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LINEAR ADVECTION WITH ILL-POSED BOUNDARY CONDITIONS VIA L^1-MINIMIZATION

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Abstract. It is proven that in dimension one the piecewise linear best L^1-approximation to the linear transport equation equipped with a set of ill-posed boundary conditions converges in $W^{1,1}_{\rm loc}$ to the viscosity solution of the equation and the boundary layer associated with the ill-posed boundary condition is always localized in one mesh cell, i.e., the "last" one.

Key Words. Finite elements, best L^1-approximation, viscosity solution, linear transport, ill-posed problem

1. Introduction

The goal of this paper is to explain a phenomenon that has been reported in [4]; namely, finite-element-based best L^1-approximations seem to converge to viscosity solutions of some classes of first-order PDE's. In particular we prove in this paper that it is indeed the case in dimension one for the linear transport equation equipped with a set of ill-posed boundary conditions.

To explain our interest for finite element best L^1-approximations and ill-posed boundary conditions, we now briefly recall the result from [4] that we exactly refer to. We denote by Ω a bounded domain of $\mathbb{R}^d$ with smooth boundary. Let $\alpha > 0$ be a real number and let $\boldsymbol{\beta} \in [\mathcal{C}^1(\overline{\Omega})]^d$ be a smooth vector field. Let u_0 be a smooth function on $\partial\Omega$, say $u_0 \in \mathcal{C}^2(\partial\Omega)$, and let $f \in W^{1,1}(\Omega)$. Following Bardos–le Roux–Nédélec, [2], we say that u is a viscosity solution to

$$\alpha u + \nabla\cdot(\boldsymbol{\beta} u) = f; \quad u|_{\partial\Omega} = u_0, \tag{1.1}$$

if $u \in \mathrm{BV}(\Omega)$, u solves the PDE, and u satisfies the boundary condition in the following sense

$$\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\mathbf{n})(u-k)(\mathrm{sg}(u-k) - \mathrm{sg}(u_0-k)) \geq 0, \quad \forall k \in \mathbb{R}, \tag{1.2}$$

where $\mathrm{sg}(t)$ is the sign of t if $t \neq 0$ and $\mathrm{sg}(0) = 0$. In the present linear case, the boundary condition amounts to enforcing $u = u_0$ on $\partial\Omega^- = \{\mathbf{x} \in \partial\Omega \,|\, \boldsymbol{\beta}(\mathbf{x})\cdot\mathbf{n}(x) < 0\}$.

Using arguments similar to those in [2] and [1], it is possible to prove that (1.1) has a unique viscosity solution provided α is large enough. The bulk of the argument consists of proving that the solution to the following problem

$$\alpha u_\epsilon + \nabla\cdot(\boldsymbol{\beta} u_\epsilon) - \epsilon\nabla^2 u_\epsilon = f; \quad u_\epsilon|_{\partial\Omega} = u_0, \tag{1.3}$$

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converges in $L^1(\Omega)$ and the limit is the so-called viscosity solution, i.e. the limit satisfies the PDE (1.1) and the boundary condition (1.2).

Despite its appearance, the problem (1.1) is not purely formal. It arises when one tries to approximate (1.3) on finite element meshes that are not refined enough. For instance, denoting by h the mesh size, whenever $\epsilon/h^2 \ll \|\boldsymbol{\beta}\|_{L^\infty}/h$, the second-order term in (1.3) is completely dominated by the first-order one, and approximating (1.3) in this circumstance amounts to trying to solve (1.1), where the boundary condition is understood in the classical sense instead of (1.2).

It has been shown in [4] that the best L^2-approximation (i.e., Least-Squares) does not converge to the right limit of (1.3) under the limiting process $\lim_{h\to 0}\lim_{\epsilon\to 0}$. The situation is quite different in $L^1(\Omega)$, since for reasons that will be detailed latter, the best L^1-approximation to (1.1) converges to the viscosity solution. Before going into the details of the proof and to illustrate this claim, we now reproduce a numerical experiment reported in [4].

