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ALTERNATING SCHEMES OF PARALLEL COMPUTATION FOR THE DIFFUSION PROBLEMS

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Abstract. In this paper, a set of new alternating segment explicit-implicit (NASEI) schemes is derived based on an one-dimensional diffusion problem. The schemes are capable of parallel computation; third-order accurate in space; and stable under a reasonable mesh condition. The numerical examples show that the NASEI schemes are more accurate than either the old ASEI or the ASCN schemes.

Key Words. New alternating segment explicit-implicit (NASEI) schemes, finite difference method; diffusion problems, parallel computation.

1. Introduction

The goal of this paper is to study appropriate finite difference schemes suitable for parallel computation. Two major types of schemes which are capable of parallel computation are: the alternating schemes ([1]-[3]), and the domain decomposition schemes ([4]-[6]). Our interest is on the alternating schemes which the NASEI schemes belong.

Before getting into the detail construction of the NASEI schemes, we like to briefly mention three closely related existing alternating schemes. They are: the Alternating Group Explicit (AGE) schemes ([1]); the Alternating Segment Explicit-Implicit (ASEI) schemes ([2]); and the Alternating Segment Crank-Nicolson (ASCN) schemes ([3]). All these three schemes are capable of parallel computation, however, their truncation errors are only second order or lower. Thus, we propose to derive the NASEI schemes, a new set of alternating schemes, which are capable of parallel computation; stable under a reasonable condition; and have truncation errors of third order.

The NASEI schemes are derived based on the following diffusion problem with periodic solution:

$$Lu \equiv \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,\ x \in \Re,\ t \in [0,T], \tag{1.1}$$

$$u(x,t) = u(x+H,t),\ x \in \Re,\ t \in [0,T], \tag{1.2}$$

$$u(x,0) = u_0(x),\ x \in \Re. \tag{1.3}$$

Here, H represents the length of one period.

The outline of the paper is as following. Six basic schemes of (1.1) are introduced in Section 2. The NASEI schemes are derived; their stability result is proved; and

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their truncation errors are obtained, all in Section 3. The numerical examples are presented in Section 4. Finally, a short conclusion remark is given in Section 5.

2. The six basic schemes

2.1. Preliminaries. The idea of using basic schemes to derive alternating schemes was first used by us in ([3]), where six basic schemes were introduced to derive a set of alternating schemes for the Dispersive equation. In this paper, we generalize this idea to the diffusion equation (1.1).

Throughout the rest of the paper, $\triangle x$ and $\triangle t$ are used to represent the spacial mesh size and time increment respectively; r represents $\frac{\triangle t}{\triangle x^2}$; and h represents the pair $(\triangle x,\, \triangle t)$. U_j^n is used to represent the approximate value of $u(x_j, t^n)$ which is shortened to u_j^n. Here $u(x,\, t)$ represents the exact solution. We assume that there exists a positive integer J, such that $J\,\triangle x = H$.

2.2. The six basic schemes. The first two basic schemes are the following explicit and implicit schemes:

$$U_j^{n+1} + \frac{r}{12}\,U_{j-2}^n - \frac{4r}{3}\,U_{j-1}^n - (1 - \frac{5r}{2})\,U_j^n - \frac{4r}{3}\,U_{j+1}^n + \frac{r}{12}\,U_{j+2}^n = 0, \tag{2.1}$$

$$\frac{r}{12}\,U_{j-2}^{n+1} - \frac{4r}{3}\,U_{j-1}^{n+1} + (1 + \frac{5r}{2})\,U_j^{n+1} - \frac{4r}{3}\,U_{j+1}^{n+1} + \frac{r}{12}\,U_{j+2}^{n+1} - U_j^n = 0. \tag{2.2}$$

The other four are four asymmetric schemes given below, their rules are displayed at the end of the paper.

$$\begin{aligned} & (1 + \frac{7r}{12})\,U_j^{n+1} - \frac{2r}{3}\,U_{j+1}^{n+1} + \frac{r}{12}\,U_{j+2}^{n+1} \\ = \; & -\frac{r}{12}\,U_{j-2}^n + \frac{4r}{3}\,U_{j-1}^n + (1 - \frac{23r}{12})\,U_j^n + \frac{2r}{3}\,U_{j+1}^n, \end{aligned} \tag{2.3}$$

$$\begin{aligned} & -\frac{2r}{3}\,U_{j-1}^{n+1} + (1 + \frac{23r}{12})\,U_j^{n+1} - \frac{4r}{3}\,U_{j+1}^{n+1} + \frac{r}{12}\,U_{j+2}^{n+1} \\ = \; & -\frac{r}{12}\,U_{j-2}^n + \frac{2r}{3}\,U_{j-1}^n + (1 - \frac{7r}{12})\,U_j^n, \end{aligned} \tag{2.4}$$

