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A POSTERIORI ERROR ESTIMATOR FOR SPECTRAL APPROXIMATIONS OF COMPLETELY CONTINUOUS OPERATORS

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Abstract. In this paper, we study numerical approximations of eigenvalues when using projection method for spectral approximations of completely continuous operators. We improve the theory depending on the ascent of $T - \mu$ and provide a new approach for error estimate, which depends only on the ascent of $T_h - \mu_h$. Applying this estimator to the integral operator eigenvalue problems, we obtain asymptotically exact indicators. Numerical experiments are provided to support our theoretical conclusions.

Key Words. completely continuous operators, projection method, eigenvalues, a posteriori error estimates

1. Spectral Approximations of Completely Continuous Operators

In this paper, we assume that X is a separable reflexive Banach space or a separable Hilbert space, $\|\cdot\|$ and $<\cdot,\cdot>$ are the norm and the adjoint pair in X, respectively. Let S^h be a sequence of finite dimensional spaces such that

$$S^{h_1} \subset S^{h_2} \quad \forall h_2 < h_1; \qquad \overline{\bigcup_{h>0} S^h} = X.$$

We will consider a completely continuous operator $T : X \to X$ and a family of finite ranked operators $T_h : X \to X$, such that

$$\|T_h - T\| \to 0 \quad (h \to 0).$$

Consider the operator eigenvalue problem: Find $\mu \in C$, $0 \neq u \in X$, such that

$$Tu = \mu u. \tag{1}$$

Also consider its discrete scheme: Find $\mu_h \in C$, $0 \neq u_h \in S^h$, such that

$$T_h u_h = \mu_h u_h. \tag{2}$$

Let μ be an eigenvalue of T with algebraic multiplicity m, let E be the spectral projection associated with T and μ, and let E_h be the spectral projection associated with T_h and the eigenvalues of T_h which converge to μ. Similarly, let E^* and E_h^* be spectral projections associated with the adjoint T^* of T and the adjoint T_h^* of T_h, respectively. Moreover, denote $R(E)$, $R(E_h)$, $R(E^*)$, and $R(E_h^*)$ the image spaces of E, E_h, E^*, and E_h^*, respectively.

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In [4], Chatelin has proved that there exist m eigenvalues of T_h (including multiplicity) $\mu_{1,h},\ \mu_{2,h},\ \ldots,\ \mu_{m,h}$ converging to μ and $\mu_{1,h}, \mu_{2,h},\ \ldots,\ \mu_{m,h}$ are not necessarily equal, neither are the ascent of μ and that of $\mu_{i,h}$. In addition, the abstract error estimates of approximate eigenvalues and eigenfunctions have been studied since 1964 by Babuška, Bramble, Chatelin, Grigorieff, Lemordant, Osborn, Stummel, Vainikko, etc. A systematic summarization is found in [1]. We will need the following lemmas [1].

Lemma 1. *There is a constant c independent of h, such that*

$$\theta(R(E), R(E_h)) \leq c \cdot \|(T - T_h)\mid_{R(E)}\| \tag{3}$$

for small h, where $(T - T_h)\mid_{R(E)}$ denotes the restriction of $T - T_h$ to $R(E)$.

Lemma 2. *Let $\varphi_1, \cdots, \varphi_m$ be any basis for $R(E)$, and $\varphi_1^*, \cdots, \varphi_m^*$ be the dual basis for $R(E^*)$. We define $\bar{\mu}_h = \frac{1}{m} \cdot \sum_{j=1}^{m} \mu_{j,h}$, then there is a constant c independent of h, such that*

$$\begin{aligned} \mid \mu - \bar{\mu}_h \mid \ &\leq\ \frac{1}{m}\sum_{j=1}^{m} |< (T - T_h)\varphi_j, \varphi_j^* >| \\ &+\ c \cdot \|(T - T_h)\mid_{R(E)}\| \|(T^* - T_h^*)\mid_{R(E^*)}\|. \end{aligned} \tag{4}$$

Lemma 3. *Let α be the ascent of $\mu - T$. Let $\varphi_1, \cdots, \varphi_m$ be any basis for $R(E)$, and $\varphi_1^*, \cdots, \varphi_m^*$ be the dual basis for $R(E^*)$. Then there is a constant c, such that*

$$\begin{aligned} \mid \mu - \mu_{j,h} \mid \ &\leq\ c\{\sum_{i,k=1}^{m} |< (T - T_h)\varphi_i, \varphi_k^* >| \\ &+\ \|(T - T_h)\mid_{R(E)}\| \|(T^* - T_h^*)\mid_{R(E^*)}\|\}^{\frac{1}{\alpha}} \\ &\quad (j = 1, 2, \cdots, m). \end{aligned} \tag{5}$$

