INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 3, Pages 348–360

A SYMMETRIC FINITE VOLUME ELEMENT SCHEME ON QUADRILATERAL GRIDS AND SUPERCONVERGENCE

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(Communicated by Zhimin Zhang)

Abstract. A symmetric finite volume element scheme on quadrilateral grids is established for a class of elliptic problems. The asymptotic error expansion of finite volume element approximation is obtained under rectangle grids, which in turn yields the error estimates and superconvergence of the averaged derivatives. Numerical examples confirm our theoretical analysis.

Key Words. quadrilateral grids, symmetric finite volume schemes, asymptotic error expansion, superconvergence.

1. Introduction

Finite volume methods are a class of important numerical methods to solve PDEs ([2, 6, 8, 9, 14]), which can be viewed as a bridge between finite element methods and finite difference methods. Due to being able to preserve some physical conservation properties locally, such as mass, momentum and energy conservation, finite volume methods are widely applied in many fields, such as computational fluid dynamics and computational physics and so on.

The standard finite volume discretizations usually generate a linear systems with asymmetric matrix for self-adjoint elliptic problems, in many cases, the symmetry is the fundamental physical principle of reciprocity. This asymmetry leads to the fact that many efficient iterative methods which are suitable for solving the symmetric linear systems, such as the conjugate gradient method, can't be employed. It is interesting to see if there exist finite volume schemes that are symmetry preserving. Recently, Aihui Zhou and Xiuling Ma([10, 11, 13]) proposed a class of symmetric finite volume schemes under the triangular grids for solving the self adjoint elliptic boundary value problems and parabolic problems, which gave a positive answer for the triangular grids. However, the answer is still open for the quadrilateral grids so far. For quadrilateral grids, the non-constant derivatives of finite volume element makes the analysis more difficult since there is no convenient weak form.

In this paper, by choosing vertex-centered type control volume properly and using finite volume element methods to discretize the balance equation, a symmetrical finite volume scheme on quadrilateral grids is established. Different from the symmetrical finite volume scheme on the triangular grid, There is no weak form available, so the convergence analysis is more difficult. Here we give a detailed analysis for rectangle grids only. The main ingredients are the bound estimate of

Received by the editors August 1, 2004 and, in revised form, March 22, 2005.
2000 *Mathematics Subject Classification.* 65N30,65F10.
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Subsidized by NASF (10376031),Research Fund for Doctoral Program of Chinese Ministry of Education and by National High-Tech ICF Committee in China.

the minimum eigenvalue for the coefficient matrix of our scheme and asymptotic expansions of the truncation error. The asymptotic error expansion of finite volume element approximation is obtained under rectangle grids, which in turn yields the error estimates and superconvergence of the averaged derivatives. Numerical examples confirm our theoretical analysis and show the efficiency of the method on general quadrilateral grids.

2. Preliminary

In this paper, we consider the following model problem,

$$\begin{cases} -\nabla(a(x)\nabla u) &= f, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \in R^2$ is a convex polygonal domain with boundary $\partial\Omega$, $x = (x_1, x_2), c_1 \leq a(x) \leq c_2$ and c_1, c_2 are two positive real numbers.

For simplicity, we introduce the notation $\lesssim$, $\gtrsim$ as same as that in paper ([3]) which means that when we write $A \lesssim B, A \gtrsim B$ then there exist two positive constant c and C such that $A \leq cB, A \geq CB$ respectively.

Let $\mathcal{P}_{1,1} = \{a_0 + a_1\xi_1 + a_2\xi_2 + a_3\xi_1\xi_2 : \ a_l \in \mathbf{R}, l = 0(1)3\}$ be the set of bilinear polynomial, and $W^{m,p}(\Omega)$ be the Sobolev space with the norms:

$$\|v\|_{m,p} = (\sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(\Omega)})^{\frac{1}{p}}, \ 1 \leq p < \infty,$$
$$\|v\|_{m,\infty} = \max_{|\alpha| \leq m} \|D^\alpha v\|_{L^\infty(\Omega)}, \ p = \infty,$$

where $\alpha = (\alpha_1, \alpha_1, ..., \alpha_n)$, $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $\alpha_i > 0, \ 1 \leq i \leq n$.

In addition, we assume that $\Omega^h = \{E_i, \ 1 \leq i \leq M\}$ is any given quadrilateral grid of Ω (shown as Fig. 1(a) and (b)), and $X = \{X_i = (x_1^i, x_2^i), \ 1 \leq i \leq N\}$ is the set of all nodes in Ω^h, where M and N are the total numbers of all partition elements and nodes respectively.

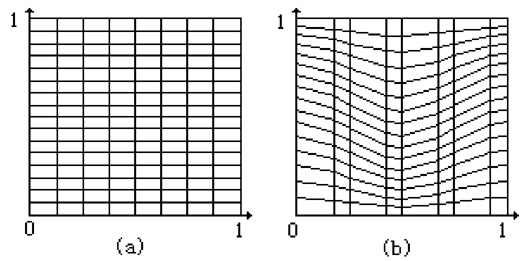

FIGURE 1. (a) uniform grids. (b) non-orthogonal grids.