Consider the 2D rectangular domain $\Omega =]0,1[^2$ and set $\partial\Omega_D = \{x=0\}\cup\{x=1\}$ and $\partial\Omega_N = \{y=0\}\cup\{y=1\}$, i.e., We want to solve the following scalar problem

$$u + \partial_x u = 1; \quad u|_{\partial\Omega_D} = 0, \tag{1.4}$$

Of course the above problem is not well-posed in the usual sense, since the outflow boundary condition is over-specified, but it is meaningful in the viscosity sense. Let $\{X_h\}_{h>0}$ be a sequence of H^1-conforming finite element spaces constructed on a shape regular mesh family and such that for all v_h in X_h, $v_h|_{\partial\Omega_D} = 0$. We show in figure 1 the best L^1-approximation and the best L^2-approximation of the above problem using a coarse mesh, $h = 1/10$. The $\mathbb{P}_1$ Lagrange interpolant of the exact solution is shown in the left panel, the best L^1-approximation is in the center panel, and the best L^2-approximation is shown in the right panel. Considering the mesh

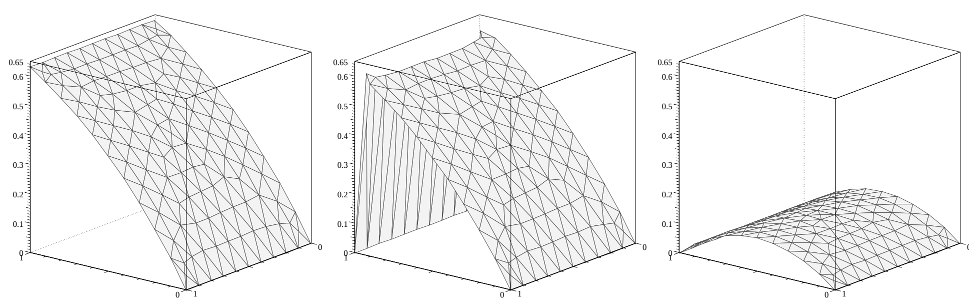

FIGURE 1. Viscosity solution to (1.4) from [4]. Left: $\mathbb{P}_1$ Lagrange interpolant of exact solution; center, L^1 solution; right, L^2 solution.

used, the best L^1-approximation is a reasonable approximation, whereas the Least-Squares solution is completely wrong. Contrary to what it looks, the two horn-like spikes observable on the graph of the L^1-solution are not over-shootings. These are perspective effects induced by the fact that the two corresponding $\mathbb{P}_1$-nodes are not aligned with the others. Given that the Least-Squares method together with its many variants is a central part for the stabilization of the Galerkin technique (see *e.g.* [3, 6, 7]), the above example gives new reasons why the Galerkin-Least-Squares method cannot generally cope properly with shocks and boundary layers without the help of shock-capturing terms [7, 5].

The rest of the paper is organized as follows. In §2 we introduce the ill-posed one-dimensional linear advection problem under scrutiny in this paper. The discrete

L^1-minimization problem is formulated. An expression for the discrete minimizer is derived in §3. The error analysis is done in §4. The main result of the paper is Theorem 7. Concluding remarks are reported in §5.

2. The one-dimensional problem

2.1. The continuous problem. In the rest of the paper we restrict ourselves to the following one dimensional differential equation:

$$\begin{cases} u(x) + \beta(x)u'(x) = f(x), & \text{in } \Omega, \\ u(0) = 0 \end{cases} \tag{2.1}$$

where β is assumed to be in $\mathcal{C}^1(\overline{\Omega})$ and f is in $L^1(\Omega)$. We assume moreover that

$$0 < \inf_{x\in\Omega} \beta(x) \tag{2.2}$$

$$\sup_{x\in\Omega} \beta'(x) < 1. \tag{2.3}$$

The condition $\beta' \leq 1$ is the one-dimensional counterpart of $\nabla\cdot\boldsymbol{\beta} \leq 1$ which is standard for (1.1). The uniqueness of a weak solution $u \in W^{1,1}(\Omega)$ to (2.1) is well known even under weaker assumptions on β and f. It is clear that u is also the viscosity solution to

$$\begin{cases} u + \beta(x)u' = f, & \text{in } \Omega, \\ u(0) = u(1) = 0, \end{cases} \tag{2.4}$$