Lemma 4. *Let μ_h be an eigenvalue of T_h such that $\lim_{h\to 0} \mu_h = \mu$. Suppose for each h, u_h is a unit vector satisfying $(\mu_h - T_h)^k u_h = 0$ for some positive integer $k \leq \alpha$. Then for any integer j with $k \leq j \leq \alpha$, we have*

$$\|u_h - P_j u_h\| \leq c \cdot \|(T_h - T)\mid_{R(E)}\|^{\frac{j-k+1}{\alpha}}, \tag{6}$$

where P_j is the projection on $N((\mu - T)^j)$ along M_j . M_j is a closed subspace of X, such that $X = N((\mu - T)^j) \oplus M_j$.

These Lemmas provide a foundation of the spectral approximate theory for completely continuous operators. We can establish *a prior* error estimates of finite element solution for differential operators and integral operators by using these Lemmas. However, we shall note that (5) and (6) depend on the ascent α of $T - \mu$, which is very difficult to determine for non-self adjoint eigenvalue problems. Furthermore, the value of the constant c is unknown in (5) and (6). So, it is inconvenient to obtain *a posteriori* error estimates.

Since Babuška and Rheinboldt published the first paper on *a posteriori* error estimates of finite element methods [2], many developments have been made in this subject. In [6], an abstract error estimate has been presented, which gives *a posteriori* error estimates to finite element approximations for self-adjoint compact operator eigenvalue problems. In the cuurent paper, we will present an abstract error estimate that can provide *a posteriori* error estimates to finite element approximations for general completely continuous (probably non-self adjoint) operator eigenvalue problems.

Let (μ_h, u_h) be an eigen-pair of T_h and l be the ascent of $T_h - \mu_h$, where $\|u_h\| = 1$. We choose u_h^* such that

$$u_h^* \in R(E_h^*),\ < u_h, u_h^* >= 1,\ < v, u_h^* >= 0,\ \forall v \in M, \tag{7}$$

where $M \subset R(E_h)$ satisfies $R(E_h) = M \oplus \{u_h\}$. Since $T_h^* - \mu_h$ and $T_h - \mu_h$ have the same ascent, we have

$$(T_h^* - \mu_h)^l u_h^* = 0. \tag{8}$$

Theorem 1. *Given u_h^* satisfying (7) and (8), there exists $u^* \in R(E^*)$ such that*

$$\|u_h^* - u^*\| \le c\|(T^* - T_h^*)\ |_{R(E^*)}\ \|^{\frac{1}{\alpha}}, \tag{9}$$

$$(\mu - \mu_h)^l < u_h, u^* >=< \sum_{i=0}^{l-1} (\mu - \mu_h)^i (\mu - T)^{l-1-i} (T - \mu_h) u_h, u^* >, \tag{10}$$

and

$$\begin{aligned} &(\mu - \mu_h)^l < u_h, u^* > \\ &= - \sum_{k=0}^{l-1} < (\sum_{m=k}^{l-1} C_m^k)(\mu_h - T)^{k+1} u_h, u^* > (\mu - \mu_h)^{l-1-k}, \end{aligned} \tag{11}$$

where $< u_h, u^* >= 1+ < u_h, u^* - u_h^* >$.

Proof. From the proof of Lemma 4 [1], there exists u^* which satisfies

$$(T^* - \mu)^l u^* = 0, \tag{12}$$

and (9). So, we have

$$< (T - \mu)^l u_h, u^* >=< u_h, (T^* - \mu)^l u^* >= 0.$$

Thus,

$$\begin{aligned} & (\mu - \mu_h)^l < u_h, u^* >=< (\mu - \mu_h)^l u_h, u^* > \\ = \ & < ((\mu - \mu_h)^l - (\mu - T)^l) u_h, u^* > \\ = \ & < \sum_{i=0}^{l-1} (\mu - \mu_h)^i (\mu - T)^{l-1-i} (T - \mu_h) u_h, u^* > . \end{aligned}$$