In order to establish the finite volume scheme, we need to introduce the dual partition $\Omega_*^h = \{b_{X_i}, \ 1 \leq i \leq N\}$ of Ω^h, where b_{X_i} be the dual element(control volume) of the node X_i shown in Fig. 2(a). In this figure, $O_{i_l}, 1 \leq l \leq 4$ is the "center" of the l-th quadrilateral element neighboring to X_i, which is mapped from the center of the reference unit square element E shown as in Fig. 2(b) by the bilinear isoparametric transformation , and $M_{i_l}, 1 \leq l \leq 4$ are midpoints of all edges connected with X_i. Additionally, for any quadrilateral element E_k, we call the restriction region D_l of dual element $b_{X_{k_l}}, 1 \leq l \leq 4$ in E_k as the l-th control sub-volume(shown as Fig. 2 (c)) of E_k, and denote the area of D_l as S_l.

For any quadrilateral grids Ω^h, we introduce the following bilinear isoparametric finite element space

$$V_h = \{v(x) : v(x) = \sum_{i=1}^{4} v_i N_i(\xi_1(x), \xi_2(x)),\ x \in E_k,\ E_k \in \Omega^h,\ v(x) \in C(\overline{\Omega})\},$$

where $v_i = v(X_i)$ and $N_l(\xi_1, \xi_2), 1 \leq l \leq 4$ are shape functions on the reference unit square element E shown in Fig. 2(b), such that

$$N_l(\xi_1, \xi_2) \in \mathcal{P}_{1,1} \text{ and } N_l(\xi_m) = \delta_{lm}, 1 \leq l, m \leq 4. \tag{2.2}$$

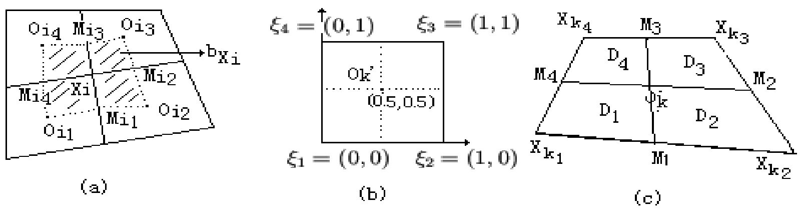

FIGURE 2. (a) dual element b_{X_i} of the node X_i. (b) reference unit square element E. (c) quadrilateral element E_k and its control sub-volumes.

3. The symmetric finite volume scheme on quadrilateral grids

In this section, we will give a symmetric finite volume scheme under the quadrilateral grid Ω^h.

For any X_i, taking the integration of equation (2.1) on the control volume b_{X_i} and using the Green's formula, we get the standard finite volume scheme: Find $u^h \in V^h$ such that

$$-\int_{\partial b_{X_i}} a(x) \frac{\partial u^h(x)}{\partial \overrightarrow{n}} ds = \int_{b_{X_i}} f dx,\ \forall\ b_{X_i} \in \Omega_*^h\ , \tag{3.1}$$

where $\overrightarrow{n}$ is the unit outer normal vector on ∂b_{X_i}.

Similarly to the finite element method, we introduce the finite volume element stiffness matrix $A^{E_k} = (a_{lm}^{E_k})_{4\times4}$ and load vector $f^{E_k} = (f_l^{E_k})_{4\times1}$ for any quadrilateral element E_k shown in Fig.2(c) where $k_l, 1 \leq l \leq 4$ are indices of four nodes on the element E_k, which express the restriction of equations (3.1) related to nodes $X_{k_l}, 1 \leq l \leq 4$ in E_k.

Noting that for any quadrilateral element E_k, we have:

$$u^h(x) := \sum_{l=1}^{4} u_{k_l}^h N_l(\xi_1(x), \xi_2(x)),\ x \in E_k, \tag{3.2}$$

where $u_{k_l}^h = u^h(x_{k_l})$ and $N_l, 1 \leq l \leq 4$ are defined by (2.2).

Substituting (3.2) into (3.1), we have

$$a_{lm}^{E_k} = -\int_{M_l \widehat{O_k M}_{l-1}} a(x) \frac{\partial N_m(x)}{\partial \overrightarrow{n}} ds,\ f_l^{E_k} = \int_{D_l} f dx,\ 1 \leq l, m \leq 4, \tag{3.3}$$

where $M_l, 1 \leq l \leq 4,\ O_k$ are midpoints of corresponding edges and "center" of E_k defined in last section respectively, and let $X_{k_5} = X_{k_1},\ M_0 = M_4$.

Remark 3.1. *We usually take the approximation* $f_l^{E_k} \doteq f(X_{k_l})S_l,\ 1 \le l \le 4.$

The key problem to establish a symmetric finite volume scheme is how to obtain the finite volume element stiffness matrix with symmetric property.

In the following, we take approximations on the edge $M_lO_kM_{l-1}, 1 \le l \le 4$ for the variable coefficient $a(x)$, the gradient of N_m and the Jacobi matrix $J(x)$ of the bilinear isoparametric transformation from E to E_k as follows:

$$a(x) \doteq a_k := a(O_k),\ \nabla N_m(x) \doteq N_{m,k} := \nabla N_m(O_k),\ J(x) \doteq J_k := J(O_k). \quad (3.4)$$

Substituting (3.4) into (3.3), and by simple calculation, we get the following approximation of $a_{lm}^{E_k}$

$$a_{lm}^{E_k} = a_k \frac{\mathbf{q_l} \cdot \mathbf{q_m}}{\det(J_k)},\ 1 \le l, m \le 4, \quad (3.5)$$

where the vectors $\mathbf{q_l} = (\frac{x_2^{k_{l-1}} - x_2^{k_{l+1}}}{2}, -\frac{x_1^{k_{l-1}} - x_1^{k_{l+1}}}{2}),\ 1 \le l \le 4.$

From (3.5), we get the symmetric property of element stiffness matrix.

