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INCREMENTAL UNKNOWNS AND GRAPH TECHNIQUES WITH IN-DEPTH REFINEMENT

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Abstract. With in-depth refinement, the condition number of the incremental unknowns matrix associated to the Laplace operator is $p(d)O(1/H^2)O(|\log_d h|^3)$ for the first order incremental unknowns, and $q(d)O(1/H^2)O((\log_d h)^2)$ for the second order incremental unknowns, where d is the depth of the refinement, H is the mesh size of the coarsest grid, h is the mesh size of the finest grid, $p(d) = \dfrac{d-1}{2}$ and $q(d) = \dfrac{d-1}{2}\dfrac{1}{12}d(d^2-1)$. Furthermore, if block diagonal (scaling) preconditioning is used, the condition number of the preconditioned incremental unknowns matrix associated to the Laplace operator is $p(d)O((\log_d h)^2)$ for the first order incremental unknowns, and $q(d)O(|\log_d h|)$ for the second order incremental unknowns. For comparison, the condition number of the nodal unknowns matrix associated to the Laplace operator is $O(1/h^2)$. Therefore, the incremental unknowns preconditioner is efficient with in-depth refinement, but its efficiency deteriorates at some rate as the depth of the refinement grows.

Key Words. finite differences, incremental unknowns, hierarchical basis, Laplace operator, Poisson equation, Chebyshev polynomials, Fejér's kernel.

1. Introduction

The incremental unknowns—first introduced by Temam [22] through approximate inertial manifolds and spatial multilevel finite-difference discretizations—are a natural tool to study the long-term dynamic behavior of nonlinear dissipative evolutionary equations. Although only dyadic and triadic refinements have been considered so far, Temam has already suggested the use of incremental unknowns with in-depth refinement, ibid., page 169.

As an example, the numerical solution of the incompressible Navier-Stokes equations [20, 21] with Dirichlet boundary value conditions on a staggered marker-and-cell (MAC) grid [16] entails the numerical solution of the (generalized) Poisson equation with Dirichlet and Neumann boundary conditions on a classical and staggered grid [13]; the incremental unknowns with dyadic refinement appear there as an efficient preconditioner. In what follows, we present an analysis of the Poisson equation: we first introduce the equation, then its spatial finite-difference discretization (variational approach), the self-similar interpolating continuous function, the

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incremental unknowns with in-depth refinement and the graph techniques. With $\Omega =]0,1[\times]0,1[$, the Poisson equation with Dirichlet boundary conditions is

$$\begin{cases} -\mathbf{\Delta u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} &= \varphi \quad \text{on } \Gamma = \partial\Omega. \end{cases}$$

We consider the preconditioned incremental unknowns matrix $\mathcal{K}^{-1}\widehat{A}_h$, where $\widehat{A}_h = S^T\widetilde{A}_hS$. Here $\widetilde{A}_h = \mathcal{P}^TA_h\mathcal{P}$, where $\mathcal{P}$ stands for the permutation matrix from hierarchical order to lexicographical order, $A_h = -\mathbf{\Delta}_h$, and $\mathbf{\Delta}_h$ is the finite-difference Laplace operator. In addition, S stands for the transfer matrix from the incremental unknowns ζ to the nodal unknowns u, i.e., $u = S\zeta$, and $\mathcal{K}$ stands for a suitable symmetric block diagonal matrix.

With in-depth refinement, the condition number of the incremental unknowns matrix associated to the Laplace operator is $p(d)O(1/H^2)O(|\log_d h|^3)$ for the first order incremental unknowns, and $q(d)O(1/H^2)O((\log_d h)^2)$ for the second order incremental unknowns, where d is the depth of the refinement, H is the mesh size of the coarsest grid, h is the mesh size of the finest grid, $p(d) = \dfrac{d-1}{2}$ and $q(d) = \dfrac{d-1}{2}\dfrac{1}{12}d(d^2-1)$. Furthermore, if block diagonal (scaling) preconditioning is used, the condition number of the preconditioned incremental unknowns matrix associated to the Laplace operator is $p(d)O((\log_d h)^2)$ for the first order incremental unknowns, and $q(d)O(|\log_d h|)$ for the second order incremental unknowns. For comparison, the condition number of the nodal unknowns matrix associated to the Laplace operator is $O(1/h^2)$. Therefore, the incremental unknowns preconditioner is efficient with in-depth refinement, but its efficiency deteriorates at some rate as the depth of the refinement grows.

Related conditioning analyses for dyadic refinement are done using a functional analytic argument [4, 3, 2, 24], whereas here we present a purely linear algebraic reasoning for in-depth refinement, following the corresponding analysis with dyadic refinement from [12].

This analysis consists in:

- describing the block-matrix structure of the matrix $(S\mathcal{K}^{-1}S^T)^{-1}$, with graph techniques;
- deriving an appropriate upper bound of the preconditioned generalized Rayleigh quotient
$$\frac{(v,(S\mathcal{K}^{-1}S^T)^{-1}v)}{(v,h^2(-\mathbf{\Delta}_h)v)};$$
- deriving an upper bound of the maximum eigenvalue of the incremental unknowns matrix $\widehat{\mathcal{A}}_h$.

Incremental unknowns with triadic refinement have been introduced by Poullet [19] for the numerical solution of the generalized Stokes equations. Moreover, computational experiments displayed therein (see page 37, Fig. 6) show that this condition number is $O((\log_3 h)^2)$, agreeing with the theoretical results presented herein (with the coarsest grid reduced to one point). No conditioning analysis is reported therein.

As usual, the symbols $(\cdot,\cdot)$ and $|\cdot|$ will denote the scalar product and norm of the Hilbert space $L^2(\Omega)$. Throughout this article, c will denote an absolute positive constant, which may be different at different occurrences.

The outline of this paper is as follows. In Section 2, we present the incremental unknowns framework: first we introduce the incremental unknowns with in-depth refinement in two space dimensions by means of a self-similar interpolating continuous function, and then we introduce some finite-difference operators to obtain an upper bound of the generalized Rayleigh quotient. In Section 3, we describe the multi-level structure of the matrix $(SS^T)^{-1}$ with graph techniques. In Section 4, we derive the condition number of the incremental unknowns matrix $\widehat{A}_h$: first we derive an upper bound of the generalized Rayleigh quotient without preconditioning, and then we derive an upper bound of the maximum eigenvalue of the incremental unknowns matrix. In Section 5, we consider block diagonal (scaling) preconditioning: first we describe the multi-level structure of the matrix $(S\mathcal{K}^{-1}S^T)^{-1}$ with graph techniques, then we derive an upper bound of the preconditioned generalized Rayleigh quotient, and in the end we derive estimates for the condition number of the incremental unknowns matrix with or without preconditioning. Finally, In Section 6, we summarize the distinctive features that are intrinsic to in-depth refinement and draw the conclusions.

2. The incremental unknowns framework

In this section, we first introduce the incremental unknowns with in-depth refinement in two space dimensions by means of a self-similar interpolating continuous function, and then we introduce some finite-difference operators to obtain an upper bound of the generalized Rayleigh quotient.

Herein, we consider the finite-difference variational approach [1]. Let n be a nonnegative integer. In two space dimensions, we consider the plane segment $\mathbf{\Omega} =]0,1[\times]0,1[$ and set up the classical uniform grid $\mathbf{\Omega}_h$, corresponding to the mesh size $h = 1/n$ in both directions, as follows:

$$\mathbf{\Omega}_h = \Omega_h \times \Omega_h, \text{ where } \Omega_h = \{x_k = kh \mid k = 1, \ldots, n-1\}.$$

We introduce the finite-difference vector space $\mathbf{V}_h$ that consists of restrictions to the plane segment $]0,1[\times]0,1[$ of step functions which are constant on the plane segments $[x_k, x_{(k+1)}[\times[x_l, x_{(l+1)}[$ for $k, l = 1, \ldots, n-1$. The vector space $\mathbf{V}_h$ is spanned by the nodal basis $\varpi_{(x_k,x_l)}, k, l = 1, \ldots, n-1$, with $\varpi_{(x_k,x_l)}$ equal to 1 on the plane segment $[x_k, x_{(k+1)}[\times[x_l, x_{(l+1)}[$ and 0 outside this plane segment. Then we can write a generic vector $u \in \mathbf{V}_h$ as follows:

$$u = \sum_{z \in \mathbf{\Omega}_h} u(z)\varpi_z. \tag{2.1}$$

The nodal values $\{u(z)\}_{z\in\mathbf{\Omega}_h}$ are the nodal unknowns of the generic vector u at the nodes of the grid $\mathbf{\Omega}_h$. This equation relates these nodal unknowns to the nodal basis.

Moreover, we introduce the finite-difference operators $\nabla_{ih}, \overline{\nabla}_{ih}$

$$\nabla_{ih}v(x) = \frac{1}{h}(v(x + he_i) - v(x)),$$
$$\overline{\nabla}_{ih}v(x) = \frac{1}{h}(v(x) - v(x - he_i)),$$

for $i = 1, 2$, where $e_1 = (1, 0), e_2 = (0, 1)$ is the canonical basis of $\mathbb{R}^2$.

The finite-difference Laplace operator reads

$$\mathbf{\Delta}_h = \sum_{i=1}^{2} \overline{\nabla}_{ih}\nabla_{ih}; \tag{2.2}$$

its finite-difference matrix reads

$$\mathbf{\Delta}_h = I \bigotimes \Delta_h + \Delta_h \bigotimes I, \tag{2.3}$$

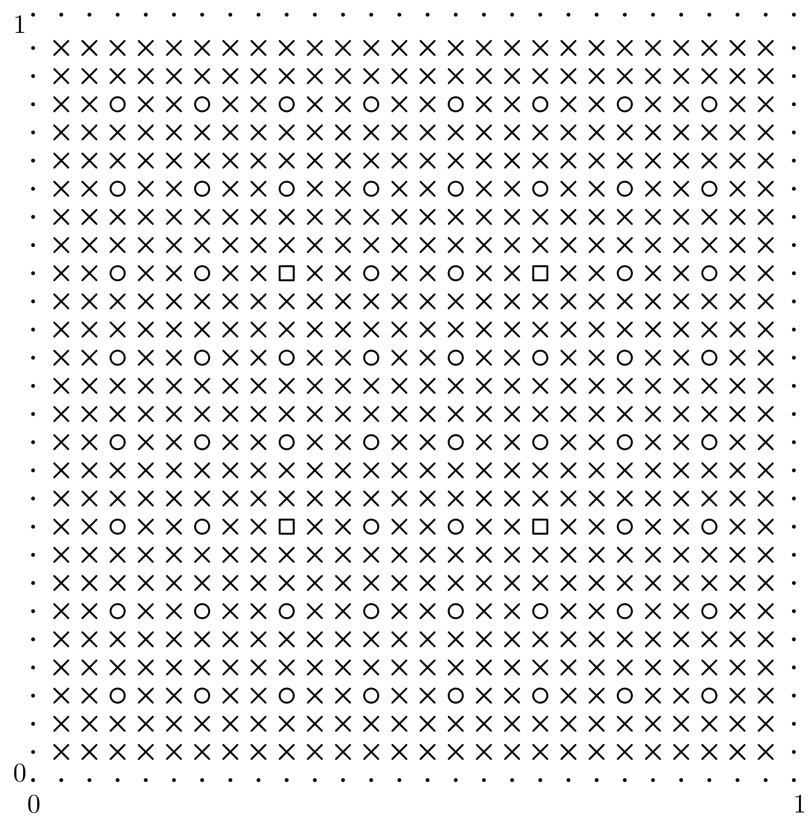

Figure 2.1: The nested sequence of grids: $d = 3, \text{ł} = 3, N = 3.$
Triadic refinement $(d = 3)$.

where Δ_h is the one-dimensional finite-difference Laplace operator, and its associated finite-difference bilinear form reads

$$((u, v))_h = \sum_{i=1}^{2} (\nabla_{ih} u, \nabla_{ih} v). \tag{2.4}$$

Now we present the multigrid-like framework used to introduce incremental unknowns. Here, we assume that $n = d^{\ell} N$, where $\ell = \text{ł} - 1$ and $d, \text{ł}, N$ are fixed integers, $d, \text{ł}, N \geq 2$. The parameter d is the depth of the refinement, the parameter ł is the number of levels, and the parameter N determines the size of the coarsest grid. For $j = \ell, \ldots, 0$, we introduce the jth-level uniform grid Ω_j corresponding to the mesh size $h_j = d^{\ell - j} h$ in both directions; therefore, we obtain the nested sequence of grids

$$\Omega_\ell \supset \Omega_{\ell-1} \supset \cdots \supset \Omega_1 \supset \Omega_0. \tag{2.5}$$

In addition, we denote by H (instead of h_0) the mesh size of the coarsest grid.

In Figure 2.1, we display the nested sequence of grids for $d = 3, \text{ł} = 3, N = 3$; this is triadic refinement ($d = 3$).

Now we propose a *hierarchical ordering* of the nodal values, the unknown values of u at the nodes of the finest grid Ω_ℓ, as follows:

- First, the nodal values of u at the nodes of the fine grid Ω_j that do not belong to the coarser grid Ω_{j-1}, for $j = \ell, \ldots, 1$.

- Finally, the nodal values of u at the nodes of the coarsest grid Ω_0.

At this point, we consider a function $\psi(t)$ with the following intrinsic properties:

- **Continuity and Compact Support**: $\psi(t)$ is continuous with compact support; there exits an absolute positive constant M such that $|\psi(t)| \leq M, \forall t$.
- **Interpolation**: $\psi(t)$ interpolates the Kronecker sequence at the integers: $\psi(0) = 1, \psi(m) = 0, \forall m \in \mathbb{Z} \setminus \{0\}$.
- **Self-similarity**: $\psi(t)$ satisfies the two-scale relation

$$\psi\left(\frac{x}{d}\right) = \psi(x) + \sum_{r=1}^{d-1} \alpha_r \left(\psi(x-r) + \psi(x+r)\right), \tag{2.6}$$

  where $\alpha_r = \psi\left(\frac{r}{d}\right), r = 1, \ldots d-1$.
- **Piecewise continuous differentiability**: $\psi(t)$ is piecewise continuously differentiable; there exists an absolute positive constant M' such that $|\psi'(t)| \leq M'$ almost everywhere.

The construction of this function $\psi(t)$ and examples giving rise to the first order incremental unknowns can be found, e.g., in [6, 7, 8, 5]. The one example giving rise to the second order incremental unknowns appears below.

