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MODELING, ANALYSIS AND DISCRETIZATION OF STOCHASTIC LOGISTIC EQUATIONS

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Abstract. The well–known logistic model has been extensively investigated in deterministic theory. There are numerous case studies where such type of nonlinearities occur in Ecology, Biology and Environmental Sciences. Due to the presence of environmental fluctuations and a lack of precision of measurements, one has to deal with effects of randomness on such models. As a more realistic modeling, we suggest nonlinear stochastic differential equations (SDEs)

$$dX(t) = [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]dt + \sigma X(t)^{\alpha}|K - X(t)|^{\beta} dW(t)$$

of Itô-type to model the growth of populations or innovations X, driven by a Wiener process W and positive real constants ρ, λ, K, μ, α, $\beta \geq 0$. We discuss well–posedness, regularity (boundedness) and uniqueness of their solutions. However, explicit expressions for analytical solution of such random logistic equations are rarely known. Therefore one has to resort to numerical solution of SDEs for studying various aspects like the time–evolution of growth patterns, exit frequencies, mean passage times and impact of fluctuating growth parameters. We present some basic aspects of adequate numerical analysis of these random extensions of these models such as numerical regularity and mean square convergence. The problem of keeping reasonable boundaries for analytic solutions under discretization plays an essential role for practically meaningful models, in particular the preservation of intervals with reflecting or absorbing barriers. A discretization of the continuous state space can be circumvented by appropriate methods. Balanced implicit methods (see Schurz, IJNAM 2 (2), p. 197-220, 2005) are used to construct strongly converging approximations with the desired monotone properties. Numerical studies can bring out salient features of the stochastic logistic models (e.g. almost sure monotonicity, almost sure uniform boundedness, delayed initial evolution or earlier points of inflection compared to deterministic model).

Key Words. logistic growth, stochastic logistic equation, properties of solutions, numerical methods, balanced implicit methods, boundedness, convergence, stability, monotonicity

1. Introduction

Logistic growth phenomenon is observed in numerous models and underlying data such as for the population of fruit flies or flour beetle in population ecology or innovation diffusion in marketing sciences or social sciences. In the continous time

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framework it is commonly believed that the evolution of the number $x = x(t)$ of certain species can be approximately modeled by the per-capita-growth rate

$$\frac{1}{x}\frac{dx}{dt} = \lambda(K - x) - \mu \tag{1}$$

where $\lambda > 0$ is some growth parameter, μ the death rate and $K > 0$ the underlying carrying capacity limited by a finite number of natural resources. Model (1) with $\mu = 0$ is also known as *Verhulst-Pearl equation*, see [34], [46]. It was used by several authors to describe the evolution of species, populations or innovations, see [12], [35], [8], [22] or [37], and specified later by [20], [21], [26] and [47] for biological applications with delay effects, among many others.

It is well-known that equation (1) has two equilibria solutions, namely a locally asymptotically unstable solution $x_1^* = 0$ and, if $\mu = 0$, a globally asymptotically stable $x_2^* = K$. Moreover, these points represent barriers for any other solution and, if $\mu = 0$, the interval $(0, K)$ is left invariant and attracting from above by the related flow of analytical solutions. Furthermore, discrete analoga are often used to motivate the existence and effects of chaos in related dynamical systems.

In reality of collecting and analyzing environmental data, these models need to be specified. In particular, due to the Heisenberg's uncertainty principle and the resulting lack of precise measurements, the logistic growth undergoes environmental and parametric noise. Recall that Heisenberg's uncertainty principle also means that two or more quantities (here our model parameters) cannot be estimated exactly, only with random deviations. Consequently, meaningful stochastic generalizations of logistic equations lead to nonlinear **s**tochastic **d**ifferential **e**quations (**SDEs**) of Itô-type

$$dX(t) = [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]\,dt + \sigma X(t)^{\alpha}|K - X(t)|^{\beta} dW(t) \tag{2}$$

driven by a standard Wiener process $(W(t) : t \geq 0)$, started at $X_0 \in \mathbb{D} = [0, K] \subset \mathbb{R}^1$, where $\rho, \lambda, K, \mu, \sigma$ are positive and α, β nonnegative real parameters. There ρ can be understood as coefficient of transition (self-innovation), λ as coefficient of immitation depending on the contact itensity with its environment, K as a somewhat "optimal" environmental carrying capacity and μ as natural death rate. However, in view of issues of practical meaningfulness, model (2) makes only sense within deterministic algebraic constraints, either given by extra boundary conditions or self-inherent properties resulting into natural barriers at 0 at least. This fact is supported by the limited availability of natural resources as known from the evolution of species in population ecology. In what follows we study **almost sure regularity** (boundedness on $\mathbb{D}$) of both exact and numerical solutions of (2) which has been mostly omitted in literature in the latter case. At the same time we are aiming at the **maintenance of** certain **convergence** orders of related standard numerical approximations towards exact solutions. In particular we shall construct a numerical solution which exclusively possesses values in $\mathbb{D}$ and is mean square converging with order $\gamma = 0.5$ towards the exact solution. Note that usual numerical methods as most–used Euler method fail to live a.s. on bounded domain $\mathbb{D}$ for any choice of constant step sizes (for examples, see [38], [39]). Besides, "higher order methods" as systematically developed by [48] can not be applied in general, since their mathematical justification requires too much boundedness and smoothness on drift and diffusion coefficients of SDEs, which is not given within the general framework of model (2). The latter statement does not mean that we do not advise to try out methods of higher order of convergence in specific situations. It is more the expression for a current lack of knowledge on qualitative behavior of them such as the control on their stability or boundary behavior and an observed lack of smoothness of subclasses preventing us from achieving higher convergence rates.

From the fluctuation-dissipation theorem of mathematical physics, we know that the relation between fluctuations and dissipation terms should be chosen as

$$\text{random fluctuation terms} \sim \sqrt{|\text{dissipation terms}|}$$

for physically most relevant models while considering fluctuations in the per-capita-growth rate. This fact supports the preference of models with parameters $\alpha \approx 1$ and $\beta \approx 0.5$. From mathematics (see the central limit theorem (CLT) in the theory of probability), we know about the approximate role of Gaussian distributions for modeling random fluctuations. From ecological applications, we are tempted to take into account any form of density-dependent randomness effecting the per-capita-growth rates, birth or death rates rather than any other forms of stochasticity. Thus, practically meaningful models as first approximations of given data should bear all these facts in mind. From this point of view, equation (2) contains the following very reasonable subclasses

$$\begin{aligned}
dX(t) &= [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]\, dt + \sigma \sqrt{|X(t)(K - X(t))|} dW(t), \\
dX(t) &= [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]\, dt + \sigma X(t) \sqrt{|K - X(t)|} dW(t), \\
dX(t) &= [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]\, dt + \sigma X(t)(K - X(t)) dW(t), \\
dX(t) &= [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]\, dt + \sigma X(t) dW(t), \\
dX(t) &= [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]\, dt + \sigma \sqrt{X(t)} dW(t), \text{ or} \\
dX(t) &= [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]\, dt + \sigma \sqrt{|K - X(t)|} dW(t).
\end{aligned}$$

These models can be interpreted as random models where diverse parameters are randomly perturbed in deterministic logistic equation by locally density-dependent, conditional Gaussian distributions. We do not believe much in density-independent perturbations due to mathematical and modeling reasons as we shall see below. However, one might think of models with other non-integer-type nonlinearity parameters α and β too. That is how we naturally arrive at a series of randomized logistic models of type (2).

Stochastic logistic equations have been investigated by a number of authors. For example, see [1] as birth-death processes, [5] in social sciences, [6], [7], [11] and [15] as population models in ecology, [25] and [30] with respect to the approximation of certain moments, [24] in biosciences, [29] in view of computation of extinction times, [31] on related quasistationary disctributions or [13] and [33] from stability point of view among many others. However, none of those authors has delt with the problem of adequate (regular and convergent) numerical approximation of stochastic logistic equations in terms of nonlinear stochastic differential equations as presented in full generality above.

The paper is organized as follows. After this introduction, Section 2 investigates the problem of regularity of analyical solutions of (2) which is relevant to determine the practical meaningfulness of presented random logistic models. Thereafter, we study how one can regularize numerical approximations in order to get regularity of strongly converging approximations of (2) in Section 3. This study is carried out through the class of balanced implicit methods without discretizing the underlying bounded state-space. Thereafter, Section 4 presents some simulation results. One of them exhibits a striking difference between the evolution of distinct stochastic calculi, the other illustrates a surprising vanishing-chaos effect during an appropriate tuning of noise parameters. This paper is finished by some summarizing remarks in Section 5 and two appendices. The first appendix (Section 6) contains a general convergence theorem of stochastic-numerical approximations on bounded connected subsets (manifolds) of $\mathbb{R}^d$. The second appendix (Section 7) deals with a proof of pathwise and moment monotonicity of exact solutions which was observed by numerical experiments.

2. Regularity and nonregularity of stochastic logistic equations

At first we recall the notion of regularity of continuous time stochastic processes as introduced in [18]. Let $\mathbb{D} \subset \mathbb{R}^d$ be a fixed closed domain. Note, for simplicity, we exclusively consider deterministic domains $\mathbb{D} \subset \mathbb{R}^1$ in this exposition.

