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SOME NEW LOCAL ERROR ESTIMATES IN NEGATIVE NORMS WITH AN APPLICATION TO LOCAL A POSTERIORI ERROR ESTIMATION

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Abstract. Here we survey some previously published results and announce some that have been newly obtained. We first review some of the results in [3] on estimates for the finite element error at a point. These estimates and analogous ones in [4] and [7] have been applied to problems in a posteriori estimates [2], [8], superconvergence [5] and others [9], [10]. We then discuss the extension of these estimates to local estimates in L_∞ based negative norms. These estimates have been newly obtained and are applied to the problem of obtaining an asymptotically exact a posteriori estimator for the maximum norm of the solution error on each element.

Key Words. Superconvergence, error estimate, a posteriori

1. Introduction

This is a survey paper whose aim is threefold. First we will discuss some of the results in [3]. There various error estimates for the finite element method for second order elliptic problems were derived. In particular, sharp error estimates for the solution and gradient at a point were given which more clearly showed the dependence of the error at the point on the solution at, and also away from the point. Some very useful inequalities for applications are so-called "asymptotic expansion inequalities". These are simple consequences of the error estimates and are the key in [2] and [8] on local a posteriori estimators, in [6] on superconvergence, in [9] on asymptotic expansions, and in [10] on Richardson extrapolation.

Our second aim is to present some newly obtained error estimates that are extensions of the estimates discussed above together with associated asymptotic expansion inequalities.

Finally our third aim is to apply these new asymptotic expansion inequalities to obtain an asymptotically exact a posteriori estimator for the maximum norm of the solution on each element. Some asymptotically exact estimators for the maximum norm of the gradient on each element that were given in [2] and [8] are very closely related.

This paper is organized as follows. Section 1(a) contains some preliminaries both for the Neumann problem we shall discuss and the finite element method we shall use. Section 1(b) contains special cases of the error estimates from [3]. Section 2 contains some new estimates. Namely, we give error estimates in local negative norms (L_∞ based) and corresponding asymptotic error expansion inequalities. Section 3 contains a discussion of the application of the results of Section 2 to a problem in a posteriori estimation.

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(a) Preliminaries for a smooth Neumann problem

Let $\Omega \subset\subset R^N$ be a domain with a smooth boundary $\partial\Omega$. Consider the boundary value problem

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f \text{ in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial n_L} = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where $\frac{\partial u}{\partial n_L}$ is the conormal derivative. The weak formulation of (1.1), (1.2) for $u \in W_2^1(\Omega)$ is

$$\begin{aligned} A(u,v) &= \int_\Omega \Big(\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i}v + c(x)uv\Big)dx \\ &= \int_\Omega fvdx \text{for all} v \in W_2^1(\Omega). \end{aligned} \tag{1.3}$$

We assume that the coefficients are smooth and $A(\cdot,\cdot)$ is uniformly elliptic and coercive, i.e., there exist constants $C_{\text{ell}} > 0$, $C_{\text{co}} > 0$ such that

$$\begin{aligned} \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j &\geq C_{\text{ell}}|\xi|^2, \text{ for all } \xi \in R^N \\ C_{\text{co}}\|v\|^2_{W_2^1(\Omega)} &\leq A(v,v) \text{ for all } v \in W_2^1(\Omega). \end{aligned} \tag{1.4}$$

It is well known that a unique solution of (1.3) exists for each $f \in (W_2^1(\Omega))'$, and if f is smooth then so is u.

Consider the approximation of u using the finite element method. To this end let $0 < h < 1$ be a parameter, $r \geq 2$ an integer, and $S_r^h(\Omega) \subset W_\infty^1(\Omega)$, be a family of finite elements. The precise assumptions on the finite element spaces will not be given here (see [3]) but roughly speaking they are satisfied by many types of commonly used finite elements. For the purpose of this presentation we take them to be any one of a variety of continuous functions, whose restriction to each set τ of a quasi–uniform partition (of roughly size h) that fits the boundary exactly and contains all polynomials of degree $r-1$ ($r = 2$ piecewise linear, $r = 3$ piecewise quadratic, etc.). Thus they can approximate functions to order h^r in L_∞ and order h^{r-1} in $W_\infty^1(\Omega)$. We now define $u_h \in S_r^h$, the finite element approximation to u, as the solution of

$$A(u_h,\varphi) = \int_\Omega f\varphi dx, \text{ for all } \varphi \in S_r^h(\Omega), \tag{1.5}$$

or

$$A(u-u_h,\varphi) = 0, \text{ for all } \varphi \in S_r^h(\Omega). \tag{1.6}$$

(b) Some known pointwise error estimates

Define, for $d > 0$ and fixed $x \in \overline{\Omega}$

$$B_d(x) = \{y \in \Omega : |x-y| < d\}. \tag{1.7}$$

The following are special cases of error estimates given in [3].