Furthermore, the conditioning analysis hereafter requires:

- $\alpha_r > 0, \qquad r = 1, \ldots d-1.$
- $\alpha_r + \alpha_{d-r} = 1, \qquad r = 1, \ldots d-1.$ This requirement implies that

$$\sum_{r=1}^{d-1} \alpha_r = \frac{d-1}{2}. \tag{2.7}$$

Now we introduce the *incremental unknowns* and we recursively define them from the finest level up to the coarsest level (the coarser level is excluded, successively). First, at the nodes of the fine grid Ω_j that do not belong to the coarser grid Ω_{j-1}, the jth-level incremental unknowns are the increment of the nodal values of u to the weighted average of the nodal values of u at the neighboring nodes in the coarser grid Ω_{j-1}, for $j = \ell, \ldots, 1$. Finally, at the nodes of the coarsest grid Ω_0 the incremental unknowns are the nodal values of u. In Figure 2.2 we display a generic node of the grid $\Omega_j \setminus \Omega_{j-1}$ inside a square. In the refinement process, the open circle nodes ($\circ$) correspond to the previous (coarser) refinement, whereas the solid circle nodes ($\bullet$) correspond to the current (finer) refinement. The explicit definition of these incremental unknowns is the following. First, for $j = \ell, \ldots, 1$, we define two kinds of incremental unknowns as follows:

Incremental unknowns on the edge of a square (the extremes excluded):

$$\begin{aligned} \zeta^{(\mathsf{current})}(x,y) &= u^{(\mathsf{current})}(x,y) \\ &\quad - \Big(\alpha_r u^{(\mathsf{previous})}(x - rh_j, y) + \alpha_{d-r} u^{(\mathsf{previous})}(x + (d-r)h_j, y)\Big), \end{aligned} \tag{2.8}$$

for $x = kh_{j-1} + rh_j, y = lh_{j-1}, k, l = 1, \ldots, d^{j-1}N - 1, r = 1, \ldots, d-1$.

$$\begin{aligned} \zeta^{(\mathsf{current})}(x,y) &= u^{(\mathsf{current})}(x,y) \\ &\quad - \Big(\alpha_s u^{(\mathsf{previous})}(x, y - sh_j) + \alpha_{d-s} u^{(\mathsf{previous})}(x, y + (d-s)h_j)\Big), \end{aligned} \tag{2.9}$$

for $x = kh_{j-1}, y = lh_{j-1} + sh_j, k, l = 1, \ldots, d^{j-1}N - 1, s = 1, \ldots, d-1$.

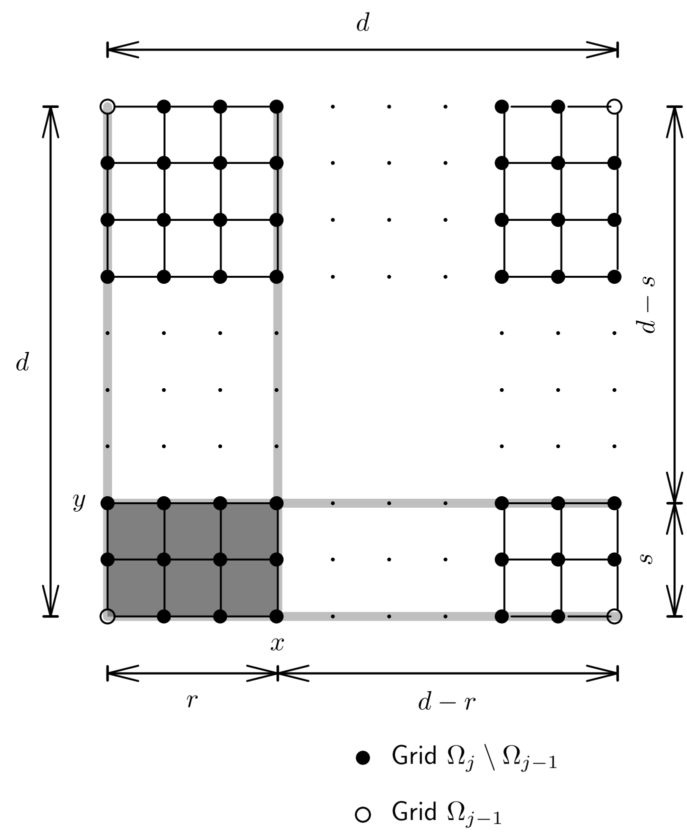

Figure 2.2: Incremental unknowns inside a square. The coarse (○) and fine (○, ●) grid points on a square.

Incremental unknowns inside a square (the edges excluded):

$$\begin{aligned}(2.10)\quad \zeta^{(\text{current})}(x,y) &= u^{(\text{current})}(x,y) \\ &- \Big(\alpha_{d-r}\alpha_{d-s}u^{(\text{previous})}(x+(d-r)h_j, y+(d-s)h_j) \\ &+ \alpha_r\alpha_s u^{(\text{previous})}(x-rh_j, y-sh_j) + \alpha_r\alpha_{d-s}u^{(\text{previous})}(x-rh_j, y+(d-s)h_j) \\ &+ \alpha_{d-r}\alpha_s u^{(\text{previous})}(x+(d-r)h_j, y-sh_j)\Big),\end{aligned}$$

for $x = kh_{j-1} + rh_j, y = lh_{j-1} + sh_j, k, l = 1, \dots, d^{j-1}N - 1, r, s = 1, \dots, d-1$.

Finally, we define the coarsest level incremental unknowns as follows:

$$(2.11)\qquad \zeta^{(\text{coarsest})}(x,y) = u^{(\text{coarsest})}(x,y),$$

for $x = kh_0, y = lh_0, k, l = 1, \dots, N-1$. Here, these incremental unknowns are the coarsest level nodal values of u.

Using the Taylor series expansion, we infer that the incremental unknowns introduced before are small quantities of order $h_j, j = 1, \dots, \ell$. In fact, because

$$\alpha_{d-r}\alpha_{d-s} + \alpha_r\alpha_s + \alpha_r\alpha_{d-s} + \alpha_{d-r}\alpha_s = 1, \qquad r, s = 1, \dots, d-1,$$

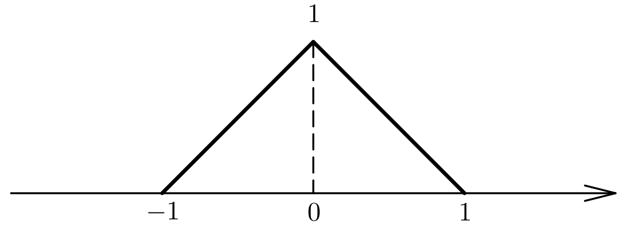

Figure 2.3: The function $\psi(t)$.
Second order incremental unknowns.

we have

$$\begin{aligned} & u(x,y) - (\alpha_{d-r}\alpha_{d-s}u(x+(d-r)h_j, y+(d-s)h_j) + \alpha_r\alpha_s u(x-rh_j, y-sh_j) \\ & \quad + \alpha_r\alpha_{d-s}u(x-rh_j, y+(d-s)h_j) + \alpha_{d-r}\alpha_s u(x+(d-r)h_j, y-sh_j)) = \\ & \quad - (\alpha_{d-r}(d-r)-\alpha_r r)\, h_j \frac{\partial u}{\partial x}(x,y) - \frac{1}{2!}\left(\alpha_{d-r}(d-r)^2 + \alpha_r r^2\right) h_j^2 \frac{\partial^2 u}{\partial x^2}(x,y) \\ & \quad - (\alpha_{d-s}(d-s)-\alpha_s s)\, h_j \frac{\partial u}{\partial y}(x,y) - \frac{1}{2!}\left(\alpha_{d-s}(d-s)^2 + \alpha_s s^2\right) h_j^2 \frac{\partial^2 u}{\partial y^2}(x,y) \\ & \qquad - (\alpha_{d-r}(d-r)-\alpha_r r)\,(\alpha_{d-s}(d-s)-\alpha_s s)\, h_j^2 \frac{\partial^2 u}{\partial x \partial y}(x,y) + O(h_j^3). \end{aligned}$$

In addition, we infer that the incremental unknowns introduced before are small quantities of order $h_j^2, j = 1, \ldots, \ell$, if and only if

$$\alpha_{d-r}(d-r) - \alpha_r r = 0, \qquad r = 1, \ldots, d-1.$$

We therefore obtain

$$\alpha_r = \frac{d-r}{d} = 1 - \frac{r}{d}, \qquad r = 1, \ldots, d-1.$$

Thus, for second order incremental unknowns, the subjacent self-similar interpolating continuous function $\psi(t)$ is linear (see Figure 2.3):

$$\psi(t) = \begin{cases} 1 - |t|, & \text{if } |t| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

As the nodal unknowns are related to the nodal basis, the incremental unknowns are related to the hierarchical basis that we introduce below.

First we introduce the hierarchical basis in one space dimension. Here only the context differentiates the nested sequence of grids (2.5) in one and two space dimensions. For $x \in \Omega_\ell$, we introduce the quasi hierarchical-basis elements $\widehat{\omega}_x^{(\ell)} = \omega_x$; for $x \in \Omega_\ell \setminus \Omega_{\ell-1}$, the hierarchical-basis elements are $\widehat{\omega}_x^{(\ell)}$. For $k = \ell - 1, \ldots, 0$ and for $x \in \Omega_k$, we introduce the quasi hierarchical-basis elements

$$\widehat{\omega}_x^{(k)} = \omega_x + \sum_{j=k+1}^{\ell} \sum_{r=1}^{d-1} \alpha_r \left(\widehat{\omega}_{x-rh_j}^{(j)} + \widehat{\omega}_{x+rh_j}^{(j)}\right);$$

for $x \in \Omega_k \setminus \Omega_{k-1}$, the hierarchical-basis elements are $\widehat{\omega}_x^{(k)}$. Furthermore, for $x \in \Omega_k$, we introduce the function $\widetilde{\psi}_x^{(k)}(t) = \psi\left(\dfrac{t-x}{h_k}\right)$, and we denote by $\widetilde{\psi}_{x,h}^{(k)}(t)$ its finite-difference discretization on the grid Ω_h.

Then using the self-similarity property of ψ, we obtain

$$\widehat{\omega}_x^{(k)}(t) = \widetilde{\psi}_{x,h}^{(k)}(t), \qquad \forall k = \ell, \ldots, 0, \forall x \in \Omega_k. \tag{2.12}$$

Therefore, all the elements of the hierarchical basis are constructed by means of one function, the self-similar interpolating continuous function $\psi(t)$, using the above formulae.

Now we introduce the hierarchical basis in two space dimensions. Using the definition of the incremental unknowns, we observe that

$$u = \sum_{z\in\mathbf{\Omega}_h} u(z)\varpi_z = \sum_{z\in\Omega_\ell\setminus\Omega_{\ell-1}} u(z)\varpi_z + \sum_{z\in\Omega_{\ell-1}} u(z)\varpi_z$$
$$= \sum_{z\in\Omega_\ell\setminus\Omega_{\ell-1}} \zeta(z)\varpi_z + \sum_{z=(x,y)\in\Omega_{\ell-1}} u(z)\widehat{\omega}_x^{(\ell-1)} \bigotimes \widehat{\omega}_y^{(\ell-1)}.$$

For $z=(x,y)\in\Omega_\ell$, we introduce the quasi hierarchical-basis elements $\widehat{\varpi}_z^{(\ell)} = \varpi_z = \omega_x \bigotimes \omega_y$; for $z\in\Omega_\ell\setminus\Omega_{\ell-1}$, the hierarchical-basis elements are $\widehat{\varpi}_z^{(\ell)}$. We obtain

$$u = \sum_{z\in\Omega_\ell\setminus\Omega_{\ell-1}} \zeta(z)\widehat{\varpi}_z^{(\ell)} + \sum_{z=(x,y)\in\Omega_{\ell-1}} u(z)\widehat{\omega}_x^{(\ell-1)} \bigotimes \widehat{\omega}_y^{(\ell-1)}.$$

Now we assume that, for $1\le k\le \ell-1$, we can write

$$u = \sum_{z\in\Omega_\ell\setminus\Omega_{\ell-1}} \zeta(z)\widehat{\varpi}_z^{(\ell)} + \cdots + \sum_{z\in\Omega_{k+1}\setminus\Omega_k} \zeta(z)\widehat{\varpi}_z^{(k+1)} + \sum_{z=(x,y)\in\Omega_k} u(z)\widehat{\omega}_x^{(k)} \bigotimes \widehat{\omega}_y^{(k)}.$$

For $z=(x,y)\in\Omega_k$, we introduce the quasi hierarchical-basis elements $\widehat{\varpi}_z^{(k)} = \widehat{\omega}_x^{(k)} \bigotimes \widehat{\omega}_y^{(k)}$; for $z\in\Omega_k\setminus\Omega_{k-1}$, the hierarchical-basis elements are $\widehat{\varpi}_z^{(k)}$. We therefore obtain

$$u = \sum_{z\in\Omega_\ell\setminus\Omega_{\ell-1}} \zeta(z)\widehat{\varpi}_z^{(\ell)} + \cdots + \sum_{z\in\Omega_{k+1}\setminus\Omega_k} \zeta(z)\widehat{\varpi}_z^{(k+1)} + \sum_{z\in\Omega_k\setminus\Omega_{k-1}} u(z)\widehat{\varpi}_z^{(k)}$$
$$+ \sum_{z=(x,y)\in\Omega_{k-1}} u(z)\widehat{\omega}_x^{(k)} \bigotimes \widehat{\omega}_y^{(k)}.$$

Furthermore, using the definition of the incremental unknowns, we observe that

$$u = \sum_{z\in\Omega_\ell\setminus\Omega_{\ell-1}} \zeta(z)\widehat{\varpi}_z^{(\ell)} + \cdots + \sum_{z\in\Omega_{k+1}\setminus\Omega_k} \zeta(z)\widehat{\varpi}_z^{(k+1)} + \sum_{z\in\Omega_k\setminus\Omega_{k-1}} \zeta(z)\widehat{\varpi}_z^{(k)}$$
$$+ \sum_{z=(x,y)\in\Omega_{k-1}} u(z)\widehat{\omega}_x^{(k-1)} \bigotimes \widehat{\omega}_y^{(k-1)}.$$

For $z=(x,y)\in\Omega_0$, the hierarchical-basis elements are

$$\widehat{\varpi}_z^{(0)} = \widehat{\omega}_x^{(0)} \bigotimes \widehat{\omega}_y^{(0)}.$$

Therefore, the two-dimensional hierarchical-basis elements $\widehat{\varpi}_{(x,y)}^{(k)}$ are twofold tensor products of the form $\widehat{\varpi}_{(x,y)}^{(k)} = \widehat{\omega}_x^{(k)} \bigotimes \widehat{\omega}_y^{(k)}$, and all the elements of the hierarchical basis are constructed by means of one function, the self-similar interpolating continuous function $\psi(t)$.

