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DISCRETIZATION METHODS FOR SEMILINEAR PARABOLIC OPTIMAL CONTROL PROBLEMS

ION CHRYSSOVERGHI

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Abstract. We consider an optimal control problem described by semilinear parabolic partial differential equations, with control and state constraints. Since this problem may have no classical solutions, it is also formulated in the relaxed form. The classical control problem is then discretized by using a finite element method in space and the implicit Crank-Nicolson midpoint scheme in time, while the controls are approximated by classical controls that are bilinear on pairs of blocks. We prove that strong accumulation points in L^2 of sequences of optimal (resp. admissible and extremal) discrete controls are optimal (resp. admissible and weakly extremal classical) for the continuous classical problem, and that relaxed accumulation points of sequences of optimal (resp. admissible and extremal relaxed) discrete controls are optimal (resp. admissible and weakly extremal relaxed) for the continuous relaxed problem. We then apply a penalized gradient projection method to each discrete problem, and also a progressively refining version of the discrete method to the continuous classical problem. Under appropriate assumptions, we prove that accumulation points of sequences generated by the first method are admissible and extremal for the discrete problem, and that strong classical (resp. relaxed) accumulation points of sequences of discrete controls generated by the second method are admissible and weakly extremal classical (resp. relaxed) for the continuous classical (resp. relaxed) problem. For nonconvex problems whose solutions are non-classical, we show that we can apply the above methods to the problem formulated in Gamkrelidze relaxed form. Finally, numerical examples are given.

Key Words. Optimal control, parabolic systems, discretization, piecewise bilinear controls, penalized gradient projection method, relaxed controls.

1. Introduction

We consider an optimal distributed control problem for systems governed by a semilinear parabolic boundary value problem, with control and state constraints. The problem is motivated, for example, by the control of a heat (or another, e.g. pollution) diffusion process involving a source, which is nonlinear in the heat and temperature, with a possibly nonconvex cost, resulting in an optimal control problem, which is not necessarily convex. The scope of this paper is the study of discretization/optimization methods generating classical controls (instead of relaxed ones used in our previous work, see [4]-[7] for the numerical solution of nonconvex optimal control problems (but with a convex control constraint set), which may

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have classical, or non-classical relaxed, solutions. The problem is therefore also formulated in relaxed form, using Young measures. The classical control problem is then discretized by using a Galerkin finite element method with continuous piecewise linear basis functions in space and the implicit Crank-Nicolson midpoint scheme in time, while the controls are approximated by classical controls that are bilinear on pairs of blocks. We have adopted the midpoint scheme since it gives good state approximation (under some smoothness) and yields a simple and purely symmetric matching backward scheme for the adjoint discretization. On the other hand, discontinuous double-blockwise bilinear controls generally give better overall approximation of smooth, and in some cases piecewise smooth, optimal controls, than blockwise constant ones (see numerical examples). They are well defined on pairs of blocks due to the midpoint scheme used, and for consistency with minimizations involving the Hamiltonian in the algorithms. We first state various useful necessary optimality conditions for the continuous classical and relaxed problems, and for the discrete problem. Under appropriate assumptions, we prove that strong accumulation points in L^2 of sequences of optimal (resp. admissible and extremal) discrete controls are optimal (resp. admissible and weakly extremal classical) for the continuous classical problem, and that relaxed accumulation points of sequences of optimal (resp. admissible and extremal relaxed) discrete controls are optimal (resp. admissible and weakly extremal relaxed) for the continuous relaxed problem. We then apply a penalized gradient projection method to each discrete problem, and also a corresponding discrete method to the continuous classical problem, which progressively refines the discretization during the iterations, thus reducing computing time and memory. Under appropriate assumptions, we prove that accumulation points of sequences generated by the fixed discretization method are admissible and extremal for the discrete problem, and that strong classical (resp. relaxed) accumulation points of sequences of discrete controls generated by the progressively refining method are admissible and weakly extremal classical (resp. relaxed) for the continuous classical (resp. relaxed) problem. For nonconvex problems whose solutions are non-classical, we show that we can apply the above methods to the problem formulated in Gamkrelidze relaxed form. Using a standard procedure, the computed Gamkrelidze controls can then be approximated by classical ones. For nonconvex problems with smooth (or in some cases piecewise smooth) classical solutions, the proposed discrete penalized gradient projection method often yields very accurate numerical results. On the other hand, and if the control constraint set convex, the Gamkrelidze formulation approach seems to give better results than pure relaxed methods proposed in previous work (see e.g. [3]) when dealing with nonconvex problems with non-classical solutions, since the approximation of the relaxed control by highly oscillating classical controls is replaced by the approximation of three, possibly piecewise smooth, classical ones. Finally, several numerical examples are given. For approximation of nonconvex optimal control and variational problems, and of Young measures, see [1]-[7], [10]-[12].

2. The Continuous Optimal Control Problem

Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary Γ, and let $I = (0, T)$, $T < \infty$, be an interval. Consider the semilinear parabolic state equation

$$y_t + A(t)y = f(x, t, y(x, t), w(x, t)) \text{ in } Q = \Omega \times I,$$

$$y(x, t) = 0 \text{ in } \Sigma = \Gamma \times I \text{ and } y(x, 0) = y^0(x) \text{ in } \Omega,$$

where $A(t)$ is the second order elliptic differential operator

$$A(t)y = -\sum_{j=1}^{d}\sum_{i=1}^{d}(\partial/\partial x_i)[a_{ij}(x,t)\partial y/\partial x_j].$$

The constraints on the control are $w(x,t) \in U$ in Q, where U is a compact subset of $\mathbb{R}^{d'}$, the state constraints are

$$G_m(w) = \int_Q g_m(x,t,y,w)dxdt = 0,\ m = 1,...,p,$$

$$G_m(w) = \int_Q g_m(x,t,y,w)dxdt \leqslant 0,\ m = p+1,...,q,$$

and the cost functional to be minimized

$$G_0(w) = \int_Q g_0(x,t,y,w)dxdt.$$

Define the set of *classical controls*

$$W = \{w : (x,t) \mapsto w(x,t)|\, w \ \text{ measurable from } \ Q \ \text{ to } \ U\},$$

and the set of *relaxed controls* (Young measures; for the theory, see [18], [15])

$$R = \{r : Q \to M_1(U)| r \text{ weakly measurable } \} \subset L_w^\infty(Q, M(U)) \equiv L^1(Q, C(U))^*,$$

where $M(U)$ (resp. $M_1(U)$) is the set of Radon (resp. probability) measures on U. The set W is endowed with the relative strong topology of $L^2(Q)$ and the set R with the relative weak star topology of $L^1(Q, C(U))^*$. The set R is convex, metrizable and compact. If we identify every classical control $w(\cdot)$ with its associated Dirac relaxed control $r(\cdot) = \delta_{w(\cdot)}$, then W may be considered as a subset of R, and W is thus dense in R. For a given function $\phi \in L^1(Q, C(U)) = L^1(\bar{Q}, C(U))$ (or equivalently, for a given Caratheodory function ϕ in the sense of Warga [18]) and $r \in R$, we shall use the notation

$$\phi(x,t,r(x,t)) := \int_U \phi(x,t,u)r(x,t)(du).$$

We denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$, by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm in $L^2(\Omega)$, by $(\cdot,\cdot)_Q$ and $\|\cdot\|_Q$ the inner product and norm in $L^2(Q)$, by $(\cdot,\cdot)_1$ and $\|\cdot\|_1$ the inner product and norm in the Sobolev space $V = H_0^1(\Omega)$, and by $< \cdot,\cdot >$ the duality bracket between the dual $V^* = H^{-1}(\Omega)$ and V. We also define the usual bilinear form associated with $A(t)$

$$a(t,y,v) = \sum_{j=1}^{d}\sum_{i=1}^{d}\int_\Omega a_{ij}(x,t)\frac{\partial y}{\partial x_i}\frac{\partial v}{\partial x_j}dx.$$

The relaxed formulation of the above optimal control problem is the following. The relaxed state equation (in weak form) is

$$< y_t, v > + a(t,y,v) = \int_\Omega f(x,t,y(x,t),r(x,t))v(x)dx,\ \forall v \in V,\ \text{a.e. in } I,$$

$$y(t) \in V \text{ a.e. in } I, y(x,0) = y^0(x) \text{ a.e. in } \Omega,$$

the control constraint is $r \in R$, and the state constraints and cost functionals are

$$G_m(r) = \int_Q g_m(x,t,y(x,t),r(x,t))dxdt,\ m = 0,...,q.$$

We suppose that the coefficients a_{ij} satisfy the ellipticity conditions

$$\sum_{j=1}^{d}\sum_{i=1}^{d} a_{ij}(x,t)z_i z_j \geqslant \alpha \sum_{i=1}^{d} z_i^2,\ \forall z_i \in \mathbb{R},\ \text{a.e. in } Q,$$

with $\alpha > 0$, $a_{ij} \in L^\infty(Q)$, which imply that

$$|a(t,y,v)| \leqslant \alpha_1 \|y\|_1 \|v\|_1,\ a(t,v,v) \geqslant \alpha_2 \|v\|_1^2,\ t \in I,\ v \in V,$$

for some $\alpha_1 \geqslant 0,\ \alpha_2 > 0$. We suppose that the function f is defined on $Q \times \mathbb{R} \times U$, measurable for fixed y, u, continuous for fixed x, t, and satisfies the condition

$$|f(x,t,y,u)| \leqslant \psi(x,t) + \beta|y|,\ \forall (x,t,y,u) \in Q \times \mathbb{R} \times U,$$

with $\psi \in L^2(Q)$, $\beta \geqslant 0$, and the Lipschitz condition

$$|f(x,t,y_1,u) - f(x,t,y_2,u)| \leqslant L|y_1 - y_2|,\ \forall (x,t,y_1,y_2,u) \in Q \times \mathbb{R} \times \mathbb{R} \times U,$$

Then, for every relaxed control $r \in R$ and $y^0 \in L^2(\Omega)$, the state equation has a unique solution $y = y_r$ such that $y \in L^2(I,V)$, $y_t \in L^2(I,V^*)$; hence y is essentially equal to a function in $C(\bar{I}, L^2(\Omega))$, and thus the initial condition makes sense.