Definition 2.1. *A continuous time stochastic process* $\{X(t), t \geq 0\}$ *is called* **regular on** $\mathbb{D}$ *(or invariant with respect to* $\mathbb{D}$*) iff*

$$\forall t \geq 0 \; : \quad \mathbb{P}(X(t) \in \mathbb{D}) \; = \; 1,$$

otherwise **nonregular with respect to** $\mathbb{D}$ *(or not invariant with respect to* $\mathbb{D}$*).*

2.1. Nonregularity of $[0, K]$ **with additive noise.** Intuitively clear, at first we rigorously show that exact solution of SDEs (2) with additive noise (i.e. $\alpha = \beta = 0$) leaves the bounded domain $\mathbb{D} = [0, K]$. That is, one has to impose algebraic constraints on SDEs (2) which would lead to the formulation of stochastic differential-algebraic equations (SDAEs). However, it is possible to avoid this problem with appropriate choice of α and β as we shall see later. This is an important fact for adequate modeling. Let us show nonregularity of intervals $[0, K]$ under purely additive noise at first. For this purpose, we make use of **Lyapunov–type methods**. Note that in $\mathbb{R}^1$ there is an alternative given by Feller's classification of boundary values (see [11], [16] and [17]), which can be carried out by evaluating the scale function and speed measure of related diffusions. We follow the Lyapunov–type analysis in order to clarify strong existence, uniqueness, boundedness and Markovian properties of (2) within an unified approach which is also very efficient in multi–dimensional situation. A comparison study between these methods is left to the reader. Now, let $\tau^{s,x} = \tau^{s,x}(\mathbb{D})$ be the random time of first exit of stochastic process X from domain $\mathbb{D}$, started in $X(s) = x \in \mathbb{D}$ at initial time $s \in [0, +\infty)$.

Theorem 2.1. *Assume that* $\{X(t), t \geq 0\}$ *satisfies SDE (2) with* $\alpha = \beta = 0$, $K > 0$, $\sigma^2 > 0$, $\rho \geq 0$, $\lambda \geq 0$, $\mu \geq 0$ *and* $X(0) \in \mathbb{D} = [0, K]$ *is independent of* σ*-algebra* $\sigma(W(t), t \geq 0)$.
Then $\{X(t), t \geq 0\}$ *is nonregular with respect to* $\mathbb{D}$*. More precisely speaking,*

$$\forall x \in \mathbb{D} \quad \forall s \geq 0 \qquad \mathbb{P}(\tau^{s,x}(\mathbb{D}) < +\infty) \; > 0\,. \tag{3}$$

Remark. $\mathbb{D}$ can be replaced by any nonempty, nonrandom interval contained in $\mathbb{R}^1$, and the result of Theorem 2.1 is still true.

In all following proofs, let $C^{1,2}(A\text{x}B)$ with real sets A and B denote the set of all real-valued functions $f : A\text{x}B \to \mathbb{R}^1$ such that f is one times continuously differentiable with respect to first coordinate on A and twice continuously differentiable with respect to second coordinate on B.

Proof. Define drift $a(x) = (\rho + \lambda x)(K - x) - \mu x$ and diffusion $b(x) = \sigma$. Introduce the Lyapunov function $V(x) = 1 + x^2, x \in \mathbb{D}$. Note, equation (2) is well–defined, has unique and bounded solution up to random time $\tau^{s,x}(\mathbb{D})$, due to Lipschitz continuity and (linear) boundedness of drift $a(x)$ and diffusion $b(x)$ on $\mathbb{D}$. Let $\mathcal{L}$ denote

$$\mathcal{L} \; = \; \frac{\partial}{\partial t} + a(x)\frac{\partial}{\partial x} + \frac{1}{2}b^2(x)\frac{\partial^2}{\partial x^2}\,. \tag{4}$$

Verify that $V \in C^2(\mathbb{D})$ and $1 \leq V(x) \leq 1 + K^2$ on $\mathbb{D}$. Compute

$$\mathcal{L}V(x) \; = \; 2\rho x(K-x) + 2\lambda x^2(K-x) - 2\mu x^2 + \sigma^2$$

which can be estimated from below by

$$\mathcal{L}V(x) \; \geq \; \begin{cases} -2\mu \cdot V(x) + 2\mu + \sigma^2 & \text{if } \mu > 0 \\ \frac{\sigma^2}{1+K^2}V(x) & \text{if } \mu = 0 \end{cases}$$

for $x \in \mathbb{D}$. Now, fix initial time $s \geq 0$ and define

$$c \; := \; \begin{cases} -2\mu & \text{if } \mu > 0, \\ \frac{\sigma^2}{1+K^2} & \text{if } \mu = 0 \end{cases}\,.$$

Introduce another Lyapunov function $W \in C^{1,2}([s,+\infty)\mathrm{x}\mathbb{D})$ by

$$W(t,x) := \exp(-c(t-s))V(x)$$

for all $t \geq s \geq 0, x \in \mathbb{D}$. It follows that

$$\mathcal{L}W(t,x) \geq \begin{cases} \exp(-c(t-s))[2\mu+\sigma^2] & \text{if } \mu > 0 \\ 0 & \text{if } \mu = 0 \end{cases}\,.$$

After applying Dynkin's formula (see [14] as applied in [18]) to get to

$$\mathbb{E}\, W(\min{(\tau^{s,x}(\mathbb{D}),t)}, X_{\min{(\tau^{s,x}(\mathbb{D}),t)}}) \geq \left\{ \begin{array}{ll} V(x) + \frac{1-\exp(-c(t-s))}{c}[2\mu+\sigma^2] & \text{if } \mu > 0 \\ V(x) & \text{if } \mu = 0 \end{array} \right\} \geq V(x),$$

where the starting value $X(s) = x \in \mathbb{D}$ is deterministic, one obtains

$$\begin{aligned} & \mathbb{E}\, \exp(-c\,(\min{(\tau^{s,x}(\mathbb{D}),t)} - s)) \\ &= \; \mathbb{E}\, \left[\exp(-c\,(\min{(\tau^{s,x}(\mathbb{D}),t)} - s)) \cdot \frac{V(X_{\min{(\tau^{s,x}(\mathbb{D}),t)}})}{V(X_{\min{(\tau^{s,x}(\mathbb{D}),t)}})}\right] \\ &\geq \; \mathbb{E}\, \left[\exp(-c\,(\min{(\tau^{s,x}(\mathbb{D}),t)} - s)) \cdot \frac{V(X_{\min{(\tau^{s,x}(\mathbb{D}),t)}})}{\sup_{y\in\mathbb{D}} V(y)}\right] \\ &= \; \mathbb{E}\, \left[\frac{W(\min{(\tau^{s,x}(\mathbb{D}),t)}, X_{\min{(\tau^{s,x}(\mathbb{D}),t)}})}{\sup_{y\in\mathbb{D}} V(y)}\right] \; \geq \; \frac{V(x)}{\sup_{y\in\mathbb{D}} V(y)} \; = \; \frac{1+x^2}{1+K^2} \end{aligned}$$

for all $t \geq s \geq 0$. By taking limit $t \to +\infty$ in this inequality, this leads to

$$\mathbb{E}\, \exp(-c\,\tau^{s,x}(\mathbb{D})) \; \geq \; \exp(-c\,s)\left(\frac{1+x^2}{1+K^2}\right) \; > \; 0.$$

Therefore we have

$$\begin{aligned} \mathbb{P}(\tau^{s,x}(\mathbb{D}) < +\infty) \; &= \; \mathbb{E}\, I_{\{\tau^{s,x}(\mathbb{D})<+\infty\}} \geq \mathbb{E}\, \left[\exp(-c\,\tau^{s,x}(\mathbb{D}))I_{\{\tau^{s,x}(\mathbb{D})<+\infty\}}\right] \\ &= \; \mathbb{E}\, \left[\exp(-c\,\tau^{s,x}(\mathbb{D}))\left(I_{\{\tau^{s,x}(\mathbb{D})<+\infty\}} + I_{\{\tau^{s,x}(\mathbb{D})=+\infty\}}\right)\right] \\ &= \; \mathbb{E}\, [\exp(-c\,\tau^{s,x}(\mathbb{D}))] \; > \; 0, \end{aligned}$$

where I_S denotes the indicator function of the set S. Hence, the exit time $\tau^{s,x}(\mathbb{D})$ must be finite with some positive probability. □

2.2. Regularity of $[0, K]$ with multiplicative noise. In contrast to previous result, there is a quite general class of SDEs (2) which provides almost surely regular stochastic processes with respect to domain $\mathbb{D} = [0, K]$ with $K \geq 1$.

Theorem 2.2. *Let $X(0) \in \mathbb{D} = [0, K]$ be independent of σ-algebra $\sigma(W(t), t \geq 0)$. Then, under the conditions that $\alpha \geq 1$, $\beta \geq 1$, $K \geq 1$, $\rho \geq 0$, $\lambda \geq 0$, $\mu \geq 0$, the stochastic process $\{X(t), t \geq 0\}$ governed by equation (2) is regular on $\mathbb{D} = [0, K]$, i.e. we have $\mathbb{P}(X(t) \in [0, K]) = 1$ for all $t \geq 0$. Moreover, regularity on $\mathbb{D}$ implies boundedness, uniqueness, continuity and Markov property of the strong solution process $\{X(t), t \geq 0\}$ of SDE (2) whenever $X(0) = 0$ (a.s.), $X(0) = K$ (a.s.) or*

$$\mathbb{E}\left[\ln(X(0)(K - X(0)))\right] > -\infty.$$

Remark. To avoid technical complications, define the diffusion coefficient $b(x)$ to be zero outside $[0, K]$. Of course, the condition α, $\beta \geq 1$ eliminates some of the SDEs as mentioned in the introduction. The case $\alpha \geq 1$ and $\beta \geq 0$ is addressed in subsections below. Note that the requirement $\alpha \geq 1$ is a reasonable one. This can be seen from the fact that modeling in population models is motivated by modeling per-capita-growth rates (modeled as in (1)). Similar argumentation applies to models in finance (asset pricing) and marketing (innovation diffusion).