¿From the above, we infer that

$$u = \sum_{k=0}^{\ell} \sum_{z\in\Omega_k\setminus\Omega_{k-1}} \zeta(z)\widehat{\varpi}_z^{(k)}, \tag{2.13}$$

where we have assumed that $\Omega_{-1} = \emptyset$. The incremental values $\{\zeta(z)\}_{z\in\mathbf{\Omega}_h}$ are the incremental unknowns of the generic vector u at the nodes of the grid $\mathbf{\Omega}_h$. This equation relates the incremental unknowns to the hierarchical basis.

We now extend every function $u : \Omega \longrightarrow \mathbb{R}$ to a function $\tilde{u} : \mathbb{R}^2 \longrightarrow \mathbb{R}$, defined as follows:

$$\tilde{u}(x,y) = \begin{cases} u(x,y), & \text{if } (x,y) \in \Omega, \\ 0, & \text{if } (x,y) \notin \Omega, \end{cases}$$

and, from now on, we always consider the extension $\tilde{u}$, while dropping the symbol $\tilde{}$.

For $r = 1, \dots, d-1$, we introduce the symmetric finite-difference operators $\mathbf{\Delta}^{(r)}_{h_j}$, defined over the axial directions of the grid as follows:

$$\mathbf{\Delta}^{(r)}_{h_j} u(x,y) = \mathbf{\Delta}^{(r)}_{xh_j} u(x,y) + \mathbf{\Delta}^{(r)}_{yh_j} u(x,y), \tag{2.14}$$

where

$$\mathbf{\Delta}^{(r)}_{xh_j} u(x,y) = \frac{1}{h_j^2}\left(u(x - rh_j, y) - 2u(x,y) + u(x + rh_j, y)\right), \tag{2.15}$$

$$\mathbf{\Delta}^{(r)}_{yh_j} u(x,y) = \frac{1}{h_j^2}\left(u(x, y - rh_j) - 2u(x,y) + u(x, y + rh_j)\right). \tag{2.16}$$

For $r, s = 1, \dots, d-1$, we introduce the symmetric finite-difference operators $\mathbf{\Theta}^{(r,s)}_{h_j}$, defined over oblique directions of the grid as follows:

$$\begin{aligned} \mathbf{\Theta}^{(r,s)}_{h_j} u(x,y) &= \frac{1}{h_j^2}\left(u(x - rh_j, y - sh_j) - 2u(x,y) + u(x + rh_j, y + sh_j)\right) \\ &\quad + \frac{1}{h_j^2}\left(u(x + rh_j, y - sh_j) - 2u(x,y) + u(x - rh_j, y + sh_j)\right). \end{aligned} \tag{2.17}$$

We also introduce the finite-difference operators $\mathbf{\nabla}^{(r,s)}_{h_j}, \overline{\mathbf{\nabla}}^{(r,s)}_{h_j}$

$$\mathbf{\nabla}^{(r,s)}_{h_j} v = \frac{1}{h_j}\left(v(x + rh_j, y + sh_j) - v(x,y)\right), \tag{2.18}$$

$$\overline{\mathbf{\nabla}}^{(r,s)}_{h_j} v = \frac{1}{h_j}\left(v(x,y) - v(x - rh_j, y - sh_j)\right). \tag{2.19}$$

We observe that

$$\mathbf{\Delta}^{(r)}_{h_j} = \overline{\mathbf{\nabla}}^{(r,0)}_{h_j}\mathbf{\nabla}^{(r,0)}_{h_j} + \overline{\mathbf{\nabla}}^{(0,r)}_{h_j}\mathbf{\nabla}^{(0,r)}_{h_j}, \qquad r = 1, \dots, d-1, \tag{2.20}$$

$$\mathbf{\Theta}^{(r,s)}_{h_j} = \overline{\mathbf{\nabla}}^{(r,s)}_{h_j}\mathbf{\nabla}^{(r,s)}_{h_j} + \overline{\mathbf{\nabla}}^{(-r,s)}_{h_j}\mathbf{\nabla}^{(-r,s)}_{h_j}, \qquad r, s = 1, \dots, d-1. \tag{2.21}$$

¿From now on, we will write $\mathbf{\Delta}_{h_j}$ instead of $\mathbf{\Delta}^{(1)}_{h_j}$, $\mathbf{\Theta}_{h_j}$ instead of $\mathbf{\Theta}^{(1,1)}_{h_j}$, and $\mathbf{\Delta}_{xh_j}, \mathbf{\Delta}_{yh_j}$ instead of $\mathbf{\Delta}^{(1)}_{xh_j}, \mathbf{\Delta}^{(1)}_{yh_j}$, respectively.

With the above notations, we have the following discrete integration-by-parts formula (see, e.g., [23, page 481]).

Lemma 2.1. *For any $u, v \in \mathbf{V}_{h_j}$, we have*

$$\left(u, \overline{\mathbf{\nabla}}^{(r,s)}_{h_j} v\right) = -\left(\mathbf{\nabla}^{(r,s)}_{h_j} u, v\right). \tag{2.22}$$

Proof. Indeed,

$$\left(u, \overline{\boldsymbol{\nabla}}_{h_j}^{(r,s)} v\right) = \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} u(kh_j, lh_j) \overline{\boldsymbol{\nabla}}_{h_j}^{(r,s)} v(kh_j, lh_j)$$

$$= \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} u(kh_j, lh_j) \frac{1}{h_j} \left(v(kh_j, lh_j) - v(kh_j - rh_j, lh_j - sh_j)\right)$$

$$= \frac{1}{h_j} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} u(kh_j, lh_j) v(kh_j, lh_j)$$

$$- \frac{1}{h_j} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} u(kh_j, lh_j) v((k-r)h_j, (l-s)h_j)$$

(by setting first $k' = k - r, l' = l - s$ and then $k = k', l = l'$)

$$= \frac{1}{h_j} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} u(kh_j, lh_j) v(kh_j, lh_j)$$

$$- \frac{1}{h_j} \sum_{k=1-r}^{d^j N-1-r} \sum_{l=1-s}^{d^j N-1-s} u(kh_j + rh_j, lh_j + sh_j) v(kh_j, lh_j)$$

(since u, v are equal to 0 outside Ω)

$$= \frac{1}{h_j} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} u(kh_j, lh_j) v(kh_j, lh_j)$$

$$- \frac{1}{h_j} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} u(kh_j + rh_j, lh_j + sh_j) v(kh_j, lh_j)$$

$$= \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \frac{1}{h_j} \left(u(kh_j, lh_j) v(kh_j, lh_j) - u(kh_j + rh_j, lh_j + sh_j)\right) v(kh_j, lh_j)$$

$$= - \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \frac{1}{h_j} \left(u(kh_j + rh_j, lh_j + sh_j) - u(kh_j, lh_j)\right) v(kh_j, lh_j)$$

$$= - \left(\boldsymbol{\nabla}_{h_j}^{(r,s)} u, v\right).$$

This proves the lemma. □

3. Incremental unknowns and graph techniques

In this section, we describe the multi-level structure of the matrix $(SS^T)^{-1}$ with graph techniques.

Hereafter, the matrices $A = (a_{(x_k,x_l),(x_{k'},x_{l'})})$ will be of order $(n-1)^2$; their associated directed graphs $G(A)$ consist of $(n-1)^2$ vertices that are the nodes of the finest grid Ω_ℓ, and arrows from one vertex to another. An arrow leads from (x_k, x_l) to $(x_{k'}, x_{l'})$ if and only if $a_{(x_k,x_l),(x_{k'},x_{l'})} \neq 0$, and this element (coefficient) is associated to that arrow; this is a direct connection.

The product $C = AB$ of two square matrices of order $(n-1)^2$ may be done graphwise. To compute the element $c_{(x_k,x_l),(x_{k'},x_{l'})}$, we take into account all the vertices $(x_{k''}, x_{l''})$ such that there exits an arrow in $G(A)$ from (x_k, x_l) to $(x_{k''}, x_{l''})$

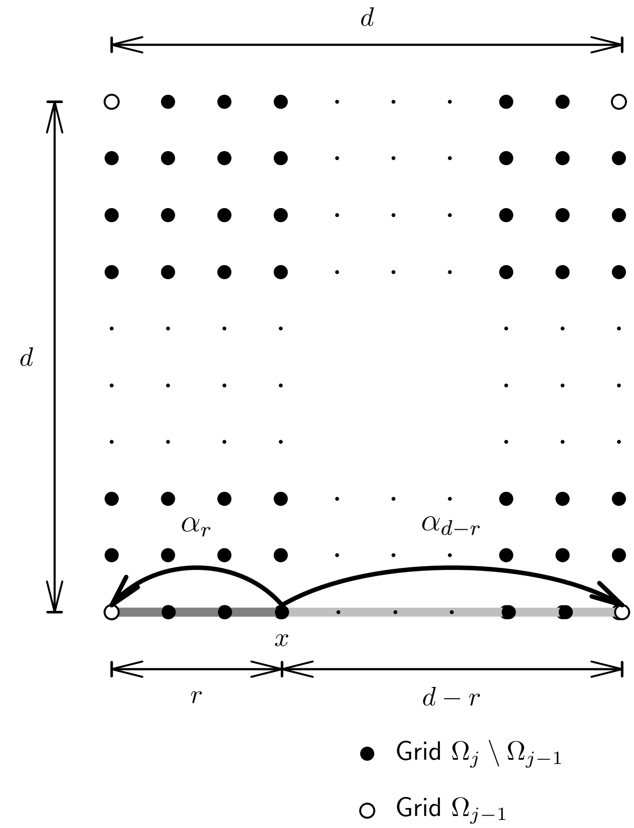

Figure 3.1: Directed graph of the matrix D_j.
Vertices on the edge of a square.

and an arrow in $G(B)$ from $(x_{k''}, x_{l''})$ to $(x_{k'}, x_{l'})$; we compute the product of the associated coefficients, and we add them for all such vertices $(x_{k''}, x_{l''})$. If there are no vertices $(x_{k''}, x_{l''})$ with such characteristics, the element $c_{(x_k,x_l),(x_{k'},x_{l'})}$ is equal to 0.

An intrinsic (i.e., invariant under permutations) description of the transfer matrix S^{-1} from the nodal unknowns to the incremental unknowns is readily done using the associated directed graph of the transfer matrix from the (previous) nodal unknowns to the (current) jth-level incremental unknowns. This is illustrated in Figure 3.1 for the generic vertices of the grid $\Omega_j \setminus \Omega_{j-1}$ on the edge of a coarse square (the extremes excluded) and in Figure 3.2 for the generic vertices of the grid $\Omega_j \setminus \Omega_{j-1}$ inside a coarse square (the edges excluded). In addition, with the indication that the generic axial coefficients are α_r and the generic oblique coefficients are $\alpha_r \alpha_s$, we have a complete definition of a square matrix D_j of order $(n-1)^2$ such that

$$S^{-1} = I - \sum_{j=1}^{\ell} D_j.$$

We display the associated directed graph of the matrix D_j^T in Figure 3.3 for the generic vertices of the grid $\Omega_j \setminus \Omega_{j-1}$ on the edge of a coarse square (the extremes excluded) and in Figure 3.4 for the generic vertices of the grid $\Omega_j \setminus \Omega_{j-1}$ inside a coarse square (the edges excluded). Now, it is immediate to see graphwise that

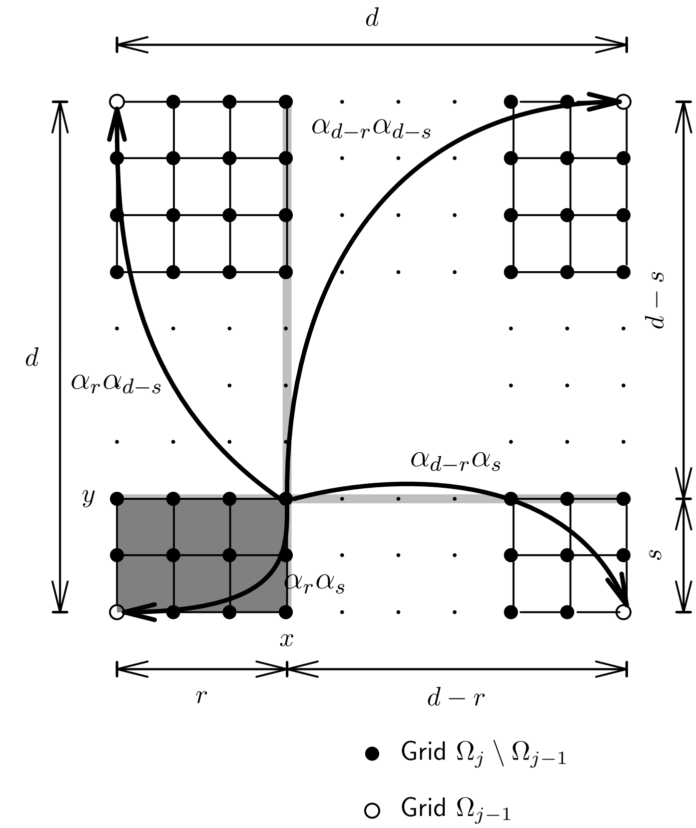

Figure 3.2: Directed graph of the matrix D_j.
Vertices inside a square.

$D_k^T D_l = 0$ for $k \neq l$; indeed, the arrows in $G(D_k^T)$ leave a node of the grid Ω_{k-1} and land on a node of the grid $\Omega_k \setminus \Omega_{k-1}$, and the arrows in $G(D_l)$ leave a node of the grid $\Omega_l \setminus \Omega_{l-1}$ and land on a node of the grid Ω_{l-1}; therefore, because $k \neq l$, the arrows in $G(D_k^T)$ and in $G(D_l)$ are not connected, and then $D_k^T D_l = 0$. ¿From there, we obtain

$$(SS^T)^{-1} = S^{-T}S^{-1} = I - \sum_{j=1}^{\ell} D_j^T - \sum_{j=1}^{\ell} D_j + \sum_{j=1}^{\ell} D_j^T D_j.$$

To compute the matrix $D_j^T D_j$, we select a generic coarse vertex in the grid Ω_{j-1}, and we consider the four nearby coarse squares. Then we need to compute the circular coefficients, associated with an arrow leaving a coarse vertex and landing on the same coarse vertex; the axial coefficients, associated with an arrow leaving a coarse vertex and landing on the nearest coarse vertices on the axes; and the oblique coefficients, associated with an arrow leaving a coarse vertex and landing on the nearest coarse vertices on the diagonals. There are no other direct connections.