We suppose now in addition that the functions g_m are defined on $Q \times \mathbb{R} \times U$, measurable for fixed y, u, continuous for fixed x, t, and satisfy

$$|g_m(x,t,y,u)| \leqslant \zeta_m(x,t) + \eta_m y^2,\ \forall (x,t,y,u) \in Q \times \mathbb{R} \times U,$$

with $\zeta_m \in L^1(Q)$, $\eta_m \geqslant 0$. The following lemma and theorem are proved in [4].

Lemma 2.1. *The operators $w \mapsto y_w$, from W to $L^2(Q)$, and $r \mapsto y_r$, from R to $L^2(Q)$, and the functionals $w \mapsto G_m(w)$ on W, and $r \mapsto G_m(r)$ on R, are continuous.*

Theorem 2.1. *Under the above assumptions, if there exists an admissible control (i.e. satisfying all the constraints), then there exists an optimal relaxed control.*

It is well known that, even when the control set U is convex, the classical problem may have no classical solutions. In order to state the various necessary conditions for optimality, we suppose in addition that the functions f,g_m,f_y,f_u, g_{my},g_{mu} are defined on $Q \times \mathbb{R} \times U'$, where U' is an open set containing the compact set U, measurable on Q for fixed $(y,u) \in \mathbb{R} \times U$ and continuous on $\mathbb{R} \times U$ for fixed $(x,t) \in Q$, and satisfy

$$|g_{my}(x,t,y,u)| \leqslant \zeta_{m1}(x,t) + \eta_{m1}|y|,\ \forall (x,t,y,u) \in Q \times \mathbb{R} \times U,$$
$$|g_{mu}(x,t,y,u)| \leqslant \zeta_{m2}(x,t) + \eta_{m2}|y|,\ \forall (x,t,y,u) \in Q \times \mathbb{R} \times U,$$

with $\zeta_{m1}, \zeta_{m2} \in L^2(Q)$, $\eta_{m1}, \eta_{m2} \geqslant 0$, and

$$|f_y(x,t,y,u)| \leqslant L_1, \quad \forall (x,t,y,u) \in Q \times \mathbb{R} \times U,$$
$$|f_u(x,t,y,u)| \leqslant \zeta(x,t) + \eta|y|, \forall (x,t,y,u) \in Q \times \mathbb{R} \times U,$$

with $\zeta \in L^2(Q)$, $\eta \geqslant 0$.

We now give some useful results concerning necessary conditions for optimality (see also [9]).

Lemma 2.2. *Dropping the index m in the functionals, for $r, r' \in R$, the directional derivative of the functional G, defined on R, is given by*

$$DG(r, r'-r) = \lim_{\varepsilon \to 0^+} \frac{G(r + \varepsilon(r'-r)) - G(r)}{\varepsilon}$$
$$= \int_Q H(x,t,y,z,r'(x,t) - r(x,t))dxdt,$$

where the Hamiltonian H is defined by

$$H(x,t,y,z,u) = zf(x,t,y,u) + g(x,t,y,u),$$

and the adjoint state $z = z_r$ satisfies the equation

$$- < z_t, v > +a(t,v,z) = (zf_y(t,y,r) + g_y(t,y,r), v),\ \forall v \in V,\ \text{a.e. in } I,$$

$$z(t) \in V \ \text{a.e. in } I, z(x,T) = 0 \ \text{a.e. in } \Omega,$$

where $y = y_r$. The mappings $r \mapsto z_r$, from R to $L^2(Q)$, and $(r,r') \mapsto DG(r,r'-r)$, from $R \times R$ to $\mathbb{R}$, are continuous.

Theorem 2.2. *If $r \in R$ is optimal for either the relaxed or the classical optimal control problem, then r is strongly extremal relaxed, i.e. there exist multipliers $\lambda_m \in \mathbb{R}$, $m = 0,...,q$, with $\lambda_0 \geqslant 0$, $\lambda_m \geqslant 0$, $m = p+1,...,q$, $\sum\limits_{m=0}^{q} |\lambda_m| = 1$, such that*

$$\sum_{m=0}^{q} \lambda_m DG_m(r, r'-r) \geqslant 0,\ \forall r' \in R,$$

$$\text{and } \lambda_m G_m(r) = 0,\ m = p+1,...,q \ \text{(transversality conditions)}\ .$$

The above inequalities are equivalent to the strong relaxed pointwise minimum principle

$$H(x,t,y(x,t),z(x,t),r(x,t)) = \min_{u \in U} H(x,t,y(x,t),z(x,t),u),\ \text{a.e. in } Q,$$

where H and z are defined with $g = \sum\limits_{m=0}^{q} \lambda_m g_m$.
If U is convex, then this minimum principle implies the weak relaxed pointwise minimum principle

$$H_u(x,t,y,z,r(x,t))r(x,t) = \min_{\phi} H_u(x,t,y,z,r(x,t))\phi(x,t,r(x,t)),\ \text{a.e. in } Q,$$

where the minimum is taken over the set $B(Q,U;U)$ of Caratheodory functions $\phi : Q \times U \to U$ (see [18]*), which in turn implies the global weak relaxed condition*

$$\int_Q H_u(x,t,y,z,r(x,t))[\phi(x,t,r(x,t)) - r(x,t)]dxdt \geqslant 0,\ \forall \phi \in B(Q,U;U).$$

A control r satisfying this condition and the above transversality conditions is called weakly extremal relaxed.

Proof. The first part of the theorem is proved using the techniques of [18] (mainly Theorem V.3.2, see also [4]). Now, the strong relaxed minimum principle can be written in the compact form, for a.a. $(x,t) \in Q$, (x,t) *fixed*

$$\int_U H(u)r(du) \leqslant H(u),\ \forall u \in U.$$

Let $\phi : Q \times U \to U$ be any Caratheodory function ($\phi \in B(Q,U;U)$). Since U is convex here, we have

$$\int_U H(u)r(du) \leqslant H(u + \varepsilon(\phi(u) - u)),\ \forall u \in U,\ \forall \varepsilon \in [0,1],$$

hence

$$\int_U H(u)r(du) \leqslant \int_U H(u + \varepsilon(\phi(u) - u))r(du).$$

By the Mean Value Theorem and the uniform continuity of H in u

$$0 \leqslant \int_U \frac{H(u+\varepsilon(\phi(u)-u))-H(u)}{\varepsilon} r(du)$$

$$= \int_U H_u(u+\varepsilon\mu(u)(\phi(u)-u))(\phi(u)-u) r(du) \quad (0 \leqslant \mu(u) \leqslant 1)$$

$$= \int_U H_u(u)(\phi(u)-u) r(du) + \alpha(\varepsilon),$$

where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$, hence

$$\int_U H_u(u)(\phi(u)-u) r(du) = H'_u(r)(\phi(r)-r) \geqslant 0,$$

for every $\phi \in B(Q,U;U)$, a.e. in Q, which is the weak relaxed minimum principle. By integration, we get the global weak relaxed condition

$$\int_Q H_u(r)(\phi(r)-r) dxdt \geqslant 0, \; \forall \phi \in B(Q,U;U).$$

$\square$

Lemma 2.3. *We suppose that U is convex, and drop the index m. For $w, w' \in W$, the directional derivative of the functional G, defined on W, is given by*

$$DG(w, w'-w) = \lim_{\varepsilon \to 0^+} \frac{G(w+\varepsilon(w'-w))-G(w)}{\varepsilon}$$

$$= \int_Q H_u(x,t,y,z)(w'-w) dxdt,$$

where the adjoint state $z = z_w$ satisfies the equation

$$-< z_t, v > + a(t,v,z) = (z f_y(t,y,w) + g_y(t,y,w), v), \; \forall v \in V, \; \text{a.e. in } I,$$

$$z(x,t) = 0 \text{ in } \Sigma, \; z(x,T) = 0 \text{ in } \Omega, \; y = y_w.$$

The mappings $w \mapsto z_w$, from W to $L^2(Q)$, and $(w,w') \mapsto DG(w,w'-w)$, from $W \times W$ to $\mathbb{R}$, are continuous.

In the above notations of DG, it is understood, depending on the notation used for the arguments, that the directional derivative is taken in the corresponding space, W or R, on which G is defined.

Theorem 2.3. *If $w \in W$ is optimal for the classical problem, then w is weakly extremal classical, i.e. there exist multipliers λ_m as in Theorem 2.2 such that*

$$\sum_{m=0}^{q} \lambda_m DG_m(w, w'-w) \geqslant 0, \; \forall w' \in W,$$

$$a\text{and } \lambda_m G_m(w) = 0, \; m = p+1, ..., q \text{ (transversality conditions).}$$

The above inequalities are equivalent to the weak classical pointwise minimum principle

$$H_u(x,t,y,z,w(x,t))w(x,t) = \min_{u \in U} H_u(x,t,y,z,w(x,t))u, \; \text{a.e. in } Q,$$

where H and z are defined with $g = \sum_{m=0}^{q} \lambda_m g_m$.