Then (10) is proved. Using binomial theorem with respect to the right-hand side, we have

$$\begin{aligned} & (\mu - \mu_h)^l < u_h, u^* > \\ = \ & < \sum_{i=0}^{l-1} (\mu - \mu_h)^i (\sum_{k=0}^{l-1-i} C_{l-1-i}^k (\mu - \mu_h)^{l-1-i-k} (\mu_h - T)^k)(T - \mu_h) u_h, u^* > \\ = \ & - < \sum_{i=0}^{l-1} \sum_{k=0}^{l-1-i} C_{l-1-i}^k (\mu - \mu_h)^{l-1-k} (\mu_h - T)^{k+1}) u_h, u^* > . \end{aligned}$$

On the right-hand side, we arrange $\mu - \mu_h$ in descending power to have (11). Since $< u_h, u_h^* >= 1$, it follows that $< u_h, u^* >= 1+ < u_h, u^* - u_h^* >$. □

Remark. *If X is a Hilbert space, let T^* and T_h^* be the Hilbert adjoints of T and T_h, respectively. Let $\varphi_1, \varphi_2, \cdots, \varphi_m$ be an orthonormal basis of $R(E)$, and let*

$\varphi_j^* = E^*\varphi_j$. Then $\varphi_1^*, \varphi_2^*, \cdots, \varphi_m^* \in N((\mu^* - T^*)^\alpha)$, and $\{\varphi_i^*\}$ is the dual basis of $\{\varphi_i\}$. Replaceing the adjoint pair $< \cdot, \cdot >$ in Banach space by the inner product $(\cdot, \cdot)$ in Hilbert space, all the former results remain valid.

2. The Galerkin Method for Integral Operator Eigenvalue Problems and Sloan Iteration

Consider (1) as an integral operator eigenvalue problem, i.e.

$$(Tu)(s) = \int_\Omega k(s,t)u(t)dt, \ \forall s \in \Omega. \tag{13}$$

We assume that the integral operator T satisfies one of the following conditions:

1. Ω is a bounded domain of R^n, $\int\limits_\Omega\int\limits_\Omega \mid k(x,y) \mid^2 dxdy < \infty$. We choose $X = L_2(\Omega)$. $T : L_2(\Omega) \to L_2(\Omega)$ is completely continuous.

2. $\Omega = [0,1]$. The kernel $k(s,t)$ belongs to the family of $C(a,\gamma)$ with respect to t; namely, for $a \geq \gamma \geq 0$, and any t with $0 \leq t \leq 1$,

$$k_t \in C^a(0,t) \cap C^a(t,1) \cap C^\gamma(0,1)$$

is uniformly valid with respect to t in $[0,1]$.

According to the Grahum - Sloan theorem [4], we have $T : L_2(\Omega) \to C(\Omega)$ is completely continuous. Moreover,$T : L_2(\Omega) \to L_2(\Omega)$ and $T : C(\Omega) \to C(\Omega)$ is completely continuous. Let $V_\mu = \{u : (T - \mu)u = 0\}$ be the eigenvector space of μ corresponding to T. Let $V_\mu^* = \{u : (T^* - \mu^*)u = 0\}$ be the eigenvector space of μ^* corresponding to T^*. From [4] Th.7.2, we see that $V_\mu \subset C^a$. Furthermore, if $k(s,t)$ belongs to the family of $C(a,\gamma)$ with respect to s, then $V_\mu^* \subset C^a$.

In this section, we will discuss the Galerkin method of the compact integral operator eigenvalue problem (1). Assume that X is a Hilbert space. Let $S^h \subset X$ be a piecewise polynomial space and $P_h : X \to S^h$ be a family of orthogonal projection operators, so that $P_h \to I$ pointwisely. The Galerkin method takes the solution of (2) as the approximate solution of (1) after choosing $T_h = P_hT$ in (2).

Let $\{\mu_h, u_h\}$ be an eigen-pair of (2). Since P_h is a family of projection operators, $P_h \to I$ pointwisely, and T is completely continuous, we have

$$\|P_hT - T\| \to 0 \ (h \to 0).$$

Thus, from [4], we conclude that μ_h converge to μ with the same algebraic multiplicity, where μ is the eigenvalue of (1).

In order to improve the accuracy, Sloan established a calculate scheme in 1976 [5]:

$$TP_hu_h^s = \mu_hu_h^s. \tag{14}$$

It is easy to prove that, if $\{\mu_h, u_h\}$ is an eigen-pair of (2), then $\{\mu_h, Tu_h\}$ is an eigen-pair of (14). Inversely, if $\{\mu_h, u_h^s\}$ is an eigen-pair of (14), then $\{\mu_h, P_hu_h^s\}$ is an eigen-pair of (2). We say that u_h^s is the Sloan iterate solution of (1).