The Circular Coefficients. First we select a (generic) neighbor fine vertex on the edge of a square. We can go there through a D_j^T-arrow with associated coefficient α_r, and we can come back through a D_j-arrow also with associated coefficient α_r. Then we compute the product of these coefficients, and we add them for the overall number of vertices: $r = 1, \ldots, d-1$. Since we have to consider

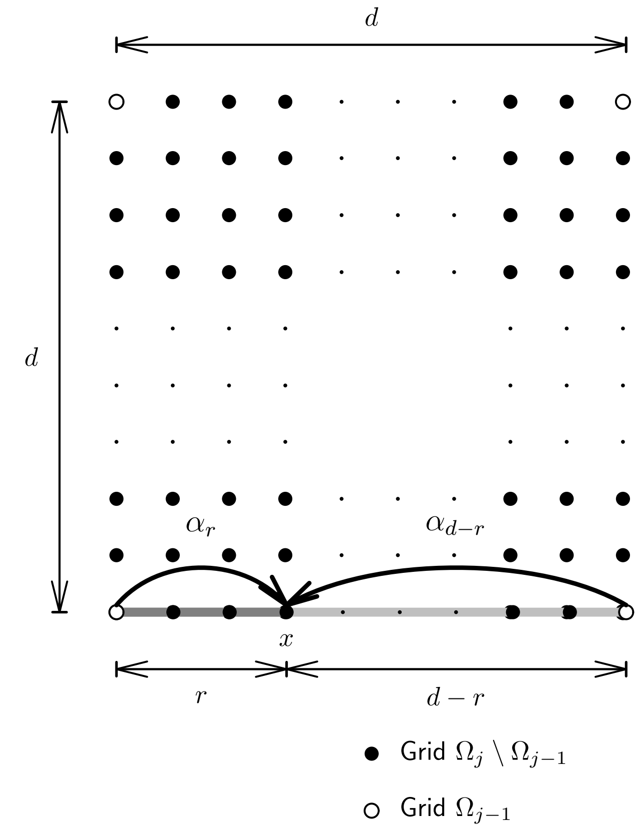

Figure 3.3: Directed graph of the matrix D_j^T.
Vertices on the edge of a square.

four edges, we obtain

$$4\sum_{r=1}^{d-1}\alpha_r\alpha_r = 4\sum_{r=1}^{d-1}\alpha_r^2.$$

Second we select a (generic) neighbor fine vertex inside a square (the edges excluded). We can go there through a D_j^T-arrow with associated coefficient $\alpha_r\alpha_s$, and we can come back through a D_j-arrow also with associated coefficient $\alpha_r\alpha_s$. Then we compute the product of these coefficients, and we add them for the overall number of vertices: $r, s = 1, \ldots, d-1$. Since we have to consider four squares, we obtain

$$4\sum_{r=1}^{d-1}\sum_{s=1}^{d-1}\alpha_r\alpha_s\alpha_r\alpha_s = 4\left(\sum_{r=1}^{d-1}\alpha_r^2\right)^2.$$

Therefore, the circular coefficients are

$$4\sum_{r=1}^{d-1}\alpha_r\alpha_r + 4\sum_{r=1}^{d-1}\sum_{s=1}^{d-1}\alpha_r\alpha_s\alpha_r\alpha_s = 4\left(\sum_{r=1}^{d-1}\alpha_r^2\right)\left(1+\sum_{r=1}^{d-1}\alpha_r^2\right).$$

The Axial Coefficients. First we select a (generic) neighbor fine vertex on the edge of a square. We can go there through a D_j^T-arrow with associated coefficient α_r, and we can go forward, to the next coarse vertex on the same edge, through a D_j-arrow with associated coefficient α_{d-r}. Then we compute the product of these

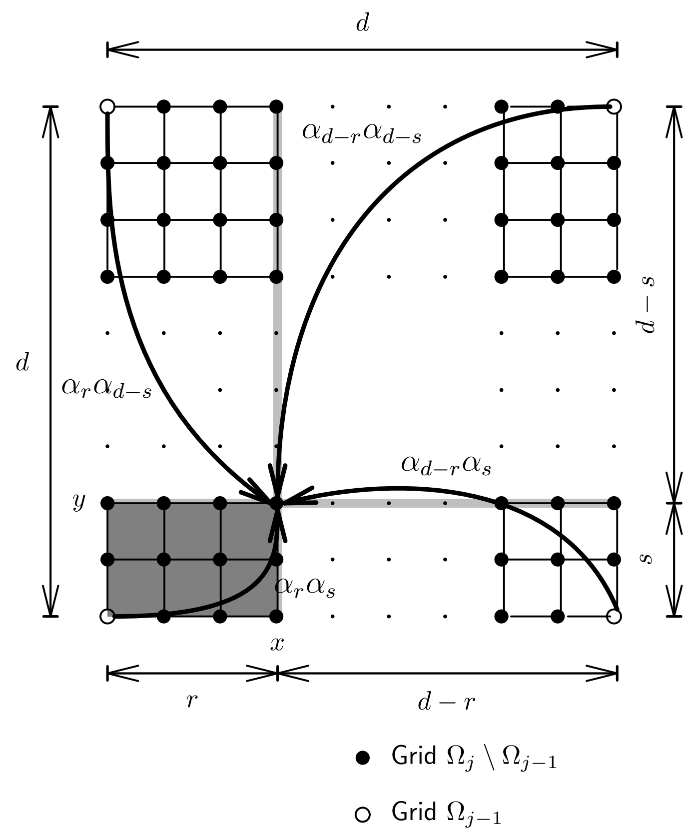

Figure 3.4: Directed graph of the matrix D_j^T.
Vertices inside a square.

coefficients, and we add them for the overall number of vertices: $r = 1, \ldots, d-1$. Since we have to consider only one edge, we obtain

$$\sum_{r=1}^{d-1} \alpha_r \alpha_{d-r} = \sum_{r=1}^{d-1} \alpha_r (1-\alpha_r) = \sum_{r=1}^{d-1} \alpha_r - \sum_{r=1}^{d-1} \alpha_r^2.$$

Second we select a (generic) neighbor fine vertex inside a square (the edges excluded). We can go there through a D_j^T-arrow with associated coefficient $\alpha_r\alpha_s$, and we can go forward, to the next coarse vertex on the same edge, through a D_j-arrow with associated coefficient $\alpha_{d-r}\alpha_s$. Then we compute the product of these coefficients, and we add them for the overall number of vertices: $r, s = 1, \ldots, d-1$. Here we have to consider two coarse squares, and we obtain

$$2\sum_{r=1}^{d-1}\sum_{s=1}^{d-1} \alpha_r\alpha_s\alpha_{d-r}\alpha_s = 2\left(\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2\right)\sum_{r=1}^{d-1}\alpha_r^2.$$

Therefore, the axial coefficients are

$$\sum_{r=1}^{d-1} \alpha_r\alpha_{d-r} + 2\sum_{r=1}^{d-1}\sum_{s=1}^{d-1}\alpha_r\alpha_s\alpha_{d-r}\alpha_s = \left(\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2\right)\left(1 + 2\sum_{r=1}^{d-1}\alpha_r^2\right).$$

The Oblique Coefficients. Here we select a (generic) neighbor fine vertex inside a square (the edges excluded). We can go there through a D_j^T-arrow with

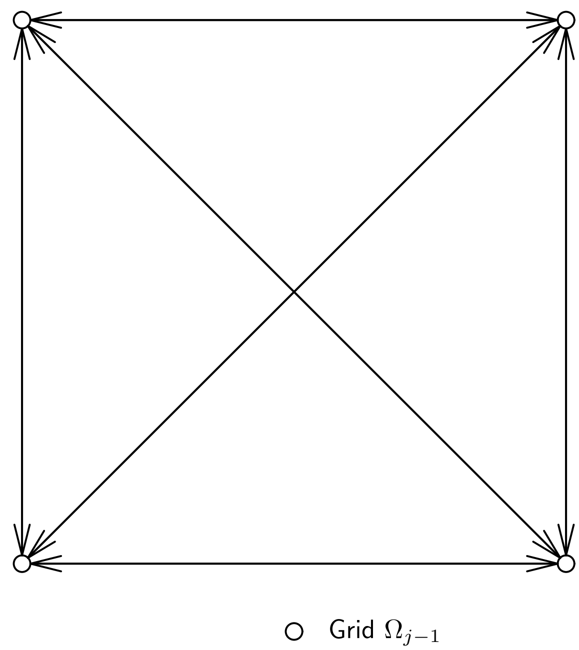

Figure 3.5: Directed graph of the matrix F_{j-1}.

associated coefficient $\alpha_r\alpha_s$, and we can go forward, to the next coarse vertex on the opposite diagonal side, through a D_j-arrow with associated coefficient $\alpha_{d-r}\alpha_{d-s}$. Then we compute the product of these coefficients, and we add them for the overall number of vertices: $r, s = 1, \ldots, d-1$. Since we have to consider only one of these coarse squares, we obtain that the oblique coefficients are

$$\sum_{r=1}^{d-1}\sum_{s=1}^{d-1} \alpha_r\alpha_s\alpha_{d-r}\alpha_{d-s} = \left(\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2\right)^2 .$$

Consequently, we conclude that

$$D_j^T D_j = 4\left(\sum_{r=1}^{d-1}\alpha_r^2\right)\left(1+\sum_{r=1}^{d-1}\alpha_r^2\right) I_{j-1} + F_{j-1}, \tag{3.1}$$

where I_{j-1} denotes the adequate identity matrix, and F_{j-1} denotes a matrix whose associated directed graph is described in Figure 3.5. Furthermore, recalling that the axial coefficients are

$$\left(\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2\right)\left(1+2\sum_{r=1}^{d-1}\alpha_r^2\right),$$

and that the oblique coefficients are

$$\left(\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2\right)^2,$$

we have a complete definition of the matrix F_{j-1}. More precisely,

(3.2)

$$F_j = h_j^2 \left(\left(\sum_{r=1}^{d-1} \alpha_r - \sum_{r=1}^{d-1} \alpha_r^2 \right) \left(1 + 2 \sum_{r=1}^{d-1} \alpha_r^2 \right) \boldsymbol{\Delta}_{h_j} + \left(\sum_{r=1}^{d-1} \alpha_r - \sum_{r=1}^{d-1} \alpha_r^2 \right)^2 \boldsymbol{\Theta}_{h_j} \right) + 4 \left(\sum_{r=1}^{d-1} \alpha_r - \sum_{r=1}^{d-1} \alpha_r^2 \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r + \sum_{r=1}^{d-1} \alpha_r^2 \right) I_j.$$

Also, it is easy to see graphwise (completing the arrows) that

$$I_j - D_j^T - D_j = h_j^2 \left(\sum_{r=1}^{d-1} \alpha_r \left(-\boldsymbol{\Delta}_{h_j}^{(r)} \right) + \sum_{r=1}^{d-1} \sum_{s=1}^{d-1} \alpha_r \alpha_s \left(-\boldsymbol{\Theta}_{h_j}^{(r,s)} \right) \right) - \left(\left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) - 1 \right) I_j - G_j \right), \tag{3.3}$$

with a square matrix G_j of order $(n-1)^2$ easily defined graphwise; we display its associated directed graph in Figure 3.6 for the generic vertices of the grid $\Omega_j \setminus \Omega_{j-1}$ on the edge of a coarse square (the extremes excluded), and in Figure 3.7 for the generic vertices of the grid $\Omega_j \setminus \Omega_{j-1}$ inside a coarse square (the edges excluded). Furthermore, recalling that the axial coefficients are α_r, α_s and the oblique coefficients are $\alpha_r \alpha_s$, we have a complete definition of the matrix G_j. To complete the definition of the finite-difference operator

$$\digamma_{h_j} = h_j^2 \left(\sum_{r=1}^{d-1} \alpha_r \left(-\boldsymbol{\Delta}_{h_j}^{(r)} \right) + \sum_{r=1}^{d-1} \sum_{s=1}^{d-1} \alpha_r \alpha_s \left(-\boldsymbol{\Theta}_{h_j}^{(r,s)} \right) \right), \tag{3.4}$$

axial and oblique (D_j^T, D_j)-arrows with their associated coefficients need, at first, to be subtracted and, afterwards, to be added (to maintain equation (3.3) unchanged) at all the vertices of the grid $\Omega_j \setminus \Omega_{j-1}$; those axial and oblique (D_j^T, D_j)-arrows with their associated coefficients account for the definition of the matrix G_j and correspond to the axial and oblique (D_j^T, D_j)-arrows with their associated coefficients defined by the finite-difference operator $-\digamma_{h_j}$, except that those axial and oblique (D_j^T, D_j)-arrows with an arrowhead landing on a coarse vertex of the grid Ω_{j-1} are discarded (see Figure 3.6 and Figure 3.7 light and straight arrows).