Proof. Similar to Theorem 2.2, using here Theorem V.2.3 in [18]. $\square$

3. The discrete optimal control problems

We suppose here that the domain Ω is a polyhedron for simplicity, that $a(t,u,v)$ is independent of t and symmetric, the functions f,f_y,f_u, g_m,g_{my},g_{mu} are continuous on $\bar{Q}\times\mathbb{R}\times U$ (possibly finitely piecewise in t), the functions ψ, ζ_m, η_m, ζ_{m1}, ζ_{m2}, η_{m1}, η_{m2} are constant, and $y^0 \in V := H_0^1(\Omega)$. For each integer $n \geqslant 0$, let $\{E_i^n\}_{i=1}^{M(n)}$ be an admissible regular quasi-uniform triangulation of $\bar{\Omega}$ into closed d-elements (e.g. d-simplices), with $h^n = \max_i[\mathrm{diam}(E_i^n)] \to 0$ as $n \to \infty$, and $\{I_j^n\}_{j=1}^{N(n)}$, with $N(n) = 2N'(n)$, a subdivision of the interval $\bar{I}$ into closed intervals $I_j^n = [t_{j-1}^n, t_j^n]$, of equal length Δt^n, with $\Delta t^n \to 0$ as $n \to \infty$. We define the *blocks* $Q_{ij}^n = E_i^n \times I_j^n$. Let $V^n \subset V$ be the subspace of functions that are continuous on $\bar{\Omega}$ and linear (i.e. affine) on each S_i^n. The set of *discrete controls* W^n is the set of controls that are bilinear (biaffine), i.e. a product of a linear function of x by a linear function of t, on the interior of each double block $Q_{i,2k-1}^n \cup Q_{i,2k}^n$, $i = 1,...,M$, $k = 1,...,N'$. A discrete control

$$w^n \approx (\bar{w}_{ij}^n)_{ij} = [\bar{w}_{i,2k-1}^n, \bar{w}_{i,2k}^n]_{ik} \in W^n$$

is uniquely determined by its (limit) values at the vertices of each block Q_{ij}^n and the midpoints $\bar{t}_{2k-1}^n$, $\bar{t}_{2k}^n$ of the consecutive pairs of intervals I_{2k-1}^n, I_{2k}^n

$$\bar{w}_{i,2k-1}^{ln}, \bar{w}_{i,2k}^{ln},\ i = 1,...,M, \quad k = 1,...,N',\ l = 1,...,d+1,$$

where l corresponds to the vertex x_i^{ln} of E_i^n. The set of *acceptable discrete controls* $W_a^n \subset W^n$ is the subset of discrete controls satisfying in addition the following linear constraints on the values at the endpoints of the double intervals $I_{2k-1}^n \cup I_{2k}^n$

$$c_1 \leqslant \bar{w}_{i,2k-1}^{ln} - (\bar{w}_{i,2k}^{ln} - \bar{w}_{i,2k-1}^{ln})/2 \leqslant c_2,\ c_1 \leqslant \bar{w}_{i,2k}^{ln} + (\bar{w}_{i,2k}^{ln} - \bar{w}_{i,2k-1}^{ln})/2 \leqslant c_2,$$

$$i = 1,...,M,\ k = 1,...,N',\ l = 1,...,d+1,$$

which guarantee that $w^n(x,t) \in U$ a.e. in Q, and on the derivatives

$$|\nabla_x w^n(x,t)| \leqslant c_x \text{ (optional) },|\partial_t w^n(x,t)| \leqslant c_t,\ \text{a.e. in } Q,$$

Since w^n is piecewise bilinear, the inequality for ∂_t a.e. in Q is equivalent to the linear constraints

$$-c_t\Delta t^n \leqslant \bar{w}_{i,2k}^{ln} - \bar{w}_{i,2k-1}^{ln} \leqslant c_t\Delta t^n,\ i = 1,...,M,\ k = 1,...,N',\ l = 1,...,d+1.$$

We also define the simplexwise linear *midsections* of w^n at each midpoint $\bar{t}_j^n$

$$\bar{w}_j^n(x) = w^n(x,\bar{t}_j^n),\ \text{a.e. in } \Omega, j = 1,...,N.$$

Remark. Note that all the results in this article remain valid (with obvious simplifications) if we define W_a^n to be the set of controls that are constant on the interior of each block Q_{ij}^n, with values in U.

For a given discrete control

$$w^n \approx (\bar{w}_j^n)_{j=1,...,N} \in W^n,\ \text{with } \bar{w}_j^n = (\bar{w}_{ij}^n)_{i=1,...,M},$$

the corresponding discrete state $y^n = (y_0^n,...,y_N^n)$ is given by the discrete state equation (implicit Crank-Nicolson midpoint scheme)

$$(1/\Delta t^n)(y_j^n - y_{j-1}^n, v^n) + a(\bar{y}_j^n, v^n) = (f(\bar{t}_j^n, \bar{y}_j^n, \bar{w}_j^n), v^n),\ \forall v^n \in V^n,\ j = 1,...,N,$$

$$(y_0^n - y^0, v^n)_1 = 0,\ \forall v^n \in V^n,\ y_j^n \in V^n,\ j = 0,...,N,$$

$$\text{with } \bar{y}_j^n = (y_{j-1}^n + y_j^n)/2,\ \bar{t}_j^n = (t_{j-1}^n + t_j^n)/2.$$

Note that the discrete state depends only on the piecewise linear midsections (functions of x) $\bar{w}_1^n, ..., \bar{w}_N^n$ of the discrete control w^n. For Δt^n sufficiently small, depending on the Lipschitz constant L of f, and for each j, this scheme has a unique solution y_j^n, which can be computed by the standard predictor-corrector method, where a regular linear system is involved, and where the corrector scheme is contractive. The discrete functionals are defined by

$$G_m^n(w^n) = \Delta t^n \sum_{j=1}^{N} \int_\Omega g_m(x, \bar{t}_j^n, \bar{y}_j^n, \bar{w}_j^n) dx.$$

The discrete control constraint is $w^n \in W_a^n$, and the discrete state constraints are *either* of the two following ones

$$\text{Case (a) } |G_m^n(w^n)| \leqslant \varepsilon_m^n, \quad m = 1, ..., p,$$

$$\text{Case (b) } G_m^n(w^n) = \varepsilon_m^n, \quad m = 1, ..., p,$$

and

$$G_m^n(w^n) \leqslant \varepsilon_m^n, \ \varepsilon_m^n \geqslant 0, \ m = p+1, ..., q,$$

where the feasibility perturbations ε_m^n are chosen numbers converging to zero, to be defined later. The straightforward proof of the following lemma is omitted.

Lemma 3.1. *The operators $w^n \mapsto y_j^n$ and the discrete functionals $w^n \mapsto G_m^n(w^n)$, defined on W_a^n, are continuous. If any of the discrete problems is feasible, then there exists an optimal control for this problem.*

Lemma 3.2. *Dropping the index m, for $w^n, w'^n \in W_a^n$, the directional derivative of the functional G^n is given by*

$$DG^n(w^n, w'^n - w^n) = \Delta t^n \sum_{j=1}^{N} (H_u(\bar{t}_j^n, \bar{y}_j^n, \bar{z}_j^n, \bar{w}_j^n), \bar{w}_j^{'n} - \bar{w}_j^n),$$

where the discrete adjoint z^n is given by the scheme

$$-(1/\Delta t^n)(z_j^n - z_{j-1}^n, v^n) + a(v^n, \bar{z}_j^n) = (\bar{z}_j^n f_y(\bar{t}_j^n, \bar{y}_j^n, \bar{w}_j^n) + g_y(\bar{t}_j^n, \bar{y}_j^n, \bar{w}_j^n), v^n),$$

$$\forall v^n \in V^n, \ j = N, ..., 1, \ z_N^n = 0, \ z_j^n \in V^n, j = N, ..., 0,$$

which has a unique solution z_{j-1}^n for Δt^n sufficiently small, and for each j. Moreover, the operator $w^n \mapsto z^n$ and the functional $(w^n, w'^n) \mapsto DG^n(w^n, w'^n - w^n)$ are continuous.

The proofs of the two following theorems parallel the continuous case and are omitted.

Theorem 3.1. *If $w^n \in W_a^n$ is optimal for the discrete problem (constraint Case (b)), then it is discrete weakly extremal classical, i.e. there exist multipliers $\lambda_m^n \in \mathbb{R}$, $m = 0, ..., q$, with $\lambda_0^n \geqslant 0$, $\lambda_m^n \geqslant 0$, $m = p+1, ..., q$, $\sum_{m=0}^{q} |\lambda_m^n| = 1$, such that*

$$\sum_{m=0}^{q} \lambda_m^n DG_m^n(w^n, w'^n - w^n) = \Delta t^n \sum_{j=1}^{N} (H_u^n(\bar{t}_j^n, \bar{y}_j^n, \bar{z}_j^n, \bar{w}_j^n), \bar{w}_j^{'n} - \bar{w}_j^n) \geqslant 0,$$

$$\forall w'^n \in W_a^n,$$

$$\text{and } \lambda_m^n [G_m(w^n) - \varepsilon_m^n] = 0, \quad m = p+1, ..., q,$$

where H^n and z^n are defined with $g = \sum_{m=0}^{q} \lambda_m^n g_m$. The above global inequality condition is equivalent to the discrete weak classical double-blockwise minimum principle

$$\int_{E_i^n} [H_u^n(x, \bar{t}_{2k-1}^n, \bar{y}_{2k-1}^n, \bar{z}_{2k-1}^n, \bar{w}_{i,2k-1}^n)\bar{w}_{i,2k-1}^n + H_u^n(x, \bar{t}_{2k}^n, \bar{y}_{2k}^n, \bar{z}_{2k}^n, \bar{w}_{i,2k}^n)\bar{w}_{i,2k}^n]dx$$

$$= \min_{w_{i,2k-1}^{\prime n}, w_{i,2k}^{\prime n}} \int_{E_i^n} [H_u^n(x, \bar{t}_{2k-1}^n, \bar{y}_{2k-1}^n, \bar{z}_{2k-1}^n, \bar{w}_{i,2k-1}^n)\bar{w}_{i,2k-1}^{\prime n}$$

$$+ H_u^n(x, \bar{t}_{2k}^n, \bar{y}_{2k}^n, \bar{z}_{2k}^n, \bar{w}_{i,2k}^n)\bar{w}_{i,2k}^{\prime n}]dx, \; i = 1, ..., M, \; k = 1, ..., N',$$

where the minimum is taken, for each i, k, over all pairs $[w_{i,2k-1}^{\prime n}, w_{i,2k}^{\prime n}]$ subject to the linear constraints on the values and derivatives defining the set W_a^n.

4. Behavior in the limit

Let $\bar{W}_a^n$ denote the set of discrete controls that are constant on the interior of each double block $Q_{i,2k-1}^n \cup Q_{i,2k}^n$, $k = 1, ..., N'$, with values in U. Clearly, $\bar{W}_a^n \subset W_a^n$. The following classical control approximation result (a) is proved similarly to the lumped parameter case (see [13]), and the second (b) is proved in [4].

Proposition 4.1. *(a) For every $w \in W$, there exists a sequence $(w^n \in \bar{W}_a^n)$ that converges to w in L^2 strongly.*
(b) For every $r \in R$, there exists a sequence $(w^n \in \bar{W}_a^n)$ that converges to w in R.