We will need the following lemmas [4].

Lemma 5. *Let S^h be a finite element space of order r and $P_h : L_2(\Omega) \to S^h$ be a family of orthogonal projection operators, $T : L_2(\Omega) \to H^{r+1}(\Omega)$, $T^* : L_2(\Omega) \to H^{r+1}(\Omega)$. Then*

$$\|u_h - Eu_h\| \leq ch^{r+1}, \tag{15}$$

$$\|u_h^s - Eu_h^s\| \leq ch^{2(r+1)}. \tag{16}$$

Comparing with u_h, the convergence order of u_h^s used by Sloan iteration has a square improvement.

Let α be the ascent of μ. For the orthogonal projection method, if $\alpha = 1$, then $R(E) = V_\mu$. Thus $\mu - \mu_h$ and $dist(u_h^s, V_\mu)$ have the same accuracy as $\mu - \hat{\mu}_h$ and $dist(u_h^s, R(E))$, respectively. If $\alpha \geq 1$, we have the following lemma.

Lemma 6. *Let $K(t,s)$ be the kernel of T. If $K(t,s)$ belongs to the family of $C(a,\gamma)$ with respect to t and s and α is the ascent of $T - \mu$ satisfying $\alpha \geq 1$. Then for the orthogonal projection method, when h is sufficient small, we have*

$$\mu - \mu_h = O(h^{\frac{2\beta}{\alpha}}), \tag{17}$$

$$dist(u_h^s, V_\mu) = O(h^{\frac{2\beta}{\alpha}}), \tag{18}$$

where $\beta = \min(a, r+1)$.

In order to do *a posteriori* error estimates to the eigen-pair (μ_h, u_h) using Sloan iteration, we need to study the difference between Eu_h and ETu_h.

If Eu_h is an eigenvector of T associated with μ, then

$$ETu_h = TEu_h = \mu Eu_h.$$

Hence, Eu_h and ETu_h are the same eigenvectors (ignoring constant coefficient).

If Eu_h is a generalized eigenvector (not an eigenvector) of T associated with μ, we have the following result.

Theorem 2. *Given $P_hTu_h = \mu_h u_h$, then*

$$ETu_h - \mu_h Eu_h = E(I - P_h)(I - P_h)Tu_h. \tag{19}$$

Proof. Note that

$$\begin{aligned} ETu_h &= E(T - P_hT + P_hT)u_h \\ &= E(T - P_hT)u_h + EP_hTu_h \\ &= E(I - P_h)(I - P_h)Tu_h + \mu_h Eu_h, \end{aligned} \tag{20}$$

then the result follows. □

From (19), we conclude that the error $ETu_h - \mu_h Eu_h$ is of higher order comparing with the error of u_h. Thus

$$\begin{aligned} u_h - Eu_h &= u_h - \mu_h^{-1}ETu_h + \mu_h^{-1}ETu_h - Eu_h \approx u_h - \mu_h^{-1}ETu_h \\ &= u_h - \mu_h^{-1}Tu_h + \mu_h^{-1}Tu_h - \mu_h^{-1}ETu_h \approx u_h - \mu_h^{-1}Tu_h. \end{aligned} \tag{21}$$

Now, consider a further problem: Given u_h a generalized eigenvector of P_hT, so that $(P_hT - \mu_h)^l u_h = 0$ $(l > 1)$. Is Tu_h a generalized eigenvector of TP_h, i.e. $(TP_h - \mu_h)^l Tu_h = 0$? If it is, what is the difference between Eu_h and ETu_h? The answer for the first question is positive. In fact, given $(P_hT - \mu_h)^l u_h = 0$, from binomial theorem and $(TP_h)^{l-k} \cdot T = T(P_hT)^{l-k}$, we have

$$\begin{aligned} (TP_h - \mu_h)^l Tu_h &= (\sum_{k=0}^{l} C_l^k (TP_h)^{l-k}(-\mu_h)^k)Tu_h \\ &= (\sum_{k=0}^{l} C_l^k (TP_h)^{l-k} T(-\mu_h)^k)u_h = T(\sum_{k=0}^{l} C_l^k (P_hT)^{l-k}(-\mu_h)^k)u_h \\ &= T(P_hT - \mu_h)^l u_h = 0. \end{aligned} \tag{22}$$

As for the difference between Eu_h and ETu_h, from (20) in Theorem 2, it is easy to conclude that

$$ETu_h - \mu_h Eu_h = E(I-P_h)(I-P_h)Tu_h + E(T_h-\mu_h)u_h. \tag{23}$$

However, to obtain more subtle estimates, we need to do further analysis in $E(T_h - \mu_h)u_h$.