Now, using equations (3.1)–(3.3), we obtain

$$\begin{aligned} - D_j^T - D_j + D_{j+1}^T D_{j+1} &= \left(I_j - D_j^T - D_j \right) + \left(-I_j + D_{j+1}^T D_{j+1} \right) \\ &= h_j^2 \left(\sum_{r=1}^{d-1} \alpha_r \left(-\boldsymbol{\Delta}_{h_j}^{(r)} \right) + \sum_{r,s=1}^{d-1} \alpha_r \alpha_s \left(-\boldsymbol{\Theta}_{h_j}^{(r,s)} \right) \right) \\ &\quad + G_j \\ &\quad + h_j^2 \left(\left(\sum_{r=1}^{d-1} \alpha_r - \sum_{r=1}^{d-1} \alpha_r^2 \right) \left(1 + 2 \sum_{r=1}^{d-1} \alpha_r^2 \right) \boldsymbol{\Delta}_{h_j} + \left(\sum_{r=1}^{d-1} \alpha_r - \sum_{r=1}^{d-1} \alpha_r^2 \right)^2 \boldsymbol{\Theta}_{h_j} \right), \end{aligned}$$

and observing that

$$(SS^T)^{-1} = D_1^T D_1 + \sum_{j=1}^{\ell-1} \left(-D_j^T - D_j + D_{j+1}^T D_{j+1} \right) + (I - D_\ell^T - D_\ell),$$

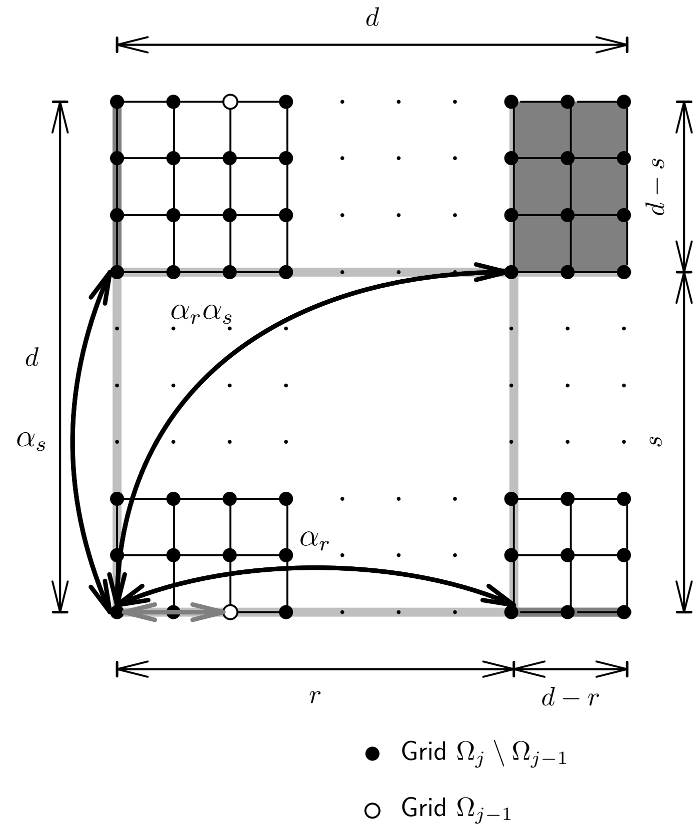

Figure 3.6: Directed graph of the matrix G_j. Vertices on the edge of a square.

we find

$$(SS^T)^{-1} = F_{h_0} + \sum_{j=1}^{\ell} F_{h_j} - (G + \widetilde{G}), \tag{3.5}$$

where

$$F_{h_0} = I_0,$$

$$F_{h_j} = h_j^2 \left(\sum_{r=1}^{d-1} \alpha_r \left(-\mathbf{\Delta}_{h_j}^{(r)} \right) + \sum_{r,s=1}^{d-1} \alpha_r \alpha_s \left(-\mathbf{\Theta}_{h_j}^{(r,s)} \right) \right), \quad \text{for } j = 1, \dots, \ell,$$

$$\begin{aligned} G &= I_0 - D_1^T D_1 + \left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) - 1 \right) I - \sum_{j=1}^{\ell} G_j \\ &= \left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) I_0 - D_1^T D_1 \right) \\ &\quad + \sum_{j=1}^{\ell} \left(\left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) - 1 \right) (I_j - I_{j-1}) - G_j \right), \end{aligned}$$

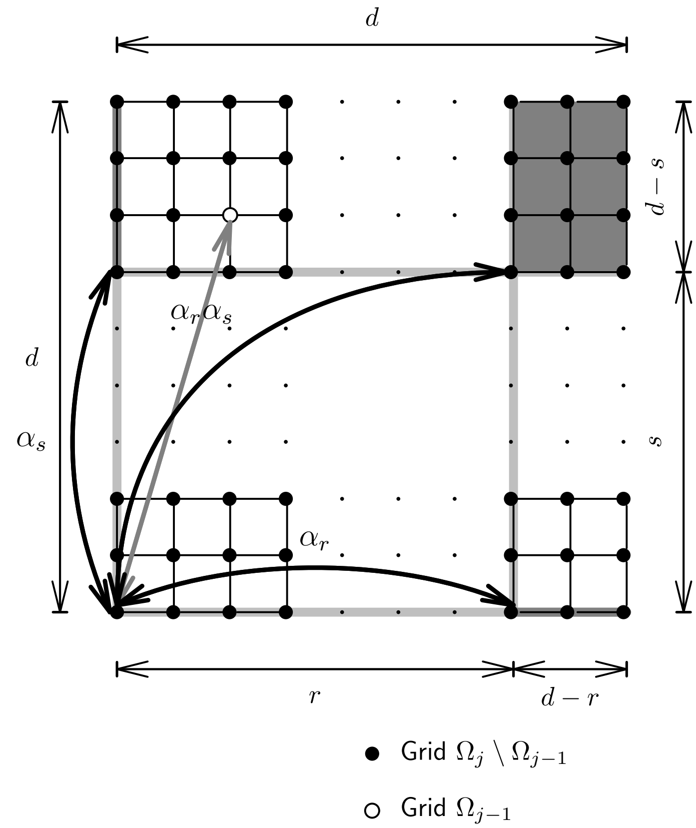

Figure 3.7: Directed graph of the matrix G_j. Vertices inside a square.

$$\widetilde{G} = \sum_{j=1}^{\ell-1} h_j^2 \left(\left(\sum_{r=1}^{d-1} \alpha_r - \sum_{r=1}^{d-1} \alpha_r^2 \right) \left(1 + 2 \sum_{r=1}^{d-1} \alpha_r^2 \right) (-\mathbf{\Delta}_{h_j}) + \left(\sum_{r=1}^{d-1} \alpha_r - \sum_{r=1}^{d-1} \alpha_r^2 \right)^2 (-\mathbf{\Theta}_{h_j}) \right).$$

Lemma 3.1. *The matrices G and $\widetilde{G}$ are positive definite.*

Proof. First, we note that the matrix

$$\left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) - 1 \right) (I_j - I_{j-1}) - G_j \tag{3.6}$$

is positive semidefinite.

Indeed, in the first place, the sum of all the associated coefficients corresponding to the axial and oblique (D_j^T, D_j)-arrows defined by the finite-difference operator $-F_{h_j}$ is

$$4 \left(\sum_{r=1}^{d-1} \alpha_r + \sum_{r=1}^{d-1} \sum_{s=1}^{d-1} \alpha_r \alpha_s \right) = 4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right).$$

In the second place, the sum of all the associated coefficients corresponding to the axial and oblique (D_j^T, D_j)-arrows discarded while defining the matrix G_j is 1. In fact, we have the following:

- For a generic vertex of the grid $\Omega_j \setminus \Omega_{j-1}$ on the edge of a square at position r within this edge, there are two axial (D_j^T, D_j)-arrows discarded. The sum of their associated coefficients is

$$\alpha_r + \alpha_{d-r} = 1, \qquad r = 1, \dots, d-1.$$

- For a generic vertex of the grid $\Omega_j \setminus \Omega_{j-1}$ inside a square (the edges excluded) at position (r, s) inside this square, there are four oblique (D_j^T, D_j)-arrows discarded. The sum of their associated coefficients is

$$\alpha_{d-r}\alpha_{d-s} + \alpha_r\alpha_s + \alpha_r\alpha_{d-s} + \alpha_{d-r}\alpha_s = 1, \qquad r, s = 1, \dots, d-1.$$

Then the Geršgorin theorem (see, e.g., [15, page 46]) implies that the matrix (3.6) is positive semidefinite if

$$\left(4\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(1+\sum_{r=1}^{d-1}\alpha_r\right)-1\right) > 0.$$

To show that, we note that $4x(1+x)-1 = 4(x-\xi)(x-\xi')$, where $\xi = -\frac{1}{2}+\frac{1}{2}\sqrt{2}, \xi' = -\frac{1}{2}-\frac{1}{2}\sqrt{2}$, and that $\sum_{r=1}^{d-1}\alpha_r = \frac{d-1}{2} > \xi = -\frac{1}{2} + \frac{\sqrt{2}}{2} > 0$.

Second, we note that the matrix

$$4\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(1+\sum_{r=1}^{d-1}\alpha_r\right) I_j - D_{j+1}^T D_{j+1} \tag{3.7}$$

is positive definite, $\forall j \geq 0$.

Indeed, using (3.1) and (3.2), we obtain

$$\begin{aligned} D_{j+1}^T D_{j+1} = h_j^2 &\left(\left(\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2\right)\left(1+2\sum_{r=1}^{d-1}\alpha_r^2\right)\boldsymbol{\Delta}_{h_j}\right. \\ &\left. + \left(\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2\right)^2 \boldsymbol{\Theta}_{h_j}\right) + 4\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(1+\sum_{r=1}^{d-1}\alpha_r\right) I_j, \end{aligned}$$

and hence

$$\begin{aligned} &4\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(1+\sum_{r=1}^{d-1}\alpha_r\right) I_j - D_{j+1}^T D_{j+1} = \\ &h_j^2\left(\left(\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2\right)\left(1+2\sum_{r=1}^{d-1}\alpha_r^2\right)(-\boldsymbol{\Delta}_{h_j}) + \left(\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2\right)^2 (-\boldsymbol{\Theta}_{h_j})\right). \end{aligned} \tag{3.8}$$

By the Geršgorin theorem, the symmetric matrices $-\boldsymbol{\Delta}_{h_j}$ and $-\boldsymbol{\Theta}_{h_j}$ are positive definite, since they are irreducibly diagonally dominant and have positive diagonal entries. Moreover, we have

$$\sum_{r=1}^{d-1}\alpha_r - \sum_{r=1}^{d-1}\alpha_r^2 = \sum_{r=1}^{d-1}\alpha_r\alpha_{d-r} > 0.$$

Hence the right-hand side matrix of equation (3.8) is positive definite, and then the matrix (3.7) is positive definite.

Now, using the definition of $\widetilde{G}$ and recalling (3.8), we have

$$\widetilde{G} = \sum_{j=1}^{\ell-1} \left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) I_j - D_{j+1}^T D_{j+1} \right), \tag{3.9}$$

and thus $\widetilde{G}$ is positive definite, too. This completes the proof of Lemma 3.1. □

We now introduce the following symmetric positive definite operators, needed in the sequel:

$$\mathcal{A}_{h_j} = h_j^2 \left(\sum_{r=1}^{d-1} \alpha_r \left(-\mathbf{\Delta}_{h_j}^{(r)} \right) \right), \qquad \text{for } j = 1, \ldots, \ell, \tag{3.10}$$

$$\mathcal{X}_{h_j} = h_j^2 \left(\sum_{r=1}^{d-1} \sum_{s=1}^{d-1} \alpha_r \alpha_s \left(-\mathbf{\Theta}_{h_j}^{(r,s)} \right) \right), \qquad \text{for } j = 1, \ldots, \ell, \tag{3.11}$$

$$\mathcal{B}_{h_j} = h_j^2 \left(-\mathbf{\Delta}_{h_j} \right), \qquad \text{for } j = 0, \ldots, \ell. \tag{3.12}$$

4. Condition number of the incremental unknowns matrix

In this section, we derive the condition number of the incremental unknowns matrix $\widehat{A}_h$. First, we derive an upper bound of the generalized Rayleigh quotient (see Subsection 4.1), and then we derive an upper bound of the maximum eigenvalue of the incremental unknowns matrix (see Subsection 4.2).

4.1. Upper bound of the generalized Rayleigh quotient. We begin with discrete inequalities relating the operators $\mathcal{X}_{h_j}$ and $\mathcal{A}_{h_j}$ and the operators $\mathcal{F}_{h_j}$ and $\mathcal{A}_{h_j}$.

Lemma 4.1. *For any $v \in \mathbf{V}_{h_j}$, we have*

$$\left(v, \mathcal{X}_{h_j} v \right) \leq 4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(v, \mathcal{A}_{h_j} v \right). \tag{4.1}$$

Proof. Using the discrete integration-by-parts formula (see Lemma 2.1), we obtain

$$\begin{aligned} \left(v, -\overline{\nabla}_{h_j}^{(r,0)} \nabla_{h_j}^{(r,0)} v \right) &= \left(\nabla_{h_j}^{(r,0)} v, \nabla_{h_j}^{(r,0)} v \right) \\ &= \frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j + rh_j, lh_j) - v(kh_j, lh_j) \right)^2, \end{aligned}$$

$$\begin{aligned} \left(v, -\overline{\nabla}_{h_j}^{(0,r)} \nabla_{h_j}^{(0,r)} v \right) &= \left(\nabla_{h_j}^{(0,r)} v, \nabla_{h_j}^{(0,r)} v \right) \\ &= \frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j, lh_j + rh_j) - v(kh_j, lh_j) \right)^2. \end{aligned}$$

Using discrete integration by parts again and the inequality $(a+b)^2 \le 2(a^2+b^2)$, we find

$$\left(v, -\overline{\nabla}_{h_j}^{(r,s)} \nabla_{h_j}^{(r,s)} v\right) = \left(\nabla_{h_j}^{(r,s)} v, \nabla_{h_j}^{(r,s)} v\right) = \frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j + rh_j, lh_j + sh_j) - v(kh_j, lh_j)\right)^2$$

(adding and subtracting the term $v(kh_j + rh_j, lh_j)$)

$$\le 2\frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j + rh_j, lh_j + sh_j) - v(kh_j + rh_j, lh_j)\right)^2 + 2\frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j + rh_j, lh_j) - v(kh_j, lh_j)\right)^2$$

(by setting in the former term first $k' = k + r$ and then $k = k'$)

$$\le 2\frac{1}{h_j^2} \sum_{k=1+r}^{d^j N-1+r} \sum_{l=1}^{d^j N-1} \left(v(kh_j, lh_j + sh_j) - v(kh_j, lh_j)\right)^2 + 2\frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j + rh_j, lh_j) - v(kh_j, lh_j)\right)^2$$

(since v is equal to 0 outside Ω)

$$\le 2\frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j, lh_j + sh_j) - v(kh_j, lh_j)\right)^2 + 2\frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j + rh_j, lh_j) - v(kh_j, lh_j)\right)^2 .$$

Thus,

$$\begin{aligned}
&\left(v, \sum_{r=1}^{d-1} \sum_{s=1}^{d-1} \alpha_r \alpha_s \left(-\overline{\nabla}_{h_j}^{(r,s)} \nabla_{h_j}^{(r,s)}\right) v\right) \\
&\quad \le 2\left(\sum_{r=1}^{d-1} \alpha_r\right) \sum_{s=1}^{d-1} \alpha_s \frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j, lh_j + sh_j) - v(kh_j, lh_j)\right)^2 \\
&\quad + 2\left(\sum_{s=1}^{d-1} \alpha_s\right) \sum_{r=1}^{d-1} \alpha_r \frac{1}{h_j^2} \sum_{k=1}^{d^j N-1} \sum_{l=1}^{d^j N-1} \left(v(kh_j + rh_j, lh_j) - v(kh_j, lh_j)\right)^2 \\
&\quad = 2\left(\sum_{r=1}^{d-1} \alpha_r\right) \left(v, \sum_{r=1}^{d-1} \alpha_r \left(-\boldsymbol{\Delta}_{h_j}^{(r)}\right) v\right).
\end{aligned}$$

Similarly, we obtain

$$\left(v, \sum_{r=1}^{d-1} \sum_{s=1}^{d-1} \alpha_r \alpha_s \left(-\overline{\nabla}_{h_j}^{(-r,s)} \nabla_{h_j}^{(-r,s)}\right) v\right) \le 2\left(\sum_{r=1}^{d-1} \alpha_r\right) \left(v, \sum_{r=1}^{d-1} \alpha_r \left(-\boldsymbol{\Delta}_{h_j}^{(r)}\right) v\right).$$

Consequently, we conclude that

$$\left(v, \mathcal{X}_{h_j} v\right) \leq 4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(v, \mathcal{A}_{h_j} v\right).$$

This proves Lemma 4.1. $\square$

Lemma 4.2. *For any* $v \in \mathbf{V}_{h_j}, v \neq 0$*, the operators* $\mathcal{F}_{h_j}$ *and* $\mathcal{A}_{h_j}$ *satisfy the relation*

$$1 < \frac{(v, \mathcal{F}_{h_j} v)}{(v, \mathcal{A}_{h_j} v)} \leq c' \left(\sum_{r=1}^{d-1} \alpha_r \right), \tag{4.2}$$

where c' *is an absolute positive constant; in fact, we have* $c' = 9$*.*

Proof. By the lemma above, for any $v \in \mathbf{V}_{h_j}, v \neq 0$, the operators $\mathcal{A}_{h_j}$ and $\mathcal{X}_{h_j}$ satisfy the relation

$$1 < \frac{(v, (\mathcal{A}_{h_j} + \mathcal{X}_{h_j})v)}{(v, \mathcal{A}_{h_j} v)} \leq 1 + 4 \left(\sum_{r=1}^{d-1} \alpha_r \right).$$

Now, since $\displaystyle\sum_{r=1}^{d-1} \alpha_r > \xi = -\frac{1}{2} + \frac{\sqrt{2}}{2} > 0$ and since $\xi^{-1} + 4 < 9$, we conclude that the discrete inequality (4.2) holds with $c' = 9$. $\square$

In what follows, we establish discrete inequalities relating the operators $(-\mathbf{\Delta}_{h_j}^{(r)})$ and $(-\mathbf{\Delta}_{h_j})$ and the operators $\mathcal{A}_{h_j}$ and $\mathcal{B}_{h_j}$.