Lemma 4.1. *(Stability) We suppose that $\Delta t^n \leqslant C(h^n)^2$, for some constant C independent of n. If Δt is sufficiently small, for every $w^n \in W_a^n$, we have the following inequalities, where the constants c are independent of n*

(i) $\|y_k^n\| \leqslant c, \quad k = 0, ..., N$, (ii) $\sum_{j=1}^{N} \left\|y_j^n - y_{j-1}^n\right\|^2 \leqslant c$,

(iii) $\Delta t^n \sum_{j=1}^{N} \left\|\bar{y}_j^n\right\|_1^2 \leqslant c$, (iv) $\Delta t^n \sum_{j=0}^{N} \left\|y_j^n\right\|_1^2 \leqslant c$.

Proof. Dropping the index n for simplicity of notation, setting $v = 2\Delta t y_j$ in the discrete equation, and using our assumptions on a, f and the Cauchy-Schwarz inequality, we have

$$\|y_j - y_{j-1}\|^2 + \|y_j\|^2 - \|y_{j-1}\|^2 + \frac{\Delta t}{2}[4a(\bar{y}_j, \bar{y}_j) + a(y_j, y_j) - a(y_{j-1}, y_{j-1})]$$
$$\leqslant 2\Delta t|(f(\bar{t}_j, \bar{y}_j, \bar{w}_j), y_j)| \leqslant c\Delta t(1 + \|y_j\| + \|y_{j-1}\|)\|y_j\|$$
$$\leqslant c\Delta t(1 + \|y_{j-1}\|^2 + \|y_j\|^2) \leqslant c\Delta t(1 + \|y_{j-1}\|^2 + \|y_j - y_{j-1}\|^2),$$

hence, for $\Delta t \leqslant 1/2c$

$$\frac{1}{2}\|y_j - y_{j-1}\|^2 + \|y_j\|^2 - \|y_{j-1}\|^2 + \frac{\Delta t}{2}[4a(\bar{y}_j, \bar{y}_j) + a(y_j, y_j) - a(y_{j-1}, y_{j-1})]$$
$$\leqslant c\Delta t(1 + \|y_{j-1}\|^2).$$

By summation over $j = 1, ..., k$, we obtain

$$\frac{1}{2}\sum_{j=1}^{k} \|y_j - y_{j-1}\|^2 + \|y_k\|^2 + 2\alpha_2\Delta t\sum_{j=1}^{k} \|\bar{y}_j\|_1^2$$

$$\leqslant \|y_0\|^2 + \alpha_1\frac{\Delta t}{2}\|y_0\|_1^2 + c\Delta t\sum_{j=1}^{k} (1 + \|y_{j-1}\|^2).$$

Since $\|y_0\|$ and $\|y_0\|_1$ remain bounded, using the discrete Bellman-Gronwall inequality (see [17]), we obtain inequality (i), and inequalities (ii), (iii) follow. By the inverse inequality (see [8]), the condition $\Delta t^n \leqslant C(h^n)^2$, and inequality (ii), we get

$$\Delta t \sum_{j=1}^{N} \|y_j - y_{j-1}\|_1^2 \leqslant \frac{\Delta t}{h^2} \sum_{j=1}^{N} \|y_j - y_{j-1}\|^2 \leqslant C \sum_{j=1}^{N} \|y_j - y_{j-1}\|^2 \leqslant c.$$

Inequality (iv) follows easily from this inequality and inequality (iii). □

For given values $v_0, ..., v_N$ in a vector space, define the piecewise constant and continuous piecewise linear functions

$$v_-(t) = v_{j-1},\ v_+(t) = v_j,\ \bar{v}(t) = (v_{j-1} + v_j)/2,\ t \in \overset{o}{I}{}_j^n,\ j = 1, ..., N,$$
$$v_\wedge(t) = v_{j-1} + \frac{t - t_{j-1}^n}{\Delta t^n}(v_j - v_{j-1}),\ t \in I_j^n,\ j = 1, ..., N.$$

Remark. Note that for any sequence $(w^n \in W_a^n)$, we have

$$|w^n(x,t) - \bar{w}^n(x,t)| = |w^n(x,t) - w^n(x, \bar{t}_j)| \leqslant \frac{\Delta t^n}{2} \sup_{t \in I_j^n} |\partial_t w^n|, x \in \Omega, t \in I_j^n,$$

and due to the constraint on the derivative $\partial_t w^n$

$$\|w^n - \bar{w}^n\|_\infty \leqslant \frac{\Delta t^n}{2} c_t \to 0,\ \text{as } n \to \infty.$$

It follows that $w^n \to w$ if and only if $\bar{w}^n \to w$, in L^2 strongly or weakly. It follows also from the definition of the weak star convergence in R that $w^n \to r$ in R ((w^n) considered as a sequence in R) if and only if $\bar{w}^n \to r$ in R.

Lemma 4.2. *(Consistency) Under the condition $\Delta t^n \leqslant C(h^n)^2$, if $w^n \to w \in W$ in L^2 strongly (resp. $w^n \to r$ in R, the w^n considered as relaxed controls), then the corresponding discrete states $y_-^n, y_+^n, \bar{y}^n, y_\wedge^n$ converge to y_w (resp. y_r) in $L^2(Q)$ strongly, as $n \to \infty$, and*

$$\lim_{n\to\infty} G_m^n(w^n) = G_m(w) (\text{ resp. } \lim_{n\to\infty} G_m^n(r^n) = G_m(r)),\ m = 0, ..., q.$$

Proof. By the above remark, we have also $\bar{w}^n \to w$ in L^2 (resp. $\bar{w}^n \to r$ in R). Since, by Lemma 4.1 (iii), y_-^n and y_+^n are bounded in $L^2(I, V)$, it follows from the definition of $y_\wedge^n$ that $y_\wedge^n$ is also bounded in $L^2(I, V)$. By extracting a subsequence, we can suppose that $y_\wedge^n \to y$ in $L^2(I, V)$ weakly (hence in $L^2(Q)$ weakly), for some y. The discrete state equation can be written in the form

$$\frac{d}{dt}(y_\wedge^n(t), v^n) = (\psi^n(t), v^n)_1, \quad \forall v^n \in V^n,\ \text{a.a. } t \in (0, T),$$

in the scalar distribution sense, where the piecewise constant function ψ^n is defined, using Riesz' representation theorem, by

$$(\psi_j^n(t), v^n)_1 = -a(\bar{y}_j^n, v^n) + (f(\bar{t}_j^n, \bar{y}_j^n, \bar{w}_j^n), v^n),\ \text{in } \overset{o}{I}{}_j^n,\ j = 1, ..., N.$$

By our assumptions, we have, for $j = 1, ..., N$

$$|(\psi_j^n, v^n)_1| \leqslant c[\|\bar{y}_j^n\|_1 \|v^n\|_1 + (1 + \|\bar{y}_j^n\|)\|v^n\|] \leqslant c(1 + \|\bar{y}_j^n\|_1)\|v^n\|_1,$$

hence

$$\|\psi_j^n\|_1 \leqslant c(1 + \|\bar{y}_j^n\|_1) \text{ and } \|\psi_j^n\|_1^2 \leqslant c(1 + \|\bar{y}_j^n\|_1^2).$$

Therefore, using Lemma 4.1 (iii)

$$\int_0^T \|\psi^n(t)\|_1^2 dt \leqslant c(1 + \int_0^T \|\bar{y}^n\|_1^2 dt) \leqslant c,$$

which shows that ψ^n belongs to $L^2(I,V)$, hence to $L^1(I,V)$. Following the proof of Lemma 5.6 in [16], it can then be shown that

$$\int_{-\infty}^{+\infty} |\tau|^{2\rho} \|\hat{y}_\wedge^n(\tau)\|^2 d\tau \leqslant c, \text{ for } \rho < 1/4,$$

where $\hat{y}_\wedge^n$ denotes the Fourier transform of $y_\wedge^n$ (extended by 0 outside $[0,T]$). By the 2[nd] Compactness Theorem in [16], p. 274, there exists a subsequence (same notation) such that $y_\wedge^n \to \tilde{y}$ in $L^2(Q)$ strongly, for some $\tilde{y}$, and we must have $\tilde{y} = y$, since $\hat{y}_\wedge^n \to y$ also in $L^2(Q)$ weakly. Since, by Lemma 4.1 ((ii) multiplied by Δt), $y_+^n - y_-^n \to 0$ in $L^2(Q)$, we have also $\bar{y}^n \to y$ in $L^2(Q)$ strongly. Finally, similarly to the proof of Lemma 4.3 in [4], we can pass to the limit in the weak discrete equation, integrated in t, using Proposition 2.1 in [3] for the nonlinear term, and show that $y = y_w$, or $y = y_r$. The last convergences follow using the same Proposition. $\square$

In the sequel, we suppose (theoretically) that there exists a constant C such that $\Delta t^n \leqslant C(h^n)^2$, for every n. Note that this condition (in fact, the inverse inequality used to derive inequality (iv)) is a worst case one. In practice, the corresponding sequences of gradients (∇y^n) constructed by the algorithms are often bounded in $L^2(Q)$, or even in $L^\infty(Q)$, and the above condition is not needed. We suppose also that the considered continuous classical or relaxed problem is feasible. The following theorem is a theoretical result concerning the behavior in the limit of optimal discrete controls.

Theorem 4.1. *In the presence of state constraints, we suppose that the sequences (ε_m^n) in the discrete state constraints (Case (a)) converge to zero as $n \to \infty$ and satisfy*

$$|G_m^n(\tilde{w}^n)| \leqslant \varepsilon_m^n, \ m = 1, ..., p, \ G_m^n(\tilde{w}^n) \leqslant \varepsilon_m^n, \ \varepsilon_m^n \geqslant 0, \ m = p+1, ..., q,$$

for every n, where $(\tilde{w}^n \in W^n \subset R)$ is a sequence converging in L^2 strongly (resp. in R) to an optimal control $\tilde{w} \in W$ (resp. $\tilde{r} \in R$) of the classical (resp. relaxed) problem, if it exists (resp. which always exists). For each n, let w^n be optimal for the discrete problem (Case (a)). Then every strong classical (resp. relaxed) accumulation point of (w^n), if it exists (resp. which always exists), is optimal for the continuous classical (resp. relaxed) problem.