Property 3.1. *The finite volume element stiffness matrix A^{E_k} defined by (3.5) is a symmetric matrix.*

Assembling all element stiffness matrices $A^{E_k}, 1 \le k \le M$ and load vectors f^{E_k} and dealing with the Dirichlet boundary condition, we get the linear system of finite volume scheme of problem (2.1) on quadrilateral grids Ω^h as follows

$$AU = f, \quad (3.6)$$

where the global finite volume stiffness matrix A and load vector f are defined by

$$A = \sum_{k=1}^{M} I_k^T A^{E_k} I_k, f = \sum_{k=1}^{M} I_k^T f^{E_k}, \quad (3.7)$$

and $I_k : \mathbf{R}^N \to \mathbf{R}^4$ is the nature inclusion related to the quadrilateral element E_k (see Fig.2(c)) such that for any $v = (v_1, v_2, \cdots, v_N) \in \mathbf{R}^N$, $I_k v = (v_{k_1}, v_{k_2}, v_{k_3}, v_{k_4})^T$, and I_{ij}^T is the adjoint of I_{ij} with respect to the inner product $(\cdot, \cdot)$.

By (3.7), it is obvious that the total stiffness matrix A is symmetric, so our finite volume scheme (3.6) of problem (2.1) is symmetric.

4. Error analysis on uniform rectangle grids

Different from the symmetric finite volume scheme on the triangulation([10, 11, 13]), the finite volume scheme on quadrilateral grids proposed here does not fall into some convenient weak form. We need to develop some new techniques for the error analysis.

For simplicity, we only consider the following model problem

$$\begin{cases} -\triangle u = f,\ x \in \Omega = [0,1] \times [0,1], \\ u|_{\partial\Omega} = 0. \end{cases} \quad (4.1)$$

Let $\Omega^h = \{E_{i,j},\ 1 \le i \le n_1,\ 1 \le j \le n_2\}$ be uniform rectangular grids with step sizes $h_l = \frac{1}{n_l}$ where $h = max\{h_1, h_2\}$, $n_l, l = 1, 2$ are partition numbers in the direction of x_l respectively. We denote that $X = \{X_{i,j} = (x_1^{i,j}, x_2^{i,j})\}_{i=1,j=1}^{n_1+1,n_2+1}$ is the set of all nodes where $x_1^{i,j} = (i-1)h_1,\ x_2^{i,j} = (j-1)h_2$, and *supp* ϕ_{ij} is the compact support set of the canonical base function of $X_{i,j}$.

By (3.5), for rectangular grids, we can write the corresponding element stiffness matrices and load vectors according to the following cases.

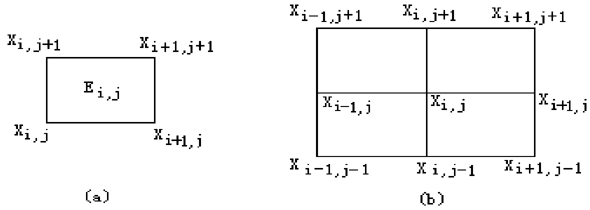

FIGURE 3. (a) a rectangular element $E_{i,j}$. (b) all nodes on the close set of supp ϕ_{ij}

Case 1. For any given "inner" rectangle element $E_{i,j},\ 2 \le i \le n_1 - 1,\ 2 \le j \le n_2 - 1$,

$$A^{E_{i,j}} = \frac{1}{2}\begin{pmatrix} \rho & \delta & -\rho & -\delta \\ \delta & \rho & -\delta & -\rho \\ -\rho & -\delta & \rho & \delta \\ -\delta & -\rho & \delta & \rho \end{pmatrix},\ f^{E_{i,j}} = \begin{pmatrix} sf^{i,j} \\ sf^{i+1,j} \\ sf^{i+1,j+1} \\ sf^{i,j+1} \end{pmatrix}, \tag{4.2}$$

where

$$\rho = \frac{1}{2}(\frac{h_1}{h_2} + \frac{h_2}{h_1})\ ,\ \delta = \frac{1}{2}(\frac{h_1}{h_2} - \frac{h_2}{h_1})\ ,\ s = \frac{h_1 h_2}{4}\ ,\ f^{i,j} = f(X_{i,j}).$$

Case 2. For any given rectangle element along the boundary, say $E_{1,1}$ (corner element, with three nodes on the boundary) or $E_{1,2}$(edge element, with two nodes on the boundary) for examples. Dealing with the corresponding *Dirichlet* boundary condition, we have

$$A^{E_{1,1}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\rho & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ f^{E_{1,1}} = \begin{pmatrix} 0 \\ 0 \\ sf^{i+1,j+1} \\ 0 \end{pmatrix}. \tag{4.3}$$

$$A^{E_{i,1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\rho & \frac{1}{2}\delta \\ 0 & 0 & \frac{1}{2}\delta & \frac{1}{2}\rho \end{pmatrix},\ f^{E_{i,1}} = \begin{pmatrix} 0 \\ 0 \\ sf^{i+1,j+1} \\ sf^{i,j+1} \end{pmatrix}. \tag{4.4}$$

A direct calculation yields the following lemma.

