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ON A FINITE DIFFERENCE SCHEME FOR A BEELER-REUTER BASED MODEL OF CARDIAC ELECTRICAL ACTIVITY

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Abstract. We investigate an explicit finite difference scheme for a Beeler-Reuter based model of cardiac electrical activity. As our main result, we prove that the finite difference solutions are bounded in the L^∞-norm. We also prove the existence of a weak solution by showing convergence to the solutions of the underlying model as the discretization parameters tend to zero. The convergence proof is based on the compactness method.

Key Words. reaction-diffusion system of Beeler-Reuter type, excitable cells, cardiac electric field, monodomain model, finite difference scheme, maximum principle, convergence

1. Introduction

The purpose of this paper is to study a finite difference scheme for a mathematical model that describes electrical activity in cardiac tissue. A spatially dependent model for this phenomenon is commonly written

$$\begin{aligned} \frac{\partial v}{\partial t} &= \nabla \cdot (M \nabla v) - I_{ion}(s, v), \\ \frac{ds}{dt} &= F(s, v), \end{aligned} \tag{1}$$

where v is the transmembrane potential, M is the (diagonal) conductivity tensor, and s is a state vector whose entries depend on the cell model. This reaction-diffusion system is commonly referred to as the monodomain model of electrophysiology, and the complexity depends on the cell model represented by the ODE system. A more general model (which is not treated here) for electrical activity in anisotropic cardiac tissue is the so-called bidomain model. In this model the cardiac muscle is viewed as a superposition of two (anisotropic) continuous media, referred to as the intracellular and the extracellular. The intracellular and extracellular media are connected by a continuous cellular membrane, and this coupling gives rise to a reaction-diffusion system of degenerate type [5], where the unknowns are the intracellular potential u_i, the extracellular potential u_e, and the transmembrane potential $v = u_i - u_e$ (i.e., the jump in the potential across the cellular membrane). Under the assumption of equal anisotropies, (that is, the ratio of the conductivity coefficients parallel and transverse to the direction of fibre is constant, both for

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the intracellular and extracellular media), the bidomain model reduces to the monodomaim model (1). In this case, the intracellular and extracellular potential u_i and u_e can be recovered by scaling the transmembrane potential v appropriately.

The cell model represented by the system of ODEs in (1) could be rather simple, as in, e.g., the well known FitzHugh-Nagumo model, or, as in recent models, very complex, see, e.g., Winslow *et al.* [10], where s contains dozens of variables, such as membrane channels and ionic concentrations. In the present paper, we investigate the monodomain model coupled to the Beeler-Reuter equations [1], which was one of the first mathematical models to be developed for describing the electrophysiology of a cardiac cell. Compared to more recent models, this is a simple yet fairly realistic description of cell dynamics due to the presence of the intracellular calcium concentration, which is important for contraction of the heart. Physiological, as well as mathematical, considerations impose certain constraints or bounds on the calcium concentration, and it is our aim here to identify actual values for these bounds by analysing a finite difference scheme. We point out that the upper bound on the calcium concentration found in this study depends greatly on the governing equation of this quantity, and does not necessarily have a significant physiological interpretation. More advanced models for describing intracellular calcium dynamics have been developed, by, e.g, Luo and Rudy [8], Noble [2], and Winslow *et al.* [10].

During the depolarization phase of the heart, the solution is rapidly changing, i.e., there are steep solution gradients present. Thus there is a need for a strict requirement on the time step in any numerical scheme that wishes to resolve this feature of the solution. We identify such a constraint when proving a maximum principle for an explicit finite difference scheme approximating the system (1) with Beeler-Reuter kinetics, which is our main result in this paper. In addition, we give a rather simple convergence proof for the finite difference scheme for a large class of initial values for the transmembrane potential v. We prove L^2 convergence of the finite difference solutions using the compactness method and the concept of weak solution. In passing, we mention that due to the sharp transition layers in the solution, it is reasonable to seek weak solutions of the system. For convergence of some numerical schemes for (1) with the simpler Hodgkin-Huxley type kinetics, see [9, 6] and the references cited therein.

The remainder of this paper is organised as follows: In Section 2 we present the mathematical model. Section 3 is devoted to an informal motivation at the continuous level of why lower and upper bounds should be available for the solutions of the reaction-diffusion system (1) with Beeler-Reuter kinetics. We give a weak formulation of the problem in Section 4, while the finite difference scheme is presented in Section 5. In Section 6 we prove the maximum principle for the scheme, while the convergence of the scheme is proved in Section 7. We conclude the paper by showing some simulation results in Section 8.

Throughout this paper we denote a generic constant that does not depend on the discretization parameters by K. The actual value of K may change from one line to the next during a computation.

2. Model description

In this section we present the mathematical model to be studied. We confine the discussion to two spatial dimensions, but the extension to higher dimensions is straightforward. We consider a bounded open domain $\Omega \subset \mathbf{R}^2$, a fixed final time

Rate	C_1	C_2	C_3	C_4	C_5	C_6	C_7
α_z	0.0005	0.083	50	0	0	0.057	1
β_z	0.0013	-0.06	20	0	0	-0.04	1
α_m	0	0	47	-1	47	-0.1	-1
β_m	40	-0.056	72	0	0	0	0
α_n	0.126	-0.25	77	0	0	0	0
β_n	1.7	0	22.5	0	0	-0.082	1
α_l	0.055	-0.25	78	0	0	-0.2	1
β_l	0.3	0	32	0	0	-0.1	1
α_d	0.095	-0.01	-5	0	0	-0.072	1
β_d	0.07	-0.017	44	0	0	0.05	1
α_f	0.012	0.008	28	0	0	0.15	1
β_f	0.0065	-0.02	30	0	0	-0.2	1

TABLE 1. Constants for the rate functions

$T > 0$, and set $\Omega_T := \Omega \times (0, T)$. The transmembrane potential satisfies

$$\chi \partial_t v = \nabla \cdot (M \nabla v) - \chi I_{ion}(v, [Ca]_i, m, n, l, f, d, z), \quad \text{for } (x,t) \in \Omega_T, \tag{2}$$

where χ is the surface to volume ratio of the cell. The initial and boundary conditions read

$$v(x, 0) = v_0, \quad \text{for } x \in \Omega, \qquad v = 0, \quad \text{on } \partial\Omega \times (0, T). \tag{3}$$

Here $I_{ion}(v, [Ca]_i, m, n, l, f, d, z)$ is the collection of membrane currents, which will be specified below. The conductivity of the tissue is represented by a scaled tensor M of the form [7]

$$M = \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^t \end{pmatrix},$$

where $\sigma^l = \sigma^l(x) \in C^1(\mathbf{R}^2)$ and $\sigma^t = \sigma^t(x) \in C^1(\mathbf{R}^2)$ are respectively the intracellular conductivities along and transverse to the direction of the fibre.