Proof. The proof is similar to that of Theorem 4.1 in [6], using here Lemma 4.2. Note that our assumption implies that the discrete problems are feasible for every n. $\square$

Lemma 4.3. *(Consistency) If $w^n \to w \in W$ in L^2 strongly, or if $w^n \to r$ in R (the w^n considered here as relaxed controls), then the corresponding discrete adjoint states $z_-^n, z_+^n, \bar{z}^n, z_\wedge^n$ converge to z_w in $L^2(Q)$ strongly, as $n \to \infty$. If $w^n \to w \in W$ and $w'^n \to w' \in W$, in L^2 strongly, then*

$$\lim_{n\to\infty} DG_m^n(w^n, w'^n - w^n) = DG_m(w, w' - w), \ m = 0, ..., q.$$

Proof. The proof is similar to that of Lemma 4.2, using also Lemma 4.2. $\square$

Next, we study the behavior in the limit of extremal discrete controls. Consider the discrete problems with state constraints (Case (b)). We shall construct sequences of perturbations (ε_m^n) converging to zero and such that the discrete problem is feasible for every n. Let $w'^n \in W^n$ be any solution of the problem without state constraints

$$c^n = \min_{w^n \in W^n} \{ \sum_{m=1}^{p} [G_m^n(w^n)]^2 + \sum_{m=p+1}^{q} [\max(0, G_m^n(w^n))]^2 \},$$

and set

$$\varepsilon_m^n = G_m^n(w'^n),\ m = 1, ..., p,\ \varepsilon_m^n = \max(0, G_m^n(w'^n)),\ \ m = p+1, ..., q.$$

Let $\tilde{v}$ be an admissible control for the continuous classical (resp. relaxed) problem, and $(\tilde{w}^n \in \bar{W}_a^n)$ a sequence converging to $\tilde{v}$ in L^2 strongly (resp. in R) (Proposition 4.1). We have

$$\lim_{n\to\infty} [G_m^n(\tilde{w}^n)]^2 = [G_m(\tilde{v})]^2 = 0,\ m = 1, ..., p,$$

$$\lim_{n\to\infty} [\max(0, G_m^n(\tilde{w}^n))]^2 = [\max(0, G_m(\tilde{v}))]^2 = 0, \quad m = p+1, ..., q,$$

which imply a fortiori that $c^n \to 0$, hence $\varepsilon_m^n \to 0$, $m = 1, ..., q$. Then clearly the discrete problem (Case (b)) is feasible for every n, for these perturbations ε_m^n. We suppose in the sequel that the ε_m^n are chosen as in the above minimum feasibility procedure. Note that we often find $c^n = 0$, for large n, due to sufficient discrete controllability, in which case the perturbations ε_m^n are equal to zero.

Theorem 4.2. *For each n, let w^n be admissible and extremal for the discrete problem (Case (b)). Then*
(i) Every strong accumulation point of the sequence (w^n) in L^2 is admissible and weakly extremal classical for the continuous classical problem,
(ii) Every relaxed accumulation point of (w^n) is admissible and weakly extremal relaxed for the continuous relaxed problem.

Proof. (i) The passage to the strong limit in the discrete principle in global form is proved similarly to Theorem 4.2 in [6], using here Proposition 2.1 in [3], Proposition 4.1 and Lemmas 4.2, 4.3.
(ii) Since R is compact and $\sum_{m=0}^{q} |\lambda_m^n| = 1$, let (w^n), (λ_m^n), $m = 0, ..., q$, be subsequences such that $w^n \to r$ in R and $\lambda_m^n \to \lambda_m$, $m = 0, ..., q$, and consider the discrete principle in global form, which can be written

$$\int_Q H_u^n(x, \bar{t}^n, \bar{y}^n, \bar{z}^n, \bar{w}^n)(\bar{w}'^n - \bar{w}^n) dxdt \geqslant 0, \quad \forall w'^n \in W_a^n.$$

For every continuous function $\phi : Q \times U \to U$, we then have

$$\int_Q H_u(x, \bar{t}^n, \bar{y}^n, \bar{z}^n, \bar{w}^n)[\phi(\bar{x}^n(x), \bar{t}^n(t), \bar{w}^n) - \bar{w}^n] dxdt \geqslant 0,$$

where we set $\bar{x}^n(x) =$ barycenter of S_i^n, for $x \in \overset{o}{S_i^n}$, $\quad i = 1, ..., M$. Passing to the limit, by Lemmas 4.2, 4.3 and Proposition 2.1 in [3], we obtain

$$\int_Q H_u(x, t, y, z, r(x,t))[\phi(x, t, r(x,t)) - r(x,t)] dxdt$$

$$= \int_Q \int_U H_u(x, t, y, z, r(x,t))[\phi(x, t, u) - u] r(du) dxdt \geqslant 0, \text{ for every such } \phi.$$

Now let $\phi : Q \times U \to U$ be any Caratheodory function, or equivalently, $\phi \in L^1(Q, C(U; U))$, and let (ϕ_k) be a sequence in $C(Q \times U; U)$ converging to ϕ in $L^1(Q, C(U; U))$. By Egorof's theorem, we can suppose that $\phi_k \to \phi$ a.e. in Q, with values in $C(U; U)$, hence a.e. in $Q \times U$, with values in U. Replacing ϕ by ϕ_k in the above inequality and using Lebesgue's dominated convergence theorem (the integrand is clearly bounded by a fixed function in $L^1(Q \times U)$), we can pass to the limit as $k \to \infty$ and obtain the global weak relaxed condition. Finally, we pass to the limit as $n \to \infty$ in the transversality conditions and the state constraints as in Theorem 4.2 in [6]. $\square$

5. Discrete penalized gradient methods

We suppose here that U is convex. Let (M_m^l), $m = 1, ..., q$, be nonnegative increasing sequences such that $M_m^l \to \infty$ as $l \to \infty$, and define the *penalized discrete functionals*

$$G^{nl}(w^n) = G_0^n(w^n) + (1/2)\{\sum_{m=1}^{p} M_m^l [G_m^n(w^n)]^2 + \sum_{m=p+1}^{q} M_m^l [\max(0, G_m^n(w^n))]^2\}.$$

Let $\gamma \geqslant 0$, $b, c \in (0,1)$, and let (β^l), (ζ_k) be positive sequences, with (β^l) decreasing and converging to zero, and $\zeta_k \leqslant 1$. The algorithm described below contains various options. In the case of the progressively refining version, we suppose that each element $E_{i'}^{n+1}$ is a subset of some element E_i^n and that either $N(n+1) = N(n)$ or $N(n+1) = \mu N(n)$, for some integer $\mu \geqslant 2$. In this case, we have $W_a^n \subset W_a^{n+1}$, and thus a control $w^n \in W_a^n$ may be considered also as belonging to W_a^{n+1}, hence the computation of states, adjoints and functional derivatives for this control, but with the possibly finer discretization $n+1$, makes sense.

Algorithm

Step 1. Set $k = 0$, $l = 1$, choose a value of n and an initial control $w_0^{n1} \in W_a^n$.

Step 2. Find $v_k^{nl} \in W_a^n$ such that

$$\begin{aligned} e_k &= DG^{nl}(w_k^{nl}, v_k^{nl} - w_k^{nl}) + \frac{\gamma}{2}\left\|\bar{v}_k^{nl} - \bar{w}_k^{nl}\right\|_Q^2 \\ &= \min_{v'^n \in W_a^n} [DG^{nl}(w_k^{nl}, v'^n - w_k^{nl}) + \frac{\gamma}{2}\left\|\bar{v}'^n - \bar{w}_k^{nl}\right\|_Q^2], \end{aligned}$$

and set $d_k = DG^{nl}(w_k^{nl}, v_k^{nl} - w_k^{nl})$.

Step 3. If $|e_k| \leqslant \beta^l$, set $w^{nl} = w_k^{nl}$, $v^{nl} = v_k^{nl}$, $d^l = d_k$, $e^l = e_k$, $l = l+1$, $[n = n+1]$, and go to Step 2.

Step 4. (*Armijo step search*) Find the lowest integer value $s \in \mathbb{Z}$, say $\bar{s}$, such that $\alpha(s) = c^s \zeta_k \in (0,1]$ and $\alpha(s)$ satisfies the inequality

$$G^{nl}(w_k^{nl} + \alpha(s)(v_k^{nl} - w_k^{nl})) - G^{nl}(w_k^{nl}) \leqslant \alpha(s) b e_k,$$

and set $\alpha_k = \alpha(\bar{s})$.

Step 5. Set $w_{k+1}^{nl} = w_k^{nl} + \alpha_k(v_k^{nl} - w_k^{nl})$, $k = k+1$, and go to Step 2.

In the above Algorithm, we consider two versions:

Version A. "$n = n+1$" is *skipped* in Step 3: n is a constant integer chosen in Step 1, i.e. we choose a *fixed discretization* and replace the discrete functionals G_m^n by the perturbed ones $\tilde{G}_m^n = G_m^n - \varepsilon_m^n$.

Version B. "$n = n+1$" is *not skipped* in Step 3: we have a *progressively refining* discrete method, i.e. $n \to \infty$ (see proof of Theorem 5.1 below), in which case we can take $n = 1$ in Step 1, hence $n = l$ in the Algorithm. This version has the advantage of reducing computing time and memory, and also of avoiding the computation of the minimum feasibility perturbations ε_m^n.