Lemma 4.1 $A^{E_{i,j}},\ 2 \le i \le n_1,\ 2 \le j \le n_2$ *are semi-positive definite, and its four eigenvalues are*

$$\lambda_1 = \lambda_2 = 0, \lambda_3 = \frac{h_1}{h_2}, \lambda_4 = \frac{h_2}{h_1}.$$

By (3.6) and (3.7), we write the linear system of finite volume scheme of problem (4.1) on the rectangle grids Ω^h as follows

$$A^h U^h = F^h, \tag{4.5}$$

where

$$A^h = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} I_{ij}^T A^{E_{i,j}} I_{ij},\ F^h = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} I_{ij}^T f^{E_{i,j}}, \tag{4.6}$$

and $I_{ij} : \mathbf{R}^N \to \mathbf{R}^4$ is the nature inclusion related to the rectangle element $E_{i,j}$.

Now we are going to the lower bound estimate of $\lambda_{\min}(A^h)$. For this purpose, we introduce the following auxiliary matrix

$$\widetilde{A}^h = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} I_{ij}^T \widetilde{A}^{E_{i,j}} I_{ij} \ , \tag{4.7}$$

where

$$(\widetilde{A}^{E_{i,j}})_{lm} := \begin{cases} 0, \text{ when } (A^{E_{i,j}})_{lm} = \delta \text{ or } \delta/2, \\ (A^{E_{i,j}})_{lm}, \text{ otherwise}, \end{cases} \quad 1 \le l, m \le 4. \tag{4.8}$$

Definition 4.1.([12]) *For any two matrices $A := (a_{ij}),\ B := (b_{ij}) \in \mathrm{R}^{n\times n}$ and vectors $x := (x_1, \cdots, x_n)^T,\ y := (y_1, \cdots, y_n)^T \in \mathrm{R}^n$, the tensor production matrix $C := A \otimes B \in \mathrm{R}^{n^2\times n^2}$ and tensor production vector $z := x \otimes y \in \mathrm{R}^{n^2}$ is defined as $C = (a_{ij}B)$ and $z = (x_1y^T, \cdots, x_ny^T)$ respectively.*

Lemma 4.2 (see [12]) *For any two matrices $A,\ B \in \mathrm{R}^{n\times n}$, let $\lambda(A)$ and $\lambda(B)$ be eigenvalues of A and B, x_A and x_B be eigenvectors corresponding to $\lambda(A)$ and $\lambda(B)$ respectively, then $\lambda(A)\lambda(B)$ and $x_C := x_A \otimes x_B$ is the eigenvalue and corresponding eigenvector of the tensor product matrix $C := A \otimes B$.*

Using Lemma 4.2, we can transfer the estimate of $\lambda_{\min}(A^h)$ to that of $\lambda_{\min}(\widetilde{A}^h)$. In the following, we always assume that Ω^h is quasi-uniform, that is, $h_1 = O(h_2)$.

Lemma 4.3. There holds

$$\lambda_{\min}(A^h) \gtrsim \lambda_{\min}(\widetilde{A}^h).$$

Proof. It is easy to see that the conclusion of this lemma holds if and only if

$$\lambda_{\min}(A^{E_{i,j}}) \gtrsim \lambda_{\min}(\widetilde{A}^{E_{i,j}}),\ 2 \le i \le n_1 - 1, 2 \le j \le n_2 - 1. \tag{4.9}$$

Since $A^{E_{i,j}}$ and $\widetilde{A}^{E_{i,j}}$ are positive definite for any "boundary" element $E_{i,j}$, in order to prove (4.9), we only need to prove that there exists a positive constant c, such that for "inner" element $E_{i,j}$,

$$A^{E_{i,j}} - c\tilde{A}^{E_{i,j}} \ge 0,$$

where matrix $A \ge 0$ means that A is semi-positive definite.

In fact

$$A^{E_{i,j}} - c\tilde{A}^{E_{i,j}} = \frac{1}{2}\begin{pmatrix} (1-c)\rho & \delta & -(1-c)\rho & -\delta \\ \delta & (1-c)\rho & -\delta & -(1-c)\rho \\ -(1-c)\rho & -\delta & (1-c)\rho & \delta \\ -\delta & -(1-c)\rho & \delta & (1-c)\rho \end{pmatrix} = A_e \otimes B_e,$$

where 2×2 matrices

$$A_e = \frac{1}{2}\begin{pmatrix} (1-c)\rho & \delta \\ \delta & (1-c)\rho \end{pmatrix}, \quad B_e = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

By simple calculation, we get $\lambda_1(B_e) = 0, \lambda_2(B_e) = 2$, and

$$\lambda_{1,2}(A_e) = \frac{1}{2}(1-c)\rho \pm \delta = (1-c)\frac{1}{2}(\frac{h_1}{h_2} + \frac{h_2}{h_1}) \pm \frac{1}{2}(\frac{h_1}{h_2} - \frac{h_2}{h_1}).$$

The assumption $h_1 = O(h_2)$ leads to that there exist positive constants C_1, C_2, such that $C_1 \le \frac{h_x}{h_y} \le C_2$, so it is easy to see that

$$\lambda_{1,2}(A_e) \ge 0,\ \text{if } 0 < c < 2\min\{\frac{C_1^2}{1+C_1^2}, \frac{C_2^2}{1+C_2^2}\}.$$

According to the fact above, we know that there exists the positive constant c such that all eigenvalues $\lambda_l(B_e)\lambda_k(A_e), 1 \le k, l \le 2$ of the matrix $A^{E_{i,j}} - c\tilde{A}^{E_{i,j}}$ are nonnegative, i.e., $A^{E_{i,j}} - c\tilde{A}^{E_{i,j}}$ is semi-positive definite, this completes the proof of the lemma. □

The formula (4.7) can be equivalently written as follows

$$\widetilde{A}^h = \sum_{i=1}^{n_1} \widetilde{A}_i,\ \widetilde{A}_i = \sum_{j=1}^{n_2} I_{ij}^T \widetilde{A}^{E_{i,j}} I_{ij}.$$

By (4.8), (4.2), (4.3), (4.4) and noting $\rho = O(1)$, we can get the following conclusion.

