-330.

[17] Volterra, V., Variations and fluctuations of the number of individuals in animal species living together, R.N. Chapman: Animal Ecology (1931) 409-448.

Department of Ecology and Evolutionary Biology, University of Tennessee, 569 Dabney Hall, 1416 Circle Drive, Knoxville, TN 37996-1610, USA

E-mail: `ddimitr1@utk.edu`

Department of Mathematics, University of Texas at Arlington, P.O. Box 19408, Arlington, TX 76019-0408, USA

E-mail: `hristo@uta.edu`

URL: `http://www.uta.edu/math/pages/faculty/hristo.htm`