Theorem 1. *Suppose $u \in W^r_\infty(\Omega)$ and $u_h \in S^h_r(\Omega)$ satisfy (1.6). Let $x \in \overline{\Omega}$ be arbitrary but fixed, and let γ_1 satisfy $0 \le \gamma_1 \le r-1$, $r \ge 2$. Then there exists a constant C independent of x, u, h, and u_h such that*

(1.8)
$$\|u-u_h\|_{W^1_\infty(B_h(x))} \le C\Big(\ln\frac{1}{h}\Big)^{\overline{\gamma}_1} h^{r-1} \sum_{|\alpha|=r} \Big\|\Big(\frac{h}{h+|x-y|}\Big)^{\gamma_1} D^\alpha u(y)\Big\|_{L_\infty(\Omega)}.$$

Here $\overline{\gamma}_1 = 0$ if $0 \le \gamma_1 < r-1$, and $\overline{\gamma}_1 = 1$ if $\gamma_1 = r-1$.

For applications, a simple consequence of (1.8) has proved very useful. This is a so–called "asymptotic expansion inequality" (see [3]) which follows from (1.8) using a Taylor expansion. Here we give one form of the simplest two term expansion for which we choose $\gamma_1 = 1-\varepsilon$ and expand each $D^\alpha u(y)$ about the point x.

Corollary 1 *Under the condition of Theorem 1 let $u \in W^{r+1}_\infty(\Omega)$, then at points of continuity x of ∇u_h*

$$|\nabla(u-u_h)(x)| \le Ch^{r-1}\Big(\sum_{|\alpha|=r} |D^\alpha u(x)| + h^{1-\varepsilon}|u|_{W^{r-1}_\infty(\Omega)}\Big). \tag{1.9}$$

The inequalities (1.8) and (1.9) give an indication of how local the finite element method can be when polution effects can be neglected. Since γ_1 can be larger for higher order elements than lower order elements, (1.8) may be interpreted as indicating that higher order elements are more local than lower order elements. The inequality (1.9) says that we may bound the error in the gradient at a point by optimal order h^r multiplied by the sum of rth order derivatives at the same point plus a global term which is higher order accurate. This again shows the local nature of the finite element method but in a simpler form. The first term on the right may be interpreted as local interpolation error and the second term the pollution error which is a higher order term.

Theorem 2. *Suppose that $u \in W^r_\infty(\Omega)$ and $u_h \in S^h_r(\Omega)$ satisfy (1.6). Let $x \in \overline{\Omega}$ be arbitrary but fixed, and let γ_0 satisfy $0 \le \gamma_0 \le r-2$, $r \ge 2$. Then there exists a constant C independent of x, u, h and u_h such that*

$$\|u-u_h\|_{L_\infty(B_h(x))} \le C\Big(\ln\frac{1}{h}\Big)^{\overline{\gamma}_0} h^{r} \sum_{|\alpha|=r} \Big\|\Big(\frac{h}{h+|y-x|}\Big)^{\gamma_0} D^\alpha u(y)\Big\|_{L_\infty(\Omega)}. \tag{1.10}$$

Here $\overline{\gamma}_0 = 0$ if $0 \le \gamma_0 < r-2$, and $\overline{\gamma}_0 = 1$ if $\gamma_0 = r-2$.

There are also asymptotic expansion inequalities for $(u-u_h)(x)$ analogous to that for $\nabla(u-u_h)(x)$.

We shall again restrict ourselves to giving the simplest case of a two term expansion. To do this we set $\gamma_0 = 1-\varepsilon$ for ε arbitrary but fixed, and again use Taylor's expansion to obtain

Corollary 2. *Under the conditions of Theorem 2, let $u \in W^{r+1}_\infty$, and $r \ge 3$. Then*

$$|(u-u_h)(x)| \le Ch^{r}\Big(\sum_{|\alpha|=r} |D^\alpha u(x)| + h^{1-\varepsilon}\|u\|_{W^{r+1}_\infty(\Omega)}\Big). \tag{1.11}$$

It is important to remark that the case $r = 2$, i.e., e.g. piecewise linears are excluded. That follows from the fact that for $r = 2$, $\gamma_0 = 0$ is the only possible choice allowed. An asymptotic expansion inequality was derived in [7] for the difference of the error at any two points of $\overline{\Omega}$. This result was applied in [8] to extend the results of [2] to the piecewise linear case.