Each $w = m, n, l, f, d, z$ models the openness of the ionic channels, and obeys

$$\frac{dw}{dt} = \alpha_w(v)(1 - w) - \beta_w(v)w, \qquad w(x, 0) = w_0, \tag{4}$$

where α_w and β_w are respectively the opening and closing rates, which are given by

$$\alpha_w(v) = \frac{C_1 e^{C_2(v+C_3)} + C_4(v + C_5)}{e^{C_6(v+C_3)} + C_7},$$

and

$$\beta_w(v) = \frac{C_1 e^{C_2(v+C_3)} + C_4(v + C_5)}{e^{C_6(v+C_3)} + C_7},$$

for given constants $C_i,\ i = 1, \ldots, 7$. See Table 1 for the precise values of these constants, such that

$$\alpha_w(v), \beta_w(v) > 0. \tag{5}$$

The charge flow through the membrane includes four individual currents. The direction of two of these point out of the cell, and these are really the same current split into two components. They represent the flow of potassium (K) ions, one time-independent and the other slowly varying in space:

$$I_K(v) = 1.4 \frac{e^{0.04(v+85)} - 1}{e^{0.08(v+53)} + e^{0.04(v+53)}}. \tag{6}$$

and the time-dependent component reads

$$I_z(v,z) = 0.8z \frac{e^{0.04(v+77)} - 1}{e^{0.04(v+35)}} \tag{7}$$

There are also two inward currents; namely, (i) the transport of sodium (Na) ions, which excites the media,

$$I_{Na}(v,m,n,l) = (g_{Na}m^3nl + g_{NaC})(v - E_{Na}), \tag{8}$$

where E_{Na} is the equilibrium potential of sodium, and (ii) the slow inward current given by

$$I_s(v,[Ca]_i,f,d) = g_s fd(v + 82.3 + 13.0287\ln([Ca]_i)), \tag{9}$$

which primarily describes the flow of calcium ions across the membrane. Here $g_{Na} = 23.0$ and $g_{NaC} = 0.003$ are the membrane conductivities of, respectively, the sodium current and the sodium-calcium exchanger current, and $g_s = 0.09$ is the conductivity related to the slow inward current. Note that we can now insert

$$I_{ion} = I_K(v) + I_z(v,z) + I_{Na}(v,m,n,l) + I_s(v,[Ca]_i,f,d)$$

into (2).

The intracellular calcium concentration is denoted by $[Ca]_i$, and is scaled like $c = 10^3[Ca]_i$ obeying the ordinary differential equation

$$\frac{dc}{dt} = 0.07(10^{-4} - c) - 10^{-4}I_s(v,c,f,d), \qquad c(x,0) = c_0. \tag{10}$$

A thorough discussion of the physiology of this model can be found in, e.g., [1, 7].

The initial data (v_0, c_0, w_0), $w = m,n,l,f,d,z$, of the system (2)-(10) are supposed to satisfy

$$\begin{aligned} &v^- \le v_0 \le v^+, \quad \text{in } \Omega,\\ &c^- \le c_0 \le c^+, \quad \text{in } \Omega\\ &0 \le w_0 \le 1, \quad \text{in } \Omega, \text{ for } w = m,n,l,f,d,z, \end{aligned} \tag{11}$$

The particular values of the constants $v^{\mp}, c^{\mp}$ will be discussed in the next section. In addition, we assume that

$$c_{0,x}, c_{0,y}, w_{0,x}, w_{0,y} \in L^2(\Omega), \quad \text{for } w = m,n,l,f,d,z. \tag{12}$$

3. Heuristic motivation of lower/upper bounds on solutions

The purpose of this section is to motivate at the continuous level why it is reasonable to expect L^∞ bounds on the (approximate) solutions of our reaction-diffusion system. Such bounds must be fully compatible with physiological characteristics of the mathematical model.

In the previous section we introduced the scaling $c = 10^3[Ca]_i$. The intracellular calcium concentration is normally of the order of 10^{-7}Molar, so it is reasonable to assume that $c^- \sim 10^{-4}$Molar. As a too large $[Ca]_i$ is toxic, we have proposed the existence of an upper bound c^+ for c, but we are not aware of any exact physiological value for this bound. Nevertheless, it is the purpose of this section to derive a c^+ suggested by the mathematical model, which will serve as an upper bound of c in the remainder of the paper.

The membrane potential of the Beeler-Reuter model lies between −85mV and about 30mV under normal conditions. However, we have not been able to identify a maximum principle for v lying between these values, so we need to weaken our assumption slightly. We assume that v is less than the equilibrium potential of the slow inward current, that is, we define

$$v^+ := E(c^-) = 7.7 - 13.0287\ln(c^-) = 127.7.$$

The lower bound is kept as

$$v^- := -85,$$

so that

$$v^- \le v \le v^+. \tag{13}$$

Similarly, it is reasonable to seek bounds on the form

$$0 \le m, n, l, f, d, z \le 1. \tag{14}$$

In the considerations presented here we will suppose that (13) and (14) hold. In the rigorous proof given in Section 6, similar discrete bounds will be justified on the finite difference scheme.

Inserting (9) into (10) yields

$$c_t = 0.07(10^{-4} - c) - g_s 10^{-4} fd\big(v - 7.7 + 13.0287\ln(c)\big), \tag{15}$$

upon noting that

$$13.0287\ln(10^{-3}c) = -90.0 + 13.0287\ln(c).$$

To carry out the analysis, define a function F by

$$F(v, c, f, d) = 0.07(10^{-4} - c) - g_s 10^{-4} fd(v - 7.7 + 13.0287\ln(c)), \tag{16}$$

such that

$$c_t = F(v, f, d, c). \tag{17}$$

Differentiating F with respect to v gives

$$\partial_v F(v, c, f, d) = -g_s 10^{-4} fd < 0,$$

since $f, d \ge 0$ by (14). Hence, F is monotonously decreasing in v, implying that we can insert for the lower and upper bounds into the right hand side of (17). This gives

$$F^-(c, f, d) \le c_t \le F^+(c, f, d), \tag{18}$$

where

$$F^-(c, f, d) := F(v^+, c, f, d) = 0.07(10^{-4} - c) - g_s 10^{-4} fd(127.7 - 7.7 + 13.0287\ln(c))$$

and

$$F^+(c, f, d) := F(v^-, c, f, d) = 0.07(10^{-4} - c) - g_s 10^{-4} fd(-92.7 + 13.0287\ln(c)).$$

We focus on finding an upper limit of c_t. Differentiating the latter with respect to f and d yields

$$\partial_f F^+(c, f, d) = -g_s 10^{-4} d(-92.7 + 13.0287\ln(c)),$$

and

$$\partial_d F^+(c,f,d) = -g_s 10^{-4} f(-92.7 + 13.0287\ln(c)).$$

These are both positive whenever

$$c \le e^{\frac{92.7}{13.0287}} = 1230, \tag{19}$$

so a limit function is found by inserting for the largest value of f and d. This yields

$$F_1^+(c) := F^+(c,1,1) = 0.07(10^{-4} - c) - g_s 10^{-4}(-92.7 + 13.0287\ln(c)),$$

so that

$$F^+(c,f,d) \le F_1^+(c),$$

and consequently c_t is restricted above by

$$c_t \le F_1^+(c). \tag{20}$$

To find a lower bound for c_t, differentiate F^- in the same manner:

$$\partial_f F^-(c,f,d) = -g_s 10^{-4} d(120.0 + 13.0287\ln(c))$$

and

$$\partial_d F^-(c,f,d) = -g_s 10^{-4} d(120.0 + 13.0287\ln(c)).$$

Since $120.0 + 13.0287\ln(c)$ is nonnegative for $c \ge c^-$, $F^-(c,f,d)$ is monotonously decreasing in f, d. Thus

$$F^-(c,f,d) \ge F^-(c,1,1) = 0.07(10^{-4} - c) - g_s 10^{-4}(120.0 + 13.0287\ln(c)) =: F_1^-(c),$$

implying

$$F_1^- \le c_t. \tag{21}$$

Combining (20) and (21), we have proved lower and upper bounds on c_t whenever (19) holds. Now, define the two curves c_t^- and c_t^+ by the ODEs

$$c_t^- = F_1^-(c), \qquad c_t^+ = F_1^+(c),$$

so that

$$c_t^- \le c_t \le c_t^+. \tag{22}$$

The curves $c^-(t), c(t), c^+(t)$ all start out at the same point, so $c^-(t) \le c(t) \le c^+(t)$ upon integrating (22). We start by finding an upper bound for $c(t)$, which essentially involves investigating $F_1^+(c)$. This function has a root in $c_r = 0.0187$, and we find that

$$F_1^+(c) < 0 \quad \text{for} \quad c > c_r \qquad \text{and} \quad F_1^+(c) > 0 \quad \text{for} \quad c < c_r. \tag{23}$$

This means that the function $c^+(t)$ is decreasing for values above c_r, and increasing for $c < c_r$. The root c_r is a bound on $c^+(t)$ and thus also for $c(t)$. We therefore set $c^+ = 0.0187$. It remains to investigate the lower bound proposed above. We have that

$$F_1^-(c) = 0 \quad \text{for} \quad c = c^-,$$

and moreover,

$$F_1^-(c) < 0 \quad \text{for} \quad c^- \le c \le c^+.$$

Hence $c^-(t)$ is a nonincreasing function in the domain of interest. So due to the second part of (23), $c(t)$ will increase from the initial value $c(0) = 10^{-4}$.