$$\begin{aligned} & \frac{r}{12}\,U_{j-2}^{n+1} - \frac{4r}{3}\,U_{j-1}^{n+1} + (1 + \frac{23r}{12})\,U_j^{n+1} - \frac{2r}{3}\,U_{j+1}^{n+1} \\ = \; & (1 - \frac{7r}{12})\,U_j^n + \frac{2r}{3}\,U_{j+1}^n - \frac{r}{12}\,U_{j+2}^n, \end{aligned} \tag{2.5}$$

$$\begin{aligned} & \frac{r}{12}\,U_{j-2}^{n+1} - \frac{2r}{3}\,U_{j-1}^{n+1} + (1 + \frac{7r}{12})\,U_j^{n+1} \\ = \; & \frac{2r}{3}\,U_{j-1}^n + (1 - \frac{23r}{12})\,U_j^n + \frac{4r}{3}\,U_{j+1}^n - \frac{r}{12}\,U_{j+2}^n. \end{aligned} \tag{2.6}$$

If $L_h^{(2.1)}$, $L_h^{(2.2)}$, $L_h^{(2.3)}$, $L_h^{(2.4)}$, $L_h^{(2.5)}$, $L_h^{(2.6)}$ are used to represent the analogous discretizied operators of L based on schemes (2.1)-(2.6), then their truncation errors at point $(x_j,\, t^n)$, derived from the Taylor series, are given below:

$$(2.7)\qquad \begin{aligned} & L_h^{(2.1)} u_j^n - [Lu]_j^n \\ = \; & \frac{r}{2} \triangle x^2 \left(\frac{\partial^4 u}{\partial x^4} \right)_j^n + O(\triangle t^2 + \frac{\triangle x^6}{\triangle t}), \end{aligned}$$

$$(2.8)\qquad \begin{aligned} & L_h^{(2.2)} u_j^n - [Lu]_j^n \\ = \; & -\frac{r}{2} \triangle x^2 \left(\frac{\partial^4 u}{\partial x^4} \right)_j^n + O(\triangle t^2 + \triangle t \,\triangle x^2 + \frac{\triangle x^6}{\triangle t}). \end{aligned}$$

$$(2.9)\qquad \begin{aligned} & L_h^{(2.3)} u_j^n - [Lu]_j^n \\ = \; & -\frac{r}{2} \triangle x \left(\frac{\partial^2 u}{\partial t \,\partial x} \right)_j^n + \frac{r}{3} \triangle x^2 \left(\frac{\partial^4 u}{\partial x^4} \right)_j^n - \frac{r}{4} \triangle x \,\triangle t \left(\frac{\partial^3 u}{\partial t^2 \,\partial x} \right)_j^n \\ & + O(\triangle t^2 + \triangle t \,\triangle x^2 + \frac{\triangle x^6}{\triangle t}), \end{aligned}$$

$$(2.10)\qquad \begin{aligned} & L_h^{(2.4)} u_j^n - [Lu]_j^n \\ = \; & -\frac{r}{2} \triangle x \left(\frac{\partial^2 u}{\partial t \,\partial x} \right)_j^n - \frac{r}{3} \triangle x^2 \left(\frac{\partial^4 u}{\partial x^4} \right)_j^n - \frac{r}{4} \triangle x \,\triangle t \left(\frac{\partial^3 u}{\partial t^2 \,\partial x} \right)_j^n \\ & + O(\triangle t^2 + \triangle t \,\triangle x^2 + \frac{\triangle x^6}{\triangle t}), \end{aligned}$$

$$(2.11)\qquad \begin{aligned} & L_h^{(2.5)} u_j^n - [Lu]_j^n \\ = \; & \frac{r}{2} \triangle x \left(\frac{\partial^2 u}{\partial t \,\partial x} \right)_j^n - \frac{r}{3} \triangle x^2 \left(\frac{\partial^4 u}{\partial x^4} \right)_j^n + \frac{r}{4} \triangle x \,\triangle t \left(\frac{\partial^3 u}{\partial t^2 \,\partial x} \right)_j^n \\ & + O(\triangle t^2 + \triangle t \,\triangle x^2 + \frac{\triangle x^6}{\triangle t}), \end{aligned}$$

$$(2.12)\qquad \begin{aligned} & L_h^{(2.6)} u_j^n - [Lu]_j^n \\ = \; & \frac{r}{2} \triangle x \left(\frac{\partial^2 u}{\partial t \,\partial x} \right)_j^n + \frac{r}{3} \triangle x^2 \left(\frac{\partial^4 u}{\partial x^4} \right)_j^n + \frac{r}{4} \triangle x \,\triangle t \left(\frac{\partial^3 u}{\partial t^2 \,\partial x} \right)_j^n \\ & + O(\triangle t^2 + \triangle t \,\triangle x^2 + \frac{\triangle x^6}{\triangle t}), \end{aligned}$$