Conclusion 4.1. *All matrices* $\widetilde{A}^{E_{k,j}},\ 1 \le j \le n_2, k = 1, n_1$ *are positive definite and* $\lambda_{min}(\widetilde{A}^{E_{k,j}}) \gtrsim 1$.

In the following, we want to re-organize $\widetilde{A}_i, 2 \le i \le n_1 - 1$ as follows

$$\widetilde{A}_i = \sum_{j=1}^{n_2} I_{ij}^T \hat{A}^{E_{i,j}} I_{ij},\ 2 \le i \le n_1 - 1, \tag{4.10}$$

where all $\hat{A}^{E_{i,j}}, 1 \le j \le n_2$ are positive definite.

For simplicity, we only give the collocating process for the case of $i = 2$. By direct verifying, we have

$$\widetilde{A}_2 = \sum_{j=1}^{n_2} I_{ij}^T \hat{A}^{E_{2,j}} I_{ij},$$

where

$$\hat{A}^{E_{2,0}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho - \frac{1}{2}\beta_0 & 0 \\ 0 & 0 & 0 & \rho - \frac{1}{2}\beta_0 \end{pmatrix},\ \hat{A}^{E_{2,n_2}} = \begin{pmatrix} \rho - \frac{1}{2}\alpha_{n_2-2} & 0 & 0 & 0 \\ 0 & \rho - \frac{1}{2}\alpha_{n_2-2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.11}$$

$$\hat{A}^{E_{2,j}} = \frac{1}{2} \begin{pmatrix} \beta_{j-1} & 0 & -\rho & 0 \\ 0 & \beta_{j-1} & 0 & -\rho \\ -\rho & 0 & \alpha_j & 0 \\ 0 & -\rho & 0 & \alpha_j \end{pmatrix},\ 2 \le j \le n_2 - 1, \tag{4.12}$$

and

$$\alpha_j = \frac{j}{j+1}\rho + 2(j+2)h^2,\ \beta_j = \rho + \frac{1}{j+1}\rho - 2(j+2)h^2. \tag{4.13}$$

Conclusion 4.2. *All matrices* $\hat{A}^{E_{2,j}}$ *defined by (4.11) and (4.12) are positive definite and* $\lambda_{min}(\hat{A}^{E_{2,j}}) \gtrsim h^2$ $, 1 \le j \le n_2$.

In fact,

(i) When j=1, by (4.13), we have

$$\hat{A}^{E_{2,1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho - \frac{1}{2}\beta_0 & 0 \\ 0 & 0 & 0 & \rho - \frac{1}{2}\beta_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2h^2 & 0 \\ 0 & 0 & 0 & 2h^2 \end{pmatrix},$$

it obvious that $\hat{A}^{E_{2,1}}$ is positive definite and $\lambda_{min}(\hat{A}^{E_{2,1}}) \gtrsim h^2$.

(ii) When $j = n_2$, we have

$$\hat{A}^{E_{2,n_2}} = \begin{pmatrix} \rho - \frac{1}{2}\alpha_{n_2-2} & 0 & 0 & 0 \\ 0 & \rho - \frac{1}{2}\alpha_{n_2-2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\omega = \frac{1}{2}(\rho - \frac{\rho}{n_2-1} - 2n_2h^2)$.

Noting that $\rho = O(1)$, $\frac{\rho}{n_2-1} = O(h)$, $n_2h^2 = O(h)$, so $\hat{A}^{E_{2,n_2}}$ is positive definite and $\lambda_{min}(\hat{A}^{E_{2,n_2}}) \gtrsim h^2$.

(iii) When $2 \le j \le n_2 - 1$, we have

$$\hat{A}^{E_{2,j}} = \frac{1}{2}\begin{pmatrix} \beta_{j-1} & 0 & -\rho & 0 \\ 0 & \beta_{j-1} & 0 & -\rho \\ -\rho & 0 & \alpha_j & 0 \\ 0 & -\rho & 0 & \alpha_j \end{pmatrix} = G \otimes D,$$

where $G = \frac{1}{2}\begin{pmatrix} \beta_{j-1} & -\rho \\ -\rho & \alpha_j \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

By computing, we have

$$\lambda_{1,2}(D) = 1,\ \lambda_{1,2}(G) = \frac{\alpha_j\beta_{j-1} - \rho^2}{\frac{\alpha_j+\beta_{j-1}}{2} \pm (\rho^2 - \alpha_j\beta_{j-1} + (\frac{\alpha_j+\beta_{j-1}}{2})^2)^{\frac{1}{2}}}.$$

Noting that

$$\alpha_j\beta_{j-1} - \rho^2 = \frac{3j+2}{j}h^2 + (j+1)(j+2)h^4 = O(h^2),\ \frac{\alpha_j + \beta_{j-1}}{2} = O(1),$$

thus

$$\lambda_{1,2}(G) \gtrsim h^2.$$

which together with Lemma 4.2 implies that $\hat{A}^{E_{2,j}}$ is positive definite and $\lambda_{min}(\hat{A}^{E_{2,j}}) \gtrsim h^2$. So the conclusion 4.2 holds.