Proof. Define drift $a(x) = (\rho+\lambda x)(K-x)-\mu x$ and diffusion $b(x) = \sigma x^{\alpha}(K-x)^{\beta}$ for $x \in [0, K]$. Take sequence of open domains $\mathbb{D}_n := (\exp(-n), K - \exp(-n)), n \in \mathbb{N}$. Then, equation (2) is well–defined, has unique, bounded and Markovian solution up to random time $\tau^{s,x}(\mathbb{D}_n)$, due to Lipschitz continuity and (linear) boundedness of drift $a(x)$ and diffusion $b(x)$ on $\mathbb{D}_n$. Now, use Lyapunov function $V \in C^2(\mathbb{D})$ defined on $\mathbb{D} = (0, K)$ via

$$V(x) \;=\; K - \ln(x(K - x))\,.$$

Note that $V(x) = K - \ln(x(K - x)) = x - \ln(x) + K - x - \ln(K - x) \geq 2$ for $x \in \mathbb{D} = (0, K)$. Now, fix initial time $s \geq 0$, introduce a new Lyapunov function $W \in C^{1,2}([s, +\infty)\text{x}\mathbb{D})$ by $W(t, x) = \exp(-c(t - s))V(x)$ for all $(t, x) \in [s, +\infty)\,\text{x}\,\mathbb{D}$, where

$$c = \frac{\rho + \lambda K + \sigma^2 K^{2\alpha+2\beta-2} + \mu}{2}.$$

Then $V \in C^2(\mathbb{D})$ and $W \in C^{1,2}([s, +\infty)\,\text{x}\,\mathbb{D}_n)$. Define $\mathcal{L}$ as infinitesimal generator as in the proof of Theorem 2.1 above (see expression (4)). Calculate

$$\mathcal{L}V(x) = \Big((\rho+\lambda x)(K-x) - \mu x\Big)\left[\frac{-1}{x} + \frac{1}{K-x}\right] + \frac{\sigma^2}{2}x^{2\alpha}(K-x)^{2\beta}\left[\frac{1}{x^2} + \frac{1}{(K-x)^2}\right]$$

for $x \in \mathbb{D} = (0, K)$. An elementary calculus-based estimate leads to $\mathcal{L}V(x) \leq c \cdot V(x)$ on $\mathbb{D}$. Consequently, we have

$$V(x) \geq 2, \qquad \inf_{y \in \mathbb{D}\setminus\mathbb{D}_n} V(y) > 1 + n, \qquad \mathcal{L}V(x) \leq c \cdot V(x) \qquad \forall x \in \mathbb{D}\,.$$

Therefore one may conclude that $\mathcal{L}W(t, x) \leq 0$, since $\mathcal{L}V(x) \leq c \cdot V(x)$. Introduce $\tau_n := \min(\tau^{s,x}(\mathbb{D}_n), t)$. After applying Dynkin's formula (averaged Itô formula), one finds that $\mathbb{E}\; W(\tau_n, X_{\tau_n}) \leq V(x)$ ($X_s = x$ is deterministic!), hence

$$\mathbb{E}\left[\exp(c(t - \tau_n))V(X_{\tau_n})\right] \;\leq\; \exp(c(t - s))V(x)\,.$$

Using this fact, $x \in \mathbb{D}_n$ (n large enough), one estimates

$$\begin{aligned} 0 \le \mathbb{P}(\tau^{s,x}((0,K)) < t) \ &\le\ \mathbb{P}(\tau^{s,x}(\mathbb{D}_n) < t) = \mathbb{P}(\tau_n < t) \ =\ \mathbb{E}\, \mathbb{I}_{\tau_n < t} \\ &\le\ \mathbb{E}\left[\exp(c(t-\tau_n)) \cdot \frac{V(X_{\tau^{s,x}(\mathbb{D}_n)})}{\inf_{y \in \mathbb{D}\setminus\mathbb{D}_n} V(y)} \cdot \mathbb{I}_{\tau_n<t}\right] \\ &\le\ \exp(c(t-s)) \cdot \frac{V(x)}{\inf_{y\in\mathbb{D}\setminus\mathbb{D}_n} V(y)} \le \exp(c(t-s)) \cdot \frac{V(x)}{1+n} \underset{n\to+\infty}{\longrightarrow} 0\,, \end{aligned}$$

for all fixed $t \in [s, +\infty)$, where $\mathbb{I}_{(.)}$ represents the indicator function of subscribed random set. Consequently

$$\mathbb{P}(\tau^{s,x}(\mathbb{D}) < t) \ =\ \lim_{n\to+\infty} \mathbb{P}(\tau^{s,x}(\mathbb{D}_n) < t) \ =\ 0,$$

for $x \in (0,K)$. After discussion of the trivial invariance behavior of $X(t)$ when $X_0 = 0$ or $X_0 = K$, (almost sure) regularity of $X(t)$ on $[0,K]$ follows immediately. Eventually, uniqueness, continuity and Markov property is obtained by a result from Khas'minskiǐ [18] (see Theorem 4.1, p. 84). □

2.3. Regularity of $[0, +\infty)$ **with multiplicative noise.** For ecological and financial applications, models based on SDEs (2) should possess regular solutions with respect to domain $\mathbb{D} = [0, +\infty)$. Such a property can be guaranteed as follows. Modify SDE (2) to Itô SDE

$$(5)\quad dX(t) = [(\rho+\lambda X(t))(K-X(t))-\mu X(t)]\,dt+\sigma I(X(t))X(t)^\alpha(K-X(t))^\beta dW(t)$$

where $I(x)$ denotes the indicator function of interval $(0, K)$.

Theorem 2.3. *Let $X(0) \in \mathbb{D} = [0, +\infty)$ be independent of σ-algebra $\sigma(W(t), t \ge 0)$. Then, under the conditions $\alpha \ge 1$, $\beta \ge 0$, $\rho \ge 0$, $\lambda \ge 0$, $\mu \ge 0$, $K \ge 1$, the stochastic process $\{X(t), t \ge 0\}$ governed by (5) is regular on $\mathbb{D} = [0, +\infty)$, i.e. we have $\mathbb{P}(X(t) \in [0, +\infty)) = 1$ for all $t \ge 0$. Moreover, regularity on $\mathbb{D}$ implies boundedness, uniqueness, continuity and Markov property of the strong solution process $\{X(t), t \ge 0\}$ of SDE (5) whenever $X(0) = 0$ (a.s.) or*

$$\mathbb{E}\,[X(0) - \ln(X(0))] < +\infty.$$

Remark. For simplicity, define the diffusion coefficient $b(x) = 0$ outside $[0, K]$.

Proof. Define drift $a(x) = (\rho + \lambda x)(K - x) - \mu x$ for $x \in \mathbb{R}^1_+$ and diffusion $b(x) = \sigma x^\alpha (K - x)^\beta$ for $x \in [0, K]$. Take sequence of open nonrandom intervals $\mathbb{D}_n := (\exp(-n), \exp(n))$ for $n \in \mathbb{N}$ with $n \ge \ln(K)$. Then, equation (5) is well–defined, and it has a unique, bounded, continuous and Markovian solution up to random time $\tau^{s,x}(\mathbb{D}_n)$, due to local Lipschitz continuity and (linear) boundedness of drift $a(x)$ and diffusion $b(x)$ on $\mathbb{D}_n$. Now, use Lyapunov function V defined on $\mathbb{D} = (0, +\infty)$ via

$$V(x) \ =\ x - \ln(x)\,.$$

Obviously, $V(x) \ge 1$ holds for all $x \in \mathbb{R}^1_+$. Now, fix initial time $s \ge 0$, introduce a new Lyapunov function W by $W(t, x) = \exp(-c(t - s))V(x)$ for all $(t, x) \in [s, +\infty) \,\mathrm{x}\, \mathbb{D}$, where

$$c = \rho(\sqrt{K}-1)^2 + \lambda\frac{(K-1)^2}{4} + \mu + \frac{\sigma^2}{2}K^{2\alpha+2\beta-2}.$$

Then $V \in C^2(\mathbb{D}_n)$ and $W \in C^{1,2}([s, +\infty) \,\mathrm{x}\, \mathbb{D}_n)$. Define $\mathcal{L}$ as infinitesimal generator as in the proof of Theorem 2.1 (see (4)) with diffusion

$$b(x) = \begin{cases} \sigma x^\alpha (K-x)^\beta & \text{if} \quad 0 \le x \le K \\ 0 & \text{if} \quad x > K \end{cases}$$

and drift coefficients $a(x) = (\rho + \lambda x)(K - x) - \mu x$ for $x \geq 0$. Calculate

$$\mathcal{L}V(x) = [(\rho + \lambda x)(K - x) - \mu x]\,\frac{x-1}{x} + \frac{\sigma^2}{2} I(x) x^{2\alpha} (K - x)^{2\beta} \frac{1}{x^2}$$

for $x \in \mathbb{D} = (0, +\infty)$, where $I(x)$ is the indicator function of $(0, K)$. Note that the estimates

$$V(x) \geq 1, \qquad \inf_{y \in \mathbb{D} \setminus \mathbb{D}_n} V(y) > n, \quad \mathcal{L}V(x) \leq c \cdot V(x) \qquad \forall x \in \mathbb{D}$$

hold. Therefore one may conclude that $\mathcal{L}W(t, x) \leq 0$, since $\mathcal{L}V(x) \leq c \cdot V(x)$. Introduce $\tau_n := \min(\tau^{s,x}(\mathbb{D}_n), t)$. After applying Dynkin's formula (averaged Itô formula), one finds that $\mathbb{E}\, W(\tau_n, X_{\tau_n}) \leq V(x)$ ($X_s = x$ is deterministic!), hence

$$\mathbb{E}\left[\exp(c(t - \tau_n))V(X_{\tau_n})\right] \;\leq\; \exp(c(t - s))V(x)\,.$$