3. A Posteriori Error Estimates

Consider the integral operator $T: L_2(\Omega) \to L_2(\Omega)$. Let μ be an eigenvalue of T, α be the ascent of $T-\mu$, and (μ_h, u_h) be an eigen-pair of $T_h(=P_hT)$ with $\|u_h\|_0 = 1$ and μ_h converges to μ. We choose $u_h^* \in R(E_h^*)$ as in (7).

Let S^h be a piecewise polynomial space of degree r and P_h be a family of projection operators on S^h. For $T_h = P_hT$, we choose

$$u^* = P_l^* u_h^*; \tag{24}$$

and for $T_h = TP_h$, we choose

$$u^* = P_l^* u_h^{*s} \ (u_h^{*s} = {\mu_h^*}^{-1} T^* u_h^*), \tag{25}$$

where P_l^* is the projection defined by the similar method in lemma 4.

Theorem 3. *Assume that $P_h : L_2(\Omega) \to S^h$ is a family of orthogonal projection operators in (2). Let $T^* : R(E^*) \subset L_2(\Omega) \to H^a(\Omega)$ be continuous, $\|Tu_h\|_a' \leq c\|u_h\|_0$, $R(E) \subset H^a(\Omega)$, $R(E^*) \subset H^a(\Omega)$, (μ_h, u_h) be the approximate eigen-pair obtained from (2), and l be the ascent of $T_h - \mu_h$. We choose $T_h = TP_h$ and u^* defined in (25). Then*

$$|\ \mu - \mu_h\ | \quad \leq \quad ch^{\frac{2\beta}{l}}, \tag{26}$$

where $\beta = min(a, r+1)$. Moreover, $\epsilon(\mu_h)$ defined below is the asymptotically exact indicator of μ_h.

$$\epsilon(\mu_h)^l = -\frac{1}{(u_h^s, u_h^{*s})}((\mu_h - T)^l u_h^s, u_h^{*s}). \tag{27}$$

Proof. To avoid over-elaborate narration, we assume that $(\mu - T^*)^{l-1}u^* \neq 0$. Let $I_r : C(\Omega) \to S^h$ be piecewise interpolate operator of degree r. For any fixed i, by orthogonality of P_h and interpolation error estimate, we have

$$\begin{aligned} & |\ ((\mu-\mu_h)^i(\mu-T)^{l-1-i}(T-\mu_h)u_h^s, u^*)\ | \\ = & |\ (\mu-\mu_h)^i((T-\mu_h)u_h^s, (\mu^*-T^*)^{l-1-i}u^*)\ | \\ = & |\ \mu-\mu_h\ |^i|\ ((I-P_h)u_h^s, (I-I_r)T^*(\mu^*-T^*)^{l-1-i}u^*)\ | \\ \leq & c\ |\ \mu-\mu_h\ |^i\ h^{2\beta}\|u_h^s\|_\beta\|T^*(\mu^*-T^*)^{l-1-i}u^*\|_\beta. \end{aligned} \tag{28}$$

Note that T, T^* are continuous, and (9), we have:

$$\begin{aligned} & \sum_{k=0}^{l-1} < (\sum_{m=k}^{l-1} C_m^k)(\mu_h - T)^{k+1}u_h^s, u^* > (\mu-\mu_h)^{l-1-k} \\ = & (1+O(\mu-\mu_h))((\mu_h-T)^l u_h^s, u^*) \\ = & (1+O(\mu-\mu_h))(((\mu_h-T)^l u_h^s, u_h^{*s}) \\ & +((I-P_h)u_h^s, (I-I_r)T^*(\mu_h^*-T^*)^{l-1}(u^*-u_h^{*s}))) \\ = & (1+O(\|(T^*-T_h^*)\ |_{R(E^*)}\ \|^{\frac{1}{\alpha}}))(1+O(\mu-\mu_h))((\mu_h-T)^l u_h^s, u_h^{*s}). \end{aligned} \tag{29}$$

$$(30)\quad \frac{1}{(u_h^s, u^*)} - \frac{1}{(u_h^s, u_h^{*s})} = \frac{(u_h^s, u_h^{*s} - u^*)}{(u_h^s, u^*)(u_h^s, u_h^{*s})} = O(\|(T^* - T_h^*) \mid_{R(E^*)} \|^{\frac{1}{\alpha}}),$$