Same results can be obtained by taking similar collocating processes for matrices $\widetilde{A}_i, 3 \le i \le n_1 - 1$. Hence together with the conclusion 4.1, we get

Lemma 4.4. *The matrix $\widetilde{A}^h$ can be decomposed into*

$$\widetilde{A}^h = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} I_{ij}^T \hat{A}^{E_{i,j}} I_{ij},$$

where $\hat{A}^{E_{i,j}}$, $1 \le i \le n_1, 1 \le j \le n_2$ are positive definite, and $\lambda_{min}(\hat{A}^{E_{i,j}}) \gtrsim h^2$.

The following lemma can be directly derived from Lemma 4.4.

Lemma4.5. *There exists*

$$\lambda_{min}(\widetilde{A}^h) \gtrsim h^2.$$

Combining Lemma 4.3 and Lemma 4.5, we can obtain the estimate of $\lambda_{\min}(A^h)$.

Lemma 4.6. *There exists*

$$\lambda_{min}(A^h) \gtrsim h^2.$$

In the following, we give the convergence analysis for the finite volume solution U^h of equation (4.5).

Let $U^h = (u_{1,1}^h, \cdots, u_{n_1+1,1}^h, \cdots, u_{1,n_2+1}^h, \cdots, u_{n_1+1,n_2+1}^h)^T$, by (4.2) and (4.6), for any node $X_{i,j}, 2 \le i \le n_1, 2 \le j \le n_2$, the equation (4.5) can be rewritten by

$$L_{ij}^h(U^h) = -\frac{\rho}{2}(4u_{i,j}^h + \sum_{l,m=\pm1} u_{i+l,j+m}^h) + \delta(-u_{i,j-1}^h - u_{i,j+1}^h + u_{i-1,j}^h + u_{i+1,j}^h) = f_{i,j}h_1h_2, \quad (4.14)$$

where $f^{i,j} = f(X_{i,j})$, $\rho = \frac{1}{2}(\frac{h_1}{h_2} + \frac{h_2}{h_1})$ and $\delta = \frac{1}{2}(\frac{h_1}{h_2} - \frac{h_2}{h_1})$.

Lemma 4.7. *If the exact solution of the problem (4.1)* $u \in C^k(\overline{\Omega}), k = 3, 4$ *, then for any node* $X_{i,j}, 2 \le i \le n_1, 2 \le j \le n_2$,

$$L_{ij}^h(U^h - U) = O(h^k)\|u\|_{k,\infty},$$

where the solution vector $U = (u_{1,1}, \cdots, u_{n_1+1,1}, \cdots, u_{1,n_2+1}, \cdots, u_{n_1+1,n_2+1})^T$ and $u_{i,j} := u(X_{i,j})$.

Proof. By the definition (4.14) of the discretization operator L_{ij}^h and taking the Taylor expansion, we have

$$L_{ij}^h(U) = -h_1h_2(u_{x_1x_1} + u_{x_2x_2})(X_{i,j}) + O(h^k)\|u\|_{k,\infty}, k = 3, 4. \tag{4.15}$$

Using (4.14), (4.15) and noting that $f^{i,j} = -h_1h_2(u_{x_1x_1} + u_{x_2x_2})(X_{i,j})$, we can derive

$$L_{ij}^h(U^h - U) = O(h^k)\|u\|_{k,\infty},$$

i.e., Lemma 4.7 holds. □

Theorem 4.1. *Let* $u \in C^k(\overline{\Omega}), k = 3, 4$*, then*

$$\|U^h - U\| = O(h^{k-2})\|u\|_{k,\infty},$$

where $\|\cdot\|$ *is the average norm defined by*

$$\|W\| = (\frac{1}{N}\sum_{i=1}^{N} w_i^2)^{\frac{1}{2}}, \forall\ W \in \mathbf{R^N}. \tag{4.16}$$

Proof. Using Lemma 4.6 and Lemma 4.7, we have

$$\begin{aligned}\|U^h - U\| &= \|(A^h)^{-1}A^h(U^h - U)\| \lesssim h^{-2}\|A^h(U^h - U)\| \\ &\lesssim h^{-2}(\tfrac{1}{N}\sum_{i=1}^{n_1+1}\sum_{j=1}^{n_2+1}|L_{ij}^h(U^h - U)|^2)^{\frac{1}{2}} \lesssim O(h^{k-2})\|u\|_{k,\infty}.\end{aligned}$$

So, the proof of Theorem 4.1. is completed. □

So far, we have obtained the optimal estimate $O(h^2)$ of the finite volume approximation solution of equation (4.5) on rectangular grids in the average norm.

Remark 5.1. *Although our analysis only applied for uniform grids but the optimal convergence rate can be observed for more general grids. See numerical example in section 6(Table 2).*

5. Asymptotic error expansion and superconvergence result

In this section, we discuss superconvergence property of the finite volume approximation solution vector U^h of equation (4.5).