Summarising, we have the following lower and upper bounds on the value of c:

$$c^- \leq c \leq c^+, \qquad c^- = 10^{-4}, \quad c^+ = 0.0187.$$

Certainly, the considerations made in this section are merely indications that it should be possible to determine appropriate lower and upper bounds on the solutions of our reaction-diffusion system. Such bounds will be derived rigorously in Section 6 at the discrete level of an explicit finite difference scheme. The convergence result given in Section 7 will then show that the same bounds hold for the exact solution of the reaction-diffusion system.

4. A weak formulation

Denote by Q a bounded open subset of $\mathbf{R}^d$, for $d \geq 1$. Denote by $C_c^\infty(Q)$ the set of all infinitely smooth functions with compact support in Q. Let $L^p(Q)$, $1 \leq p < \infty$ be the space of functions $u : Q \to \mathbf{R}$ for which the associated norm $\|u\|_{L^p(Q)}$ is finite, where

$$\|u\|_{L^p(Q)} = \left\{ \int_Q |u(x,y)|^p \, dx \, dy \right\}^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \|u\|_{L^\infty(Q)} = \sup_{(x,y) \in Q} |u(x,y)|.$$

We use $H_0^1(Q)$ to denote the Sobolev space of functions $u : Q \to \mathbf{R}$ that vanish at the boundary and for which $u, u_{x_1}, u_{x_2} \in L^2(Q)$. The dual space of $H_0^1(Q)$ is denoted by $H^{-1}(Q)$. If X is a Banach space, $a < b$, and $1 \leq p \leq \infty$, then $L^p(a,b;X)$ denotes the space of all functions $u : (a,b) \to X$ such that

$$\|u(\cdot)\|_X \in L^p(a,b).$$

We denote by $C([a,b];X)$ the space of continuous functions $t \mapsto u(t) \in X$ on $[a,b]$ equipped with the norm $L^\infty(a,b;X)$. We refer to [3] for more information about the functional spaces just introduced.

In the present study, $a = 0, b = T$ and $X = H_0^1(\Omega)$, and we search for solutions v, m, n, l, z, f, d, c in $L^2(0,T;H_0^1(\Omega))$ that solve the Beeler-Reuter equations (2)-(10) in the weak sense. For convenience of notation, we use ω to denote the vector

$$\omega = (m, n, l, f, d, z).$$

We will use the following notion of weak solution for the Beeler-Reuter model.

Definition 4.1 (weak solution)**.** *The functions (v, ω, c) are said to be a weak solution of the Beeler-Reuter model (2)-(10) if*

1. (Regularity)

$$v \in L^\infty(\Omega_T) \cap L^2(0,T;H_0^1(\Omega)), \quad w, c \in L^\infty(\Omega_T), \quad w = m, n, l, f, d, z,$$

and, in addition, $c(x,t) > \underline{c} > 0$ for a.e. $(x,t) \in \Omega_T$.

2. (Initial data)

$$\lim_{t \downarrow 0} \|v(\cdot,t) - v_0(\cdot)\|_{L^2(\Omega)} = 0,$$
$$\lim_{t \downarrow 0} \|w(\cdot,t) - w_0(\cdot)\|_{L^2(\Omega)} = 0, \quad w = m, n, l, f, d, z,$$
$$\lim_{t \downarrow 0} \|c(\cdot,t) - c_0(\cdot)\|_{L^2(\Omega)} = 0.$$

3. (Differential equations)

$$\int_0^T \int_\Omega \left[\chi v \phi_t + (M\nabla v)\cdot\nabla\phi - \chi \mathcal{I}(v,c,\omega)\phi\right] dx\,dy\,dt = 0, \quad \forall \phi \in C_c^\infty(\Omega_T),$$

where we have set

$$\mathcal{I}(v,c,\omega) = I_K(v) + I_{Na}(v,m,n,l) + I_z(v,z) + I_s(v,c,f,d),$$

Moreover,

$$\int_0^T \int_\Omega \left[w\phi_t + (\alpha_w(1-w) - \beta_w w)\phi\right] dx\,dy\,dt = 0, \quad w = m,n,l,f,d,z, \quad \forall \phi \in C_c^\infty(\Omega_T)$$

and

$$\int_0^T \int_\Omega \left[c\phi_t + \big(0.07(1-c) - I_s(v,c,f,d)\big)\phi\right] dx\,dy\,dt = 0, \quad \forall \phi \in C_c^\infty(\Omega_T).$$

Remark 4.1. If $(v,\omega,c,)$ is a weak solution in the sense of Definition 4.1, it follows that $v_t \in L^2(0,T;H_0^{-1}(\Omega))$ and hence $v \in C(0,T;L^2(\Omega))$, see [3]. In addition, it follows from the definition of a weak solution that $\frac{dc}{dt}, \frac{dw}{dt} \in L^\infty(\Omega_T)$, for $w = m,n,l,f,d,z$.

The existence of a weak solution is a consequence of the convergence result for our finite difference scheme given in Section 7. We now demonstrate that the weak solution is unique, using a standard argument [3]. To this end, let $(\tilde{v},\tilde{\omega},\tilde{c})$, $\tilde{\omega} = (\tilde{m},\tilde{n},\tilde{l},\tilde{f},\tilde{d},\tilde{z})$, be another weak solution of the problem, and define the difference

$$(\overline{v},\overline{\omega},\overline{c}) = (v-\tilde{v},\omega-\tilde{\omega},c-\tilde{c}).$$

Then in the weak sense, $(\overline{v},\overline{\omega},\overline{c})$ satisfies

$$\chi\overline{v}_t = \nabla\cdot(M\nabla\overline{v}) - \chi\overline{F}, \quad \overline{v}|_{t=0} = 0, \quad \overline{v}|_{\partial\Omega\times(0,T)} = 0, \tag{24}$$

where we have set

$$\overline{F} = \mathcal{I}(v,c,\omega) - \mathcal{I}(\tilde{v},\tilde{c},\tilde{\omega}).$$

The difference of the individual currents are Lipschitz continuous, and thus satisfy

$$|I_K(v) - I_K(\tilde{v})| \le k_1|v-\tilde{v}|,$$

and similarly for I_{Na}, I_z, I_s. This fact implies $\overline{F} \le k_1|\overline{v}+\overline{c}+\overline{m}+\overline{n}+\overline{l}+\overline{f}+\overline{d}+\overline{z}|$, for some finite constant k_1. Similarly, for the governing ODEs of the gating variables we have

$$\frac{d\overline{w}}{dt} = \overline{H}_w, \quad \overline{w}|_{t=0} = 0, \tag{25}$$

where

$$\overline{H}_w = (1-w)\alpha_w(v) - w\beta_w(v) - (1-\tilde{w})\alpha_{\tilde{w}}(\tilde{v}) + \tilde{w}\beta_{\tilde{w}}(\tilde{v}),$$

and there holds $|\overline{H}_w| \le k_2|\overline{v}+\overline{w}|$, $w = m,n,l,f,d,z$, for some finite constant k_2.

Finally,

$$\frac{d\overline{c}}{dt} = 0.07(1-\overline{c}) - \overline{G}, \quad \overline{c}|_{t=0} = 0, \tag{26}$$

where

$$\overline{G} = I_s(v,c,f,d) - I_s(\tilde{v},\tilde{c},\tilde{f},\tilde{d}),$$

and clearly $\overline{G} \leq k_3|\overline{v} + \overline{c} + \overline{f} + \overline{d}|$ for some finite constant k_3.