To alleviate the notation we define the linear operator

$$L : W^{1,1}(\Omega) \ni v \longmapsto v + \beta v' \in L^1(\Omega). \tag{2.5}$$

2.2. The discrete problem. We now want to compute a best L^1-approximation to the viscosity solution to (2.4). To this end, let $\mathcal{T}_h = \sum_{i=0}^{n} I_i$ be a mesh of Ω composed of $n+1$ cells. Let $x_0, x_1, \ldots x_n$ be the vertices of this mesh so that $x_0 = 0$, $x_{n+1} = 1$ and for each cell I_i we have $I_i = [x_i, x_{i+1}]$. We set $h_i = x_{i+1} - x_i > 0$, $i = 0, 1, \ldots, n$, and define $h = \max_i h_i$. The midpoint of each cell is denoted by $x_{i+1/2} = \frac{x_{i+1}+x_i}{2}$.

We now define the following approximation space

$$X_h = \{v_h \in \mathcal{C}(\overline{\Omega});\ v_h|_{I_i} \in \mathbb{P}_1, \forall I_i \in \mathcal{T}_h;\ v_h(0) = v_h(1) = 0\}. \tag{2.6}$$

Note that the functions in X_h are zero at both ends of the interval Ω. Upon defining the functional

$$J(v_h) = \int_0^1 |L(u_h)(x) - f(x)|\, dx, \tag{2.7}$$

we consider the following problem: Seek $u_h \in X_h$ such that

$$J(u_h) = \min_{v_h \in X_h} J(v_h). \tag{2.8}$$

Owing to the linearity of L, J is convex; as a result, the above problem has at least one minimizer and all local minimizers are global. A peculiarity of the above minimization problem is that uniqueness of the minimizer is not guaranteed, but as already shown in [4], this is not a serious issue if we can prove that all the minimizers converge to a single limit.

At this time, we do not know how to deal with the above minimization without resorting to quadratures to approximate the integral. As a result, we introduce a second-order discretization, that is, we use the midpoint rule to approximate the integral over each mesh cell. We then replace the functional J by the following one

$$J_h(v_h) := \sum_{i=0}^{n} h_i \left| L(v_h)(x_{i+1/2}) - f_i \right|, \tag{2.9}$$

where we have set $f_i := h_i^{-1} \int_{I_i} f(x)dx$ and $\beta_i := \beta(x_{i+1/2})$. Problem (2.8) is then replaced by the following one: Seek $u_h \in X_h$ such that

$$J_h(u_h) = \min_{v_h \in X_h} J_h(v_h). \tag{2.10}$$

3. Computation of the minimizer

In this section, we derive an explicit formula to compute the solution to Problem (2.10).

Let us denote $R_i(v_h) = h_i \big(L(v_h)(x_{i+1/2}) - f_i\big)$ for all v_h in X_h. Observe that

$$J_h(v_h) = \sum_{i=0}^{n} |R_i(v_h)|.$$

Let $j \in \{0, \dots, n\}$ and let v_h^j be a discrete function in X_h solving the following linear system

$$R_i(v_h^j) = 0 \text{ for all } i \neq j. \tag{3.1}$$

Lemma 1. *For each $j \in \{0, \dots, n\}$, there is a unique v_h^j solving (3.1).*

Proof. Let us construct v_h^j. Upon introducing the notation $v_i^j = v_h^j(x_i)$ and

$$\delta_i = \frac{h_i}{\beta_i + \frac{h_i}{2}}, \tag{3.2}$$

it is clear that for any $i \le j$ we can compute v_{i+1}^j using the left boundary $v_0^j = 0$ together with

$$v_{i+1}^j = (1 - \delta_i) v_i^j + \delta_i f_i. \tag{3.3}$$

Therefore, for all $1 \le i \le j$ we derive

$$v_i^j = \sum_{k=0}^{i-1} f_k \delta_k \phi_{i,k}, \tag{3.4}$$

where we have defined

$$\phi_{i,i-1} = 1 \quad \text{and} \quad \phi_{i,k} = \prod_{s=k+1}^{i-1} (1 - \delta_s) \text{ for any } k < i - 1 \tag{3.5}$$