The above six schemes are consistent with (1.1) as long as $\triangle t = O(\triangle x^{\beta})$, where β is in the range of $(1,\;\; 6)$. In reality, we tend to take β as low as possible to minimize the computing cost. It will be rare and extremely unusual to take $\triangle t$ the order of $O(\triangle x^6)$ or higher. Therefore, we can view these six schemes as always consistent with (1.1).

3. The NASEI schemes

3.1. The schemes. We now derive the NASEI schemes. Assume that $J = K(l+l')$, we divide the spacial grid points $x_1, x_2, ..., x_J$ into K sections at each time level, where K, $l(\geq 1)$, $l('\geq 5)$ are positive integers. Each section consists of explicit and implicit segments. At the odd time levels, each explicit segment consists of l points, and each implicit segment consists of l' points. At the even time levels, each explicit segment consists of $l'-4$ points, and each implicit segment consists of $l+4$ points.

At the time level t^{n+1}, U_j^{n+1} in the explicit segments are computed according to the explicit scheme (2.1). The U_j^{n+1} for the implicit segments, where $j=i+1,\cdots,i+k$ for some i and $k=l'$ or $l+4$, are computed according to (2.3), (2.4), (2.5) and (2.6) at points x_{i+1}, x_{i+2}, x_{i+k-1}, x_{i+k} respectively. For the rest of the points at these implicit segments, they are computed according to the implicit scheme (2.2). This arrangement results in the following linear system:

$$(I+\frac{r}{12}P_k)\begin{bmatrix} U_{i+1}^{n+1} \\ U_{i+2}^{n+1} \\ U_{i+3}^{n+1} \\ \vdots \\ U_{i+k-2}^{n+1} \\ U_{i+k-1}^{n+1} \\ U_{i+k}^{n+1} \end{bmatrix} = \begin{bmatrix} -\frac{r}{12}U_{i-1}^n+\frac{4r}{3}U_i^n+(1-\frac{23r}{12})U_{i+1}^n+\frac{2r}{3}U_{i+2}^n \\ -\frac{r}{12}U_i^n+\frac{2r}{3}U_{i+1}^n+(1-\frac{7r}{12})U_{i+2}^n \\ U_{i+3}^n \\ \vdots \\ U_{i+k-2}^n \\ (1-\frac{7r}{12})U_{i+k-1}^n+\frac{2r}{3}U_{i+k}^n-\frac{r}{12}U_{i+k+1}^n \\ \frac{2r}{3}U_{i+k-1}^n+(1-\frac{23r}{12})U_{i+k}^n+\frac{4r}{3}U_{i+k+1}^n-\frac{r}{12}U_{i+k+2}^n \end{bmatrix},$$

where

$$P_k=\begin{bmatrix} 7 & -8 & 1 & & & & \\ -8 & 23 & -16 & 1 & & & \\ 1 & -16 & 30 & -16 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -16 & 30 & -16 & 1 \\ & & & 1 & -16 & 23 & -8 \\ & & & & 1 & -8 & 7 \end{bmatrix}_{k\times k}.$$

Since the implicit and explicit segments are independent of each other, they can be solved in parallel.

Generally speaking, the computation is arranged according to the following rule:

Explicit – **Implicit**– $\cdots$ – **Implicit** **at odd time levels**,

Implicit – **Explicit**– $\cdots$ – **Explicit** **at even time levels**.

A flow chart of this rule is displayed at the end of the paper, where □ denotes the four asymmetric schemes (2.3)-(2.6); ● denotes the implicit scheme (2.2); and ○ denotes the explicit scheme (2.1).

Thus the NASEI schemes can be expressed into the following vector form:

$$(3.1)\qquad \begin{cases} (I+\frac{r}{12}G_1)\,U^{n+1}=(I-\frac{r}{12}G_2)\,U^n, \\ \\ (I+\frac{r}{12}G_2)\,U^{n+2}=(I-\frac{r}{12}G_1)\,U^{n+1}, \end{cases} \qquad n=0,2,4,...$$

where $U^n = (U_1^n, ..., U_J^n)^T$, $G_1 = \text{diag}(Q_l, P_{l'}, ..., Q_l, P_{l'})$ and

$$G_2 = \begin{bmatrix}
30 & -16 & 1 & & & & & & & & & 1 & -16 \\
-16 & 30 & -16 & 1 & & & & & & & & & 1 \\
1 & -16 & 30 & -16 & 1 & & & & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & \\
 & & 1 & -16 & 30 & -16 & 1 & & & & & & \\
 & & & 1 & -16 & 23 & -8 & & & & & & \\
 & & & & 1 & -8 & 7 & & & & & & \\
 & & & & & & & Q_{l'-4} & & & & & \\
 & & & & & & & & P_{l+4} & & & & \\
 & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & Q_{l'-4} & & \\
1 & & & & & & & & & & & 7 & -8 \\
-16 & 1 & & & & & & & & & & -8 & 23
\end{bmatrix}_{J \times J.}$$