By using standard energy arguments [3] for each of (24)-(26) and adding together the resulting inequalities, we get

$$e(t) \leq e(0) + k_1 \int_0^t e(s)\, ds,$$

where k is a finite constant

$$e(t) = \int_\Omega \left((\bar{v}(x,y,t))^2 + \sum_{\bar{w}=\bar{m},\bar{n},\bar{l},\bar{f},\bar{d},\bar{z}} (\bar{w}(x,y,t))^2 + (\bar{c}(x,y,t))^2 \right) dx\, dy.$$

We apply Gronwall's inequality [3] and find that $e(t) = 0$ for all t since $e(0) = 0$.

5. An explicit finite difference scheme

In this section we present the numerical scheme that will be analysed in the following sections. The numerical solution is defined at grid points $x_i = ih$, $y_j = jh$, $i, j = 0, \dots, N$, for some integer N satisfying $Nh = 1$. Similarly, we let the time spacing be $\Delta t > 0$, so that $t_k = k\Delta t$ for $k = 0, \dots, \kappa$, with κ being an integer satisfying $\kappa \Delta t = T$, is a discretization of $[0, T]$. The approximate solutions are denoted by

$$v_{i,j}^k \approx v(x_i, y_j, t_k), \quad w_{i,j}^k \approx w(x_i, y_j, t_k), \quad w = m, n, l, d, f, z, \quad c_{i,j}^k \approx c(x_i, y_j, t_k).$$

We consider the following explicit finite difference scheme for the transmembrane potential:

(27)
$$\begin{aligned} v_{i,j}^{k+1} = v_{i,j}^k &+ \gamma\big(\sigma_{i+\frac{1}{2},j}^l (v_{i+1,j}^k - v_{i,j}^k) - \sigma_{i-\frac{1}{2},j}^l (v_{i,j}^k - v_{i-1,j}^k)\big) \\ &+ \gamma\big(\sigma_{i,j+\frac{1}{2}}^t (v_{i,j+1}^k - v_{i,j}^k) - \sigma_{i,j-\frac{1}{2}}^t (v_{i,j}^k - v_{i,j-1}^k)\big) \\ &- \Delta t\big(I_K(v_{i,j}^k) + I_z(v_{i,j}^k, z_{i,j}^k) + I_{Na}(v_{i,j}^k, m_{i,j}^k, n_{i,j}^k, l_{i,j}^k) + I_s(v_{i,j}^k, c_{i,j}^k, f_{i,j}^k, d_{i,j}^k)\big), \end{aligned}$$

for $i, j = 1, \dots, N-1$, $k = 0, \dots, \kappa - 1$, where we have set $\gamma = \frac{\Delta t}{\chi h^2}$ and

$$\sigma_{i+\frac{1}{2},j}^{l,t} = \frac{1}{2}(\sigma_{i+1,j}^{l,t} + \sigma_{i,j}^{l,t}), \qquad \sigma_{i,j+\frac{1}{2}}^{l,t} = \frac{1}{2}(\sigma_{i,j+1}^{l,t} + \sigma_{i,j}^{l,t}).$$

At the initial and boundary nodes, we set

$$\begin{aligned} &v_{i,j}^0 = \frac{1}{h^2} \int_{(x_i,x_{i+1})\times(y_j,y_{j+1})} v_0(x,y)\, dx\, dy, \quad i, j = 1, \dots, N-1, \\ &v_{i,0}^k = v_{i,N}^k = 0 \text{ for } i = 0, \dots, N,\ k = 0, \dots, \kappa, \\ &v_{0,j}^k = v_{N,j}^k = 0 \text{ for } j = 0, \dots, N,\ k = 0, \dots, \kappa, \end{aligned} \tag{28}$$

The schemes for the gating variables read

$$w_{i,j}^{k+1} = w_{i,j}^k + \Delta t \alpha_{w,i,j}^k (1 - w_{i,j}^k) - \beta_{w,i,j}^k w_{i,j}^k, \qquad i, j = 0, \dots, N, \tag{29}$$

where $w = m, n, l, d, f, z$, and $\alpha_{w,i,j}^k = \alpha_w(v_{i,j}^k)$ and $\beta_{w,i,j}^k = \beta_w(v_{i,j}^k)$. In addition, we impose the initial condition $w_{i,j}^0 = w_0$ for all relevant i, j. Moreover, the scheme for the calcium concentration reads

$$c_{i,j}^{k+1} = c_{i,j}^k + \Delta t 0.07(10^{-4} - c_{i,j}^k) - \Delta t 10^{-4} I_s(v_{i,j}^k, c_{i,j}^k, f_{i,j}^k, d_{i,j}^k), \tag{30}$$

and $c_{i,j}^0 = c_0$ for all relevant i, j.

Finally, we remark that the discrete initial data satisfy the following lower and upper bounds:

$$
\begin{aligned}
v^- &\le v^0_{i,j} \le v^+ \\
0 &\le w^0_{i,j} \le 1, \qquad w = m, n, l, f, d, z, \\
c^- &\le c^0_{i,j} \le c^+,
\end{aligned} \tag{31}
$$

for all $i, j = 0, \dots, N$, where the constants $v^{\mp}$ and $c^{\mp}$ have been introduced in Section 3.

It is our aim in the next section to prove that these bounds remain valid at later times for the solutions produced by the difference schemes.

6. A maximum principle

Motivated by the findings in Section 3, we now seek to prove upper and lower bounds for the discrete solutions (as stated in Theorem 6.1). Introduce the constants

$$
\alpha^+ = \max_{w=m,n,l,f,d,z} \big(\max_{v^- \le v \le v^+} \alpha_w(v) \big), \quad \beta^+ = \max_{w=m,n,l,f,d,z} \big(\max_{v^- \le v \le v^+} \beta_w(v) \big),
$$
$$
\sigma^l_+ = \max_{i,j} \sigma^l_{i+\frac{1}{2},j} \quad \text{and} \quad \sigma^t_+ = \max_{i,j} \sigma^t_{i,j+\frac{1}{2}} \quad ,
$$

and define

$$
B := \max_{0 \le z \le 1, v^- \le v \le v^+} \partial_v I_z(v, z) = \max_{0 \le z \le 1, v^- \le v \le v^+} 0.8z \frac{0.04}{e^{0.04(v+35)}} < \infty. \tag{32}
$$

Theorem 6.1 (Maximum principle)**.** *Let $v^k_{i,j}, w^k_{i,j}, c^k_{i,j}$, $w = m, n, l, f, d, z$, solve the system (27)-(30). Moreover, assume that the initial data satisfy (12). If the time step constraint*

$$
\Delta t \le \min \left[\frac{\chi h^2}{2\sigma^l_+ + 2\sigma^t_+ + \chi(B + g_s + g_{Na} + g_{NaC})h^2}, \frac{1}{\alpha^+ + \beta^+} \right] \tag{33}
$$

is met, then the following lower and upper bounds hold:

$$
\begin{aligned}
&v^- \le v^k_{i,j} \le v^+, && \text{for } i, j = 0, \dots, N,\ k = 0, \dots, \kappa, \\
&0 \le w^k_{i,j} \le 1, && \text{for } i, j = 0, \dots, N,\ k = 0, \dots, \kappa,\ w = m, n, l, f, d, z, \\
&c^- \le c^k_{i,j} \le c^+, && \text{for } i, j = 0, \dots, N,\ k = 0, \dots, \kappa,
\end{aligned} \tag{34}
$$

where the constants $v^{\mp}$ and $c^{\mp}$ have been introduced in Section 3.

Proof. The statement is true for $k = 0$ thanks to (31). The proof goes by induction on the time level, so we begin by assuming that statement (34) is satisfied for a given $k \ge 0$. The goal is then to prove that it holds for time level $k + 1$.