Using this fact, $x \in \mathbb{D}_n$ (n large enough), one estimates

$$\begin{aligned}
0 \leq \mathbb{P}(\tau^{s,x}((0, K)) < t) \;&\leq\; \mathbb{P}(\tau^{s,x}(\mathbb{D}_n) < t) = \mathbb{P}(\tau_n < t) \;=\; \mathbb{E}\, \mathbb{I}_{\tau_n < t} \\
\leq\;& \mathbb{E}\left[\exp(c(t - \tau_n)) \cdot \frac{V(X_{\tau^{s,x}(\mathbb{D}_n)})}{\inf_{y \in \mathbb{D} \setminus \mathbb{D}_n} V(y)} \cdot \mathbb{I}_{\tau_n < t}\right] \\
\leq\;& \exp(c(t - s)) \cdot \frac{V(x)}{\inf_{y \in \mathbb{D} \setminus \mathbb{D}_n} V(y)} \leq \exp(c(t - s)) \cdot \frac{V(x)}{n} \quad \underset{n \to +\infty}{\longrightarrow} \quad 0\,,
\end{aligned}$$

for all fixed $t \in [s, +\infty)$, where $\mathbb{I}_{(.)}$ represents the indicator function of subscribed random set. Consequently

$$\mathbb{P}(\tau^{s,x}(\mathbb{D}) < t) \;=\; \lim_{n \to +\infty} \mathbb{P}(\tau^{s,x}(\mathbb{D}_n) < t) \;=\; 0,$$

for $x \in \mathbb{D} = (0, +\infty)$. After recognizing the trivial invariant solution of $X(t) = 0$ when $X_0 = 0$, regularity of $X(t)$ on $\mathbb{D} = [0, +\infty)$ follows immediately. Eventually, uniqueness, continuity and Markov property is obtained by a result from Khas'minskiǐ [18] (see Theorem 4.1, p. 84). □

Remark. As a by-product of proofs of Theorems 2.2 and 2.3, we have proved that the solutions $X(t)$ started in the open intervals $\mathbb{D} = (0, K)$ and $\mathbb{D} = (0, +\infty)$ never hit the boundaries of these intervals in a finite time t. That is, from biologically oriented point of view, the solutions of those models under related specified type of environmental and parametric noise are almost surely persistent - an important qualitative property in biological context.

Eventually, consider (2) with the ecologically motivated subclass of Itô SDEs

$$dX(t) = [(\rho + \lambda X(t))(K - X(t)) - \mu X(t)]\,dt + \sigma[X(t)]^{\alpha}|K - X(t)|^{\beta} dW(t) \tag{6}$$

with $\alpha = 2.0$ and $\beta = 0$ or $\alpha = 1.5$ and $\beta = 0.5$ or $\alpha = 1.0$ and $\beta = 0.5$. More general, we may refer to the case $1 \leq \alpha + \beta \leq 2$ below.

Theorem 2.4. *Let $X(0) \in \mathbb{D} = [0, +\infty)$ be independent of σ-algebra $\sigma(W(t), t \geq 0)$. Then, under the conditions $\rho \geq 0$, $2\lambda \geq \sigma^2$, $\mu \geq 0$, $K \geq 1$, $\alpha \geq 1$, $1 \leq \alpha + \beta \leq 2$, the stochastic process $\{X(t), t \geq 0\}$ governed by (6) is regular on $\mathbb{D} = [0, +\infty)$, i.e. we have $\mathbb{P}(X(t) \in [0, +\infty)) = 1$ for all $t \geq 0$. Moreover, regularity on $\mathbb{D}$ implies boundedness, uniqueness, continuity and Markov property of the strong solution process $\{X(t), t \geq 0\}$ of SDE (6) whenever $X(0) = 0$ (a.s.) or*

$$\mathbb{E}\,[X(0) - \ln(X(0))] < +\infty.$$

Remark. The condition $2\lambda \geq \sigma^2$ may be interpreted as the "small noise case". Interestingly, the equation $2\lambda = \sigma^2$ is well-known to exhibit a bifurcation point (threshold case) for stability of linear SDEs (see [3], [39]). If $2\lambda \geq \sigma^2$ then the linear system is (mean square) stable. Some kind of stability is required in biological and financial modeling due to limited resources. So, it seems to be reasonable to restrict to a stable case. Moreover, to prevent our model from explosions and to control the uniform boundedness of some moments under the case $1 \leq \alpha + \beta$, we need to require that $2\lambda \leq \sigma^2$ (resulting from the specific Lyapunov function V below). This can be easily checked also by numerical simulations. Currently, we have no hope to tackle the case $2\lambda < \sigma^2$ and $\alpha + \beta \geq 2$ for SDE (6) (also supported by a lack of meaningfulness in most applications).

Proof. Similar to proof before. Just note that the infinitesimal generator $\mathcal{L}$ satisfies

$$\mathcal{L}V(x) = [(\rho + \lambda x)(K - x) - \mu x]\,\frac{x-1}{x} + \frac{\sigma^2}{2}x^{2\alpha-2}|K - x|^{2\beta}$$

for $x \in \mathbb{D} = (0, +\infty)$, where $V(x) = x - \ln(x) \geq 1$. Clearly, we recognize the dissipative character of $\mathcal{L}V(x)$ with $\max_{x\in\mathbb{D}} \mathcal{L}V(x) < +\infty$ under $2\lambda \geq \sigma^2$ and $1 \leq \alpha + \beta \leq 2$. Moreover, under latter parameter conditions, one may estimate its values by $\mathcal{L}V(x) \leq c \leq cV(x)$ with constant

$$c = \rho(\sqrt{K} - 1)^2 + \lambda\frac{(K-1)^2}{4} + \mu + \frac{\sigma^2}{2}K^{2\alpha+2\beta-2} + g(x_{max})$$

where $g(x) = \lambda(K - x)(x - 1) + \sigma^2 x^{2\alpha-2}(x - K)^{2\beta}/2$ for all $x \geq K \geq 1$. In fact, we have $x_{max} = K/2 + \lambda/(2\lambda - \sigma^2)$ for $\alpha = 1.5$, $\beta = 0.5$. It remains to repeat the same steps using W as in proof of Theorem 2.3 to complete the proof. □

3. Numerical regularization (almost sure invariance and monotonicity)

Numerical regularization (the preservation of invariance of certain subsets under disretization while keeping convergence orders of related standard methods) is generally aiming at the construction of convergent and appropriately bounded numerical approximations for SDEs. First, we introduce the notion of regular discrete time processes.

Definition 3.1. *A random sequence* $(Y_i)_{i\in\mathbb{N}}$ *is called* **regular on** *(or invariant with respect to given domain)* $\mathbb{D} \subset \mathbb{R}^d$ *iff* $\mathbb{P}(Y_i \in \mathbb{D}) = 1$ *for all* $i \in \mathbb{N}$*, otherwise* **nonregular** *(not invariant with respect to* $\mathbb{D}$*).*

Throughout this work we only consider such random sequences which have a direct link to numerical solution of SDEs. That is that one interpretes random values Y_i as values of an approximation Y for exact solution X at times $t_i \in [0, T]$. For example, **Balanced Implicit Methods (BIMs)** (see [28], [38], [43]) provide schemes to construct such sequences. For other numerical methods and details, e.g. see [2], [4], [9], [10], [19], [27], [32], [39], [40], [41], [42], [45] and [48]. BIMs turn out to be somehow efficient to guarantee both convergence towards exact solution and some algebraic constraints on numerical solutions, i.e. to guarantee numerical regularity. For general exposition in this respect, see [38] and [39]. The following BIM solves the problem of numerical regularization on bounded domain $\mathbb{D} = [0, K]$, at least in the case of $\alpha \geq 1, \beta \geq 1$. Take

$$(7)\quad Y_{n+1} = \begin{cases} Y_n + \Big((\rho + \lambda Y_n)(K - Y_n) - \mu Y_n\Big)\Delta_n + \sigma Y_n^{\alpha}(K - Y_n)^{\beta}\Delta W_n \\ +\Big(\mu\Delta_n + C(K)Y_n^{\alpha-1}(K - Y_n)^{\beta-1}|\sigma\Delta W_n|\Big)(Y_n - Y_{n+1}), \end{cases}$$

where $C = C(K)$ is an appropriate positive constant and $Y_0 \in \mathbb{D} = [0, K]$ (a.s.). Then one finds the following assertion.