Here Q_n ($n = l$ or $l' - 4$) is a $n \times n$ zero matrix.

3.2. Stability. We first quote the following Kellogg Lemmas ([7]) in order to prove the stability result of the NASEI schemes.

Lemma 3.1. *Let C and C^* be an operator and its conjugate transpose. If θ is a positive constant and $(C + C^*)$ is a non-negative definite operator, then $(\theta\, I + C)^{-1}$ exists and satisfies:*

$$\|(\theta I + C)^{-1}\|_2 \leq \theta^{-1}.$$

Here I represents the identity operator.

Lemma 3.2. *Under the same conditions of previous lemma, the following inequality is also true.*

$$\|(\theta I - C)(\theta I + C)^{-1}\|_2 \leq 1.$$

The stability result for the NASEI schemes is stated in the next theorem.

Theorem 3.1. *The NASEI schemes are stable if $r = \dfrac{\triangle t}{\triangle x^2}$ is a bounded positive constant.*

Proof: By eliminating U^{n+1} from (3.1), we obtain that

$$U^{n+2} = T\, U^n = T^2\, U^{n-2} = \cdots = T^n\, U^0.$$

Here

$$T = \Big(I + \frac{r}{12} G_2\Big)^{-1} \Big(I - \frac{r}{12} G_1\Big) \Big(I + \frac{r}{12} G_1\Big)^{-1} \Big(I - \frac{r}{12} G_2\Big).$$

Since G_1 and G_2 are both nonnegative definite, according to the Kellogg Lemmas 3.1 and 3.2, we have the following result for any positive constant r:

$$\begin{aligned}
\|T^n\|_2 \leq\ & \Big\| \Big((I + \frac{r}{12} G_2)^{-1} (I - \frac{r}{12} G_1)(I + \frac{r}{12} G_1)^{-1} (I - \frac{r}{12} G_2) \Big)^n \Big\|_2 \\
\leq\ & \Big\| (I + \frac{r}{12} G_2)^{-1} \Big\|_2^n \Big\| (I - \frac{r}{12} G_1)(I + \frac{r}{12} G_1)^{-1} \Big\|_2^n \Big\| (I - \frac{r}{12} G_2) \Big\|_2^n \\
\leq\ & \Big\| (I + \frac{r}{12} G_2)^{-1} \Big\|_2^n \Big\| (I - \frac{r}{12} G_2) \Big\|_2^n \\
\leq\ & \Big\| (I - \frac{r}{12} G_2) \Big\|_2^n \\
\leq\ & 1 + 4r.
\end{aligned}$$

The stability result of the NASEI schemes is therefore proved. ∎

3.3. Truncation errors. The actual computation involves three pairs of schemes which are used alternately between two consecutive time levels. These pairs are: (2.1)∨(2.2), (2.3)∨(2.5) and (2.4)∨(2.6). It is sufficient to analyze the truncation errors for the first section of $(l+l')$ points which is from point 1 to point $l+l'$. The analyzes for the other sections are identical. The first section of $l+l'$ points includes three pairs of truncation errors.

The first pair is between the explicit scheme (2.1) and the implicit scheme (2.2), where the explicit segment from point 1 to point l is considered first here. A Taylor series expansion for (2.1) at point $(x_j,\, t^{n+1})$ gives the following result:

$$L_h^{(2.1)} u_j^n - [Lu]_j^{n+1} = \frac{r}{2} \triangle x^2 \left(\frac{\partial^4 u}{\partial x^4} \right)_j^{n+1} + O(\triangle t^2 + \triangle t \, \triangle x^2 + \frac{\triangle x^6}{\triangle t}). \tag{3.1}$$

By comparing this result with the error expression of the implicit scheme (2.8), we see that the terms involving $\triangle x^2$ are canceled every other time level. Hence, from point 1 to point l , the truncation errors are $O(\triangle t^2 + \triangle t \, \triangle x^2 + \frac{\triangle x^6}{\triangle t})$.