First consider the scheme for $c^k_{i,j}$, which can be written as

$$
c^{k+1}_{i,j} = c^k_{i,j} + \Delta t F(v^k_{i,j}, c^k_{i,j}, d^k_{i,j}, f^k_{i,j}), \tag{35}
$$

where F is given by (16). Inserting for the upper limit of F (see Section 3) we get

$$
c^{k+1}_{i,j} \le c^k_{i,j} + \Delta t F^+_1(c^k_{i,j}), \quad i, j = 0, \dots, N.
$$

Define a function G by

$$
G(c) = c + \Delta t F^+_1(c).
$$

We observe that for sufficiently small Δt, i.e. for

$$\Delta t \leq \frac{c}{0.07c + g_s 10^{-4} 13.0287}, \tag{36}$$

$G' \geq 0$ and thus G is nondecreasing. Notice that for reasonable h, (36) is implied by (33). Now, due to assumption (34) and the fact that $F_1^+(c^+) = 0$, we have

$$c_{i,j}^{k+1} \leq G(c_{i,j}^k) \leq G(c^+) = c^+ + \Delta t F_1^+(c^+) = c^+,$$

for all relevant i, j. Hence we conclude that

$$c_{i,j}^{k+1} \leq c^+, \quad \forall i, j = 0, \ldots, N.$$

In the same manner, using the lower bound of F, we obtain

$$c_{i,j}^{k+1} \geq c_{i,j}^k + \Delta t F_1^-(c_{i,j}^k), \quad i, j = 0, \ldots, N.$$

Define a function D by

$$D(c) = c + \Delta t F_1^-(c),$$

and note that for Δt satisfying (36), $D' \geq 0$ and thus D is nondecreasing. Since $F_1^-(c^-) = 0$, we deduce

$$c_{i,j}^{k+1} \geq D(c_{i,j}^k) \geq D(c^-) = c^- + \Delta t F_1^-(c^-) \geq c^-,$$

and hence we have proved

$$c^- \leq c_{i,j}^{k+1} \leq c^+ \quad \forall i, j = 0, \ldots, N.$$

We proceed by investigating the scheme (27) for the transmembrane potential. First consider

$$I_K(v_{i,j}^k) = 1.4 \frac{e^{0.04(v_{i,j}^k + 85)} - 1}{e^{0.08(v_{i,j}^k + 53)} + e^{0.04(v_{i,j}^k + 53)}}, \tag{37}$$

which satisfies

$$0 \leq I_K(v_{i,j}^k) \leq 1.8, \quad i, j = 0, \ldots, N. \tag{38}$$

Inserting the lower bound of I_K into (27) gives

$$\begin{aligned} v_{i,j}^{k+1} \leq v_{i,j}^k &+ \gamma \sigma_{i+\frac{1}{2},j}^l v_{i+1,j}^k + \gamma \sigma_{i,j+\frac{1}{2}}^t v_{i,j+1}^k + \gamma \sigma_{i-\frac{1}{2},j}^l v_{i-1,j}^k \\ &+ \gamma \sigma_{i,j-\frac{1}{2}}^t v_{i,j-1}^k - \gamma(\sigma_{i-\frac{1}{2},j}^l + \sigma_{i+\frac{1}{2},j}^l + \sigma_{i,j-\frac{1}{2}}^t + \sigma_{i,j+\frac{1}{2}}^t) v_{i,j}^k \\ &- \Delta t \big(I_z(v_{i,j}^k, z_{i,j}^k) + I_{Na}(v_{i,j}^k, m_{i,j}^k, n_{i,j}^k, l_{i,j}^k) + I_s(v_{i,j}^k, c_{i,j}^k, f_{i,j}^k, d_{i,j}^k) \big), \end{aligned}$$

for $i, j = 1, \ldots, N-1$. As $\partial_c I_s(v, c, f, d) = \frac{1.1726}{c}$, the current I_s is nondecreasing in c. Hence

$$I_s(v_{i,j}^k, c^-, f_{i,j}^k, d_{i,j}^k) \leq I_s(v_{i,j}^k, c_{i,j}^k, f_{i,j}^k, d_{i,j}^k) \leq I_s(v_{i,j}^k, c^+, f_{i,j}^k, d_{i,j}^k),$$

for all relevant i, j. Furthermore, there holds $\partial_f I_s(v, c^-, f, d) = g_s d(v - E(c^-)) \leq 0$ and $\partial_d I_s \leq 0$. Consequently,

$$I_s(v_{i,j}^k, c^-, 1, 1) \leq I_s(v_{i,j}^k, c_{i,j}^k, f_{i,j}^k, d_{i,j}^k) \leq I_s(v_{i,j}^k, c^+, 0, 0), \quad i, j = 0, \ldots, N.$$

If we insert the lower bound of I_s and the expressions for I_z and I_{Na}, we arrive at the following inequality:

$$
\begin{aligned}
v_{i,j}^{k+1} \leq v_{i,j}^k &+ \gamma\sigma^l_{i+\frac{1}{2},j} v^k_{i+1,j} + \gamma\sigma^t_{i,j+\frac{1}{2}} v^k_{i,j+1} - \gamma(\sigma^l_{i-\frac{1}{2},j} + \sigma^l_{i+\frac{1}{2},j} + \sigma^t_{i,j-\frac{1}{2}} \\
&+ \sigma^t_{i,j+\frac{1}{2}}) v^k_{i,j} + \gamma\sigma^l_{i-\frac{1}{2},j} v^k_{i-1,j} + \gamma\sigma^t_{i,j-\frac{1}{2}} v^k_{i,j-1} \\
&- 0.8\Delta t z^k_{i,j} \frac{e^{0.04(v^k_{i,j}+77)} - 1}{e^{0.04(v^k_{i,j}+35)}} - g_s \Delta t (v^k_{i,j} - E(c^-)) \\
&- \Delta t \left(g_{Na} (m^k_{i,j})^3 n^k_{i,j} l^k_{i,j} + g_{NaC} \right) E_{Na},
\end{aligned}
$$

or if we rearrange slightly

$$
\begin{aligned}
v_{i,j}^{k+1} \leq \gamma\sigma^l_{i+\frac{1}{2},j} v^k_{i+1,j} &+ \gamma\sigma^t_{i,j+\frac{1}{2}} v^k_{i,j+1} - \gamma(\sigma^l_{i-\frac{1}{2},j} + \sigma^l_{i+\frac{1}{2},j} + \sigma^t_{i,j-\frac{1}{2}} + \sigma^t_{i,j+\frac{1}{2}}) v^k_{i,j} \\
&+ \gamma\sigma^l_{i-\frac{1}{2},j} v^k_{i-1,j} + \gamma\sigma^t_{i,j-\frac{1}{2}} v^k_{i,j-1}) + \big(1 - g_s\Delta t \\
&- \Delta t (g_{Na} (m^k_{i,j})^3 n^k_{i,j} l^k_{i,j} + g_{NaC})\big) v^k_{i,j} \\
&- 0.8\Delta t z^k_{i,j} \frac{e^{0.04(v^k_{i,j}+77)} - 1}{e^{0.04(v^k_{i,j}+35)}} + g_s \Delta t E(c^-) \\
&+ \Delta t (g_{Na} \left(m^k_{i,j})^3 n^k_{i,j} l^k_{i,j} + g_{NaC} \right) E_{Na},
\end{aligned}
$$

for $i, j = 1, \ldots, N-1$,

Next define $v_{lx} = v^k_{i-1,j}$, $v_{ly} = v^k_{i,j-1}$, $v_{rx} = v^k_{i+1,j}$, $v_{ry} = v^k_{i,j+1}$, $v = v^k_{i,j}$, and

$$
\begin{aligned}
&H(v_{lx}, v_{ly}, v, v_{rx}, v_{ry}, m, n, l, z) \\
&= \gamma(\sigma^l_{i+\frac{1}{2},j} v_{rx} + \sigma^l_{i-\frac{1}{2},j} v_{lx} + \sigma^t_{i,j-\frac{1}{2}} v_{ly} + \sigma^t_{i,j+\frac{1}{2}} v_{ry}) \\
&\quad + \left[1 - \gamma(\sigma^l_{i-\frac{1}{2},j} + \sigma^l_{i+\frac{1}{2},j} + \sigma^t_{i,j-\frac{1}{2}} + \sigma^t_{i,j+\frac{1}{2}}) - \Delta t (g_s + g_{Na} m^3 n l + g_{NaC})\right] v \\
&\quad - 0.8\Delta t z \frac{e^{0.04(v+77)} - 1}{e^{0.04(v+35)}} + \Delta t g_s E(c^-) + \Delta t (g_{Na} m^3 n l + g_{NaC}) E_{Na}.
\end{aligned}
$$