If $\gamma > 0$, we have a *penalized gradient projection* method, in which case we can easily see "by completing the square" that Step 2 amounts to finding, independently for each $i = 1, ..., M,\ k' = 1, ..., N'$, the projection $[\bar{v}^{nl}_{ki,2k'-1}, \bar{v}^{nl}_{ki,2k'}]$ of the pair of functions in $[L^2(S^n_i)]^2$

$$\bar{u}^{nl}_{ki,2k'-1} = \bar{w}^{nl}_{ki,2k'-1} - (1/\gamma)H^n_u(\bar{t}^n_{2k'-1}, \bar{y}^{nl}_{i,2k'-1}, \bar{z}^{nl}_{i,2k'-1}, \bar{w}^{nl}_{i,2k'-1}),$$

$$\bar{u}^{nl}_{ki,2k'} = \bar{w}^{nl}_{ki,2k'} - (1/\gamma)H^{'n}_u(\bar{t}^n_{2k'}, \bar{y}^{nl}_{i,2k'}, \bar{z}^{nl}_{i,2k'}, \bar{w}^{nl}_{i,2k'}),$$

onto the convex subset of $[L^2(S^n_i)]^2$ of pairs of bilinear functions $[\bar{v}^{'nl}_{i,2k'-1}, \bar{v}^{'nl}_{i,2k'}]$ satisfying the linear acceptability constraints defining W^n_a, which in turn reduces to the minimization of a quadratic function of the coefficients of the controls $\bar{v}^{'nl}_{i,2k'-1}, \bar{v}^{'nl}_{i,2k'}$ on a convex set. The parameter γ is chosen here experimentally to yield a good convergence rate. If $\gamma = 0$, the above Algorithm is a *penalized conditional gradient (Frank-Wolfe)* method, and Step 2 reduces similarly to the minimization of a linear function on a convex set, for each i, k'. On the other hand, since clearly $d_k \leqslant e_k \leqslant 0$ and $b \in (0, 1)$, by the definition of the directional derivative the Armijo step α_k in Step 4 can be found for every k.

A (continuous classical or relaxed, or discrete) extremal control is called *abnormal* if there exist multipliers as in the corresponding optimality conditions, with $\lambda_0 = 0$ (or $\lambda^n_0 = 0$). A control is admissible *and* abnormal extremal in exceptional, degenerate, situations (see [18]).

With w^{nl} defined in Step 3, define the sequences of *multipliers*

$$\lambda^{nl}_m = M^l_m G^n_m(w^{nl}),\ m = 1, ..., p,\ \lambda^{nl}_m = M^l_m \max(0, G^n_m(w^{nl})),\ m = p+1, ..., q,$$

Theorem 5.1. *(i) In Version B, let (w^{nl}) be a subsequence, considered as a sequence in R, of the sequence generated by the Algorithm in Step 3 that converges to some r in the compact set R, as $l \to \infty$ (hence $n \to \infty$). If the sequences (λ^{nl}_m) are bounded, then r is admissible and weakly extremal relaxed for the continuous relaxed problem.*
(ii) In Version B, if (w^{nl}) is a subsequence of the sequence generated by the Algorithm in Step 3 that converges to some $w \in W$ in L^2 strongly, as $l \to \infty$ (hence $n \to \infty$). If the sequences (λ^{nl}_m) are bounded, then w is admissible and weakly extremal classical for the continuous classical problem.
(iii) In Version A, let (w^{nl}), n fixed, be a subsequence of the sequence generated by the Algorithm in Step 3 that converges to some $w^n \in W^n_a$ as $l \to \infty$. If the sequences (λ^{nl}_m) are bounded, then w^n is admissible and extremal for the fixed discrete problem.
(iv) In any of the three convergence cases (i), (ii), (iii), suppose that the (discrete or continuous) limit problem has no admissible, abnormal extremal, controls. If the limit control is admissible, then the sequences of multipliers are bounded, and this control is extremal as above.

Proof. We shall first show that $l \to \infty$ in the Algorithm. Suppose, on the contrary, that l, hence n (in both versions A, B), remains constant after a finite number of iterations in k, and so we drop here the indices l and n. Let us show that then $e_k \to 0$. Since W^n_a is compact, let $(w_k)_{k\in K},\ (v_k)_{k\in K}$ be subsequences of the sequences generated in Steps 2 and 5 such that $w_k \to \tilde{w},\quad v_k \to \tilde{v}$, in W^n_a, as $k \to \infty,\quad k \in K$. By Step 2, $d_k \leqslant e_k \leqslant 0$ for every k, hence

$$e = \lim_{k\to\infty,\, k\in K} e_k = DG(\tilde{w}, \tilde{v} - \tilde{w}) + \frac{\gamma}{2}\|\tilde{v} - \tilde{w}\|^2_Q \leqslant 0,$$

$$d = \lim_{k\to\infty,\, k\in K} d_k = DG(\tilde{w}, \tilde{v} - \tilde{w}) \leqslant \lim_{k\to\infty,\, k\in K} e^k = e \leq 0.$$

Suppose that $e < 0$, hence $d < 0$. The function $\Phi(\alpha) = G(w+\alpha(v-w))$ is continuous on $[0, 1]$. Since the directional derivative $DG(w, v - w)$ is linear w.r.t. $v - w$, Φ is differentiable on $(0,1)$ and has derivative $\Phi'(\alpha) = DG(w+\alpha(v-w), v-w)$. Using the Mean Value Theorem, we have, for each $\alpha \in (0,1]$

$$G(w_k + \alpha(v_k - w_k)) - G(w_k) = \alpha DG(w_k + \alpha'(v_k - w_k), v_k - w_k),$$

for some $\alpha' \in (0, \alpha)$. Therefore, for $\alpha \in [0,1]$, by the continuity of DG (Lemma 3.1)

$$G(w_k + \alpha(v_k - w_k)) - G(w_k) = \alpha(d + \varepsilon_{k\alpha}),$$

where $\varepsilon_{k\alpha} \to 0$ as $k \to \infty$, $k \in K$, and $\alpha \to 0^+$. Now, we have $d_k = d + \eta_k$, where $\eta_k \to 0$ as $k \to \infty$, $k \in K$, and since $b \in (0,1)$

$$d + \varepsilon_{k\alpha} \leqslant b(d + \eta_k) = bd_k,$$

for $\alpha \in [0, \bar\alpha]$, for some $\bar\alpha > 0$, and $k \geqslant \bar k$, $k \in K$. Hence

$$G(w_k + \alpha(v_k - w_k)) - G(w_k) \leqslant \alpha b d_k \leqslant \alpha b e_k,$$

for $\alpha \in [0, \bar\alpha]$, for some $\bar\alpha > 0$, and $k \geqslant \bar k$, $k \in K$. It follows from the choice of the Armijo step α_k in Step 4 that $\alpha_k \geqslant c\bar\alpha$, for $k \geqslant \bar k$, $k \in K$. Hence

$$G(w_{k+1}) - G(w_k) = G(w_k + \alpha_k(v_k - w_k)) - G(w_k) \leqslant \alpha_k b e_k \leqslant c\bar\alpha b e_k \leqslant c\bar\alpha b e/2,$$

for $k \geqslant \bar k$, $k \in K$. It follows that $G(w_k) \to -\infty$ as $k \to \infty$, $k \in K$, which contradicts the fact that $G(w_k) \to G(\tilde w)$ as $k \to \infty$, $k \in K$, by the continuity of the discrete functional (Lemma 3.1). Therefore, we must have $e = 0$, and $e_k \to e = 0$, for the whole sequence, since the limit 0 is unique. But Step 3 then implies that $l \to \infty$, which is a contradiction. Therefore, $l \to \infty$. This shows also that $n \to \infty$ in Version B.

(i) Let (w^{nl}) be a subsequence (same notation), considered as a sequence in R, of the sequence generated in Step 3, that converges to some accumulation point $r \in R$ as $l, n \to \infty$. Suppose that the sequences (λ_m^{nl}) are bounded and (up to subsequences) that $\lambda_m^{nl} \to \lambda_m$. By Lemma 4.2, we have

$$0 = \lim_{l\to\infty} \frac{\lambda_m^{nl}}{M_m^l} = \lim_{l\to\infty} G_m^n(w^{nl}) = G_m(r), \quad m = 1, ..., p,$$

$$0 = \lim_{l\to\infty} \frac{\lambda_m^{nl}}{M_m^l} = \lim_{l\to\infty} [\max(0, G_m^n(w^{nl}))] = \max(0, G_m(r)),\ m = p+1, ..., q,$$

which show that r is admissible. Now, by Steps 2 and 3 we have, for every $v'^n \in W_a^n$

$$\begin{aligned} &DG^{nl}(w^{nl}, v'^n - w^{nl}) + (\gamma/2)\left\|\bar v'^n - \bar w^{nl}\right\|_Q^2 \\ &= DG_0^n(w^{nl}, v'^n - w^{nl}) + \sum_{m=1}^{p} \lambda_m^{nl} DG_m^n(w^{nl}, v'^n - w^{nl}) \\ &+ \sum_{m=p+1}^{q} \lambda_m^{nl} DG_m^n(w^{nl}, v'^n - w^{nl}) + (\gamma/2)\left\|\bar v'^n - \bar w^{nl}\right\|_Q^2 \\ &= \int_Q H_u^{nl}(x, \bar t^n, \bar y^{nl}, \bar z^{nl}, \bar w^{nl})(\bar v'^n - \bar w^{nl}) dxdt + (\gamma/2)\int_Q \left|\bar v'^n - \bar w^{nl}\right|^2 dxdt \geqslant e^l, \end{aligned}$$

with involved multipliers λ_m^{nl}. Choosing any continuous function $\phi : \bar Q \times U \to U$ and setting $\bar x^n(x) =$ barycenter of S_i^n, for $x \in \overset{o}{S}{}_i^n$, $\quad i = 1, ..., M$, we have

$$\int_Q H_u^{nl}(x, \bar t^n, \bar y^{nl}, \bar z^{nl}, \bar w^{nl})[\phi(\bar x^n(x), \bar t^n(t), \bar w^{nl}(x,t)) - \bar w^{nl}(x,t)] dxdt$$

$$+(\gamma/2)\int_Q [\phi(\bar x^n(x), \bar t^n(t), \bar w^{nl}(x,t)) - \bar w^{nl}(x,t)]^2 dxdt \geqslant e^l, \text{ for every such } \phi,$$

Using Lemmas 4.2, 4.3 and Proposition 2.1 in [3], we can pass to the limit in this inequality as $l, n \to \infty$ and obtain

$$\int_Q H_u(x,t,y,z,r(x,t))[\phi(x,t,r(x,t)) - r(x,t)]dxdt$$

$$+(\gamma/2)\int_Q [\phi(x,t,r(x,t)) - r(x,t)]^2 dxdt \geqslant 0, \text{ for every such } \phi,$$

with involved multipliers λ_m. Replacing ϕ by $u + \mu(\phi - u)$, with $\mu \in (0,1]$, dividing by μ, and then taking the limit as $\mu \to 0^+$, we obtain the weak relaxed condition

$$\int_Q H_u(x,t,y,z,r(x,t))[\phi(x,t,r(x,t)) - r(x,t)]dxdt \geqslant 0, \text{ for every such } \phi,$$

with multipliers λ_m, which holds also by density for every Caratheodory function ϕ. If $G_m(r) < 0$, for some index $m \in [p+1, q]$, then for sufficiently large l we have $G_m^{nl}(w^{nl}) < 0$ and $\lambda_m^l = 0$, hence $\lambda_m = 0$, i.e. the transversality conditions hold. Therefore, r is weakly extremal relaxed.