Similarly, for $j + 1 \le i \le n$, we have

$$v_i^j = \frac{1}{1 - \delta_i} v_{i+1}^j - \frac{\delta_i}{1 - \delta_i} f_i.$$

Therefore, for any $j + 1 \le i \le n$, we derive

$$v_i^j = - \sum_{k=i}^{n} f_k \delta_k \phi_{i,k}, \tag{3.6}$$

where we have defined

$$\phi_{i,k} = \prod_{s=i}^{k} \frac{1}{(1 - \delta_s)} \text{ for any } k \ge i \tag{3.7}$$

Note that (3.5) and (3.7) completely define the weights $\{\phi_{i,k}\}$, and (3.4) and (3.6) entirely define v_h^j. □

Lemma 2. *Under the above assumptions*

$$\min_{v_h \in X_h} J_h(v_h) = \min_{0 \le j \le n} J_h(v_h^j).$$

Proof. This is a standard property of discrete ℓ^1-approximations, see e.g., [8, Prop 6.7, p.135]. □

This result is the key feature of L^1-minimization which is of interest to us. We are going to show that it can be rephrased as follows. The function that minimizes J_h enforce the n residuals $R_0, \dots R_{n-1}$ to be zero and does not care about the $n+1$th one, R_n. As a result the error made in the cell n does not pollute what happens in the other cells. This behavior is characteristic of L^1-minimization. Any other type of L^p-minimization with $p > 1$ tends to equilibrate the error among $n+1$ cells. Heuristically speaking L^1-minimization solves the PDE in the cells that are important and forget about the others.

Proposition 3. *Under the above assumptions*

$$\min_{v_h \in X_h} J_h(v_h) = J_h(v_h^n).$$

Proof. We are going to prove the claim by showing that $J_h(v_h^n) = \min_{0 \le j \le n} J_h(v_h^j)$ and by using Lemma 2. Let $j \in \{0, \dots n\}$. We have the following representation

$$J_h(v_h^j) = R_j(v_h^j) = (\beta_j + \tfrac{1}{2}h_j)\left|-v_h^j(x_{j+1}) + (1-\delta_j)v_h^j(x_j) + \delta_j f_j\right|.$$

Using (3.4) and (3.6) we obtain

$$v_h^j(x_j) = \sum_{k=0}^{j-1} f_k \delta_k \phi_{j,k} \qquad \text{and} \qquad v_h^j(x_{j+1}) = -\sum_{k=j+1}^{n} f_k \delta_k \phi_{j+1,k}.$$

Using the above value of $v_h^j(x_j)$, we derive

$$(1-\delta_j)v_h^j + \delta_j f_j = (1-\delta_j)\sum_{k=0}^{j-1} f_k \delta_k \phi_{j,k} + \delta_j f_j. \tag{3.8}$$

Upon observing that

$$\phi_{j+1,k} = (1-\delta_j)\phi_{j,k} \quad \text{for any} \quad k = 0, 1 \dots, n \tag{3.9}$$

and that $\phi_{j+1,j} = 1$ is a special case of (3.9), we infer

$$(1-\delta_j)u_j + \delta_j f_j = \sum_{k=0}^{j} f_k \delta_k \phi_{j+1,k}.$$

Therefore,

$$J_h(v_h^j) = \left(\beta_j + \tfrac{1}{2}h_j\right)\left|\sum_{k=0}^{n} f_k \delta_k \phi_{j+1,k}\right| \tag{3.10}$$

Let us now compare $J_h(v_h^j)$ and $J_h(v_h^{j-1})$ for $1 \le j \le n$. Using (3.9) we derive

$$J_h(v_h^j) = \frac{\beta_j + \frac{1}{2}h_j}{\beta_{j-1} + \frac{1}{2}h_{j-1}} |1-\delta_j| J_h(v_h^{j-1})$$