Then

$$
v^{k+1}_{i,j} \leq H(v^k_{i+1,j}, v^k_{i,j+1}, v^k_{i,j}, v^k_{i-1,j}, v^k_{i,j-1}, m^k_{i,j}, n^k_{i,j}, l^k_{i,j}, z^k_{i,j}).
$$

Differentiating H gives

$$
\frac{\partial H}{\partial v_{lx}} = \gamma\sigma^l_{i-\frac{1}{2},j}, \frac{\partial H}{\partial v_{ly}} = \gamma\sigma^t_{i,j-\frac{1}{2}}, \frac{\partial H}{\partial v_{rx}} = \gamma\sigma^l_{i+\frac{1}{2},j}, \frac{\partial H}{\partial v_{ry}} = \gamma\sigma^t_{i,j+\frac{1}{2}},
$$

which all are strictly positive. Moreover,

(39)

$$
\begin{aligned}
\frac{\partial H}{\partial v} = 1 - \gamma(\sigma^l_{i-\frac{1}{2},j} &+ \sigma^l_{i+\frac{1}{2},j} + \sigma^t_{i,j-\frac{1}{2}} + \sigma^t_{i,j+\frac{1}{2}}) - \Delta t 0.8 z \frac{0.04}{e^{0.04(v+35)}} - g_s \Delta t \\
&- \Delta t (g_{Na} (m^k_{i,j})^3 n^k_{i,j} l^k_{i,j} + g_{NaC}) \\
\geq 1 - \gamma(\sigma^l_{i-\frac{1}{2},j} &+ \sigma^l_{i+\frac{1}{2},j} + \sigma^t_{i,j-\frac{1}{2}} + \sigma^t_{i,j+\frac{1}{2}}) - \Delta t (B + g_s + g_{Na} + g_{NaC}),
\end{aligned}
$$

where we have inserted the upper bounds for $m^k_{i,j}, n^k_{i,j}, l^k_{i,j}, z^k_{i,j}$ and the lower bound of $v^k_{i,j}$ to derive the last inequality.

Suppose h and Δt are chosen such that $\partial_v H \geq 0$, i.e., according to the first part of (33). Then H is nondecreasing in all variables, so for each interior point we get

$$\begin{aligned}
v_{i,j}^{k+1} &\leq H(v_{i+1,j}^k, v_{i,j+1}^k, v_{i,j}^k, v_{i-1,j}^k, v_{i,j-1}^k, m_{i,j}^k, n_{i,j}^k, l_{i,j}^k, z_{i,j}^k) \\
&\leq H(v^+, v^+, v^+, v^+, v^+, m_{i,j}^k, n_{i,j}^k, l_{i,j}^k, z_{i,j}^k) \\
&= v^+ - 0.8\Delta t z_{i,j}^k \frac{e^{0.04(v^+ + 77)} - 1}{e^{0.04(v^+ + 35)}} \\
&\qquad - \Delta t g_s(v^+ - E(c^-)) - \Delta t(g_{Na}(m_{i,j}^k)^3 n_{i,j}^k l_{i,j}^k + g_{NaC})(v^+ - E_{Na}) \\
&\leq v^+ - \Delta t g_{NaC}(v^+ - E_{Na}) \leq v^+,
\end{aligned}$$

where $m_{i,j}^k, n_{i,j}^k, l_{i,j}^k, z_{i,j}^k$ have been replaced by zero. This implies the upper bound $v_{i,j}^{k+1} \leq v^+$ for $i, j = 1, \ldots, N-1$. A lower bound is found in a similar manner: $v_{i,j}^{k+1} \geq v^-$ for $i, j = 1, \ldots, N-1$. Hence, in view of (31), we have proved the desired bounds for $v_{i,j}^{k+1}$ for all $i, j = 0, \ldots, N$.

Summarising, we have proved that both $c_{i,j}^{k+1}$ and $v_{i,j}^{k+1}$ satisfy the desired lower and upper bounds, provided that the first part of (33) holds. As for the second part of this time step criterion, we refer to [6], which contains proof of bounds for the gating variables in the one-dimensional case. The two-dimensional case is very similar, and therefore omitted. □

7. Convergence to a weak solution

The transition layer of a propagating wave front can be extremely narrow, which means that the transmembrane potential v can become discontinuous-like. This fact motivates using a concept of weak solutions for the Beeler-Reuter model, which will make it possible to give a rather easy proof of convergence for the finite difference scheme, for any initial potential v_0 satisfying the lower/upper bounds in (11). The convergence proof is based on the compactness method; that is, we derive a series of *a priori* estimates and use a general L^2 compactness criterion to deduce subsequential convergence of the numerical approximations to a weak solution. The uniqueness of the weak solution then ensures that the entire sequence of numerical approximations converge to the unique weak solution, and not just a subsequence.

For the convergence analysis we need to replace the finite difference solutions, which are defined only at the grid points (x_i, y_j, t_k), by functions that are defined a.e. in Ω_T. A simple way of doing that is to introduce the following piecewise constant functions:

$$\begin{aligned}
v_h(x,y,t) &= v_{i,j}^k, \qquad (x,y,t) \in I_{i,j} \times [t_k, t_{k+1}), \\
w_h(x,y,t) &= w_{i,j}^k, \qquad (x,y,t) \in I_{i,j} \times [t_k, t_{k+1}), \qquad w = m, n, l, f, d, z, \\
c_h(x,y,t) &= c_{i,j}^k, \qquad (x,y,t) \in I_{i,j} \times [t_k, t_{k+1}),
\end{aligned}$$

where we have defined $I_{i,j} := [x_i, x_{i+1}) \times [y_j, y_{j+1})$. We suppress the dependency on Δt, since this is implicit through the choice h and the time step constraint (33). This time step constraint is assumed to hold throughout this section, so that Theorem 6.1 holds.

To prove convergence of the finite difference scheme, the main technical tool will be the following L^2 compactness lemma (the proof of which is essentially an application of the Frechet-Kolmogorov compactness criterion and can be found in, e.g., [4]);

Lemma 7.1 (Compactness criterion)**.** *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^2(\Omega \times (0,T))$. Suppose that the following three conditions hold:*

1. *There exist constants $\underline{K}_1, \overline{K}_1$, independent of n, such that*

$$\underline{K}_1 \le u_n(x,t) \le \overline{K}_1 \qquad \text{for a.e. } (x,t) \in \Omega \times (0,T).$$

2. *There exists a constant K_2, independent of n, such that*

$$\|u_n(\cdot+\zeta,\cdot) - u_n(\cdot,\cdot)\|_{L^2(\Omega_\zeta \times (0,T))} \le K_2 \sqrt{|\zeta|\,(|\zeta| + \Delta_n)}, \qquad \zeta \in \mathbf{R}^2,$$

for some sequence $\{\Delta_n\}_{n=1}^{\infty} \subset (0,\infty)$ tending to zero. Here $\Omega_\zeta = \{x \in \Omega : x + \zeta \in \Omega\}$.