Theorem 3.1. *Assume that the initial value $Y_0 \in [0, K]$ (a.s.) is independent of σ-algebra $\sigma(W(t), t \geq 0)$ and $K > 0$, $\rho \geq 0$, $\lambda \geq 0$, $\mu \geq 0$. The numerical solution $(Y_n)_{n \in \mathbb{N}}$ governed by (7) is regular on $\mathbb{D} = [0, K]$ if additionally*

$$+\infty > C(K) \geq K > 0,\ \alpha \geq 1,\ \beta \geq 1,\ 0 < \Delta_n \leq \frac{1}{\rho + \lambda K} \quad (\forall n \in \mathbb{N}).$$

Proof. Use induction on $n \in \mathbb{N}$. Then, after explicit rewriting of (7), one finds

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{\Big[(\rho + \lambda Y_n)\Delta_n + \sigma Y_n^{\alpha}(K - Y_n)^{\beta-1}\Delta W_n\Big](K - Y_n) - \mu Y_n}{1 + \mu\Delta_n + C(K)Y_n^{\alpha-1}(K - Y_n)^{\beta-1}|\sigma\Delta W_n|} \\ &\leq Y_n + \delta_n \cdot (K - Y_n) \ \leq\ K \end{aligned}$$

where

$$\delta_n = \frac{(\rho + \lambda Y_n)\Delta_n + \sigma Y_n^{\alpha}(K - Y_n)^{\beta-1}\Delta W_n}{1 + \mu\Delta_n + C(K)Y_n^{\alpha-1}(K - Y_n)^{\beta-1}|\sigma\Delta W_n|},$$

since $\delta_n \leq 1$ if $Y_n \in [0, K]$, $C(K) \geq K$ and $\Delta_n \leq 1/(\rho + \lambda K)$. Otherwise, nonnegativity of Y_{n+1} follows from the identity

$$Y_{n+1} = \frac{Y_n + (\rho + \lambda Y_n)(K - Y_n)\Delta_n + Y_n^{\alpha}(K - Y_n)^{\beta-1}((K - Y_n)\sigma\Delta W_n + C(K)|\sigma\Delta W_n|)}{1 + \mu\Delta_n + \sigma C(K)Y_n^{\alpha-1}(K - Y_n)^{\beta-1}|\Delta W_n|}$$

if $C = C(K) \geq K$. Consequently, we have $\mathbb{P}\,(\,0 \leq Y_n \leq K\,) \ =\ 1$ for all $n \in \mathbb{N}$. □

Remark. The boundedness of these sequences turns out to be essential for both interpretability within the framework of modeling issues and proof of rates of convergence. Note, *stochastic adaptation* of step sizes would form an alternative to deterministic step size selection as above. For example, for regularity, it suffices to require $\Delta_n < 1/[\rho + \lambda Y_n - \mu]_+$ for all $n \in \mathbb{N}$. However, then one has to find a truncation procedure to guarantee finiteness of corresponding algorithms to reach given terminal times T! This is particularly important for adequate long term simulations on computers.

The sequence $Y = (Y_n)_{n \in \mathbb{N}}$ following (7) is also regular on $\mathbb{D}$ under the conditions of Theorem 2.4, except for the condition $\alpha \geq 1$ is replaced by $\alpha \in [0, 1)$. However, the weights $c(x) = |\sigma|C(K)x^{\alpha-1}(K - x)^{\beta-1}$ are unbounded in this case. One even obtains regularity and boundedness of numerical increments here, but we may suspect to loose convergence speed with such methods. So the open question arises how to maintain standard convergence rates and almost sure regularity of numerical methods when $\alpha \in [0, 1)$. Who knows the right answer? (At least, the case $0.5 \leq \alpha < 1$ would be physically relevant.)

Mean square convergence of numerical sequences is examined along any sequence $(\eta = \eta^{\Delta}([0, T]))_{\Delta > 0}$ of nonrandom partitions of fixed, finite time–intervals $[0, T]$ when maximum step size $\Delta = \max \Delta_i = \max\{|t_{i+1} - t_i| : t_i, t_{i+1} \in \eta\}$ tends to zero. The **criterion of mean square convergence** is given by

$$(8) \quad \forall T > 0\ \exists K(T)\ \forall \Delta < \delta\ \forall \eta = \eta^{\Delta}([0, T]) \quad \sup_{t_i \in \eta} \mathbb{E}\,|X(t_i) - Y_i|^2 \ \leq\ K(T)\,\Delta^{2\gamma}$$

where γ is said to be the (least) **order** (rate) **of mean square convergence** of numerical sequence $Y = (Y_i)_{i \in \mathbb{N}}$.

Theorem 3.2. *The numerical approximation $(Y_n)_{n \in \mathbb{N}}$ governed by (7) is mean square converging with order $\gamma = 0.5$ towards the exact solution of (2), at least when $\alpha, \beta \geq 1$ and $Y_0 = X(0) \in \mathbb{D} = [0, K]$ (a.s.).*

Proof. Take any $\eta^{\Delta}([0,T])$ with $\Delta \leq \delta = \min\{1, 1/(\rho+\lambda K)\}$. Note that drift and diffusion coefficient of SDE (2) are uniformly bounded and Lipschitz continuous on $\mathbb{D} = [0,K]$. Thus, in view of Theorem 2.2 establishing the a.s. invariance property on $\mathbb{D} = [0,K]$, classical requirements for existence and uniqueness of strong solutions of SDEs are satisfied, cf. [3], [11] and [17]. Let $X_{t,x}(t+h)$ denote the solution of SDE (2) at time $t+h$ where $0 \leq h \leq \Delta$, started in $X(t) = X_{t,x}(t) = x$ at any time $t \in [0, T-h]$, for any $x \in \mathbb{D}$. In a similar notation, let $Y_{t,x}(t+h)$ be the integral representation of one–step approximation belonging to scheme (7), started in $Y_{t,x}(t) = x$ at any time $t \in [0, T-h]$. Note that

$$\begin{aligned} X_{t,x}(t+h) &= x + \int_t^{t+h} \Big[(\rho+\lambda X_{t,x}(s))(K - X_{t,x}(s)) - \mu X_{t,x}(s)\Big] ds \\ &\quad + \sigma \int_t^{t+h} [X_{t,x}(s)]^{\alpha}[K - X_{t,x}(s)]^{\beta} dW(s), \\ Y_{t,x}(t+h) &= x + \frac{\Big[(\rho+\lambda x)(K-x) - \mu x\Big]h + \sigma x^{\alpha}(K-x)^{\beta}(W(t+h)-W(t))}{1 + \mu h + C(K)x^{\alpha-1}(K-x)^{\beta-1}|\sigma(W(t+h)-W(t))|}. \end{aligned}$$

Obviously, both numerical approximation (7) and exact solution (2) leave domain $\mathbb{D}$ invariant, thanks to previously seen theorems. Furthermore, both have uniformly bounded second moments on any finite time–interval. Note that one can prove mean square Hölder-continuity of all solutions of (2) with exponent 0.5, i.e.

$$\mathbb{E}\,|X_{t,x}(s) - x|^2 \quad \leq \quad C_0\,|s-t|$$

for all $0 \leq t \leq s \leq T$ with $|s-t| \leq 1$, where $C_0 = C_0(T)$ is a real constant satisfying

$$0 \;\leq\; C_0 \;\leq\; 4\Big(\rho^2 K^2 + \lambda^2 \frac{K^4}{16} + \mu^2 K^2 + \sigma^2 K^{2(\alpha+\beta)}\Big)$$

Now, using this fact and almost sure invariance on $\mathbb{D} = [0,K]$, verify that

$$|\mathbb{E}\,(X_{t,x}(t+h) - Y_{t,x}(t+h))| \quad \leq \quad C_1\,h^{p_1}, \tag{9}$$

$$\mathbb{E}\,|X_{t,x}(t+h) - Y_{t,x}(t+h)|^2 \quad \leq \quad C_2\,h^{2p_2} \tag{10}$$

for all $x \in [0,K]$ and $h \leq 1$, where $p_1 \geq 1.5 = p_2 + 0.5 > p_2 = 1.0$ as local rates, $C_1 \geq 0$ and $C_2 \geq 0$ are real constants. For example, we arrive at

$$C_1 \leq \frac{2}{3}\sqrt{C_0}\Big(\mu + |\rho - \lambda K| + 2\lambda K\Big) + \mu\Big((\rho+\mu)K + \lambda\frac{K^2}{4}\Big) + C(K)K^{\alpha+\beta-1}|\sigma|(\rho+\lambda K+\mu).$$

A similar estimate can be found for C_2. Consequently, we may apply a generalization (see Theorem 6.1 in appendix, and more general Theorem 3.1 in [42] with $V(x) = 1 + x^2$ and $\mathbb{D} = [0,K]$) of Milstein's general mean square convergence theorem (i.e. Theorem 1 in [27]) to the case of SDEs on bounded manifolds in order to verify $p = 0.5$ as global rate (order) of mean square convergence of (7) towards exact solution of (2). Consequently, this completes the proof of Theorem 3.2. □

Now, consider the balanced implicit methods (BIMs)

$$Y_{n+1} \;=\; \begin{cases} Y_n + \Big((\rho+\lambda Y_n)(K - Y_n) - \mu Y_n\Big)\Delta_n + \sigma I(Y_n) Y_n^{\alpha}(K - Y_n)^{\beta}\Delta W_n \\ +\Big(\mu + (\rho/K + \lambda)[Y_n - K]_+\Big)\Delta_n (Y_n - Y_{n+1}) \\ +\Big(I(Y_n)Y_n^{\alpha-1}(K - Y_n)^{\beta}|\sigma \Delta W_n|\Big)(Y_n - Y_{n+1}) \end{cases} \tag{11}$$

where $I(Y_n)$ is the indicator function of the interval $(0,K)$ and $[a]_+$ denotes the nonnegative part of inscribed expression a. In passing, we note that more simple deterministic weights such as $c_0(x) = \mu$ would also suffice to guarantee nonnegativity for related BIMs. However, this choice implies random step size restrictions such as $\Delta_n \leq 1/[\rho + \lambda Y_n - \mu]_+$ which are hard to handle in the verification process of convergence rates. We are able to circumvent this problem of random step sizes by the following theorem.