We now consider the same pair for the implicit segment from point $l+3$ to point $l+l'-2$. A Taylor series expansion at point (x_j, t^{n+1}) for (2.2) gives the following result:

$$L_h^{(2.2)} u_j^n - [Lu]_j^{n+1} = -\frac{r}{2} \triangle x^2 \left(\frac{\partial^4 u}{\partial x^4} \right)_j^{n+1} + O(\triangle t^2 + \frac{\triangle x^6}{\triangle t}). \tag{3.2}$$

The above result is compared to the truncation error of the explicit scheme (2.7) at point $(x_j,\, t^n)$, we see again that the terms involving $\triangle x^2$ are canceled every other time level. Thus, from point $(l+3)$ to point $(l+l'-2)$, the truncation errors are also $O(\triangle t^2 + \triangle t \, \triangle x^2 + \frac{\triangle x^6}{\triangle t})$.

The pair (2.3) and (2.5), at points $l+1$ and $l+l'-1$, are alternately used every other time level. The truncation error at point $(x_j,\, t^{n+1})$ for (2.3) is:

$$\begin{aligned} L_h^{(2.3)} u_j^n - [Lu]_j^{n+1} = \; & -\frac{r}{2} \triangle x \left(\frac{\partial^2 u}{\partial t \, \partial x} \right)_j^{n+1} + \frac{r}{3} \triangle x^2 \left(\frac{\partial^4 u}{\partial x^4} \right)_j^{n+1} \\ & + \frac{r}{4} \triangle x \triangle t \left(\frac{\partial^3 u}{\partial t^2 \partial x} \right)_j^{n+1} + O(\triangle t^2 + \triangle t \, \triangle x^2 + \frac{\triangle x^6}{\triangle t}). \end{aligned} \tag{3.3}$$

The above result is compared to the truncation error of (2.11) at point (x_j, t^n), the terms involving $\triangle x$ and $\triangle x^2$ are obviously canceled every other time level. So the truncation error at point $l+1$ is order $O(\triangle t \, \triangle x)$. Similarly, we can prove that the truncation error at point $l+l'-1$ is also order $O(\triangle t \, \triangle x)$.

Similar work conducted for the pair (2.4)∨ (2.6) shows that the truncation errors for points $l+2$ and $l+l'$ are order $O(\triangle t \, \triangle x)$. Since the stability result in **Theorem 3.1** requires that $r = \frac{\triangle t}{\triangle x^2}$ is a bounded positive constant, we can assume that $\triangle t = r \, \triangle x^2$. Thus, the truncation errors for the NASEI schemes are order $O(\triangle x^3)$ in space.

4. Numerical examples

The numerical examples of the NASEI schemes are based on the following model problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ x \in \Re,\ t \in [0,T], \tag{4.1}$$

$$u(x,t) = u(x+2,t),\ x \in \Re,\ t \in [0,T], \tag{4.2}$$

$$u(x,0) = \cos(\pi x),\ x \in \Re, \tag{4.3}$$

where the exact solution is $u(x,t) = e^{-\pi^2 t}\cos(\pi x)$, and the period is 2.

The error bounds of the NASEI schemes are examined first. We divide the spacial grid points into four sections; take l' to be $\frac{J}{4}+2$; and l to be $l'-4$. The L^2 errors are defined to be $e_h = \|U-u\|_{L^2}$, and are computed based on the following two different sets of r, T, and four different $\triangle x$:

$$r = 1,\ T = 0.01; \qquad r = 5,\ T = 0.01; \qquad \triangle x = \frac{2}{100}, \frac{2}{200}, \frac{2}{400}, \frac{2}{800}.$$

The results are listed in Table 1 at the end of paper. It is not hard to see that the L^2 errors of the NASEI schemes are order 3 in space, which confirms our earlier results in Section 3.

Next, we compare the accuracy of the NASEI schemes to those of the old ASEI schemes and the ASCN schemes. The computation is based on the same two sets of r and T; and $\triangle x$ is taken to be 0.02. The absolute errors (**ae**) and the percentage errors (**pe**) for these three schemes are listed in Tables 2-3 and plotted at the end of paper. Evidently, the results show that the NASEI schemes are more accurate than either the old ASEI schemes or the ASCN schemes.

5. Conclusions

In this paper, a set of new alternating segment explicit-implicit (NASEI) schemes is derived for an one-dimensional diffusion equation of periodic solution. These schemes are designed to alternate between explicit and implicit segments at any two consecutive time levels. They are capable of parallel computation, and have truncation errors of third order in space which are higher than those of similar schemes. The schemes are proved to be stable under reasonable condition. Numerical examples are also presented.