Substituting (28) and (30) into (10), we get (26). Substituting (29) and (30) into (11), we have

$$(\mu - \mu_h)^l = (1 + O(\|(T^* - T_h^*) \mid_{R_{(E^*)}} \|^{\frac{1}{\alpha}}))^2 (1 + O(\mu - \mu_h))\epsilon(\mu_h)^l,$$

which indicates that $e(\mu_h)$ is an asymptotically exact indicator of μ_h. □

Based on (11), a more subtle error indicator can be obtained. In particular, let $\epsilon(\mu_h)$ be the initial value. Then, we use iteration method to obtain the root e of the following equation:

$$(31)\quad e^l < u_h^s, u_h^{*s} >= -\sum_{k=0}^{l-1} < (\sum_{m=k}^{l-1} C_m^k)(\mu_h - T)^{k+1} u_h^s, u_h^{*s} > e^{l-1-k}.$$

On the other hand, suppose that $\Omega = [0, 1]$ and $T : L_2(\Omega) \to C(\Omega)$ is completely continuous. Let $P_h : C(\Omega) \to S^h$ be a piecewise polynomial interpolative operator of degree r. We choose $T_h = P_h T$. Then the discrete scheme (2) is obtained by the collocation method.

Similarly as Theorem 3, we are able to establish the following result.

Theorem 4. *Suppose that $\|Tu_h\|_a' \le c\|u_h\|_0$, $R(E) \subset H^a(\Omega)$, $R(E^*) \subset H^a(\Omega)$, μ_h is the collocated solution of μ, and l is the ascent of $T_h - \mu_h$. We choose $T_h = P_h T$ and u^* defined in (24). Then*

$$(32)\quad \mid \mu - \mu_h \mid \quad \le \quad ch^{\frac{\beta}{l}},$$

where $\beta = min(a, r + 1)$. Moreover, $\epsilon(\mu_h)$ defined as below is the asymptotically exact indicator of μ_h.

$$(33)\quad \epsilon(\mu_h)^l = -\frac{1}{(u_h, u_h^*)}((\mu_h - T)^l u_h, u_h^*).$$

4. Numerical Experiments

We consider the integral equation problem

$$(34)\quad \int_0^1 K(s, t)u(t)dt = \mu u(s),$$

where $K(s, t) = \frac{1}{2} \mid s - t \mid$, for $s \le t$; $K(s, t) = 2 \mid s - t \mid$, for $t \le s$.

We note that the first order derivative of the kernel is discontinuous. The eigenvalues of the largest and the third largest modulus are $\mu_1 = 0.36031939951656$ and $\mu_3 = -0.10038687851114$, respectively. We shall use Galerkin method to approximate μ_1 and μ_3. Then, we use $e(\mu_h)$ in Theorem 3 and in Theorem 4 to estimate the errors.

Partition $[0, 1]$ into n equal subintervals. Let $S^h \subset C^0$ be a piecewise linear polynomial space, $\{l_i\}_0^n$ are the nodal base functions. An expansion method for (34) is then applied by approximating the eigenfunctions of

$$(35)\quad u(s) \simeq u_h(s) = \sum_{i=0}^{n} z_i l_i(s)$$

and by approximating the eigenvalue μ with μ_h.

Define the residual function $r_h(s)$ as

$$r_h(s) = \int_0^1 K(s,t)u_h(t)dt - \mu_h u_h(s).$$

The particular expansion method is determined by the restrictions imposed on the residual function. We need to determine the coefficients z_i, $i = 0, 1, 2, \dots, n$ in (35) so that $r_h(s)$ is small in some measure. The same as in the Ritz-Galerkin method, we require that $r_h(s)$ be orthogonal to each of the functions $l_i(s)$, $i = 0, 1, 2, \dots, n$, namely,

$$\int_0^1 l_i^*(s)r_h(s)ds = \int_0^1 l_i^*(s)[\int_0^1 K(s,t)u_h(t)dt - \mu_h u_h(s)]ds = 0, \quad i = 0, 1, \dots, n,$$

where $h = 1/n$. This leads to a generalized matrix eigenvalue problem

$$Lz = \mu_h Mz,$$

where

$$L = (l_{ij}), \quad l_{ij} = \int_0^1 \int_0^1 l_i^*(s)K(s,t)l_j(t)dsdt, \;\; i, j = 0, 1, \dots, n;$$

$$M = (m_{ij}), \quad m_{ij} = \int_0^1 l_i^*(s)l_j(s)ds, \;\; i, j = 0, 1, 2, \dots, n.$$

The entries of L and M can be calculated explicitly. We list them in the Appendix.