Theorem 3.3. *Assume that the initial value $Y_0 \in [0, +\infty)$ (a.s.) is independent of σ-algebra $\sigma(W(t), t \geq 0)$ and $K > 0$, $\rho \geq 0$, $\lambda \geq 0$, $\mu \geq 0$. Then the numerical solution $(Y_n)_{n \in \mathbb{N}}$ governed by (11) is regular on $\mathbb{D} = [0, +\infty)$ for all step sizes if additionally $\alpha \geq 1$ and $\beta \geq 0$.*

Proof. Use induction on $n \in \mathbb{N}$. Suppose that $Y_n \geq 0$ (a.s.). Then, after explicit rewriting of (11), one finds that nonnegativity of Y_{n+1} follows from the identity $Y_{n+1} =$

$$\frac{Y_n + ((\rho + \lambda Y_n)(K - Y_n) + (\rho/K + \lambda) Y_n [Y_n - K]_+) \Delta_n + I(Y_n) Y_n^{\alpha} (K - Y_n)^{\beta} (\sigma \Delta W_n + |\sigma \Delta W_n|)}{1 + (\mu + (\rho/K + \lambda)[Y_n - K]_+) \Delta_n + I(Y_n) Y_n^{\alpha - 1} (K - Y_n)^{\beta} |\sigma \Delta W_n|}$$

if $\mu, \rho, \lambda, \beta \geq 0$ and $\alpha \geq 1$. Consequently, we have $\mathbb{P}(Y_n \geq 0) = 1$ for all $n \in \mathbb{N}$, provided that $Y_0 \geq 0$ (a.s.). ☐

Theorem 3.4. *Theorem 3.3 remains valid under the substitution of expressions $I(Y_n)(K - Y_n)^{\beta}$ by terms $|K - Y_n|^{\beta}$ in methods (11).*

Remark. The proof of this modification is carried out as for Theorem 3.3, hence the details can be omitted here.

4. Some simulation results for stochastic logistic equations

Here we carry out some simulation studies related to stochastic logistic equations.

4.1. Stochastic innovation diffusion in marketing. Consider models of adoption of products with parameters which are close to those in the deterministic model due to Mahajan and Wind [22]. There, for example for a data set belonging to a sale of room air conditioners, one has estimated parameters as $\rho = 0.0094$, $\lambda = 0.3748/K$, $\mu = 0$, based on the maximum adoption $K = 1.87 \cdot 10^7$. Throughout our simulations we will use the same coefficients ρ of self-innovation and λ of immitation, but we slightly reduce the carrying capacity to $K = 18700$, just for computational simplification. The initial number of adoption of room air conditioners is supposed to be $x_0 = 50$. Let us perturb these models by random noise terms as in SDE (2) with $\alpha = \beta = 1.0$. This is a natural idea due to always present statistical and measurement errors.

For the given parameter configuration, in Figures 1 and 2 the temporal evolutions of dynamics of SDE (2) are plotted. Figure 1 displays a collection of trajectories for the model (2) with additive noise (i.e. $\alpha = \beta = 0$) stopped at 90 % level of saturation constant $K = 18700$. Figure 1 is generated by forward Euler method with equidistant step sizes, whereas Figure 2 by BIMs (7) with $C(K) = K$ and equidistant step sizes $\Delta = 10^{-4}$. Figure 2 shows the mean evolution of adoption and its confidence intervals using Stratonovich and Itô interpretation compared with that of deterministic adoption. We observe a significant difference between different calculis and to deterministic adoption process. Thus, nonlinear SDEs on bounded manifolds. can be very sensitive to the choice of stochastic calculi. In general, one notices a faster initial adoption under stochasticity compared to that of deterministic model while $X_0 < K/2$. Besides, one can prove that "stochastic equilibration" (i.e. the process in which steady states are asymptotically reached) within Itô interpretation takes place below deterministic equilibrium $x^* = K$. Note, this is converse to that of Stratonovich interpretation, due to the positive difference of drift functions for initial values smaller than threshold value $\alpha K/(\alpha+\beta)$ which is equal to $K/2$ in above simulation studies. For more details on proof of pathwise and moment monotonicity, see Theorem 7.1 in appendix B. In passing, stochasticity also leads to earlier time T^* of inflection (= time–point where derivative of adoption takes its maximum), i.e. earlier peak sales (in mean sense), which represent an important assertion on the effect of environmental noise on the adoption of products for the choice of marketing policies and strategies. Similar results can be observed for models of populations under the presence of different types of environmental noise in ecological modeling.

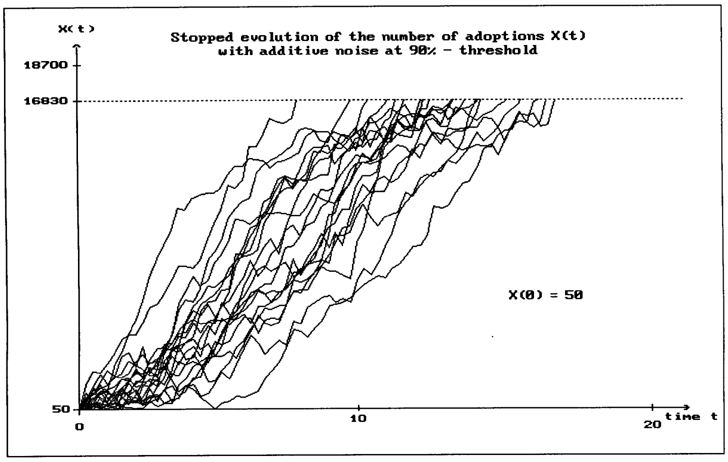

Figure 1. Stopped trajectories for additive noise with large intensity $\sigma = 1000$.

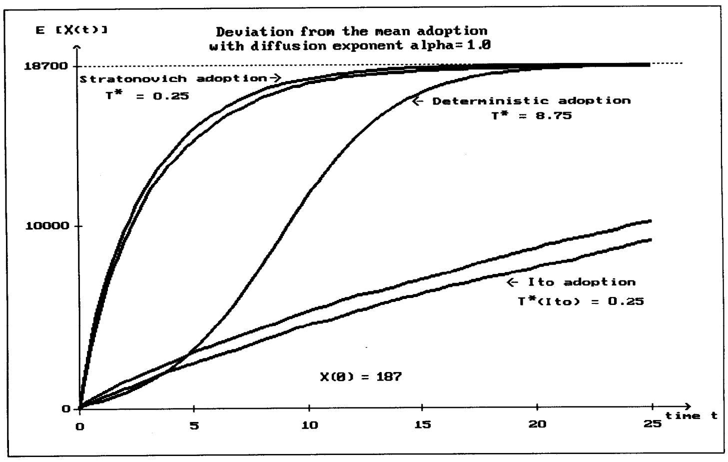

Figure 2. Mean adoption, confidence intervals and times T^* of inflection with exponent $\alpha = \beta = 1.0$ and $\sigma = 0.02$.

4.2. How to remove chaos by parametric random noise. From dynamical system theory, we know about the appearence of chaos during discretization of logistic equations. Interestingly, one can remove the presence of chaotic oscillations with appropriate methods, the proper choice of step sizes or under the appropriate modeling of present noise terms. To illustrate the latter fact, consider Itô-interpreted stochastic logistic equations (6) to model the noisy per-capita-growth rate by

$$dX(t) = \lambda X(t)(K - X(t))dt + \sigma X(t)\sqrt{|K - X(t)|}dW(t)$$

started at $X_0 = 50$, where $\lambda = 0.25$, $\sigma = 0.1$, $K = 1000$, $\alpha = 1$, $\beta = 0.5$. Notice that the quantity $2\lambda - \sigma^2 = 0.49 > 0$, hence Theorem 2.4 provides the existence of unique, continuous, Markovian solutions which are well-defined on $\mathbb{D} = [0, +\infty)$ for nonrandom initial values. Now, these models are discretized by the forward Euler method and BIMs

$$Y_{n+1} = \begin{cases} Y_n + \lambda Y_n(K - Y_n)\Delta_n + \sigma Y_n \sqrt{|K - Y_n|}\Delta W_n \\ +\Big((\lambda K I_{[0,K]}(Y_n) + \lambda[Y_n - K]_+)\Delta_n + \sqrt{|K - X_n|}|\sigma \Delta W_n|\Big)(Y_n - Y_{n+1}) \end{cases}$$

where $I_{[0,K]}(x)$ denotes the indicator function of subscribed interval $[0, K]$. Note that Theorems 3.3 and 3.4 confirm the regularity of BIMs with respect to the domain $[0, +\infty)$. This fact can be concluded also from its equivalent representation

$$Y_{n+1} = Y_n + Y_n \left(\frac{\lambda(K - Y_n)\Delta_n + \sigma\sqrt{|K - Y_n|}\Delta W_n}{1 + (\lambda K I_{[0,K]}(Y_n) + \lambda[Y_n - K]_+)\Delta_n + \sqrt{|K - X_n|}|\sigma \Delta W_n|} \right).$$

Therefore, $Y_n > 0$ and $Y_{n+1} < 2Y_n$ for all $n \in \mathbb{N}$, provided that $Y_0 > 0$ (a.s.). Moreover, we have 0 and K as trivial steady states (equilibria) and, if additionally Y_0 is independent of all increments ΔW_n then

$$\mathbb{E}\,[Y_{n+1}] \leq \mathbb{E}\,[Y_n](1 + \lambda K \Delta_n) \leq \mathbb{E}\,[Y_0]\exp(\lambda K t_{n+1})$$

as a very crude estimate of its uniformly bounded first moments. One can even prove mean square convergence of these BIMs with rate 0.5 by using the axiomatic approach along the control functions $V(x) = x^p$ as presented in [42]. This can be easily checked while assuming vanishing noise terms outside of intervals $[0, K]$.