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FIGURE 1. Diagram of formulae (2.3) - (2.6)

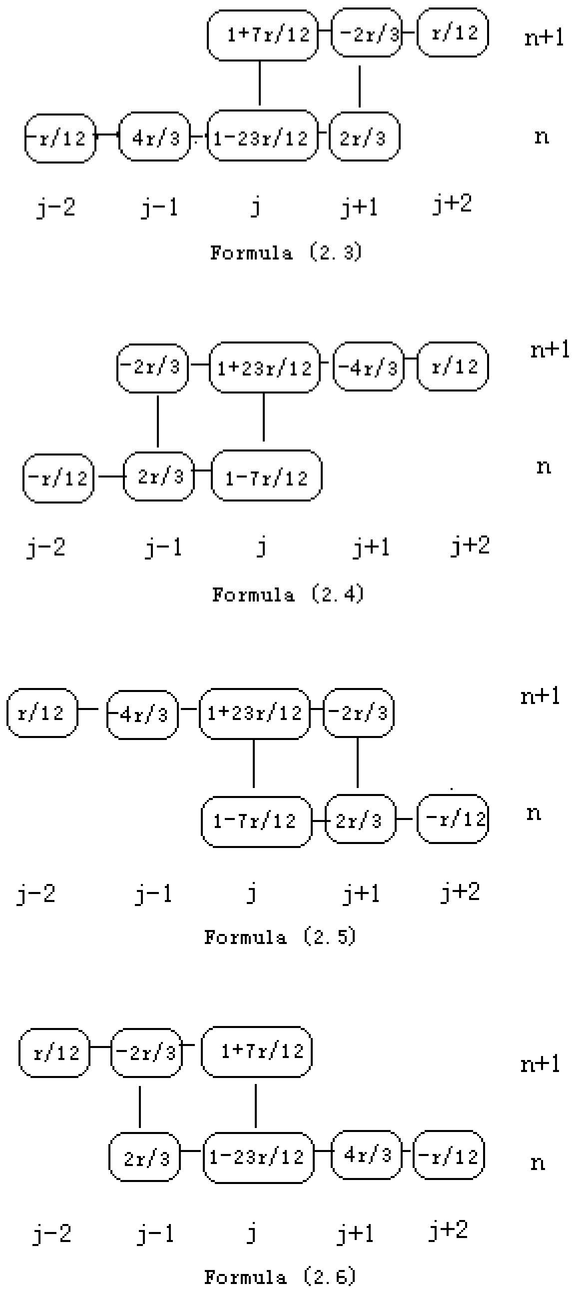

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FIGURE 2. **Diagram of the NASEI schemes for (1.1)-(1.2)**

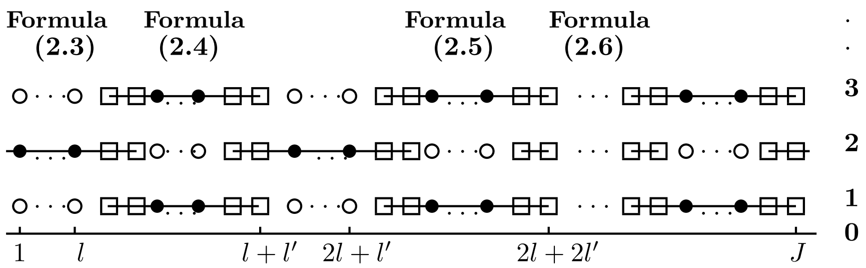

TABLE 1. **Convergence rates for the NASEI schemes**

J	$r = 1,\ T = 0.01$		$r = 5,\ T = 0.1$	
	$e_h * 10000000$	$e_h/\triangle x^3$	$e_h * 100000$	$e_h/\triangle x^3$
100	317	3.96	86.2	108
200	37.0	3.70	10.5	105
400	4.63	3.70	1.30	104
800	0.581	3.72	0.162	104

TABLE 2. **Absolute and percentage errors(I)**

	$\triangle x = 2/100,\ r = 1,\ T = 0.01$				
	$x = 0.2$	$x = 0.6$	$x = 1.0$	$x = 1.4$	$x = 1.8$
NASEI schemes					
ae	$.739 \times 10^{-5}$	$.285 \times 10^{-4}$	$.387 \times 10^{-5}$	$.278 \times 10^{-4}$	$.276 \times 10^{-5}$
pe	$.101 \times 10^{-4}$	$.102 \times 10^{-3}$	$.427 \times 10^{-5}$	$.994 \times 10^{-4}$	$.376 \times 10^{-5}$
Old ASEI schemes					
ae	$.134 \times 10^{-4}$	$.398 \times 10^{-4}$	$.322 \times 10^{-4}$	$.542 \times 10^{-4}$	$.251 \times 10^{-4}$
pe	$.183 \times 10^{-4}$	$.142 \times 10^{-3}$	$.355 \times 10^{-4}$	$.193 \times 10^{-3}$	$.342 \times 10^{-4}$
ASCN schemes					
ae	$.141 \times 10^{-4}$	$.310 \times 10^{-4}$	$.172 \times 10^{-4}$	$.139 \times 10^{-4}$	$.277 \times 10^{-4}$
pe	$.193 \times 10^{-4}$	$.111 \times 10^{-3}$	$.190 \times 10^{-4}$	$.495 \times 10^{-4}$	$.378 \times 10^{-4}$
Exact solution	.733	−.280	−.906	−.280	.733