(ii) Let (w^{nl}) be a subsequence (same notation) of the sequence generated in Step 3 that converges to some $w \in W$ in L^2 strongly as $l, n \to \infty$. The admissibility of w is proved as in (i). Now, let any $v' \in W$ and, by Proposition 4.1, $(v'^n \in \bar{W}_a^n \subset W_a^n)$ a sequence converging to v'. Let, as above, (λ_m^{nl}) be subsequences such that $\lambda_m^l \to \lambda_m$. By Step 2, we have

$$\int_Q H_u^{'nl}(x, \bar{t}^n, \bar{y}^{nl}, \bar{z}^{nl}, \bar{w}^{nl})(\bar{v}'^n - \bar{w}^{nl})dxdt + (\gamma/2)\int_Q \left|\bar{v}'^n - \bar{w}^{nl}\right|^2 dxdt \geqslant e^l,$$

with multipliers λ_m^{nl}. Using Proposition 2.1 in [3] and Lemmas 4.2, 4.3, we can pass to the limit as $l, n \to \infty$ and obtain

$$\int_Q H_u(x,t,y,z,w)(v' - w)dxdt + (\gamma/2)\int_Q |v' - w|^2 dxdt \geqslant 0, \quad \forall v' \in W.$$

It follows as in (i) that

$$\int_Q H_u(x,t,y,z,w)(v' - w)dxdt \geqslant 0, \quad \forall v' \in W,$$

with multipliers λ_m as in the optimality conditions, similarly to (i). The transversality conditions are derived as in (i).

(iii) The admissibility of the limit control w^n is proved as in (i). Passing here to the limit in the inequality resulting from Step 2 as $l \to \infty$, for n fixed, and using Lemmas 3.1 and 3.2, we obtain, similarly to (i)

$$\sum_{m=0}^{q} \lambda_m^n D\tilde{G}_m^n(w^n, v'^n - w^n) = \sum_{m=0}^{q} \lambda_m^n DG_m^n(w^n, v'^n - w^n) \geqslant 0, \ \forall v'^n \in W_a^n,$$

with multipliers as in the optimality conditions, and the discrete transversality conditions

$$\lambda_m^n \tilde{G}_m^n(w^n) = \lambda_m^n [G_m^n(w^n) - \varepsilon_m^n] = 0, \ m = p+1, ..., q,$$

(iv) In either of the three above convergence cases, suppose that the limit control is admissible and that the limit problem has no admissible, abnormal extremal, controls. Suppose that the multipliers are not all bounded. Then, dividing the corresponding inequality resulting from Step 2 by the greatest multiplier norm and passing to the limit for a subsequence, we see that we obtain an optimality inequality where the first multiplier is zero, and that the limit control is abnormal extremal, a contradiction. Therefore, the sequences of multipliers are bounded, and by (i), (ii), or (iii), this limit control is extremal as above. □

One can easily see that Theorem 5.1 remains valid if we replace e_k by d_k in Step 4 of the Algorithm. In practice, by choosing moderately growing sequences (M_m^l) and a sequence (β^l) relatively fast converging to zero, the resulting sequences of multipliers (λ_m^{nl}) are often bounded, or even convergent.

When directly applied to nonconvex optimal control problems whose solutions are non-classical relaxed controls, methods generating classical controls often yield very poor convergence (highly oscillating controls). For this reason, we propose here an alternative approach, which uses the Gamkrelidze formulation. For simplicity, we suppose that there are no state constraints, and also that $U = [c_1, c_2] \subset \mathbb{R}$, which is usually the case. Consider the relaxed problem, with state equation

$$y_t + A(t)y = f(x,t,y(x,t),r(x,t)) \text{ in } Q,\ y = 0 \text{ in } \Sigma,\ y(x,0) = y^0(x) \text{ in } \Omega,$$

control constraint $r \in R$, and cost functional

$$G(r) = \int_Q g(x,t,y(x,t),r(x,t))dxdt.$$

Since U is an interval and f, g continuous in u, for each (x,t) fixed, the set

$$S(x,t) = \left\{ \begin{bmatrix} f(x,t,y(x,t),u)) \\ g(x,t,y(x,t),u) \end{bmatrix} | \, u \in U \right\}$$

is a continuous arc in $\mathbb{R}^2$, hence a connected set. For each (x,t), the vector

$$\begin{bmatrix} f(x,t,y(x,t),r(x,t)) \\ g(x,t,y(x,t),r(x,t)) \end{bmatrix} \in \mathbb{R}^2$$

belongs to the convex hull of $S(x,t)$, and hence (see [14]) can be represented as

$$\begin{bmatrix} f(x,t,y,r)) \\ g(x,t,y,r)) \end{bmatrix} = \beta(x,t) \begin{bmatrix} f(x,t,y,u) \\ g(x,t,y,u) \end{bmatrix} + [1-\beta(x,t)] \begin{bmatrix} f(x,t,y,v) \\ g(x,t,y,v) \end{bmatrix},$$

with $u(x,y), v(x,t) \in U$, $\beta(x,t) \in [0,1]$, and by Filippov's Selection Theorem (see [18]), we can suppose that these three functions are measurable. Therefore, the control r yields the same state y as the Gamkrelidze control $r_G := \beta\delta_u + (1-\beta)\delta_v$. Conversely, every such a control r_G is clearly a relaxed control r that yields the same state. Therefore, the above relaxed control problem is equivalent to the following extended classical one, with the 3-dimensional controlled state equation

$$y_t + A(t)y = \beta(x,t)f(x,t,y(x,t),u(x,t)) + [1-\beta(x,t)]f(x,t,y(x,t),v(x,t)) \text{ in } Q,$$
$$y = 0 \text{ in } \Sigma,\ y(x,0) = y^0(x) \text{ in } \Omega,$$

control constraints

$$(u(x,y), v(x,t), \beta(x,t)) \in U \times U \times [0,1] \text{ in } Q,$$

and cost functional

$$\mathbf{G}(\beta,u,v) = \int_Q \{\beta(x,t)g(x,t,y(x,t),u(x,t)) + [1-\beta(x,t)]g(x,t,y(x,t),v(x,t))\}dxdt.$$

We can therefore apply the gradient methods described above to this classical problem. The Gamkrelidze relaxed controls thus computed can then be approximated by sub-blockwise (w.r.t. t) constant classical controls using a simple procedure (see e.g. [7] and Example (d) below). In the general case, i.e. if U is not convex, one can use methods generating relaxed controls to solve such nonconvex problems (see [5], [7]).

6. Numerical examples

Let $\Omega = I = (0,1)$.
a) *Classical optimal control, control constraints.* Define the reference control and state

$$\tilde{w}(x,t) = \begin{cases} -1, \ 0 \leqslant t < 0.25, \\ \min\left(0.6, -0.8 + 1.8 \cdot 4x(1-x)\left(\frac{t-0.25}{0.75}\right)^2\left(2-\frac{t-0.25}{0.75}\right)\right), \ 0.25 \leqslant t \leqslant 1 \end{cases}$$

$$\tilde{y}(x,t) = x(1-x)e^t,$$

and consider the following optimal control problem, with state equation

$$y_t - y_{xx} = [x(1-x)+2]e^t + \sin y - \sin\tilde{y} + w - \tilde{w} \text{ in } Q,$$

$$y = 0 \text{ in } \Sigma, \ y(x,0) = \tilde{y}(x,0) \text{ in } \Omega,$$

control constraint set $U = [-1, 0.6]$, and cost functional

$$G_0(w) = 0.5\int_Q [(y-\tilde{y})^2 + (w-\tilde{w})^2]dxdt.$$

Clearly, the optimal control and state are $\tilde{w}$ and $\tilde{y}$. The discrete gradient projection method, without penalties, was applied to this problem, with $M = N = 128$, gradient projection parameter $\gamma = 0.5$, Armijo parameters $b = c = 0.5$, zero initial control, and $c_t = 10$ (constraint on ∂_t). After 6 iterations, we obtained the results