Lemma 4.3. *For any* $v \in \mathbf{V}_{h_j}$*, we have*

$$\left(v, (-\mathbf{\Delta}_{h_j}^{(r)})v\right) \leq r^2 \left(v, (-\mathbf{\Delta}_{h_j})v\right), \quad r = 1, \ldots, d-1. \tag{4.3}$$

Proof. Indeed, adding and subtracting terms and using the triangle inequality, we obtain

$$\begin{aligned} |v(kh_j + rh_j, lh_j) - v(kh_j, lh_j)| \leq \sum_{\tilde{r}=0}^{r-1} |v(kh_j + (\tilde{r}+1)h_j, lh_j) \\ - v(kh_j + \tilde{r}h_j, lh_j)|. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we infer that

$$\begin{aligned} (v(kh_j + rh_j, lh_j) - v(kh_j, lh_j))^2 &\leq r \sum_{\tilde{r}=0}^{r-1} (v(kh_j + (\tilde{r}+1)h_j, lh_j) \\ &\quad - v(kh_j + \tilde{r}h_j, lh_j))^2 \\ &= r \sum_{\tilde{r}=0}^{r-1} (v((k+\tilde{r})h_j + h_j, lh_j) - v((k+\tilde{r})h_j, lh_j))^2. \end{aligned}$$

Therefore, we conclude that

$$(4.4)\quad \left(v, -\overline{\nabla}_{h_j}^{(r,0)}\nabla_{h_j}^{(r,0)} v\right) = \frac{1}{h_j^2}\sum_{k=1}^{d^jN-1}\sum_{l=1}^{d^jN-1}\left(v(kh_j + rh_j, lh_j) - v(kh_j, lh_j)\right)^2$$

$$\leq \frac{1}{h_j^2}\sum_{k=1}^{d^jN-1}\sum_{l=1}^{d^jN-1} r\sum_{\tilde{r}=0}^{r-1}\left(v((k+\tilde{r})h_j + h_j, lh_j) - v((k+\tilde{r})h_j, lh_j)\right)^2$$

$$= r\sum_{\tilde{r}=0}^{r-1}\frac{1}{h_j^2}\sum_{k=1}^{d^jN-1}\sum_{l=1}^{d^jN-1}\left(v((k+\tilde{r})h_j + h_j, lh_j) - v((k+\tilde{r})h_j, lh_j)\right)^2$$

(by setting first $k' = k + \tilde{r}$ and then $k = k'$)

$$= r\sum_{\tilde{r}=0}^{r-1}\frac{1}{h_j^2}\sum_{k=1+\tilde{r}}^{d^jN-1+\tilde{r}}\sum_{l=1}^{d^jN-1}\left(v(kh_j + h_j, lh_j) - v(kh_j, lh_j)\right)^2$$

(since v is equal to 0 outside Ω)

$$\leq r\sum_{\tilde{r}=0}^{r-1}\frac{1}{h_j^2}\sum_{k=1}^{d^jN-1}\sum_{l=1}^{d^jN-1}\left(v(kh_j + h_j, lh_j) - v(kh_j, lh_j)\right)^2 = r^2|\nabla_{1h_j}v|^2.$$

Similarly, we find

$$(4.5)\qquad \left(v, -\overline{\nabla}_{h_j}^{(0,r)}\nabla_{h_j}^{(0,r)} v\right) \leq r^2|\nabla_{2h_j}v|^2.$$

Now, adding inequalities (4.4) and (4.5), we obtain the conclusion (4.3) of the lemma. □

Lemma 4.4. *For any $v \in \mathbf{V}_{h_j}, v \neq 0$, the operators $\mathcal{A}_{h_j}$ and $\mathcal{B}_{h_j}$ satisfy the relation*

$$(4.6)\qquad \left(v, \mathcal{A}_{h_j}v\right) \leq \left(\sum_{r=1}^{d-1}\alpha_r r^2\right)\left(v, \mathcal{B}_{h_j}v\right).$$

Proof. This lemma promptly follows from the previous lemma. □

Now we ask the following question: can inequality (4.3) be improved? To answer this, we switch to periodic boundary conditions, we compute the eigenvalues of the operator $\mathbf{\Delta}_{h_j}^{(r)}$, and then we write the corresponding discrete inequalities relating the operators $(-\mathbf{\Delta}_{h_j}^{(r)})$ and $(-\mathbf{\Delta}_{h_j})$ and the operators $\mathcal{A}_{h_j}$ and $\mathcal{B}_{h_j}$.

Hereafter, we assume that N is even.

Lemma 4.5. *For periodic boundary conditions, the eigenvalues of $\mathbf{\Delta}_{h_j}^{(r)}$ are*

$$(4.7)\qquad -\frac{4}{h_j^2}\left(\sin^2(r\rho\pi h_j) + \sin^2(r\mu\pi h_j)\right), \qquad 0 \leq \rho, \mu \leq d^jN/2.$$

Proof. First, we are going to find the eigenvalues of $\mathbf{\Delta}_{h_j}^{(r)}$ for periodic boundary conditions in one space dimension. To do so, we introduce the finite-difference operator

$$(4.8)\qquad \Delta_{h_j}^{(r)}u(x) = \frac{1}{h_j^2}\left(u(x - rh_j) - 2u(x) + u(x + rh_j)\right),$$

and we write Δ_{h_j} instead of $\Delta_{h_j}^{(1)}$. Then using the Taylor series expansion of $u(x - rh_j)$ and $u(x + rh_j)$ around x and the Maclaurin series expansion for cosh, we infer that

$$\Delta_{h_j}^{(r)} = \frac{2}{h_j^2}\left(\cosh(rh_j\frac{d}{dx}) - I\right). \tag{4.9}$$

In particular, since

$$\Delta_{h_j} = \frac{2}{h_j^2}\left(\cosh(h_j\frac{d}{dx}) - I\right),$$

we obtain

$$h_j\frac{d}{dx} = \cosh^{-1}(I + \frac{h_j^2}{2}\Delta_{h_j}). \tag{4.10}$$

Replacing expression (4.10) of $h_j\dfrac{d}{dx}$ in (4.9), we find

$$\Delta_{h_j}^{(r)} = \frac{2}{h_j^2}\left(\cosh(r\cosh^{-1}(I + \frac{h_j^2}{2}\Delta_{h_j})) - I\right). \tag{4.11}$$

Now we consider the Chebyshev polynomials and recall some of their properties (see, e.g., [18, page 96]):

$$\forall x \in [-1,1]: \qquad T_r(x) = \cos(r\cos^{-1}(x)), \qquad \forall r \in \mathbb{N}\cup\{0\}, \tag{4.12}$$

$$\forall x \in [1,+\infty): \qquad T_r(x) = \cosh(r\cosh^{-1}(x)), \qquad \forall r \in \mathbb{N}\cup\{0\}. \tag{4.13}$$

Then using formula (4.11) and property (4.13), we obtain

$$\Delta_{h_j}^{(r)} = \frac{2}{h_j^2}\left(T_r(I + \frac{h_j^2}{2}\Delta_{h_j}) - I\right). \tag{4.14}$$

For $-d^jN/2 \le \xi \le d^jN/2 - 1$, we introduce the following functions:

$$a^\rho(\xi) = \sin(2\pi\rho\xi h_j), \qquad -d^jN/2 \le \rho \le -1, \tag{4.15}$$

$$b^\rho(\xi) = \cos(2\pi\rho\xi h_j), \qquad 0 \le \rho \le d^jN/2 - 1, \tag{4.16}$$

and we note that

$$\Delta_{h_j}a^\rho = -\frac{4}{h_j^2}\sin^2(\pi\rho h_j)a^\rho, \qquad -d^jN/2 \le \rho \le -1, \tag{4.17}$$

$$\Delta_{h_j}b^\rho = -\frac{4}{h_j^2}\sin^2(\pi\rho h_j)b^\rho, \qquad 0 \le \rho \le d^jN/2 - 1. \tag{4.18}$$

Thus,

$$\left(I + \frac{h_j^2}{2}\Delta_{h_j}\right)a^\rho = \cos(2\pi\rho h_j)a^\rho, \qquad -d^jN/2 \le \rho \le -1, \tag{4.19}$$

$$\left(I + \frac{h_j^2}{2}\Delta_{h_j}\right)b^\rho = \cos(2\pi\rho h_j)b^\rho, \qquad 0 \le \rho \le d^jN/2 - 1. \tag{4.20}$$

Therefore, using (4.14), (4.19), and (4.20), we conclude that the eigenvalues of $\Delta_{h_j}^{(r)}$ are

$$\frac{2}{h_j^2}\left(T_r(\cos(2\pi\rho h_j)) - 1\right), \qquad 0 \le \rho \le d^jN/2. \tag{4.21}$$

Using property (4.12), we obtain

$$T_r(\cos(2\pi\rho h_j)) = \cos(r\cos^{-1}(\cos(2\pi\rho h_j))) = \cos(r2\pi\rho h_j), \tag{4.22}$$

and thus the eigenvalues of $\Delta_{h_j}^{(r)}$ are

$$\frac{2}{h_j^2}\left(T_r(\cos(2\pi\rho h_j)) - 1\right) = -\frac{4}{h_j^2}\sin^2(r\pi\rho h_j), \qquad 0 \le \rho \le d^j N/2. \tag{4.23}$$

Now using the Taylor series expansion, the Maclaurin series expansion for cosh, and the one-dimesional formulae (4.11) and (4.14), we obtain

$$\begin{aligned}\boldsymbol{\Delta}_{h_j}^{(r)} u(x,y) &= \frac{2}{h_j^2}\left(\cosh(r\cosh^{-1}(I + \frac{h_j^2}{2}\boldsymbol{\Delta}_{xh_j})) - I\right)\\ &\quad + \frac{2}{h_j^2}\left(\cosh(r\cosh^{-1}(I + \frac{h_j^2}{2}\boldsymbol{\Delta}_{yh_j})) - I\right)\\ &= \frac{2}{h_j^2}\left(T_r(I + \frac{h_j^2}{2}\boldsymbol{\Delta}_{xh_j}) - I\right) + \frac{2}{h_j^2}\left(T_r(I + \frac{h_j^2}{2}\boldsymbol{\Delta}_{yh_j}) - I\right).\end{aligned} \tag{4.24}$$

Using the one-dimensional formulae (4.19), (4.20) and (4.23), and the fact that the eigenvectors of $\boldsymbol{\Delta}_{h_j}^{(r)}$ are the twofold tensor product of the eigenvectors of $\Delta_{h_j}^{(r)}$ given by (4.15) and (4.16), we conclude that the eigenvalues of $\boldsymbol{\Delta}_{h_j}^{(r)}$ are given by formulae (4.7).