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FIGURE 3. **Absolute errors (I)** $\triangle x = 0.02$, $r = 1$, $T = 0.01$

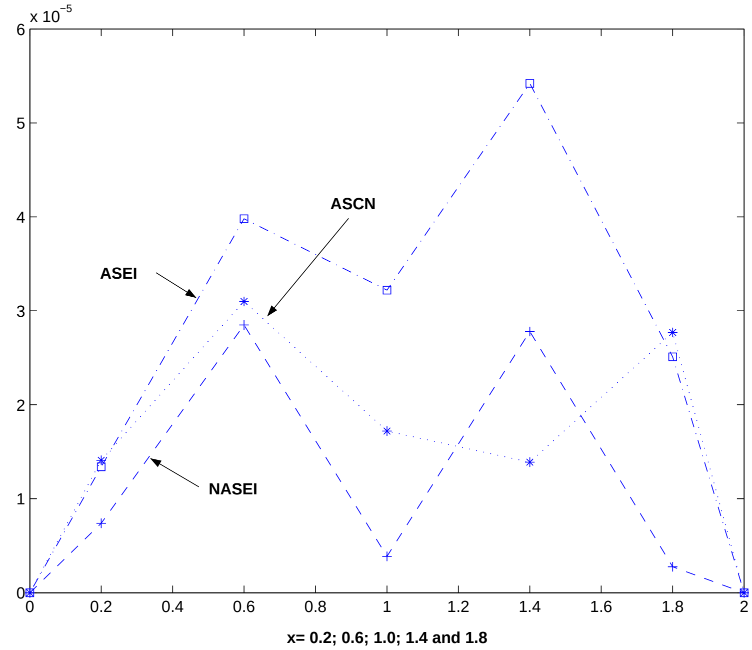

FIGURE 4. **Absolute errors (II)** $\triangle x = 0.02$, $r = 5$, $T = 0.1$

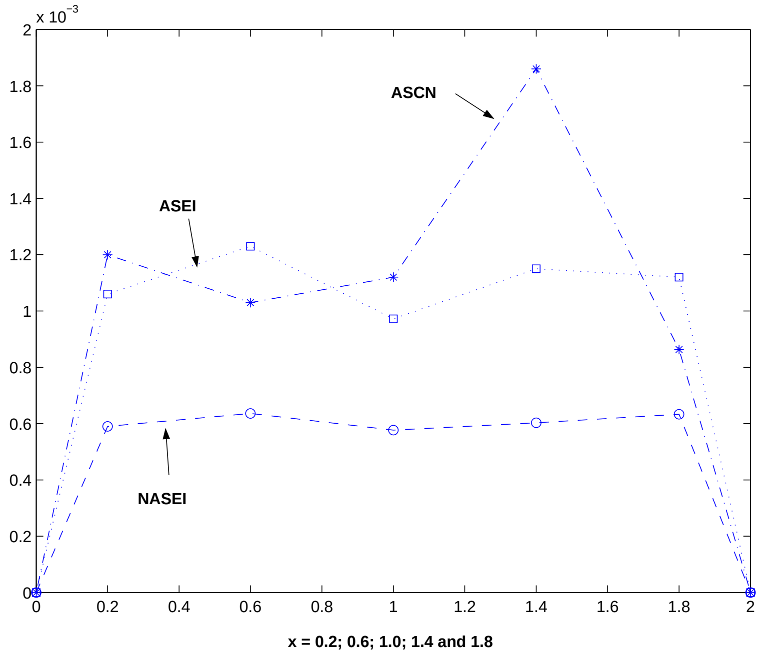

FIGURE 5. **Percentage errors (I)** $\triangle x = 0.02, r = 1, T = 0.01$

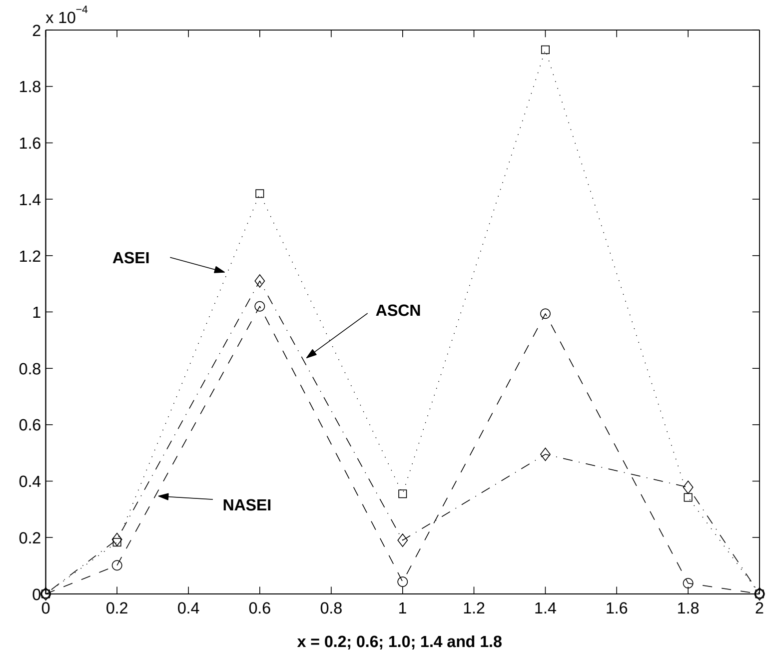

FIGURE 6. **Percentage errors (II)** $\triangle x = 0.02, r = 5, T = 0.1$

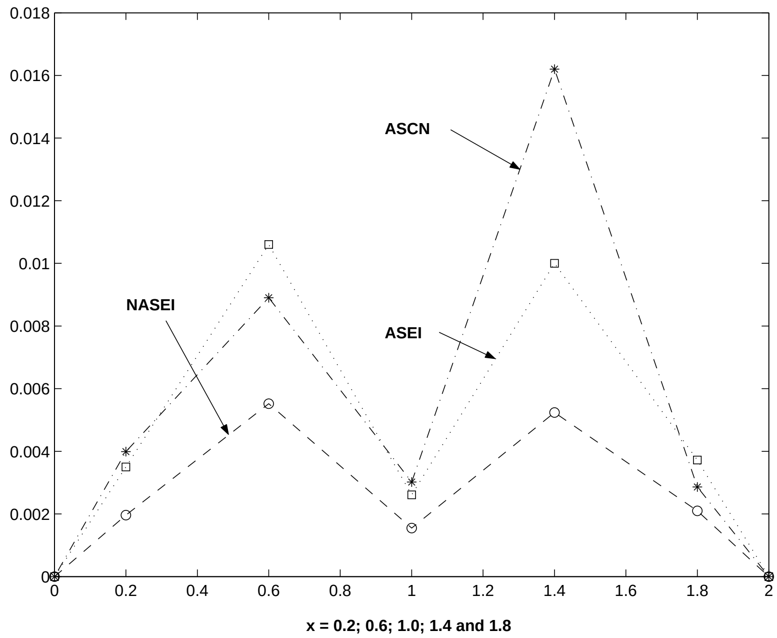

Table 3. **Absolute and percentage errors(II)**

	$\triangle x = 2/100,\ r = 5,\ T = 0.1$				
	$x = 0.2$	$x = 0.6$	$x = 1.0$	$x = 1.4$	$x = 1.8$
NASEI schemes					