2. New local estimates in negative norms

In this section we shall present some new local error estimates for the problem (1.6) in the $W_\infty^{-s}(B_d(x))$ norm, where x is any point in $\overline{\Omega}$, $d > 0$, and s is any real number $0 < s \leq r - 2$. The norm is defined by

$$(2.1) \qquad \|u\|_{W_\infty^{-s}(B_d(x))} = \sup_{\psi \in C_0^\infty(B_d(x)),\ \|\psi\|_{W_\infty^s(B_d(x))=1}} \int u\psi dx.$$

These results are applied in the next section. The proofs of the results given below are lengthy and will appear elsewhere [6].

Let us motivate these results by first noticing that it follows from (1.8) and (1.10) that

$$(2.2) \qquad \begin{aligned} \| \quad & u - u_h\|_{W_\infty^1(B_d(x))} \\ & \leq C\Big(\ln \tfrac{1}{h}\Big)^{\overline{\gamma}_1} h^{r-1} \Big(\textstyle\sum_{|\alpha|=r} \Big\| \Big(\frac{h}{h+\rho(y,B_d(x))} \Big)^{\gamma_1} D^\alpha u(y) \Big\|_{L_\infty(\Omega)} \Big) \end{aligned}$$

and

$$(2.3) \qquad \begin{aligned} \| \quad & u - u_h\|_{L_\infty(B_d(x))} \\ & \leq C\Big(\ln \tfrac{1}{h}\Big)^{\overline{\gamma}_0} h^{r} \Big(\textstyle\sum_{|\alpha|=r} \Big\| \Big(\frac{h}{h+\rho(y,B_d(x))} \Big)^{\gamma_0} D^\alpha u(y) \Big\|_{L_\infty(\Omega)} \Big) \end{aligned}$$

where $\rho(y, B_d(x)) = \text{distance}(y, B_d(x))$, where we remind the reader that $0 \leq \gamma_s \leq r - 2 + s$, $s = 1, 0$.

Modulo a logarithmic factor that arises because of technical reasons when the negative norms W_∞^{-s} are estimated, the result we shall now state could be guessed at by formally extrapolating (2.2) and (2.3) to $W_\infty^{-s}(B_d)$.

Theorem 3. *Let $u \in W_\infty^r(\Omega)$ and $u_h \in S_r^h(\Omega)$ satisfy (1.6). Let $d > 0$, then for any point $x \in \overline{\Omega}$ and any real $0 < s \leq r - 2$*

$$(2.4) \qquad \begin{aligned} \| \quad & u - u_h\|_{W_\infty^{-s}(B_d(x))} \\ & \leq Ch^{r+s}\Big(\ln \tfrac{1}{h}\Big)^{\overline{\gamma}_{-s}} \Big(\textstyle\sum_{|\alpha|=r} \Big\| \Big(\frac{h}{h+\rho(y,B_d(x))} \Big)^{\gamma_{-s}} D^\alpha u(y) \Big\|_{L_\infty(\Omega)} \Big), \end{aligned}$$

where $0 \leq \gamma_{-s} \leq r - 2 - s$, and $\overline{\gamma}_{-s} = 2$ if $\gamma_{-s} = r - 2 - s$ and $\overline{\gamma}_{-s} = 1$ otherwise. C is a constant that is independent of x, h, u_h, and u.

Before discussing this result let us derive a two term asymptotic expansion inequality that will be useful in the application to a posteriori estimates in the next section.

Corollary 3. *Suppose that the conditions of Theorem 3 hold and in addition $u \in W_\infty^{r+s}(\Omega)$ for some $0 < s \leq 1$ then*

$$(2.5) \qquad \|u - u_h\|_{W_\infty^{-s}(B_d(x))} \leq Ch^{r+s}\Big(\ln \frac{1}{h}\Big)\Big(\sum_{|\alpha|=r} |D^\alpha u(x)| + d^{\widehat{s}}\|u\|_{W_\infty^{r+s}(\Omega)}\Big)$$

where $\widehat{s} = s$ if $0 < s < 1$, and $\widehat{s} = 1 - \varepsilon$ if $s = 1$.

Remark 2.1. *If a standard duality argument is used to get an upper bound for $W_\infty^{-s}(B_d(x))$ then one obtains*

$$(2.6) \qquad \|u - u_h\|_{W_\infty^{-s}(B_d(x_0))} \leq \|u - u_h\|_{W_\infty^{-s}(\Omega)} \leq Ch^{r+s}\Big(\ln \frac{1}{h}\Big)\|u\|_{W_\infty^r(\Omega)}.$$

The inequality (2.6) is obviously sharper than (2.7) and more local in nature.

3. An asymptotically exact a posteriori estimator for $\mathbf{u - u_h}$ on each element on irregular meshes

In [2] and [8] a class of asymptotically exact a posteriori estimators was investigated for the gradient error on each element. Here we turn briefly to the problem of measuring the error for $u - u_h$ a posteriori. This work is new and details will be given in [6]. The key features of all of these works are the locality and the asymptotic exactness of the estimators under reasonable conditions.