Lemma 5.1. *Under the assumption of Theorem 4.1, we have*

$$\|U^h\| \lesssim \|F\|,$$

where $F = (f^{1,1}, \cdots, f^{n_1+1,1}, \cdots, f^{1,n_2+1}, \cdots, f^{n_1+1,n_2+1})^T$ *and* $f^{i,j} := f(X_{i,j})$*,* U^h *is the finite volume solution vector defined in the last section and* $\|.\|$ *is the average norm.*

Proof. By (4.5), Lemma 4.6 and the Cauchy inequality, we have

$$h^2(U^h)^TU^h \lesssim (U^h)^TA^hU^h \lesssim h^2((U^h)^T(U^h))^{\frac{1}{2}}(F^TF)^{\frac{1}{2}},$$

i.e.,

$$(U^h)^T(U^h)^{\frac{1}{2}} \lesssim (F^TF)^{\frac{1}{2}},$$

which leads to

$$\|U^h\| \lesssim \|F\|.$$

Thus, we complete the proof of Lemma 5.1. □

Using the similar process of Lemma 4.7, we can get the following Lemma:

Lemma 5.2. *Let the exact solution of the problem (4.1) $u \in C^k(\overline{\Omega}), k = 5, 6$, then for any node $X_{i,j}, 2 \le i \le n_1, 2 \le j \le n_2$, we have*

$$L_{ij}^h(U^h - U) = h_1^2 h_2^2 g(X_{i,j}) + O(h^k)\|u\|_{k,\infty},$$

where

$$\begin{aligned} g &= \frac{-2}{4! h_1^2 h_2^2}\left(h_1^4 \frac{\partial^4 u}{\partial x_1^4} + h_2^4 \frac{\partial^4 u}{\partial x_2^4} + 6h_1^2 h_2^2 \frac{\partial^4 u}{\partial x_2^2 \partial x_1^2}\right) \\ &= \frac{-2}{4! h_1^2 h_2^2}\left((h_1^4 - 3h_1^2 h_2^2)\frac{\partial^4 u}{\partial x_1^4} + (h_2^4 - 3h_1^2 h_2^2)\frac{\partial^4 u}{\partial x_2^4} - 3h_1^2 h_2^2 \Delta f\right). \end{aligned} \tag{5.0}$$

In the following, we first give the error asymptotic expansion of the finite volume solution U^h in the average norm. For this purpose, we introduce the following auxiliary problem

$$\begin{cases} -\triangle w = g, \ x := (x_1, x_2) \in \Omega, \\ \qquad w = 0, \ x \in \partial\Omega. \end{cases} \tag{5.1}$$

One assumption is as follows

(A0) $f \in C^2(\overline{\Omega})$ and Δf equals to zero at all corners of $\overline{\Omega}$.

The assumption (A0) together with the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ implies that the function g also equals to zero at all corners of $\overline{\Omega}$, so by the regularity theory of the elliptic PDE, we know that if $g \in C^{2-\varepsilon}(\overline{\Omega})$, then

$$w \in C^{4-\varepsilon}(\overline{\Omega}) \text{ and } \|w\|_{4-\varepsilon,\infty} \le \|g\|_{2-\varepsilon,\infty}, \forall \varepsilon > 0.$$

In particular, we have

$$w \in C^3(\overline{\Omega}) \text{ and } \|w\|_{3,\infty} \le \|g\|_{1,\infty}. \tag{5.2}$$

Similar to (4.5), the finite volume solution vector W^h of problem (5.1) satisfies

$$A^h W^h = h_1 h_2 G, \tag{5.3}$$

where $G = (g^{1,1}, \cdots, g^{n_1+1,1}, \cdots, g^{1,n_2+1}, \cdots, g^{n_1+1,n_2+1})^T$ and $g^{i,j} = g(X_{i,j})$.

By Theorem 4.1, (5.0) and (5.2), we have

$$\begin{aligned} \|W - W^h\| &= O(h)\|w\|_{3,\infty} = O(h)\|g\|_{1,\infty} \\ &= O(h)(\|u\|_{5,\infty} + \|\Delta f\|_{1,\infty}). \end{aligned} \tag{5.4}$$

Combining Lemma 5.2 with (5.3), when $u \in C^5(\overline{\Omega})$, we have:

$$A^h(U - U^h - h_1 h_2 W^h) = O(h^5)\Psi\|u\|_{5,\infty}, \tag{5.5}$$

where $\Psi = (1, ..., 1)^T$.

Lemma 4.6 together with (5.5) implies that

$$\|U - U^h - h_1 h_2 W^h\| = O(h^3)\|u\|_{5,\infty}. \tag{5.6}$$

Applying (5.4) and (5.6), we can get the following asymptotic error expansion.

Theorem 5.1. *Under the assumption* (**A0**)*, let $u \in C^5(\overline{\Omega})$ and $f \in C^3(\overline{\Omega})$, then we have*

$$\|U - U^h - h_1 h_2 W\| = O(h^3)(\|u\|_{5,\infty} + \|\Delta f\|_{1,\infty}).$$

Furthermore, using this asymptotic error expansion, we will give superconvergence results of the averaged derivatives by proper combinations of the finite volume solution U^h.

For this purpose, we introduce the center difference operators $\delta_l, l = 1, 2$ along directions of x_l respectively,

$$\delta_1 v_{i,j} = \frac{v_{i+1,j} - v_{i-1,j}}{2h_1},\ \delta_2 v_{i,j} = \frac{v_{i,j+1} - v_{i,j-1}}{2h_2},$$

where $v_{l,m} = v(X_{l,m})$.