Some simulation results for constant step sizes Δ are depicted in Figure 3 and Figure 4. We have used standard C-program runs on a LINUX operating system in order to implement the numerical algorithm of forward Euler and suggested BIMs. The random numbers $\Delta W_n \in \mathcal{N}(0, \Delta)$ are generated by the well-known Polar-Marsaglia method and the generated output in pairs of independent random numbers is used subsequently. The linearly interpolated trajectories of deterministic and 4 realizations of random paths using different sets of pairwise independent random pseudo-numbers ΔW_n are plotted in Figures 3 and 4. Figure 3 uses the same equidistant step size $\Delta = 0.01$ for all paths and Figure 4 is produced with the same equidistant $\Delta = 0.001$.

Clearly, we see that Figure 3 exhibits the chaotic structure of deterministic paths with sufficiently large step sizes while using forward Euler methods. In contrast to this, the random model using the suggested BIMs with sufficiently tuned noise intensities shows a clear lock-in behavior into the equilibrium $K = 1000$ (also called steady state) as it should happen with large step sizes too, and hence the spurious chaotic character can not be seen in the random paths such as those of appropriately chosen BIMs. This type of stabilizing effect seems to be new to the so far known literature. If noise intensity σ is very small then the chaotic regime would dominate as in the deterministic mode. If noise intensity is too large then "sudden explosions" may occur. So the tuning of proper noise intensity range is essential to observe this effect. Of course, due to the nonlinearity of logistic equations, this observation depends on the choice of step sizes and initial values as well as the interplay with

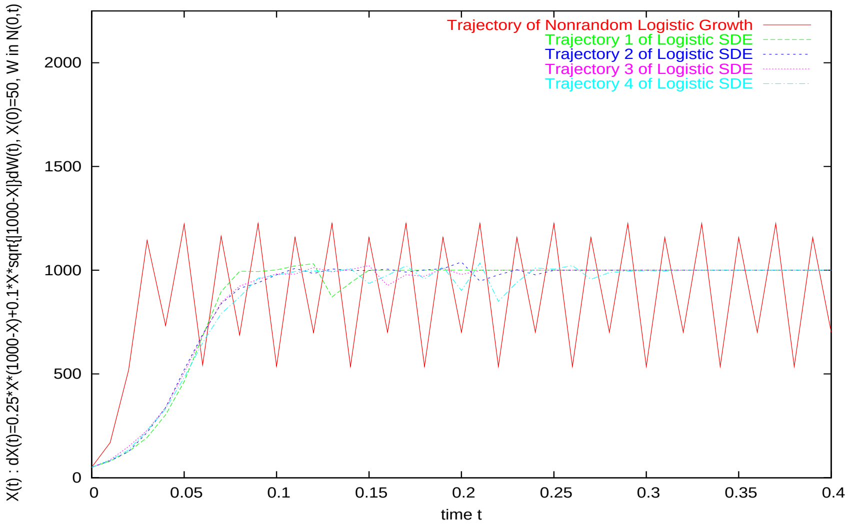

Figure 3. Asymptotic locking into the steady state $K = 1000$ of logistic equations perturbed by parametric noise compared to deterministic chaotic oscillations around $K = 1000$ with equidistant step size $\Delta = 0.01$.

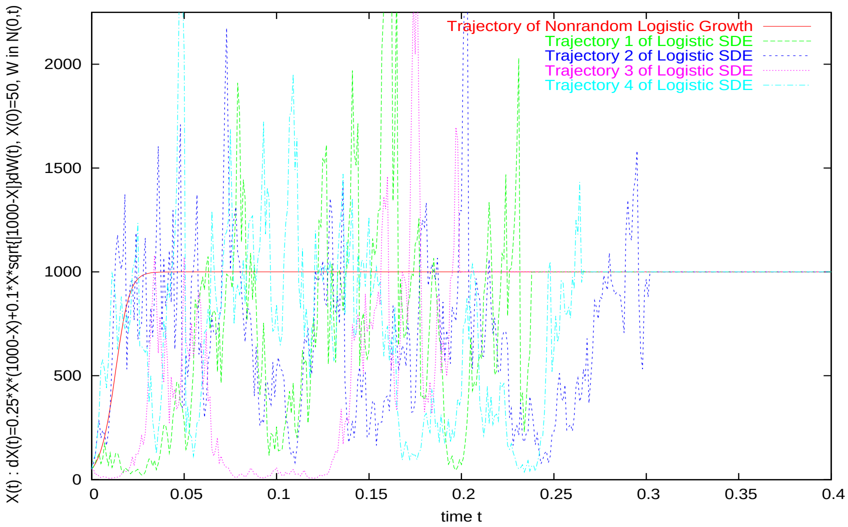

Figure 4. Asymptotic locking into the steady state $K = 1000$ of deterministic logistic equation compared to stochastic logistic equations with delayed locking behavior using equidistant step size $\Delta = 0.001$.

other parameters in a very complicated fashion. Figure 4 shows what typically happens for very small step sizes Δ (i.e. step sizes $\Delta < 1/(\lambda K)$). In this case, the lock-in into the equilibrium $K = 1000$ is clearly recognized, as we expect from the consistency of our discrete and continuous models both in deterministic and random settings. Moreover, we observe a delayed locking into the steady state $K = 1000$ in the presence of parametric white noise in Figure 4. We also obtain some empirical evidence that pathwise persistence of random logistic models with our specific form of parametric white noise perturbations is observed there for all integration times $t_n \geq 0$, despite of possible short periods of occasional stay in the neighborhood of its trivial solution 0.

5. Summary and Remarks

For adequate modeling of diffusion of innovation **in Marketing Sciences**, for interest rates in **Mathematical Finance**, for population dynamics in **Ecology**, etc., one has to consider the **problem of regular stochastic processes**, both in continuous and discrete time. In general it leads to the mathematical treatment of **s**tochastic **d**ifferential **a**lgebraic **e**quations (**SDAEs**). Classification of boundary conditions (e.g. involving an analysis with scale function and speed measure of diffusion processes, see [11], [16] and [17]) and stochastic Lyapunov–type methods are the appropriate tools to study and explain the behavior of stochastic dynamics with inherent algebraic constraints. For adequate numerical treatment, the class of **B**alanced **I**mplicit **M**ethods (**BIMs**) seems to be quite promising in order to guarantee both boundedness, stability and convergence with acceptable rates of convergence while avoiding the problem of appropriate discretization of the underlying continuous state space, removing the appearence of deterministic chaos and suppressing nondesirable effects of spurious numerical solutions.

In this paper we omitted to discuss the case $2\lambda < \sigma^2$ for SDE (6) under the remaining conditions of Theorem 2.4. The real problem is to show uniform boundedness of some moments by appropriate Lyapunov-type functions (or functionals). This delicate problem and its motivation is left to the research in the future.

More detailed studies concerning (stochastic) diffusion of innovation or population growth can easily bring out further interesting issues, e.g. effects of pulsing policies on species (i.e. pulsing parameters ρ, λ, μ, K or σ). Some mathematical clarification of well–posedness, regularity and adequate numerical solutions remains open for future research of stochastic logistic-type equations (e.g. when both α and $\beta \in (0,1)$). Besides, a laborious comparison study with real data is necessary for practical evaluation of the herein suggested models. Another interesting task would be to get more clarification on the problem **Stochastic versus Deterministic Modeling**. Also, a generalization to multi–dimensional models, to more complex domains $\mathbb{D} \subset \mathbb{R}^d$ and to stochastically changing boundary conditions is left to future. Summarizing results, it is definitively worth to **consider effects of uncertainty in models of marketing, social and ecological sciences**, not only for replication of the very erratic behavior of nature, also to get new insights.

6. Appendix A: A general theorem on mean square convergence

Under classical requirements on linear growth boundedness and smoothness of drift and diffusion coefficients of SDEs, there is a general theorem of Milstein [27] which admits to verify global rates (orders) of mean square convergence. This statement is given when both discrete and continuous time stochastic processes are living on the whole $\mathbb{R}^d$, i.e. without any algebraic constraints. In a straight forward way one can generalize this theorem to the following one in case of algebraic restrictions under further presumption that one–step mappings of both numerical method and underlying continuous time dynamics leave one and the same bounded manifold $\mathbb{D}$ invariant for any choice of sufficiently small integration step sizes. Also, it is an immediate consequence of the axiomatic approach to qualitative analysis of numerical approximation of stochastic processes as presented in Schurz [42].

For its general formulation, let $\mathbb{D}$ be any closed subdomain of $\mathbb{R}^d$. Fix a finite terminal time $T > 0$. Let $X_{t,x}(t+h)$ and $Y_{t,x}(t+h)$ be the integral representations of exact and numerical solution as above, resp. Without loss of generality, $\|\cdot\|$ denotes the Euclidean vector norm.