For each element τ, our estimator will be defined in terms of a set τ_H, $\tau \subseteq \tau_H \subset \Omega$, by

$$E(\tau) = \|u_h - P_H u_h\|_{L_\infty(\tau_H)}. \tag{3.1}$$

Here u_h is the finite element method solution of (1.6) and $P_H u_h$, $H \geq h$ is an approximate identity operator with the following properties. Let $\tau \subseteq \tau_H \subset \Omega$, then for $v \in C^{r+1}(\tau_H)$

$$\|v - P_H v\|_{L_\infty(\tau_H)} \leq C_{P_H} H^{r+1} \|v\|_{W_\infty^{r+1}(\tau_H)}. \tag{3.2}$$

Furthermore for $v \in C(\tau_H)$

$$\|P_H v\|_{L_\infty(\tau_H)} \leq C_{P_H} H^{-s} \|v\|_{W_\infty^{-s}(\tau_H)} \text{ for } s = \frac{1}{2}, 1. \tag{3.3}$$

An example of an operator satisfying (3.2) and (3.3) is the following:

Let τ_H be a simply connected union of simplices, $\tau \subset \tau_H$, having the property that there exist constants $0 < K_1 < K_2$ such that

$$B_{K_1 H} \subset \tau_H \subset B_{K_2 H}. \tag{3.4}$$

Then for $v \in L_2(\tau_H)$, we take $P_H v$ to be the L_2 projection of v onto P_r the space of polynomials of degree $\leq r$ restricted to τ_H. The $\|\nabla u_h - \nabla P_H u_h\|_{L_\infty(\tau)}$ was one example of an asymptotically exact estimator for $\|\nabla u - \nabla u_h\|_{L_\infty(\tau)}$ given in [2], [8]. The approach and results given here are analogous to the approach and results given there.

We are now in a position to state our main result. For technical reasons we must separate the case $r = 3$ from the case $r \geq 4$ and for the sake of simplicity restrict ourselves to stating the result in the case $r \geq 4$.

Theorem 4. *Let $r \geq 4$ and suppose that P_H satisfies (3.2) and (3.3). Let $0 < \varepsilon < 1$ be arbitrary but fixed. There exists a constant $C_1 = C_1(r, N, a_{ij}, b_j, c, c_{co}, c_{ell}, c_{P_H}, \varepsilon)$ such that if u and u_h satisfy (1.6), then for h sufficiently small the following two alternatives hold for $E(\tau) = \|u_h - P_H u_h\|_{L_\infty(\tau)}$ on each element τ. Let m be defined by*

$$m = C_1\Big(\frac{H^{r+1}}{h^{r+1-\varepsilon}} + \frac{h \ln \frac{1}{h}}{H} + h^\varepsilon\Big). \tag{3.5}$$

Alternative I. *Suppose the nondegeneracy condition*

$$h^{1-\varepsilon} \|u\|_{W_\infty^{r+1}(\Omega)} \leq |u|_{W_\infty^r(\tau)} \tag{3.6}$$

holds. Then if $H(h, \varepsilon)$ is chosen so that $m < 1$

$$\frac{1}{1+m} E(\tau) \leq \|u - u_h\|_{L_\infty(\tau)} \leq \frac{1}{1-m} E(\tau) \tag{3.7}$$

and $E(\tau)$ is an equivalent estimator. If $H(h, \varepsilon)$ is chosen so that $m \to 0$ as $h \to 0$ then $E(\tau)$ is an asymptotically exact estimator.

Alternative II. *If (3.6) does not hold, i.e., if*

$$|u|_{W^r_\infty(\Omega)} < h^{1-\varepsilon}\|u\|_{W^{r+1}_\infty(\Omega)} \tag{3.8}$$

then $\|u - u_h\|_{L_\infty(\tau)}$ *is small*

$$\|u - u_h\|_{L_\infty(\tau)} \le C_1 h^{r+1-\varepsilon}\|u\|_{W^{r+1}_\infty(\Omega)}$$

and so is the estimator

$$E(\tau) \le (C_1 + m)h^{r+1-\varepsilon}\|u\|_{W^{r+1}_\infty(\Omega)}.$$

A proof of this result together with treatment of the case $r = 3$ will be given elsewhere. We only remark here that the asymptotic expansion inequalities, where the exponent is different from zero, together with the non–degeneracy condition (3.6) insure that the predominant error in the finite element method on an element τ is equivalent to the interpolation error on τ. Hence it is reasonable that, even though the approximate solution is globally determined, local averaging makes sense.

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