Let $(u_{i,j})_{x_l} := \frac{\partial u}{\partial x_l}|_{x=X_{i,j}}$, and

$$U_{x_l} = ((u_{2,2})_{x_l}, \cdots, (u_{n_1,2})_{x_l}, \cdots, (u_{2,n_2})_{x_l}, \cdots, (u_{n_1,n_2})_{x_l})^T,$$

$$\delta_l U = (\delta_l u_{2,2}, \cdots, \delta_l u_{n_1,2}, \cdots, \delta_l u_{2,n_2}, \cdots, \delta_l u_{n_1,n_2})^T.$$

It is obvious that

$$\|U_{x_l} - \delta_l U\| = O(h^2)\|u\|_{3,\infty},\ \|\delta_l W\| = \|w\|_{1,\infty}. \tag{5.7}$$

where the average norm $\|.\|$ is different from the average norm defined by (4.16), which satisfies that for any $W = (w_{2,2}, \cdots, w_{2,n_2}, \cdots, w_{n_1,2}, \cdots, w_{n_1,n_2})^T \in \mathbf{R^N}$,

$$\|W\| = (\frac{1}{N}\sum_{i=2}^{n_1}\sum_{j=2}^{n_2} w_{i,j}^2)^{\frac{1}{2}},$$

where $N = (n_1 - 1)(n_2 - 1)$ is the total number of all inner nodes.

Using Theorem 5.1, we draw the conclusion on superconvergence.

Theorem 5.2. *Under the assumption* (**A0**), *let* $u \in C^5(\overline{\Omega})$ *and* $f \in C^3(\overline{\Omega})$, *then we have*

$$\|U_{x_l} - \delta_l U^h\| = O(h^2)(\|u\|_{5,\infty} + \|\Delta f\|_{1,\infty}),\ l = 1, 2.$$

Proof. Using Theorem 5.1 and (5.7), we have

$$\begin{aligned}\|U_{x_l} - \delta_l U + \delta_l U - \delta_l U^h\| &\lesssim \|U_{x_l} - \delta_l U\| + \|\delta_l U - \delta_l U^h\| \\ &\lesssim h^2\|\delta_l W\| + O(h^2)(\|u\|_{5,\infty} + \|\Delta f\|_{1,\infty}) \\ &\lesssim h^2(\|u\|_{5,\infty} + \|\Delta f\|_{1,\infty}).\end{aligned}$$

□

6. Numerical experiments

In this section, we give some numerical experiments for the convergence and superconvergence rate in the average norm.

Firstly, we take $\Omega = [0, 1] \times [0, 1]$ and quadrilateral grids which are shown in Fig. 1(a)(uniform grids) and (b)(nonorthogonal grids). Let $a(x_1, x_2) = x_1 + x_2 + 1$ and $u(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2) + 2$ be the exact solution of problem(2.1).

Now we present some numerical results of convergence order in Table 1 and Table 2 where $\gamma = \frac{\|u-u^{2h}\|}{\|u-u^h\|}$ and PCG methods are used to solve the corresponding discrete system, which can improve the efficiency.

$n_1 \times n_2$	$\|u - u^h\|$(uniform grid)	γ
8×16	2.117925e-2	
16×32	5.451602e-3	3.9
32×64	1.389064e-3	3.9
64×128	3.509974e-4	4.0

Table 1

$n_1 \times n_2$	$\|u - u^h\|$(nonorthogonal grids)	γ
8×16	3.130141e-2	
16×32	7.820736e-3	4.0
32×64	1.976474e-3	3.9
64×128	4.983725e-4	4.0

Table 2

Next for problem (4.1), we consider rectangle grids shown in Fig. 1(a), in table 3, we present numerical results about one order forward difference approximation $\Delta_l u^h_{i,j}$ to approximate $u_{x_l}(X_{i,j})$ where $\Delta_1 u^h_{i,j} = \frac{u^h_{i+1,j} - u^h_{i,j}}{h_1}$ and $\Delta_2 u^h_{i,j} = \frac{u^h_{i,j+1} - u^h_{i,j}}{h_2}$. In Table 4, we use the superconvergence formula $\delta_l u^h_{i,j}$(one order center difference approximation) to approximate $u_{x_l}(X_{i,j})$.

$n_1 \times n_2$	$\|u_{x_1} - \Delta_1 u^h\|$	γ_1	$\|u_{x_2} - \Delta_2 u^h\|$	γ_1
8×16	3.473220e-1		1.803291e-1	
16×32	1.626199e-1	2.1	8.214462e-1	2.2
32×64	7.905709e-2	2.0	3.963300e-2	2.0
64×128	3.902271e-2	2.0	1.952444e-2	2.0

Table 3

$n_1 \times n_2$	$\|u_{x_1} - \delta_1 u^h\|$	$\widetilde{\gamma}_1$	$\|u_{x_2} - \delta_2 u^h\|$	$\widetilde{\gamma}_1$
8×16	9.622503e-3		4.238013e-2	
16×32	2.475406e-3	4.0	1.029606e-2	4.0
32×64	6.252850e-4	4.0	2.546117e-3	4.0
64×128	1.570580e-4	4.0	6.334658e-4	4.0

Table 4

Table 1 and Table 2 show the second order convergence for the finite volume approximate solution and Table 3 and Table 4 show the first order convergence of the derivatives and second order convergence of averaged derivatives respectively, which confirms our theoretical analysis.

Acknowledgments

The author thank Prof. Aihui Zhou and Dr. Xiuling Ma for the valuable discussions with us.

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