3. *There exists a constant K_3, independent of n, such that*

$$\|u_n(\cdot,\cdot+\tau) - u_n(\cdot,\cdot)\|_{L^2(\Omega \times (0,T-\tau))} \le K_3 \sqrt{\tau}, \qquad \tau \in (0,T).$$

Then there exists a subsequence of $\{u_n\}_{n=1}^{\infty}$ that converges in $L^2(\Omega \times (0,T))$ to a limit u belonging to $L^2(0,T;H_0^1(\Omega))$ and $\underline{K}_1 \le u(x,t) \le \overline{K}_1$ for a.e. $(x,t) \in \Omega \times (0,T)$.

Lemma 7.2 (Spatial translation estimate on transmembrane potential)**.** *There exist constants $K_1, K_2 > 0$, independent of h, such that*

$$\Delta t\, h^2 \sum_{k=0}^{\kappa} \sum_{i,j=0}^{N-1} \left[\left(v_{i+1,j}^k - v_{i,j}^k\right)^2 + \left(v_{i,j+1}^k - v_{i,j}^k\right)^2 \right] \le K_1 h^2 \tag{40}$$

$$\|v_h(\cdot+\zeta,\cdot) - v_h(\cdot,\cdot)\|_{L^2(\Omega_\zeta \times (0,T))} \le K_2 \sqrt{|\zeta|\,(|\zeta| + h)}, \qquad \forall\, \zeta \in \mathbf{R}^2, \tag{41}$$

where $\Omega_\zeta = \{x \in \Omega : x + \zeta \in \Omega\}$.

Proof. A discrete energy estimate is classical for the explicit difference scheme for the homogeneous version of (2). The argument used to prove it applies also to the full (non-homogeneous) equation (2) thanks to the uniform boundedness of $v_{i,j}^k, c_{i,j}^k, w_{i,j}^k$. Consequently, the discrete energy estimate (40) holds. For the argument that turns (40) into (41), see [4]. □

Lemma 7.3 (Spatial translation estimate on gating variables)**.** *There exist constants K_1, K_2, independent of h, such that for $w = m, n, l, f, d, z$*

$$h^2 \sum_{i,j=0}^{N-1} \left[\left(w_{i+1,j}^k - w_{i,j}^k\right)^2 + \left(w_{i,j+1}^k - w_{i,j}^k\right)^2 \right] \le K_1 h^2, \qquad k = 0, \dots, \kappa, \tag{42}$$

$$\|w_h(\cdot+\zeta,\cdot) - w_h(\cdot,\cdot)\|_{L^2(\Omega_\zeta \times (0,T))} \le K_2 \sqrt{|\zeta|(|\zeta| + h)} \qquad \forall \zeta \in \mathbf{R}^2. \tag{43}$$

Proof. The proof of (42) is similar to the one-dimensional case given in [6], and therefore omitted. As for (43), we again refer to [4]. □

We can find a similar result for the calcium concentration, as stated in the next lemma.

Lemma 7.4 (Spatial translation estimate on calcium concentrations)**.** *There exist constants K_1 and K_2, independent of h such that*

$$h^2 \sum_{i,j=0}^{N-1} \left[\left(c_{i+1,j}^k - c_{i,j}^k\right)^2 + \left(c_{i,j+1}^k - c_{i,j}^k\right)^2 \right] \le K_1 h^2, \qquad k = 0, \dots, \kappa, \tag{44}$$

$$\|c_h(\cdot+\zeta,\cdot) - c_h(\cdot,\cdot)\|_{L^2(\Omega_\zeta \times (0,T))} \le K_2 \sqrt{|\zeta|(|\zeta| + h)} \qquad \forall \zeta \in \mathbf{R}^2. \tag{45}$$

Proof. Recall that the scheme for $c_{i,j+1}^k$ can be written in the form (35), (16). Set

$$C_{i,j}^k = \frac{c_{i+1,j}^k - c_{i,j}^k}{h}.$$

Then

$$C_{i,j}^{k+1} = C_{i,j}^k + \Delta t \frac{F(v_{i+1,j}^k, c_{i+1,j}^k, d_{i+1,j}^k, f_{i+1,j}^k) - F(v_{i,j}^k, c_{i,j}^k, d_{i,j}^k, f_{i,j}^k)}{h}. \tag{46}$$

Set

$$V_{i,j}^k = \frac{v_{i+1,j}^k - v_{i,j}^k}{h}, \quad D_{i,j}^k = \frac{d_{i+1,j}^k - d_{i,j}^k}{h}, \quad F_{i,j}^k = \frac{f_{i+1,j}^k - f_{i,j}^k}{h}.$$

Observe that (here we use that $c_{i,j}^k$ is uniformly bounded away from zero)

$$\left| \frac{F(v_{i+1,j}^k, c_{i+1,j}^k, d_{i+1,j}^k, f_{i+1,j}^k) - F(v_{i,j}^k, c_{i,j}^k, d_{i,j}^k, f_{i,j}^k)}{h} \right| \leq K \left| V_{i,j}^k + C_{i,j}^k + D_{i,j}^k + F_{i,j}^k \right|, \tag{47}$$

for some constant K that is independent of h.

Multiplying (46) by $C_{i,j}^k$ and using the formula

$$\left(C_{i,j}^{k+1} - C_{i,j}^k\right) C_{i,j}^k = \frac{1}{2}\left[\left(C_{i,j}^{k+1}\right)^2 - \left(C_{i,j}^k\right)^2\right] - \frac{1}{2}\left(C_{i,j}^{k+1} - C_{i,j}^k\right)^2,$$

we find

$$\begin{aligned} &\left(C_{i,j}^{k+1}\right)^2 - \left(C_{i,j}^k\right)^2 \\ &= \left(C_{i,j}^{k+1} - C_{i,j}^k\right)^2 + 2\Delta t C_{i,j}^k \frac{F(v_{i+1,j}^k, c_{i+1,j}^k, d_{i+1,j}^k, f_{i+1,j}^k) - F(v_{i,j}^k, c_{i,j}^k, d_{i,j}^k, f_{i,j}^k)}{h}. \end{aligned} \tag{48}$$

Using the scheme and (47), we estimate

$$\left(C_{i,j}^{k+1} - C_{i,j}^k\right)^2 \leq K(\Delta t)^2 \left((V_{i,j}^k)^2 + (C_{i,j}^k)^2 + (D_{i,j}^k)^2 + (F_{i,j}^k)^2\right),$$

where we have used the basic inequality $\left(\sum_{\ell=1}^n a_\ell\right)^2 \leq C \sum_{\ell=1}^n (a_\ell)^2$ for any real numbers $a_1, \dots, a_n$. Using this and (47) one more time, we derive from (48)

$$\begin{aligned} &\left(C_{i,j}^{k+1}\right)^2 - \left(C_{i,j}^k\right)^2 \\ &\leq K\Delta t \left((V_{i,j}^k)^2 + (C_{i,j}^k)^2 + (D_{i,j}^k)^2 + (F_{i,j}^k)^2\right), \end{aligned} \tag{49}$$

where we have also used the basic inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for any two real numbers a, b.

Set $A_k = h^2 \sum_{i,j=0}^{N-1} \left(C_{i,j}^k\right)^2$. It then follows from (49) in a straightforward way that

$$A_\kappa \leq A_0 + K\Delta t\, h^2 \sum_{k=0}^{\kappa} \sum_{i,j=0}^{N-1} \left[(V_{i,j}^k)^2 + (D_{i,j}^k)^2 + (F_{i,j}^k)^2\right] + K\Delta t \sum_{k=0}^{\kappa} A_k.$$

Keeping in mind Lemmas 7.2 and 7.3, the middle term is uniformly bounded and then a bound on A_κ follows from an application of the discrete Gronwall inequality. Similarly, we can prove that $h^2 \sum_{i,j=0}^{N-1} \left(\frac{c_{i,j+1}^\kappa - c_{i,j}^\kappa}{h}\right)^2$ is uniformly bounded. Consequently, (44) holds for $k = \kappa$. The same argument shows that (44) holds for $k = 1, \dots, \kappa - 1$.