Theorem 6.1. *Assume that* $\mathbb{E}\,\|X(0)\|^2 < +\infty$ *and there are real, positive constants* $C_1 = C_1(T), C_2 = C_2(T)$, $p_1 \geq p_2 + 0.5, p_2 \geq 0.5$ *such that for all* $x \in \mathbb{D}$, *for all* h *with* $0 \leq h \leq \delta \leq 1$, *for all* $t \in [0, T-h]$, *we have*

$$\mathbb{P}(X_{t,x}(t+h) \in \mathbb{D}) \;=\; \mathbb{P}(Y_{t,x}(t+h) \in \mathbb{D}) \;=\; 1\,, \tag{12}$$

$$\|\mathbb{E}\,(X_{t,x}(t+h) - Y_{t,x}(t+h))\| \;\leq\; C_1\, h^{p_1} \qquad \text{and} \tag{13}$$

$$\mathbb{E}\,\|X_{t,x}(t+h) - Y_{t,x}(t+h)\|^2 \;\leq\; C_2\, h^{2p_2} \tag{14}$$

Furthermore, assume that numerical solution has uniformly bounded second moments for sufficiently small step sizes, and $X_0 = Y_0 \in \mathbb{D}$.

Then the numerical solution Y^{Δ} *belonging to one–step approximation* $Y_{t,x}(t+h)$ *is mean square converging with global order* $p = p_2 - 0.5$ *towards the exact solution of Itô-type SDE*

$$dX(t) \;=\; a(t, X(t))\,dt \,+\, \sum_{j=1}^{m} b^j(t, X(t)) dW_t^j$$

under linear–polynomial growth and Lipschitz continuity of its drift $a(t,x)$ *and diffusion functions* $b^j(t,x)$ *on the domain* $\mathbb{D}$, *where* $(W_t^j)_{j=1,2,\ldots,m}$ *are* m *mutually independent Wiener processes (also independent of random variable* $X(0) = X_0 \in \mathbb{D}$*).*

Proof. (Sketch of main steps)**.** Choose time–discretization $\eta = \eta^{\Delta}([0,T])$ with Δ sufficiently small. Now, use invariance of both exact and numerical solution with respect to one and the same closed, bounded manifold $\mathbb{D}$. Then, under this invariance property, the remaining part is analogously done as in [27], [42] and [41], by telescoping of the mean square error and local comparison of numerical one–step integral representations with truncated and stopped Itô–Taylor expansion which originates in nonstopped integral form from Wagner and Platen [48]. Therefore one finds a positive real constant C_3 such that

$$\forall t \in \eta^{\Delta}([0,T]) \quad \mathbb{E}\,\|X_{0,x}(t) - Y_{0,x}(t)\|^2 \;\leq\; C_3\, \Delta^{2p} \tag{15}$$

for all $X_{0,x} = Y_{0,x} = x \in \mathbb{D}$, and Δ sufficiently small, where $p = 0.5$ is global rate (order) of mean square convergence. Note that C_1, C_2 and C_3 are only positive real constants which may depend on finite terminal time T, but not on intermediate time t, not on Δ or h. □

Remark. We have to acknowledge that the most challenging problem to verify global mean square convergence on closed subdomains of $\mathbb{R}^d$ seems to be the almost sure invariance property of both numerical method and continuous time solution with respect to one and the same related invariant subdomain. Of course, in the example of logistic equations above it was not so difficult to find this invariant manifold. In Theorem 6.1 it suffices to require local Lipschitz continuity if a uniformly bounded functional V controlling the solutions is provided, cf. [42].

7. Appendix B: Monotonicity of solutions of stochastic logistic equations

Momentwise and pathwise monotonicity of solutions of stochastic logistic equations with respect to the modeling of different stochastic calculi can be proven. Interestingly, this initial-value depending observation depends on the pathwise crossing of critical threshold value $\alpha K/(\alpha+\beta) > 0$. More precisely speaking, consider

θ-calculus-interpreted SDE

$$(16)\quad dX(t) = [(\rho+\lambda X(t))(K-X(t)) - \mu X(t)]\,dt + \sigma X(t)^{\alpha}(K-X(t))^{\beta} \underset{\theta}{(*)}\ dW(t)$$

driven by a given standard Wiener process $W(t)$, started in $X_0 \in \mathbb{D} = [0,K] \subset \mathbb{R}^1$, where $\theta \in [0,1]$ is the calculus parameter such that $\theta = 0$ represents Itô-type and $\theta = 0.5$ Stratonovich-based calculus. For details on stochastic θ-calculus, see [44] (there α is used instead of θ as calculus parameter). Let $\tau(t) = \min(\tau, t)$ where τ is the first hitting time of threshold value $\alpha K/(\alpha+\beta)$.

Theorem 7.1. *Assume that* $0 \le X_0^{(\theta_1)} \le X_0^{(\theta_2)} \le K$ *(a.s.),* $\rho, \lambda, \mu \ge 0$, $K \ge 1$, $0 \le \theta_1 \le \theta_2$, $\alpha, \beta \ge 1$, *and* $X_0^{(\theta_i)}$ *are independent of* σ*-algebra* $\sigma(W(t), t \ge 0)$. *Then, if*

$$\mathbb{E}\,[V(X_0^{(\theta_i)})] < +\infty \quad (i = 1,2)$$

with $V(x) = K - \ln(x(K-x))$ *for* $x \in \mathbb{D} = (0,K)$*, there are unique, continuous, Markovian strong solutions of (16). Moreover, for all* $t \ge 0$*, we have*

$$0 \le \mathbb{E}\,[X_{\tau(t)}^{(\theta_1)}] \le \mathbb{E}\,[X_{\tau(t)}^{(\theta_2)}] \le \frac{\alpha K}{\alpha+\beta}, \quad \mathbb{P}\Big(0 \le X_{\tau(t)}^{(\theta_1)} \le X_{\tau(t)}^{(\theta_2)} \le \frac{\alpha K}{\alpha+\beta}\Big) = 1$$

for $0 < X_0 < \alpha K/(\alpha+\beta)$ *(a.s.), and*

$$\frac{\alpha K}{\alpha+\beta} \le \mathbb{E}\,[X_{\tau(t)}^{(\theta_2)}] \le \mathbb{E}\,[X_{\tau(t)}^{(\theta_1)}] \le K, \quad \mathbb{P}\Big(\frac{\alpha K}{\alpha+\beta} \le X_{\tau(t)}^{(\theta_2)} \le X_{\tau(t)}^{(\theta_1)} \le K\Big) = 1$$

for $K > X_0 > \alpha K/(\alpha+\beta)$ *(a.s.), whereas* 0 *and* K *are the trivial equilibrium solutions (steady states).*

Proof. The trivial steady state behavior of solutions $X(t) = X_0$ for $X_0 = 0$ or $X_0 = K$ is obvious, hence this case can be exlcuded from further discussion. At first, note that $V(x) = K - \ln(x(K-x)) = x - \ln(x) + K - x - \ln(K-x) \ge 2$ for $x \in \mathbb{D} = (0,K)$ (as seen in proof of Theorem 2.2). Second, we may transform θ-calculus-integrated SDE (16) into the equivalent Itô-type interpreted SDE

$$(17)\qquad dX(t) = \begin{cases} \Big[(\rho+\lambda X(t))(K-X(t)) - \mu X(t)\Big]dt \\ +\Big[\theta\frac{\sigma^2}{2}X(t)^{2\alpha-1}(K-X(t))^{2\beta-1}(\alpha(K-X(t)) - \beta X(t))\Big]dt \\ +\sigma X(t)^{\alpha}(K-X(t))^{\beta} dW(t) \end{cases}$$

in order to use the standard theory of Itô diffusion processes. It is clear that local unique, continuous, Markovian solutions exist up to the stopping time at the boundary of any open interval $I \subset (0,K)$. Next, we show that

$$\mathcal{L}V(x) \;=\; \left\{ \begin{array}{l} -(\rho+\lambda x)\frac{K-x}{x} + \rho + \lambda x + \mu - \mu\frac{x}{K-x} \\ \theta\frac{\sigma^2}{2}x^{2\alpha-1}(K-x)^{2\beta-1}(-\alpha\frac{K-x}{x} + \beta + \alpha - \beta\frac{x}{K-x}) \\ +\frac{\sigma^2}{2}x^{2\alpha}(K-x)^{2\beta}(\frac{1}{x^2} + \frac{1}{(K-x)^2}) \end{array} \right\} \le cV(x)$$

where we may take $c = (\rho+\lambda K)/2 + \sigma^2 K^{2\alpha+2\beta-2}(\theta(\alpha+\beta)+1)/4$. Now, existence of global solutions with regularity on $[0,K]$ follows as in the proof of Theorem 2.2. Monotonicity of first moments is concluded form pathwise monotonicity directly. Pathwise monotonicity follows from Proposition 2.18 of Karatzas and Shreve [17] (p. 293) applied to the $[0,K]$-regular diffusion process (17) by noting that the difference of the drift coefficients $a_\theta(x)$ of θ_2-interpreted SDE and that of θ_1-interpreted SDE is positive on $(0, \alpha K/(\alpha+\beta))$ and negative on $(\alpha K/(\alpha+\beta), K)$, namely we have

$$a_{\theta_2}(x) - a_{\theta_1}(x) = (\theta_2 - \theta_1)\frac{\sigma^2}{2}x^{2\alpha-1}(K-x)^{2\beta-1}(\alpha(K-x)+\beta x)\begin{cases} > 0 & \text{if} \quad x < \frac{\alpha K}{\alpha+\beta} \\ < 0 & \text{if} \quad x > \frac{\alpha K}{\alpha+\beta} \end{cases}$$

for $x \in (0, K)$ since $\theta_2 \geq \theta_1$. This completes the proof of Theorem 7.1. □

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