Thus, Lemma 4.5 is proved. □

We now introduce Fejér's kernel (see, e.g., [17]), needed to obtain improved discrete inequalities relating the operators $\boldsymbol{\Delta}_{h_j}^{(r)}$ and $\boldsymbol{\Delta}_{h_j}$ and the operators $\mathcal{A}_{h_j}$ and $\mathcal{B}_{h_j}$:

$$K_r(x) = \begin{cases}\left(\dfrac{\sin(rx)}{\sin(x)}\right)^2 & \text{if } x \neq 0\\ r^2 & \text{if } x = 0.\end{cases} \tag{4.25}$$

Lemma 4.6. *For any $v \in \mathbf{V}_{h_j}$, we have*

(4.26)
$$\left(v, (-\boldsymbol{\Delta}_{h_j}^{(r)})v\right) \le K_r(\pi h_j)\left(v, (-\boldsymbol{\Delta}_{h_j})v\right) \lll r^2\left(v, (-\boldsymbol{\Delta}_{h_j})v\right), \quad r = 1, \dots, d-1.$$

Proof. Indeed, because the eigenvalues of $\boldsymbol{\Delta}_{h_j}^{(r)}$ are given by formulae (4.7) with the same set of orthonormal eigenvectors, to prove the left inequality in (4.26) it suffices to observe that for $\rho, \mu \neq 0$,

$$\begin{aligned}\frac{\sin^2(r\rho\pi h_j) + \sin^2(r\mu\pi h_j)}{\sin^2(\rho\pi h_j) + \sin^2(\mu\pi h_j)} &\le \max\left\{\frac{\sin^2(r\rho\pi h_j)}{\sin^2(\rho\pi h_j)}, \frac{\sin^2(r\mu\pi h_j)}{\sin^2(\mu\pi h_j)}\right\}\\ &\le \frac{\sin^2(r\pi h_j)}{\sin^2(\pi h_j)} = K_r(\pi h_j).\end{aligned}$$

In addition, the right inequality in (4.26) promptly follows from the properties of Fejér's kernel (ibid., page 6). In Figure 4.1, we sketch $K_r(x)$ for a few values of r and we plot the vertical straight line $x = \pi h_j$. The intersection of this line with $K_r(x)$ gives the value $K_r(\pi h_j) \lll r^2$. □

Lemma 4.7. *For any $v \in \mathbf{V}_{h_j}, v \neq 0$, the operators $\mathcal{A}_{h_j}$ and $\mathcal{B}_{h_j}$ satisfy the relation*

$$\left(v, \mathcal{A}_{h_j}v\right) \le \left(\sum_{r=1}^{d-1}\alpha_r K_r(\pi h_j)\right)\left(v, \mathcal{B}_{h_j}v\right) \lll \left(\sum_{r=1}^{d-1}\alpha_r r^2\right)\left(v, \mathcal{B}_{h_j}v\right). \tag{4.27}$$

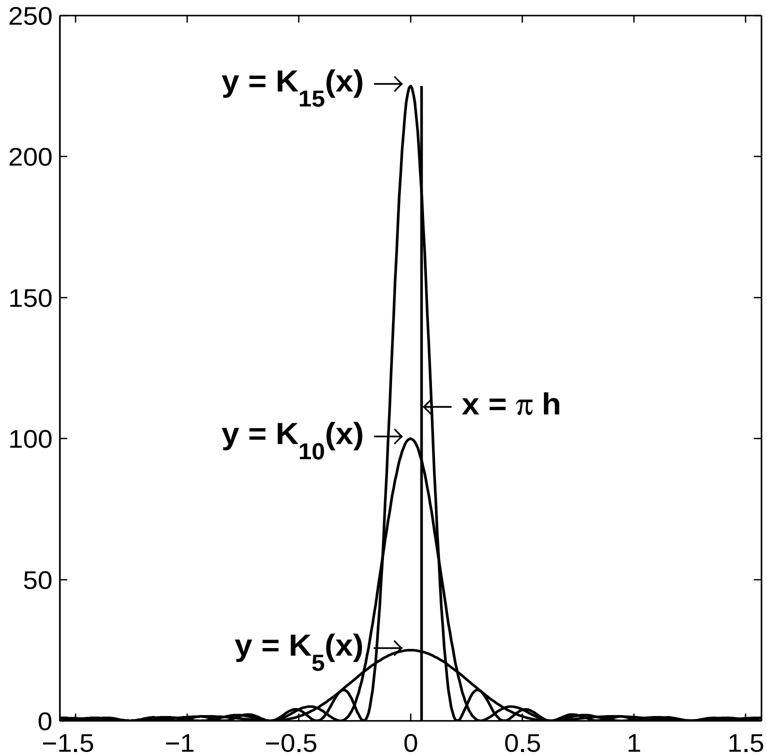

Figure 4.1: The Fejér's kernel for increasing values of r.

Proof. This lemma promptly follows from the previous one. □

We now leave the periodic boundary-conditions constraint and with the same method used to prove the discrete inequality established by Elman and Zhang [9, page 205] and restated in [14, page 354], one can show the following result.

Lemma 4.8. *For any $v \in \mathbf{V}_h, v \neq 0$, the operators $\mathcal{B}_{h_j}$ and $\mathcal{B}_{h_\ell}$ satisfy the relation*

$$\left(v, \mathcal{B}_{h_j} v\right) \leq c''(\ln d)(\ell - j)\left(v, \mathcal{B}_{h_\ell} v\right), \qquad j = 0, \dots, \ell - 1, \tag{4.28}$$

where c'' is an absolute positive constant.

The inequalities above allow us to state the following upper bound.

Lemma 4.9. *An upper bound of the generalized Rayleigh quotient without preconditioning is given by*

$$\max_{v \neq 0} \frac{(v, (SS^T)^{-1} v)}{(v, \mathcal{B}_{h_\ell} v)} \leq \left(\sum_{r=1}^{d-1} \alpha_r\right)\left(\sum_{r=1}^{d-1} \alpha_r r^2\right)(\ln d) O(\frac{1}{H^2})\ell^2. \tag{4.29}$$

If the coarsest grid is reduced to one point, this upper bound becomes

$$\max_{v \neq 0} \frac{(v, (SS^T)^{-1} v)}{(v, \mathcal{B}_{h_\ell} v)} \leq \left(\sum_{r=1}^{d-1} \alpha_r\right)\left(\sum_{r=1}^{d-1} \alpha_r r^2\right)(\ln d) c \ell^2. \tag{4.30}$$

Proof. Since the matrix $G + \widetilde{G}$ is positive definite (see Lemma 3.1), from equation (3.5) we obtain

$$\frac{(v,(SS^T)^{-1}v)}{(v,\mathcal{B}_{h_\ell}v)} \le \frac{(v,F_{h_0}v)}{(v,\mathcal{B}_{h_\ell}v)} + \sum_{j=1}^{\ell}\frac{(v,F_{h_j}v)}{(v,\mathcal{B}_{h_\ell}v)}.$$

Then we observe that

$$\begin{aligned}\max_{v\neq 0}\frac{(v,(SS^T)^{-1}v)}{(v,\mathcal{B}_{h_\ell}v)} &\le \max_{v\neq 0}\frac{(v,F_{h_0}v)}{(v,\mathcal{B}_{h_0}v)}\max_{v\neq 0}\frac{(v,\mathcal{B}_{h_0}v)}{(v,\mathcal{B}_{h_\ell}v)}\\ &\quad+\sum_{j=1}^{\ell}\max_{v\neq 0}\frac{(v,F_{h_j}v)}{(v,\mathcal{A}_{h_j}v)}\max_{v\neq 0}\frac{(v,\mathcal{A}_{h_j}v)}{(v,\mathcal{B}_{h_j}v)}\max_{v\neq 0}\frac{(v,\mathcal{B}_{h_j}v)}{(v,\mathcal{B}_{h_\ell}v)}\\
&\le \max_{v\neq 0}\frac{(v,F_{h_0}v)}{(v,\mathcal{B}_{h_0}v)}c''(\ln d)\ell + c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)\sum_{j=1}^{\ell}\max_{v\neq 0}\frac{(v,\mathcal{B}_{h_j}v)}{(v,\mathcal{B}_{h_\ell}v)}\\
&\le \max_{v\neq 0}\frac{(v,F_{h_0}v)}{(v,\mathcal{B}_{h_0}v)}c''(\ln d)\ell + c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)\sum_{j=1}^{\ell-1}c''(\ln d)(\ell-j)\\ &\quad+c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)\\
&\le \max_{v\neq 0}\frac{(v,F_{h_0}v)}{(v,\mathcal{B}_{h_0}v)}c''(\ln d)\ell + c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)c''(\ln d)\frac{(\ell-1)\ell}{2}\\ &\quad+c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right).\end{aligned}$$

We therefore obtain

$$\max_{v\neq 0}\frac{(v,(SS^T)^{-1}v)}{(v,\mathcal{B}_{h_\ell}v)} \le \left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)(\ln d)C\ell^2, \tag{4.31}$$

where

$$C = \frac{1}{3}\max\left\{\max_{v\neq 0}\frac{(v,F_{h_0}v)}{(v,h_0^2(-\mathbf{\Delta}_{h_0})v)}c'', c'c'', \frac{c'}{\ln 2}\right\},$$

and since $F_{h_0} = I_0$ and (see [15, page 53])

$$\max_{v\neq 0}\frac{(v,v)}{(v,h_0^2(-\mathbf{\Delta}_{h_0})v)} = O(\frac{1}{h_0^2}),$$

we conclude that (4.29) holds true. □

4.2. Upper bound of the maximum eigenvalue.

Lemma 4.10. *An upper bound of the maximum eigenvalue $\lambda_{\max}$ of the incremental unknowns matrix $\widehat{A}_h$ is $c\ell$ for the first order incremental unknowns and c for the second order incremental unknowns.*

Proof. Since the two-dimensional hierarchical-basis elements $\widehat{\varpi}^{(k)}_{(x,y)}$ are twofold tensor products of the form $\widehat{\varpi}^{(k)}_{(x,y)} = \widehat{\omega}^{(k)}_x \bigotimes \widehat{\omega}^{(k)}_y$ and since the scalar product on $\mathbf{V}_h$ splits into one-dimensional scalar products over tensor products [11], one-dimensional scalar-product computations provide the coefficients of the incremental unknowns matrix $\widehat{A}_h$. In order to find these coefficients, we note the following:

(1) For $k = 0, \ldots, \ell$ and for $x \in \Omega_k$, we have

$$((\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_x))_h \approx \int_{x-h_j}^{x+h_j} \frac{1}{h_k}\psi'\left(\frac{t-x}{h_k}\right)\frac{1}{h_j}\psi'\left(\frac{t-x}{h_j}\right) dt, \quad \text{for } j \geq k,$$

and hence

$$|((\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_x))_h| \lessapprox \frac{2}{h_k} M'^2 = \frac{1}{h}\left(\frac{c}{d^{\ell-k}}\right), \quad \text{for } j \geq k. \tag{4.32}$$

(2) For $k = 0, \ldots, \ell$, $x \in \Omega_k$, and for a kth-level neighbor y of x, we have

$$((\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_y))_h \approx \pm \int_{y\pm h_j}^{y} \frac{1}{h_k}\psi'\left(\frac{t-x}{h_k}\right)\frac{1}{h_j}\psi'\left(\frac{t-y}{h_j}\right) dt, \quad \text{for } j \geq k,$$

and then

$$|((\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_y))_h| \lessapprox \frac{1}{h_k} M'^2 = \frac{1}{h}\left(\frac{c}{d^{\ell-k}}\right), \quad \text{for } j \geq k. \tag{4.33}$$

(3) For $k = 0, \ldots, \ell$, $x \in \Omega_k$, $j > k$, and for a jth-level node z inside the support of $\widehat{\omega}^{(k)}_x$, with $z \neq x$, we have

$$((\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_z))_h \approx \int_{z-h_j}^{z+h_j} \frac{1}{h_k}\psi'\left(\frac{t-x}{h_k}\right)\frac{1}{h_j}\psi'\left(\frac{t-z}{h_j}\right) dt, \quad \text{for } j > k,$$

and thus

$$|((\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_z))_h| \lessapprox \frac{2}{h_k} M'^2 = \frac{1}{h}\left(\frac{c}{d^{\ell-k}}\right), \quad \text{for } j > k. \tag{4.34}$$

For the second order incremental unknowns, we obtain

$$((\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_z))_h = 0, \quad \text{for } j > k. \tag{4.35}$$

(4) For $k = 0, \ldots, \ell$ and for $x \in \Omega_k$, we have

$$(\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_x) \approx \int_{x-h_j}^{x+h_j} \psi\left(\frac{t-x}{h_k}\right)\psi\left(\frac{t-x}{h_j}\right) dt, \quad \text{for } j \geq k,$$

and hence

$$|(\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_x)| \lessapprox 2h_j M^2 = h\left(cd^{\ell-j}\right), \quad \text{for } j \geq k. \tag{4.36}$$

(5) For $k = 0, \ldots, \ell$, $x \in \Omega_k$, and for a kth-level neighbor y of x, we have

$$(\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_y) \approx \pm \int_{y\pm h_j}^{y} \psi\left(\frac{t-x}{h_k}\right)\psi\left(\frac{t-y}{h_j}\right) dt, \quad \text{for } j \geq k,$$

and then

$$|(\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_y)| \lessapprox h_j M^2 = h\left(cd^{\ell-j}\right), \quad \text{for } j \geq k. \tag{4.37}$$

(6) For $k = 0, \ldots, \ell$, $x \in \Omega_k$, $j > k$, and for a jth-level node z inside the support of $\widehat{\omega}^{(k)}_x$, we have

$$(\widehat{\omega}^{(k)}_x, \widehat{\omega}^{(j)}_z) \approx \int_{z-h_j}^{z+h_j} \psi\left(\frac{t-x}{h_k}\right)\psi\left(\frac{t-z}{h_j}\right) dt, \quad \text{for } j > k,$$

and thus

$$|(\widehat{\omega}_x^{(k)}, \widehat{\omega}_z^{(j)})| \lessapprox 2h_j M^2 = h\left(cd^{\ell-j}\right), \quad \text{for } j > k. \tag{4.38}$$

Now, let $\widehat{\varpi}_{(x,y)}^{(k)}$ be a fixed kth-level hierarchical basis; its support is a square with center at (x, y) (see Figure 4.2 light shading). The jth-level hierarchical basis $\widehat{\varpi}_{(x',y')}^{(j)}$, with $j \geq k$ such that $((\widehat{\varpi}_{(x,y)}^{(k)}, \widehat{\omega}_{(x',y')}^{(j)}))_h \neq 0$, has support a square with center at (x', y') on the support of $\widehat{\varpi}_{(x,y)}^{(k)}$. The approximate inequalities (4.32)–(4.38) imply

$$\left|((\widehat{\varpi}_{(x,y)}^{(k)}, \widehat{\omega}_{(x',y')}^{(j)}))_h\right| \leq c\frac{1}{d^{j-k}}, \quad \text{for } j \geq k.$$

To count the functions $\widehat{\omega}_{(x',y')}^{(j)}$ such that $((\widehat{\varpi}_{(x,y)}^{(k)}, \widehat{\omega}_{(x',y')}^{(j)}))_h \neq 0$, we notice that there are at most $O((d^{j-k-1}(d-1))^2)$ jth-level hierarchical basis (a two-dimensional count) for the first order incremental unknowns, and there are at most $O(d^{j-k-1}(d-1))$ jth-level hierarchical basis (a one-dimensional count) for the second order incremental unknowns. Here the center (x', y') must be either on the edges or on the axes of the support of $\widehat{\varpi}_{(x,y)}^{(k)}$ (see Figure 4.2 dark lines), because of property (4.35).