□

Lemma 7.5 (Temporal translation estimate)**.** *There exist three constants K_1, K_2, K_3, independent of h, such that*

$$\|v_h(\cdot, \cdot + \tau) - v_h(\cdot, \cdot)\|_{L^2(\Omega\times(0,T-\tau))} \leq K_1\sqrt{\tau},$$
$$\|w_h(\cdot, \cdot + \tau) - w_h(\cdot, \cdot)\|_{L^2(\Omega\times(0,T-\tau))} \leq K_2\tau, \qquad w = m, n, l, f, d, z,$$
$$\|c_h(\cdot, \cdot + \tau) - c_h(\cdot, \cdot)\|_{L^2(\Omega\times(0,T-\tau))} \leq K_3\tau.$$

Proof. Equipped with Lemmas 7.2, 7.3 and 7.4, this follows from the difference schemes along the lines of the standard arguments given in [4]. □

Theorem 7.1 (Convergence result)**.** *Let $v^k_{i,j}, c^k_{i,j}, w^k_{i,j}$, $w = m, n, l, f, d, z$, solve the system (27)-(30). Moreover, assume that the initial data satisfy* (11), (12) *and that the time step constraint* (33) *is fulfilled. Then, passing to subsequences if necessary,*

$$v_h \to v,\ w_h \to w,\ c_h \to c \text{ in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T \text{ as } h \to 0, \tag{50}$$

for some limit functions $v, w, c \in L^2(0, T; H^1_0(\Omega))$, $w = m, n, l, f, d, z$, satisfying the following lower and upper bounds: $v^- \leq v(x, t) \leq v^+$, $w^- \leq w(x, t) \leq w^+$, $c^- \leq c(x, t) \leq c^+$ for a.e. $(x, t) \in \Omega_T$, where the constants $v^\mp$ and $c^\mp$ were introduced in Section 3. Moreover, these limit functions constitute a weak solution of the Beeler-Reuter model (2)-(10).

Proof. In view of the maximum principle in Theorem 6.1 and Lemmas 7.2 and 7.5, the proof of (50) is a straightforward application of Lemma 7.1. The arguments needed to prove that the limit functions satisfy the second and third parts of of Definition 4.1 are standard and thus omitted (for similar arguments, see [4]). □

Remark 7.1. Observe that in Theorem 7.1 we have constructed gating variables w and calcium concentration c with more regularity than is strictly required by Definition 4.1.

8. Numerical experiments

To illustrate the theoretical results from Section 6, we present here some numerical experiments. The simulations are done on a square grid $[0, 1] \times [0, 1]$ with the spatial discretization as described in Section 5. We choose $\kappa = 5.0$ so that the time runs over $t \in [0, 5.0]$ms. Hence we only cover the upstroke, followed by a few milliseconds of the repolarization phase. This is the most interesting part from a numerical point of view, since the solution gradients are extremely steep in this time period. To calculate the time step restriction, we find those values of the maximum opening and closing rates of the gates that are equal to those of the m-gate, admitting the fastest upstroke. These values are $\alpha^+ = 82.02$ and $\beta^+ = 82.84$. Moreover, the value of B is, by definition (32), $\partial_v I_z(-85, 1) = 0.2364$, and the surface to volume ratio is set to $\chi = 2000$. In addition, inserting for the maximum conductivities, $\sigma^l_+ = 3.0$ and $\sigma^t_+ = 3.0$ into the first part of (33), we get

$$\begin{aligned}\Delta t &\leq \min\left[\frac{2000h^2}{2\sigma^l_+ + 2\sigma^t_+ + 2000(0.2364 + 0.09 + 23.0 + 0.003)h^2}, \frac{1}{\alpha^+ + \beta^+}\right] \\ &\approx \min\left[\frac{2000h^2}{12.0 + 2023.33h^2}, 5.9 \cdot 10^{-3}\right].\end{aligned}$$

For $h > 6 \cdot 10^{-3}$, the dominating restriction arises from the gate equations, whereas for a smaller mesh parameter, it becomes equal to the partial differential equation.

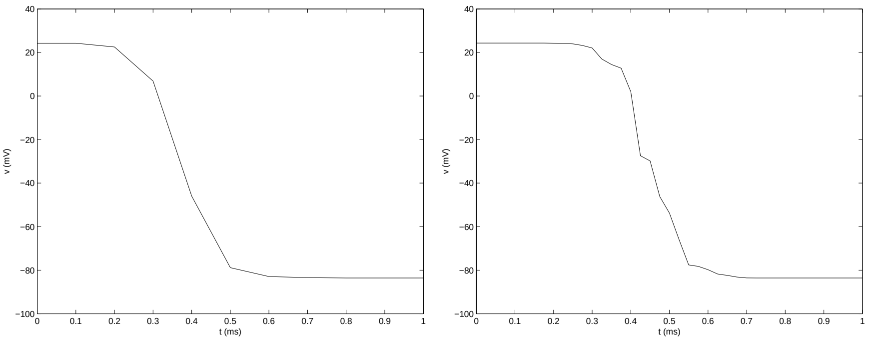

Figure 1. Plots of the transmembrane potential for $h = \{\frac{1}{11}, \frac{1}{41}\}$ at $t = 5$ms.

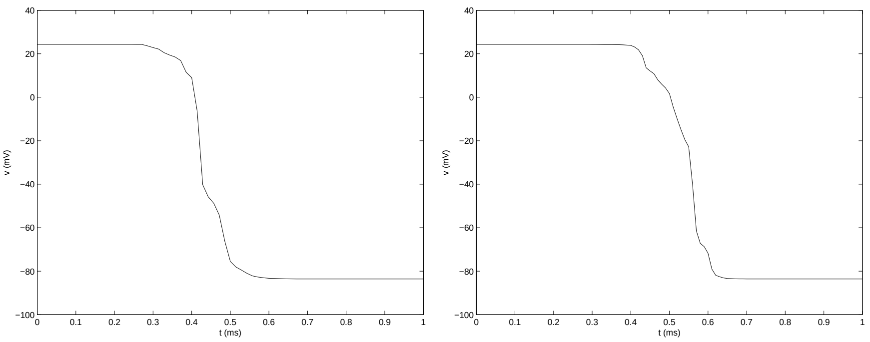

Figure 2. Plots of the transmembrane potential for $h = \{\frac{1}{71}, \frac{1}{101}\}$ at $t = 5$ms.

In our simulations, the transmembrane potential is initially set to $v^0 = 25.0$mV for $(x, y) \in [0, 1] \times [0, 0.36]$ and $v^0 = -85.0$mV in the remainder of the domain. The gate variables and the scaled calcium concentration have initial values $y_0 = (m, n, l, f, d, z)_0 = (0, 1, 1, 1, 0, 0)$ and $c_0 = 0.5 \cdot 10^{-4}$.

In Figure 1, we have extracted the solution at $x = 0.5$, such that the potential $v(0.5, y, 5)$ for $y = [0, 1]$ at time $t = 5$ms is shown. We present four plots with $h = \{\frac{1}{11}, \frac{1}{41}, \frac{1}{71}, \frac{1}{101}\}$. Observe that the solution is well-behaved and that the maximum value $v^+ = 25$ mV is never exceeded. The solution on the coarsest grid (leftmost plot) has a poor propagation speed due to the low spatial resolution. As is seen in the subsequent plots, the wave propagates at a higher speed with decreasing h. It should be noted that condition (33) guarantees numerical stability of the solutions. In practical computations however, this is observed to be too strict. For instance, choosing $h = \frac{1}{41}$, one can use a time step up to $\Delta t = 0.36$ms before severe nonphysical oscillations occur.

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