Let r_j be the ratio J_j/J_{j-1}. Using the definition of δ_j, we obtain

$$r_j = \frac{\beta_j + \frac{1}{2}h_j}{\beta_{j-1} + \frac{1}{2}h_{j-1}} |1-\delta_j| = \frac{|\beta_j - \frac{1}{2}h_j|}{\beta_{j-1} + \frac{1}{2}h_{j-1}}.$$

On the one hand, the condition (2.2) guarantees that if h is small enough, $\beta_j \geq \frac{1}{2}h_j$, i.e.,

$$r_j = \frac{\beta_j - \frac{1}{2}h_j}{\beta_{j-1} + \frac{1}{2}h_{j-1}}$$

On the other hand, the condition (2.3) implies

$$\beta_j = \beta_{j-1} + \int_{x_{i-1/2}}^{x_{1+1/2}} \beta'(x)dx < \beta_{j-1} + \tfrac{1}{2}(h_j + h_{j-1}),$$

which means $r_j < 1$. In conclusion, $J_h(v_h^j) < J_h(v_h^{j-1})$, which yields

$$0 \leq J_h(v_h^n) < \cdots < J_h(v_h^0).$$

This completes the proof. □

The above results shows that the L^1-minimizer approximates the solution to the initial value problem (2.1) on the cells $\{I_0, I_1, \ldots, I_{n-1}\}$, and does not solve anything on the last cell I_n. Let us denote by u_h the minimizer, then the node values of u_h are

$$u_h(x_i) = \sum_{k=0}^{i-1} f_k \delta_k \phi_{i,k}, \quad 1 \leq i \leq n. \tag{3.11}$$

4. Error analysis

We establish consistency, stability, and prove convergence in this section. The main result is reported in Theorem 7. Henceforth, c is generic constant that does not depend on h and the value of which may vary at each occurrence.

Corollary 4. *Let u_h solve (2.10), then u_h satisfies a maximum principle*

$$\|u_h\|_{L^\infty} \leq \frac{1}{\beta_{\min}} \|f\|_{L^1}. \tag{4.1}$$

Proof. Let $i \in \{1, \ldots, n-1\}$. Using the representation (3.11) for $u_{i+1} := u_h(x_{i+1})$ together with $0 < \phi_{i+1,k} \leq 1$ and $0 < \delta_k \leq h_k/\beta_{\min}$ for all $0 \leq k \leq i$, we obtain

$$|u_{i+1}| \leq \sum_{k=0}^{i} |f_k|\delta_k \leq \frac{1}{\beta_{\min}} \sum_{k=0}^{i} |f_k| h_k = \frac{1}{\beta_{\min}} \sum_{k=0}^{i} |\int_{I_k} f(t)dt| \leq \frac{1}{\beta_{\min}} \|f\|_{L^1}.$$

The result follows from the fact that for piecewise linear functions we have $\sup_{I_k} |u_h| \leq \max(|u_k|, |u_{k+1}|)$. □

Lemma 5 (Consistency). *Let u_h solve (2.10), then there is $c > 0$ independent of h such that*

$$\|L(u_h)\|_{L^1(0,x_n)} \leq c\|f\|_{L^1}, \qquad \forall f \in L^1(\Omega), \tag{4.2}$$

$$\|L(u_h) - f\|_{L^1(0,x_n)} \leq c\, h \|f\|_{\mathrm{BV}[0,1]}, \qquad \forall f \in \mathrm{BV}[0,1]. \tag{4.3}$$

Proof. (1) Let us first establish (4.2). Since u_h is piecewise linear we have

$$L(u_h)\big|_{I_i} = u_i + (u_{i+1} - u_i)\frac{x - x_i}{h_i} + \beta(x)\frac{u_{i+1} - u_i}{h_i}.$$

After using (3.3) in the above, we obtain

$$L(u_h)\big|_{I_i} = f_i + u_i - f_i + \frac{\delta_i}{h_i}(f_i - u_i)(x - x_i + \beta(x)).$$