Table 1

h	i	$\mu_{i,h}$	$e_{i,h}$	$e(\mu_{i,h})$	$\frac{e(\mu_{i,h})}{e_{i,h}}$
$\frac{1}{2}$	1	0.35900382	1.31558318e-003	1.32848276e-003	1.0098052
	3	-0.103039524	2.65264576e-003	2.83573653e-003	1.0690219
$\frac{1}{4}$	1	0.36022636	9.30373241e-005	9.32672751e-005	1.0024716
	3	-0.09988222	-5.04659611e-004	-5.01813644e-004	0.9943606
$\frac{1}{8}$	1	0.36031352	5.88772391e-006	5.88102704e-006	1.0006445
	3	-0.10036284	-2.40342271e-005	-2.40195864e-005	0.9993908
$\frac{1}{16}$	1	0.36031903	3.67640473e-007	3.67700370e-007	1.0001629
	3	-0.10038551	-1.36615773e-006	-1.36595929e-006	0.9998547
$\frac{1}{32}$	1	0.36031938	2.29766842e-008	2.29776188e-008	1.0000407
	3	-0.10038680	-8.32024585e-008	-8.31994260e-008	0.9999636

The results and errors are shown in Table 1. Here, the ascent of $T_h - \mu_{i,h}$ is 1, $\mu_{i,h}$ depicts the approximate eigenvalues of μ_i, $e_{i,h} = \mu_i - \mu_{i,h}$, and $e(\mu_{i,h})$ is the error obtained by using the composite Simpson's rule and extrapolation for (27) with $l = 1$. In Table 1, we see that $\mu_{i,h}$ converges to μ_i when h is decreasing. Moreover, the two errors are extremely close with ratio approaches one. Therefore, $e(\mu_{i,h})$ is an asymptotically exact indicator for $\mu_{i,h}$.

We can also use collocation method to solve this problem. We seek approximated eigen-pairs (λ_h, u_h), which satisfy

$$\int_0^1 K(s_i,t)u_h(t)dt = \mu_h u_h(s_i), \;\; i = 0, 1, , \dots, n. \tag{36}$$

Let $u_h(t) = \sum\limits_{i=0}^{n} z_i l_i(t)$, then

$$\int_0^1 K(s_i,t) \sum_{j=0}^{n} z_j l_j(t) dt = \mu_h \sum_{j=0}^{n} z_j l_j(s_i), \tag{37}$$

namely,

$$\sum_{j=0}^{n} z_j \int_0^1 K(s_i,t) l_j(t) dt = \mu_h \sum_{j=0}^{n} z_j l_j(s_i), \ \ i = 0, 1, \ldots, n. \tag{38}$$

Then, the former problem became a matrix eigenvalue problem

$$Az = \mu_h z,$$

where $z = (z_0, z_1, \ldots, z_n)^T$, and a_{ij}'s are listed in the Appendix.

Table 2

h	i	$\mu_{i,h}$	$e_{i,h}$	$e(\mu_{i,h})$	$\frac{e(\mu_{i,h})}{e_{i,h}}$
$\frac{1}{16}$	1	0.36113254	-8.13139257e-004	-8.14368623e-004	1.0015119
	3	-0.09958146	-8.05423027e-004	-8.062210e-004	1.0009898
$\frac{1}{32}$	1	0.36052281	-2.03408883e-004	-2.03486522e-004	1.0003817
	3	-0.10018395	-2.02929937e-004	-2.02981895e-004	1.0002560
$\frac{1}{64}$	1	0.36037026	-5.08600245e-005	-5.08648986e-005	1.0000958
	3	-0.10033605	-5.08301419e-005	-5.08334448e-005	1.0000650
$\frac{1}{128}$	1	0.36033212	-1.27154946e-005	-1.27157999e-005	1.0000240
	3	-0.10037416	-1.27136278e-005	-1.27138358e-005	1.0000164
$\frac{1}{256}$	1	0.36032258	-3.17890420e-006	-3.17892330e-006	1.0000060
	3	-0.10038370	-3.17878753e-006	-3.17880058e-006	1.0000041
$\frac{1}{512}$	1	0.36032019	-7.94727956e-007	-7.94729154e-007	1.0000015
	3	-0.10038608	-7.94720664e-007	-7.94721486e-007	1.0000010

The results and errors are shown in Table 2. Here, the ascent of $T_h - \mu_{i,h}$ is 1 and $e(\mu_{i,h})$ is the error obtained by (33) with $l = 1$. From Table 2, we see that $e(\mu_{i,h})$ and the true error $\mu_{i,h}$ are extremely close. Thus, $e(\mu_{i,h})$ provides an asymptotically exact indicator for $\mu_{i,h}$.