$G_0^n(w_k) = 2.4 \cdot 10^{-8}$, $e_k = -1.2 \cdot 10^{-17}$, $\varepsilon_k = 2.6 \cdot 10^{-5}$, $\eta_k = 1.1 \cdot 10^{-2}$, $\zeta_k = 6.7 \cdot 10^{-3}$,

where e_k is defined in Step 2 of the Algorithm, ε_k, η_k are the state and control max-errors at the vertices of the simplices and the end points of the double intervals $I_{2k'-1} \cup I_{2k'}$, and ζ_k the control max-error at the vertices of the simplices and the midpoints of $I_{2k'-1}, I_{2k'}$ (the control max-errors are in fact of order 10^{-4} outside a narrow surface-folding strip, see Figure 1). Actually, the constraint on $\partial_t w_k$ was found to be inactive here, due to the piecewise smoothness of the optimal control $\tilde{w}$, with mild derivative $\partial_t \tilde{w}$. Figure 1 shows the computed control $w_k \approx \tilde{w}$.
b) *Classical optimal control, strictly active control constraints.* Choosing the set $U = [-0.7, 0.3]$, the control constraints being now strictly active for the method *and* for the problem, and zero initial control, we obtained after 6 iterations the control shown in Figure 2 and the values $G_0^n(w_k) = 1.49338170, \cdot 10^{-2}$, $e_k = -3.7 \cdot 10^{-19}$.
c) *Classical optimal control, control and state constraints.* With the heat state equation (and the above boundary conditions)

$$y_t - y_{xx} = y + 3w \text{ in } Q,$$

the set $U = [-0.7, 0.5]$, the additional state constraint

$$G_1(w) = \int_Q y(x,t)dxdt = 0,$$

and with the cost of Example (a), we obtained, after 99 iterations in k of the penalized gradient projection method, the control and state shown in Figures 3 and 4 and the values $G_0^n(w_k) = 0.13075793$, $G_1^n(w_k) = -5.1 \cdot 10^{-4}$, $e_k = -8.8 \cdot 10^{-6}$. Since here the state equation and the equality constraint are linear in (y,w), and the cost is convex in (y,w), the optimality conditions are also sufficient, and therefore the method actually approximates the optimal control.
d) *Relaxed optimal control, control constraints, Gamkrelidze formulation.* Defining the state equation (with the above boundary conditions)

$$y_t - y_{xx} = [x(1-x)+2]e^t + w - \begin{cases} \phi(x,t), & \text{in } \Omega \times (0, 0.5) \\ \psi(x,t), & \text{in } \Omega \times [0.5, 1) \end{cases}, \text{ with}$$

$$\psi(x,t)=2(0.5-t)(0.25-x(1-x)),\ \phi(x,t)=2(0.5-t)(0.25+x(1-x)),$$

the convex constraint set $U=[-1,1]$, and the *nonconvex* cost functional

$$G_0(w)=\int_Q 0.5(y-\tilde{y})^2dx\,dt+\int_0^{0.5}\int_\Omega (w-\phi)^2dx\,dt+\int_{0.5}^1\int_\Omega (-w^2)dx\,dt,$$

it is easily verified that the unique optimal relaxed control is

$$\tilde{r}(x,t)=\begin{cases}\delta_{\phi(x,t)}, & \text{in } \Omega\times(0,0.5)\ (\text{one}-\text{atomic})\\ \tilde{\beta}(x,t)\delta_{-1}+[1-\tilde{\beta}(x,t)]\delta_1, & \text{in } \Omega\times[0.5,1)\ (\text{two}-\text{atomic})\end{cases}$$

where $\tilde{\beta}(x,t)=(1-\psi(x,t))/2$ and δ_α denotes the Dirac measure concentrated at the point $\alpha\in U$, the optimal state is $y=\tilde{y}$, and the optimal relaxed cost $G_0(\tilde{r})=-0.5$. Note that this cost can be approximated as closely as desired using a classical control (W is dense in R), but cannot be attained for such a control. We reformulated the problem in Gamkrelidze form (see end of Section 5), with three classical controls $u,v\in[-1,1]$, $\beta\in[0,1]$, and applied the conditional gradient method (i.e. with $\gamma=0$), for iterations 1 to 180, and then the gradient projection method (with $\gamma=0.5$), for iterations 181 to 200. This was done because in this special example the pure gradient projection method does not improve the control iterates at the boundary of $\bar{Q}$ for $t\in[0.5,1]$, since the optimal values -1, 1 of the controls u,v there are quickly found almost exactly, hence β disappears in the cost for $t\in[0.1,1]$, and the adjoint is anyway zero on this boundary. With initial controls $u_0=-0.4$, $v_0=0.4$, $\beta_0=0.5$, and $c_t=20$, we obtained the controls $u_k\approx v_k\approx\phi$, in $\Omega\times(0,0.5)$, $u_k\approx-1$, $v_k\approx 1$, in $\Omega\times[0.5,1)$, with max errors $\leqslant 7.9\cdot 10^{-4}$ (Figure 5 shows v_k), the state $y_k\approx\tilde{y}$ with max error $\leqslant 6.4\cdot 10^{-4}$, the control β_k shown in Figure 6 (note that β_k is arbitrary in $\Omega\times(0,0.5)$, since the controls u_k,v_k are almost equal there, and that the state and cost are not too sensitive to the values of β_k in $\Omega\times[0.5,1)$, given that $u_k\approx-1$, $v_k\approx 1$ there), the cost $G_0^n(\beta_k,u_k,v_k)=-0.499999988$ and $e_k=-7.1\cdot 10^{-10}$. The Gamkrelidze relaxed control corresponding to (u_k,v_k,β_k) can then be approximated by the classical control w_k which takes, for each i,j, respectively the values w_k,u_k on the two sub-blocks of $\overset{o}{Q}{}^n_{ij}$:

$$\overset{o}{E}{}^n_i\times((j-1)\Delta t^n,(j-1+\beta_{k,ij})\Delta t^n),\ \overset{o}{E}{}^n_i\times((j-1+\beta_{k,ij})\Delta t^n,j\Delta t^n).$$

Finally, the progressively refining version of each algorithm was also applied to the above problems, with successive discretizations $M=N=32,64,128$, in three nearly equal iteration periods, and yielded results of similar accuracy, but required less than half the computing time. This shows that finer discretizations become progressively more efficient as the control iterate gets closer to the extremal control, while coarser ones in the early iterations have not much influence on the final results.

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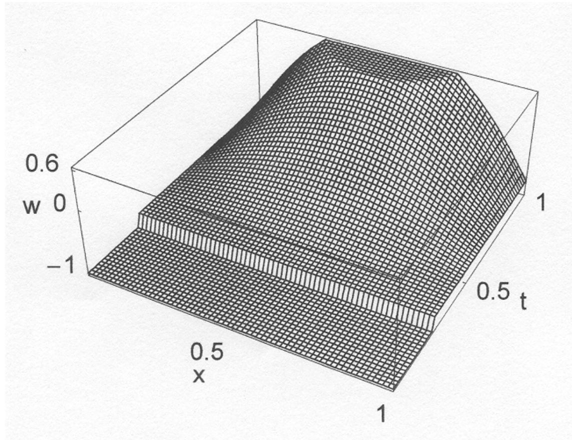

FIGURE 1.

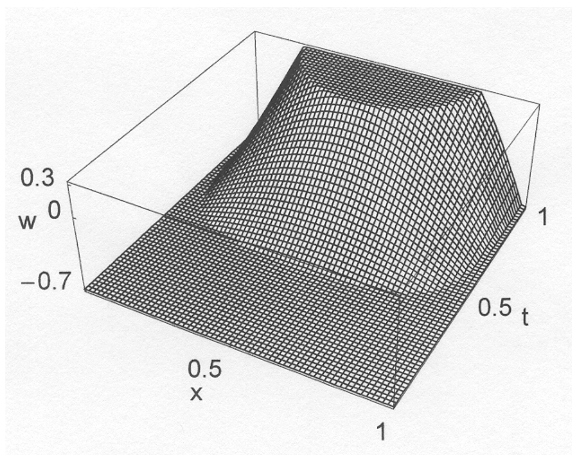

FIGURE 2.

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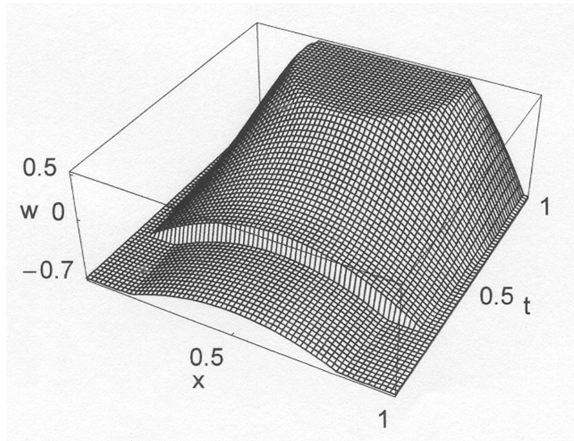

FIGURE 3.

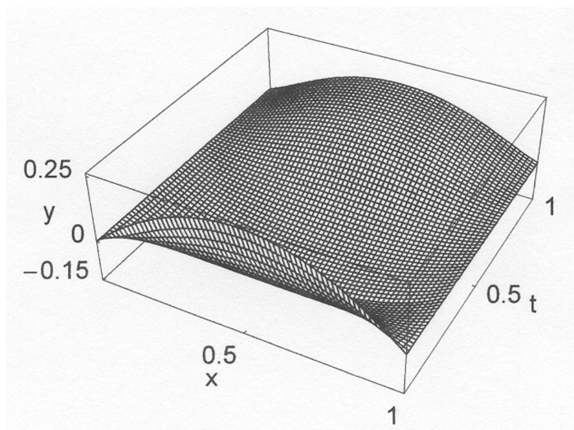

FIGURE 4.

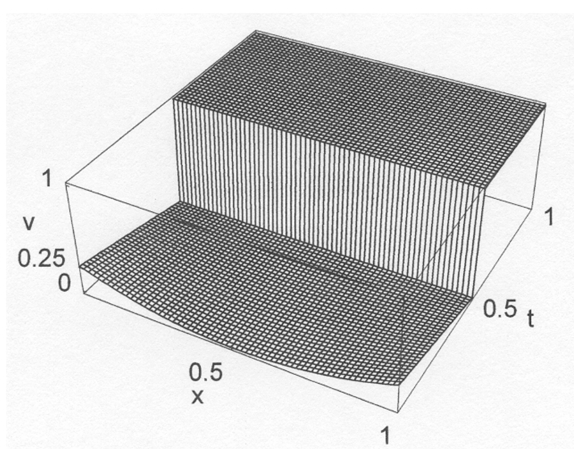

FIGURE 5.

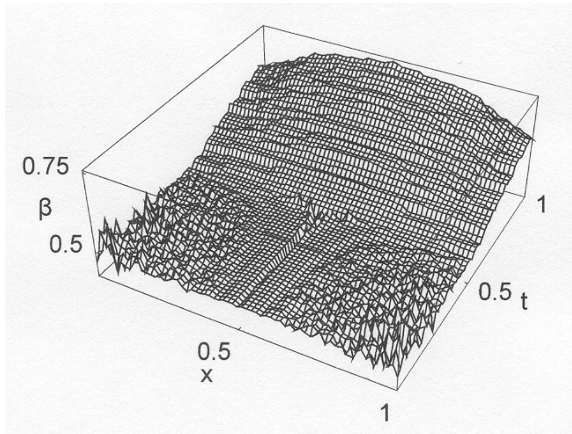

FIGURE 6.

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Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece
E-mail: `ichris@central.ntua.gr`