Since the incremental unknowns matrix $\widehat{A}_h$ has the block structure $\widehat{A}_h = [\widehat{A}_{k,j}]_{k,j}$, where $\widehat{A}_{k,j} = [((\widehat{\varpi}_{(x,y)}^{(k)}, \widehat{\varpi}_{(x',y')}^{(j)}))_h]_{\widehat{\varpi}_{(x,y)}^{(k)}, \widehat{\varpi}_{(x',y')}^{(j)}}$, the analysis before implies:

- for the first order incremental unknowns:

$$\sum_{\widehat{\varpi}_{(x',y')}^{(j)}} \left\{((\widehat{\varpi}_{(x,y)}^{(k)}, \widehat{\varpi}_{(x',y')}^{(j)}))_h\right\}^2 \leq c(d^{j-k-1}(d-1))^2\left(\frac{1}{d^{j-k}}\right)^2 \leq c;$$

- for the second order incremental unknowns:

$$\sum_{\widehat{\varpi}_{(x',y')}^{(j)}} \left\{((\widehat{\varpi}_{(x,y)}^{(k)}, \widehat{\varpi}_{(x',y')}^{(j)}))_h\right\}^2 \leq cd^{j-k-1}(d-1)\left(\frac{1}{d^{j-k}}\right)^2 \leq c\frac{1}{d^{j-k}}.$$

Therefore, an upper bound of the square of the Euclidean norm of any row of the matrix $\widehat{A}_{k,j}$ is c for the first order incremental unknowns, and $c\frac{1}{d^{j-k}}$ for the second order incremental unknowns; on the other hand, any column of the matrix $\widehat{A}_{k,j}$ has at most six nonzero elements. It then follows from linear algebra lemmas (see, e.g., [9, page 197]) that an upper bound of the norm $\|\widehat{A}_{k,j}\|_2^2$ is c for the first order incremental unknowns, and $c\frac{1}{d^{j-k}}$ for the second order incremental unknowns, and thus an upper bound of the Euclidean norm $\|\widehat{A}_h\|_2$ of the incremental unknowns matrix $\widehat{A}_h$ is $c\ell$ for the first order incremental unknowns, and c for the second order incremental unknowns. Since $\lambda_{\mathsf{max}} = \|\widehat{A}_h\|_2$, the lemma follows.

□

5. Block diagonal (scaling) preconditioning

In view of the results above (see (4.29) and (4.30)), when the coarsest grid is not reduced to one point, we will use left orientation block diagonal (scaling) preconditioning [13] to solve the incremental unknowns linear systems. An upper bound

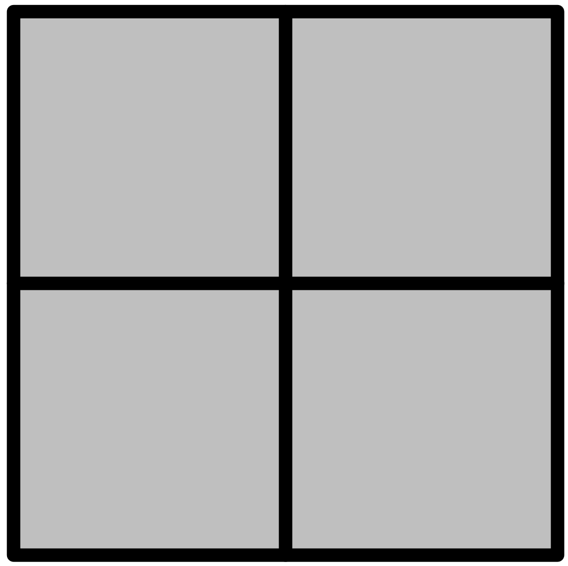

Figure 4.2: The support of $\widehat{\varpi}^{(k)}_{(x,y)}$: ultra light shading; the center of $\widehat{\varpi}^{(j)}_{(x',y')}$: dark line.

of the preconditioned generalized Rayleigh quotient will be derived. The preconditioning matrix $\mathcal{K}$ for the incremental unknowns matrix $\widehat{A}_h$ will be as follows:

$$\mathcal{K} = \left[\begin{array}{c|ccccc} \mathcal{L} & & & & \\ \hline & \mathcal{L}_1 & & & \\ & & \mathcal{L}_2 & & \\ & & & \ddots & \\ & & & & \mathcal{L}_\ell \end{array}\right],$$

$$\mathcal{L} = h_0^2(-\boldsymbol{\Delta}_{h_0}), \qquad \mathcal{L}_j = k_j\left(I_j - I_{j-1}\right), j = 1, \ldots, \ell,$$

where

$$k_j = \begin{cases} \dfrac{1}{(\ell - j)} & \text{for } j = 1, \ldots, \ell - 1, \\ 1 & \text{for } j = \ell. \end{cases}$$

The associated directed graph $G(\mathcal{K})$ is strongly connected at the coarsest level, and each node communicates only with itself at the fine levels. Furthermore, noting that the coefficients at the coarsest level are the coefficients of the coarsest level matrix $\mathcal{L}$ and that the circular coefficients at the jth-level are the coefficients $k_j > 0$, we have a complete definition of a square matrix $\mathcal{M}$ of order $(n-1)^2$ such that

$$\mathcal{K} = \mathcal{M} + \sum_{j=1}^{\ell} k_j\left(I_j - I_{j-1}\right).$$

Now we describe the multi-level structure of the matrix $(S\mathcal{K}^{-1}S^T)^{-1}$ with graph techniques. First, multiplying graphwise, we obtain

$$\mathcal{K}S^{-1} = \mathcal{M} + \sum_{j=1}^{\ell} k_j\left\{(I_j - I_{j-1}) - D_j\right\},$$

and thus

$$S^{-T}\mathcal{K}S^{-1} = \mathcal{M} + \sum_{j=1}^{\ell} k_j \left\{(I_j - I_{j-1}) - D_j\right\} + \sum_{j=1}^{\ell} k_j \left(-D_j^T + D_j^T D_j\right),$$

or

$$S^{-T}\mathcal{K}S^{-1} = \mathcal{M} + \sum_{j=1}^{\ell} k_j \left(I_j - D_j^T - D_j\right) + \sum_{j=1}^{\ell} k_j \left(-I_{j-1} + D_j^T D_j\right).$$

Recalling (3.3), we obtain

$$(S\mathcal{K}^{-1}S^T)^{-1} = S^{-T}\mathcal{K}S^{-1} = \mathcal{F}_{h_0} + \sum_{j=1}^{\ell} k_j \mathcal{F}_{h_j} - \overline{G}, \tag{5.1}$$

where

$$\mathcal{F}_{h_0} = \mathcal{M},$$

$$\mathcal{F}_{h_j} = h_j^2 \left(\sum_{r=1}^{d-1} \alpha_r \left(-\mathbf{\Delta}_{h_j}^{(r)}\right) + \sum_{r,s=1}^{d-1} \alpha_r\alpha_s \left(-\mathbf{\Theta}_{h_j}^{(r,s)}\right) \right), \quad \text{for } j = 1, \dots, \ell,$$

$$\overline{G} = \sum_{j=1}^{\ell} k_j \left(\left(\left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) - 1 \right) I_j - G_j \right) + I_{j-1} - D_j^T D_j \right).$$

Lemma 5.1. *The matrix $\overline{G}$ is positive definite.*

Proof. Indeed, since

$$\begin{aligned} &\left(\left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) - 1 \right) I_j - G_j \right) + I_{j-1} - D_j^T D_j = \\ &\qquad \left(\left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) - 1 \right) (I_j - I_{j-1}) - G_j \right) \\ &\qquad\qquad + \left(4 \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(1 + \sum_{r=1}^{d-1} \alpha_r \right) I_{j-1} - D_j^T D_j \right), \end{aligned} \tag{5.2}$$

and since the right-hand side of the equation above is the sum of the positive semidefinite matrix (3.6) and of the positive definite matrix (3.7), it follows that the matrix $\overline{G}$ itself is positive definite. □

The results above enable us to state the following:

Lemma 5.2. *An upper bound of the preconditioned generalized Rayleigh quotient is given by*

$$\max_{v \neq 0} \frac{(v, (S\mathcal{K}^{-1}S^T)^{-1}v)}{(v, \mathcal{B}_{h_\ell} v)} \leq \left(\sum_{r=1}^{d-1} \alpha_r \right) \left(\sum_{r=1}^{d-1} \alpha_r r^2 \right) (\ln d) c\ell. \tag{5.3}$$

Proof. Since the matrix $\overline{G}$ is positive definite, using equation (5.1) and the procedure in the proof of Lemma 4.9, we obtain

$$\max_{v\neq 0}\frac{(v,(S\mathcal{K}^{-1}S^T)^{-1}v)}{(v,\mathcal{B}_{h_\ell}v)} \le \max_{v\neq 0}\frac{(v,\mathcal{M}v)}{(v,\mathcal{B}_{h_0}v)}c''(\ln d)\ell$$
$$+ c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)\sum_{j=1}^{\ell-1}k_j c''(\ln d)(\ell-j) + c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)$$
$$= \max_{v\neq 0}\frac{(v,h_0^2(-\boldsymbol{\Delta}_{h_0})v)}{(v,h_0^2(-\boldsymbol{\Delta}_{h_0})v)}c''(\ln d)\ell$$
$$+ c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)c''(\ln d)(\ell-1) + c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)$$
$$= c''(\ln d)\ell + c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)c''(\ln d)(\ell-1) + c'\left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)$$
$$\le \left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right)(\ln d)c\ell.$$

□

Remark 5.1. Since $\lambda_{\max} = \|\mathcal{K}^{-1}\widehat{A}_h\|_2$ and since $\|\mathcal{K}^{-1}\widehat{A}_h\|_2 \le \|\mathcal{K}^{-1}\|_2\|\widehat{A}_h\|_2$, an upper bound of the maximum eigenvalue $\lambda_{\max}$ of the incremental unknowns matrix $\mathcal{K}^{-1}\widehat{A}_h$ is $\|\mathcal{K}^{-1}\|_2 \times c\ell$ for the first order incremental unknowns, and $\|\mathcal{K}^{-1}\|_2 \times c$ for the second order incremental unknowns. Thus, the block diagonal (scaling) preconditioner deteriorates the upper bound.

We are now in a position to give the main result.

Theorem 5.3. *With in-depth refinement, the condition number of the incremental unknowns matrix associated to the Laplace operator is* $p(d)O(1/H^2)O(|\log_d h|^3)$ *for the first order incremental unknowns, and* $q(d)O(1/H^2)O((\log_d h)^2)$ *for the second order incremental unknowns, where* d *is the depth of the refinement,* H *is the mesh size of the coarsest grid,* h *is the mesh size of the finest grid,* $p(d) = \dfrac{d-1}{2}$ *and* $q(d) = \dfrac{d-1}{2}\dfrac{1}{12}d(d^2-1)$*. Furthermore, if block diagonal (scaling) preconditioning is used, the condition number of the preconditioned incremental unknowns matrix associated to the Laplace operator is* $p(d)O((\log_d h)^2)$ *for the first order incremental unknowns, and* $q(d)O(|\log_d h|)$ *for the second order incremental unknowns.*

On the other hand, the condition number of the nodal unknowns matrix associated to the Laplace operator is $O(1/h^2)$.

Proof. In the first place, we observe that, as the depth of the refinement grows, the quantity (see (2.7))

$$Q = \left(\sum_{r=1}^{d-1}\alpha_r\right)\left(\sum_{r=1}^{d-1}\alpha_r r^2\right) = \frac{d-1}{2}\left(\sum_{r=1}^{d-1}\alpha_r r^2\right) \tag{5.4}$$

deteriorates the upper bound of the preconditioned generalized Rayleigh quotient (see (5.3)).

For the second order incremental unknowns, we have

$$\sum_{r=1}^{d-1} \alpha_r r^2 = \sum_{r=1}^{d-1} \frac{d-r}{d} r^2 = \frac{1}{12} d(d^2-1),$$

and hence

$$Q = \frac{d-1}{2} \frac{1}{12} d(d^2-1).$$

For the first order incremental unknowns, we may require the self-similar interpolating continuous function ψ to satisfy:

$$\sum_{r=1}^{d-1} \alpha_r r^2 \leq c, \tag{5.5}$$

and then

$$Q \leq \frac{d-1}{2} c.$$

In the second place, we point out that the smallest eigenvalue λ_{min} of the incremental unknowns matrix $\mathcal{K}^{-1}\widehat{A}_h$ is (see, e.g., [18, page 87])

$$\frac{1}{\lambda_{\mathsf{min}}} = \max_{v \neq 0} \frac{(v, (S\mathcal{K}^{-1}S^T)^{-1}v)}{(v, \mathcal{B}_{h_\ell} v)}. \tag{5.6}$$

In the third place, we note that $h \leq \frac{1}{d^\ell}$, and thus $\ell \leq \mid \log_d h \mid$.

Finally, since the condition number $\mathrm{cond}(\mathcal{K}^{-1}\widehat{A}_h)$ of the incremental unknowns matrix $\mathcal{K}^{-1}\widehat{A}_h$ is $\mathrm{cond}(\mathcal{K}^{-1}\widehat{A}_h) = \frac{\lambda_{\mathsf{max}}}{\lambda_{\mathsf{min}}}$, the theorem readily follows from Lemma 4.9, Lemma 5.2, and Remark 5.1.

□

6. Distinctive features

As a conclusion to this article, we summarize the distinctive features that are intrinsic to in-depth refinement:

- The upper bound of the maximum eigenvalue of the incremental unknowns matrix $\widehat{A}_h$ is $c\ell$ for the first order incremental unknowns, and c for the second order incremental unknowns. This result—worse for the first order incremental unknowns—relies primarily upon the behavior of the self-similar interpolating continuous function $\psi(t)$: it is linear for the second order incremental unknowns and it may have a rather unpredictable behavior for the first order incremental unknowns (see, e.g., [6, 7, 8, 5]). Ultimately, this result relies upon the positioning of the supports of the hierarchical basis relative to one another.
- The upper bound of the generalized Rayleigh quotient is the same both for the first and second order incremental unknowns. As the depth of the refinement grows, the quantity

$$Q = \left(\sum_{r=1}^{d-1} \alpha_r\right) \left(\sum_{r=1}^{d-1} \alpha_r r^2\right) \tag{6.1}$$

deteriorates this estimate.

For periodic boundary conditions, this quantity may be replaced both in Lemma 4.9 and Lemma 5.2 by (see Lemma 4.7)

$$\widetilde{Q} = \left(\sum_{r=1}^{d-1} \alpha_r\right)\left(\sum_{r=1}^{d-1} \alpha_r K_r(\pi h)\right) \lll \left(\sum_{r=1}^{d-1} \alpha_r\right)\left(\sum_{r=1}^{d-1} \alpha_r r^2\right). \tag{6.2}$$

For the second order incremental unknowns, we obtain

$$\widetilde{Q} \lll \frac{d-1}{2}\frac{1}{12}d(d^2-1).$$

For the first order incremental unknowns, we may require the self-similar interpolating continuous function $\psi(t)$ to satisfy a weaker requirement:

$$\sum_{r=1}^{d-1} \alpha_r K_r(\pi h) \leq c. \tag{6.3}$$

Therefore, the incremental unknowns preconditioner is efficient with in-depth refinement, but its efficiency deteriorates at some rate as the depth of the refinement grows.

Computational experiments with dyadic refinement can be found in [10]; computational experiments with in-depth refinement are the subject of a separate work.

TEXdraw and Matlab have been used for the figures in this paper.

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