We also gives

$$L(u_h)\big|_{I_i} = f_i + \frac{\delta_i}{h_i}(u_i - f_i)\left(\frac{h_i}{\delta_i} - (x - x_i + \beta(x))\right).$$

Therefore, we have the estimate

$$\|L(u_h)\|_{L^1(0,x_n)} \leq R_1 + R_2$$

where

$$\begin{aligned} R_1 &:= \sum_{i=0}^{n-1} \int_{I_i} |f_i|\, dx \leq \|f\|_{L^1}, \\ R_2 &:= \sum_{i=0}^{n-1} \frac{\delta_i}{h_i}|u_i - f_i| \int_{I_i} \left|\beta(x_{i+1/2}) - \beta(x) + \frac{h_i}{2} - (x - x_i)\right| dx. \end{aligned}$$

We now continue by estimating R_2. Using the mean-value Theorem on β, we infer

$$\begin{aligned} (4.4)\quad R_2 &\leq \sum_{i=0}^{n-1} \frac{\delta_i}{h_i}\left(|u_i| + |f_i|\right) \int_{I_i} \left|\beta'(\xi_i)(x_{i+1/2} - x) + \frac{h_i}{2} - (x - x_i)\right| dx \\ &= \sum_{i=0}^{n-1} \frac{\delta_i}{h_i}\left(|u_i| + |f_i|\right) \int_{I_i} \left|(\beta'(\xi_i) + 1)(x_{i+1/2} - x)\right| dx \\ &\leq \|\beta' + 1\|_\infty \sum_{i=0}^{n-1} \frac{\delta_i}{h_i}\left(|u_i| + |f_i|\right) \int_{I_i} \left|x_{i+1/2} - x\right| dx \\ &\leq c \sum_{i=0}^{n-1} \delta_i h_i\left(|u_i| + |f_i|\right) \leq c\, h \sum_{i=0}^{n-1} \delta_i\left(|u_i| + |f_i|\right). \end{aligned}$$

Note that $\delta_i \leq \dfrac{h_i}{\beta_{\min}}$ and by applying (4.1) we obtain

$$\sum_{i=0}^{n-1} \delta_i\left(|u_i| + |f_i|\right) \leq \frac{1}{\beta_{\min}}\left(\frac{1}{\beta_{\min}}\|f\|_{L^1} + \|f\|_{L^1}\right),$$

which in turn yields

$$R_2 \leq c\, h\|f\|_{L^1}.$$

After combining the estimates for R_1 and R_2, we obtain the desired result.

(2) We now establish (4.3). By proceeding as in step (1), we obtain

$$(L(u_h) - f)(x)\big|_{I_i} = f_i - f(x) + \frac{\delta_i}{h_i}(u_i - f_i)\left(\frac{h_i}{\delta_i} - (x - x_i + \beta(x))\right).$$

This yields the estimate

$$\|L(u_h) - f\|_{L^1(0,x_n)} \leq R_3 + R_2$$

where R_2 is defined above and

$$R_3 := \sum_{i=0}^{n-1} \int_{I_i} |f_i - f(x)|\, dx.$$

If f is in $\mathrm{BV}[0,1]$, we clearly have $R_3 \le c\, h \|f\|_{\mathrm{BV}[0,1]}$. Combining this bound with that on R_2 already obtained in step (1), we deduce the desired result. □

Lemma 6 (Stability). *There is $c > 0$ such that for all n and all $\phi \in W^{1,1}(0, x_n)$ with $\phi(0) = 0$*

$$\|\phi\|_{W^{1,1}(0,x_n)} \le c\|L(\phi)\|_{L^1(0,x_n)}.$$

Proof. Using the chain rule for functions in $W^{1,1}$, we infer

$$|\phi| + \beta(x)\frac{d}{dx}|\phi| = \mathrm{sgn}(\phi) L(\phi), \tag{4.5}$$

and integrating over $I_h := [0, x_n]$ gives

$$\int_{I_h} |\phi| dx + \int_{I_h} \beta \frac{d}{dx}|\phi| dx \le \int_{I_h} |L(\phi)| dx.$$