We note that both methods result in asymptotically exact error indicators. Therefore, accurate error estimator can be accessed by *a posteriori* error estimates with these error indicators.

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References

[1] I. Babuška and J.E. Osborn, *Eigenvalue problems, Handbook of Numerical Analysis* Vol.2, North-Holland, 1991.

[2] I. Babuška and W.C. Rheinboldt, A posteriori error estimates for the finite element method, *Int. J. Numer. Methods Eng.*, 12:1597–1615, 1978.

[3] I. Babuška, T. Strouboulis, C. S. Upadhyay, S. K. Gangaraj, and K. Copps, Validation of *a posteriori* error estimators by numerical approach, *Int. J. Numer. Methods Eng.*, 37:1073–1124, 1994.

[4] F. Chatelin, *Spectral Approximations of Linear Operators*, Academic Press, New York, 1983.

[5] I. H. Sloan, Iterated Galerkin method for eigenvalue problems, *SIAM J. Num. Anal.* 13:753–760, 1976.

[6] Yidu Yang, Jimin Shi, Zhenbao Lin, and Tao Lu, An abstact error estimate for eigenvalue finite element approximation, *Math. Numer. Sinica*, 20(4): 359-366, 1998.

Appendix

$$l_{00} = \frac{1}{12n^3} = l_{nn}; \quad l_{ii} = \frac{7}{12n^3}, \ \ i = 1, 2, \ldots, n-1;$$

$$l_{10} = \frac{11}{16n^3} = l_{n,n-1}; \quad l_{i,i-1} = \frac{97}{48n^3}, \ \ i = 2, 3, \ldots, n-1;$$

$$l_{01} = \frac{3}{16n^3} = l_{n-1,n}; \quad l_{i,i+1} = \frac{25}{48n^3}, \ \ i = 1, 2, \ldots, n-2;$$

$$l_{0j} = \frac{3j-1}{12n^3}, \ \ j = 2, \ldots, n-1; \quad l_{0n} = \frac{3n-2}{24n^3};$$

$$l_{n0} = \frac{3n-2}{6n^3}; \quad l_{nj} = \frac{3n-3j-1}{3n^3}, \ \ j = 1, 2, \ldots, n-2;$$

$$l_{i0} = \frac{3i-1}{3n^3}, \ \ i = 2, \ldots, n-1; \quad l_{in} = \frac{3n-3i-1}{12n^3}, \ \ i = 1, 2, \ldots, n-2;$$

$$l_{ij} = \frac{2(i-j)}{n^3}, \ \ i-j = 2, 3, \ldots, n-2; \quad l_{ij} = \frac{j-i}{2n^3}, \ \ j-i = 2, 3, \ldots, n-2.$$

$$m_{00} = \frac{1}{3n} = m_{nn}; \quad m_{ii} = \frac{2}{3n}, \ \ i = 1, 2, \ldots, n-1;$$

$$m_{i,i-1} = m_{i,i+1} = \frac{1}{6n}; \quad m_{i,j} = 0, \ \ |i-j| > 1.$$

$$a_{00} = \frac{1}{12n^2}, \quad a_{nn} = \frac{1}{3n^2}, \quad a_{ii} = \frac{5}{12n^2}, \ \ i = 1, 2, \ldots, n-1;$$

$$a_{i0} = \frac{3i-1}{3n^2}, \ \ i = 1, 2, \ldots, n; \quad a_{in} = \frac{3n-3i-1}{12n^2}, \ \ i = 0, 1, \ldots, n-1;$$

$$a_{ij} = \frac{j-i}{2n^2}, \ \ i = 0, 1, \ldots, j-1, \quad a_{ij} = \frac{2(i-j)}{n^2}, \ \ i = j+1, j+2, \ldots, n,$$

for $j = 1, 2, \ldots, n-1$.

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