ae	$.590 \times 10^{-3}$	$.636 \times 10^{-3}$	$.577 \times 10^{-3}$	$.603 \times 10^{-3}$	$.633 \times 10^{-3}$
pe	$.196 \times 10^{-2}$	$.552 \times 10^{-2}$	$.155 \times 10^{-2}$	$.524 \times 10^{-2}$	$.210 \times 10^{-2}$
Old ASEI schemes					
ae	$.106 \times 10^{-2}$	$.123 \times 10^{-2}$	$.972 \times 10^{-3}$	$.115 \times 10^{-2}$	$.112 \times 10^{-2}$
pe	$.350 \times 10^{-2}$	$.106 \times 10^{-1}$	$.261 \times 10^{-2}$	$.100 \times 10^{-1}$	$.372 \times 10^{-2}$
ASCN schemes					
ae	$.120 \times 10^{-2}$	$.103 \times 10^{-2}$	$.112 \times 10^{-2}$	$.186 \times 10^{-2}$	$.863 \times 10^{-3}$
pe	$.399 \times 10^{-2}$	$.890 \times 10^{-2}$	$.302 \times 10^{-2}$	$.162 \times 10^{-1}$	$.286 \times 10^{-2}$
Exact solution	$.302$	$-.115$	$-.373$	$-.115$	$.302$