We integrate by parts and obtain

$$\int_{I_h} (1 - \beta'(x))|\phi| dx + \beta(x_n)|\phi(x_n)| \le \int_{I_h} |L(\phi)| dx.$$

The assumptions (2.3) implies $1 - \beta'(x) \ge c_0 > 0$. Therefore we conclude

$$\int_{I_h} |\phi| dx \le \frac{1}{c_0} \int_{I_h} |L(\phi)| dx.$$

Moreover, using (2.2) we infer

$$|\phi'| \le \tfrac{1}{\beta_{\min}} |\beta(x)\phi'| \le \tfrac{1}{\beta_{\min}} (|L(\phi)| + |\phi|),$$

which yields the desired result. □

The main result of this section is the following

Theorem 7 (Convergence). *Let u be the solution to (2.1) (i.e., the viscosity solution of (2.4)). There is c independent of n and h such that for all $f \in \mathrm{BV}[0,1]$*

$$\|u - u_h\|_{W^{1,1}(0,x_n)} \le ch\|f\|_{\mathrm{BV}[0,1]}.$$

If f is only in $L^1(\Omega)$, we have $\lim_{h\to 0} \|u - u_h\|_{W^{1,1}(0,x_n)} = 0$.

Proof. (1) Use the stability estimate in Lemma 6 together with the consistency estimate (4.3) to deduce

$$\begin{aligned} \|u - u_h\|_{W^{1,1}(0,x_n)} &\le c\|L(u - u_h)\|_{L^1(0,x_n)} \\ &= c\|f - L(u_h)\|_{L^1(0,x_n)} \le c\, h\|f\|_{\mathrm{BV}[0,1]}. \end{aligned} \tag{4.6}$$

(2) Let us assume that f is in $L^1(\Omega)$ only. Let $\epsilon > 0$ be a real number. By density, there is $f^\epsilon \in \mathrm{BV}[0,1]$ such that $\|f - f^\epsilon\|_{L^1} \le \epsilon$. Let u_h^ϵ be the L^1-minimizer corresponding to the source term f^ϵ. By linearity of (3.11), it is clear that $u_h^\epsilon - u_h$ is the minimizer corresponding to the source term $f^\epsilon - f$. Then,

using the stability estimate in Lemma 6, the consistency estimate (4.2), together the bound established in step (1), we infer

$$\begin{aligned}
&\|u-u_h\|_{W^{1,1}(0,x_n)} \le c\|L(u-u_h)\|_{L^1(0,x_n)}\\
&\le c(\|Lu-Lu^\epsilon)\|_{L^1(0,x_n)} + \|Lu^\epsilon - Lu_h^\epsilon\|_{L^1(0,x_n)} + \|L(u_h^\epsilon - u_h)\|_{L^1(0,x_n)})\\
&\le c(\|f-f^\epsilon\|_{L^1(0,x_n)} + \|f^\epsilon - Lu_h^\epsilon\|_{L^1(0,x_n)} + \|f^\epsilon - f\|_{L^1(0,x_n)})\\
&\le c(\epsilon + h\|f^\epsilon\|_{\mathrm{BV}[0,1]}).
\end{aligned}$$

This implies

$$\limsup_{h\to 0} \|u-u_h\|_{W^{1,1}(0,x_n)} \le c\,\epsilon,$$

which in turn implies $\lim_{h\to 0}\|u-u_h\|_{W^{1,1}(0,x_n)} = 0$, since ϵ is arbitrary. □

5. Conclusions

Of course the result presented in this paper is unfortunately very partial, since it is restricted to the one-dimensional situation and linear finite elements. Moreover, we are aware that it relies too much on technicalities such as an explicit computation of the minimizer of (2.10). A better understanding of the situation should come from a more abstract handling of the problem, which eludes us at the present time. Nevertheless, to the best of our knowledge Theorem 7 is the first theoretical result that explains why L^1-best approximation of linear advection equations equipped with ill-posed boundary equations converge to viscosity solutions. As such, this result tends to confirm the findings in [4], i.e., L^1-approximation techniques are promising and are worth exploring further.

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