ERROR ESTIMATES AND SUPERCONVERGENCE OF MIXED FINITE ELEMENT FOR QUADRATIC OPTIMAL CONTROL

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Abstract. In this paper we present a priori error analysis for mixed finite element approximation of quadratic optimal control problems. Optimal a priori error bounds are obtained. Furthermore super-convergence of the approximation is studied.

Key Words. optimal control, mixed finite element methods, error estimates, and superconvergence.

1. Introduction

Finite element approximation of optimal control problems plays a very important role among the numerical methods for these problems. The literature in this aspect is huge. There have been extensive studies in convergence of the standard finite element approximation of optimal control problems; see, for example, [1], [2], [10], [14], [15], [24], and [32]. For optimal control problems governed by linear state equations, a priori error estimates of the standard finite element approximation were established long ago; see, for example, [8] and [23]. It is, however, much more difficult to obtain such error estimates for control problems where the state equations are nonlinear or where there are inequality state constraints. For a class of nonlinear optimal control problems with equality constraints, a priori error estimates were established in [12]. Some important flow controls are included in this class of problems. A priori error estimates have also been obtained for a class of state constrained control problems in [31], though the state equation is assumed to be linear. In [20] this assumption has been removed by reformulating the control problem as an abstract optimization problem in some Banach spaces and then applying nonsmooth analysis. In fact, the state equation there can be a variational inequality. Some recent progress in a priori error estimates can be found in [3], and in [18], [21] and [22] for a posteriori error estimates. Systematic introduction of the finite element method for PDEs and optimal control problems can be found in, for example, [5], [13], [26], and [30].

In many control problems, the objective functional contains gradient of the state variables. Thus accuracy of gradient is important in numerical approximation of the state equations. Traditionally in such cases mixed finite element methods should be used for discretisation of the state equations. In computational optimal control,
mixed finite element methods are not as widely used as in engineering simulations. In particular there doesn’t seem to exist much work on theoretical analysis of mixed finite element approximation of optimal control problems in the literature.

In this paper we study error estimates and super-convergence of mixed finite element schemes for quadratic optimal control problems. The problem that we are interested in is the following optimal control problem:

\[
\begin{align*}
\min_{u \in K \subset L^2(\Omega_U)} & \frac{1}{2} \left\{ \int_\Omega |p - p_0|^2 + \int_\Omega (y - y_0)^2 + \int_{\Omega_U} u^2 \right\} \\
\text{div} p & = f + Bu \quad \text{in } \Omega, \\
p & = -A \nabla y, \quad \text{in } \Omega, \\
y & = 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where the bounded open set \( \Omega \subset \mathbb{R}^2 \) is a convex polygon or has smooth boundary \( \partial \Omega \), \( \Omega_U \) is a bounded open set in \( \mathbb{R}^2 \) with Lipschitz boundary \( \partial \Omega_U \), \( K \) is a closed convex set in \( L^2(\Omega_U) \). Further specifications on data will be given later. The coefficient matrix \( A \in L^\infty(\Omega; \mathbb{R}^{2 \times 2}) \) is symmetric and uniformly elliptic, i.e., \( A(x) \) is a symmetric and positive definite \( 2 \times 2 \)-matrix, with eigenvalues \( \lambda_j(x) \in \mathbb{R} \) satisfying

\[
0 < c_A \leq \lambda_1(x), \quad \lambda_2(x) \leq C_A
\]

for almost all \( x \in \Omega \).

In this paper we adopt the standard notation \( W^{m,p}(\Omega) \) for Sobolev spaces on \( \Omega \) with a norm \( || \cdot ||_{m,p} \) given by \( ||\phi||_{m,p} = \sum_{|\alpha| \leq m} ||D^\alpha \phi||_{L^p(\Omega)} \), a semi-norm \( | \cdot |_{m,p} \) given by \( |\phi|_{m,p} = \sum_{|\alpha| = m} ||D^\alpha \phi||_{L^p(\Omega)} \). We set \( W^{m,p}_0(\Omega) = \{ \phi \in W^{m,p}(\Omega) : \phi|_{\partial \Omega} = 0 \} \).

For \( p = 2 \), we denote \( H^m(\Omega) = W^{m,2}(\Omega) \) and \( || \cdot ||_m = || \cdot ||_{m,2} \). In addition \( C \) denotes a general positive constant independent of \( h \).

2. Mixed finite element approximation of optimal control problems

Let

\[
V = H(\text{div}; \Omega) = \{ v \in (L^2(\Omega))^2, \text{ div} v \in L^2(\Omega) \},
\]

endowed with the norm given by

\[
||v||_{\text{div}} = ||v||_{H(\text{div}; \Omega)} = \left( ||v||^2_{0,\Omega} + ||\text{div} v||^2_{0,\Omega} \right)^{1/2},
\]

and

\[
W = L^2(\Omega).
\]

We denote

\[
U = L^2(\Omega_U).
\]

To consider the mixed finite element approximation of our optimal control problems, we need a weak formulation for the state equation (1.2)-(1.4). We recast (1.1)-(1.4) in the following weak form: (CCP) find \( (p, y, u) \in V \times W \times U \) such that

\[
\min_{u \in K \subset L^2(\Omega_U)} \frac{1}{2} \left\{ \int_\Omega |p - p_0|^2 + \int_\Omega (y - y_0)^2 + \int_{\Omega_U} u^2 \right\} \\
(A^{-1} p, v) - (y, \text{div} v) = 0, \quad \forall \ v \in V, \\
(\text{div} p, w) = (f + Bu, w), \quad \forall \ w \in W,
\]

where

\[
(\cdot, \cdot) = \int_\Omega \cdot \cdot.
\]
where the inner product in \( L^2(\Omega) \) or \( L^2(\Omega)^2 \) is indicated by \((\cdot, \cdot)\). \( K \) is a closed convex set in \( U \) and \( B \) is a continuous linear operator from \( U \) to \( L^2(\Omega) \). We further assume that \( p_0 \in L^2(\Omega)^2 \), \( y_0 \in L^2(\Omega) \), and \( f \in L^2(\Omega) \). It is well known (see, e.g., [19]) that the convex control problem (CCP) (2.4)-(2.6) has a unique solution \((p, y, u)\), and that a triplet \((p, y, u)\) is the solution of (CCP) (2.4)-(2.6) if and only if there is a co-state \((q, z) \in V \times W\) such that \((p, y, q, z, u)\) satisfies the following optimal conditions: (CCP-OPT)

\[
\begin{align*}
(A^{-1}p, v) - (y, \text{div}v) &= 0, \\
(\text{div}p, w) &= (f + Bu, w), \\
(A^{-1}q, v) - (z, \text{div}v) &= -(p - p_0, v), \\
(\text{div}q, w) &= (y - y_0, w), \\
(u + B^*z, \bar{u} - w)_U &\geq 0,
\end{align*}
\]

for all \( v \in V \), \( w \in W \), and \( \bar{u} \in K \), where \( B^* \) is the adjoint operator of \( B \), and \((\cdot, \cdot)_U \) is the inner product of \( U \). In the rest of the paper, we shall simply write the product as \((\cdot, \cdot)\) whenever no confusion is caused.

For ease of exposition we will assume that \( \Omega \) and \( \Omega_U \) are both polygons. Let \( T_h \) and \( T_h(\Omega_U) \) be regular (in the sense of [5]) triangulations or rectangulations of \( \Omega \) and \( \Omega_U \), respectively. They are assumed to satisfy the minimum angle condition which says that there is a positive constant \( C \) such that for all \( T \in T_h \) \((T_U \in T_h(\Omega_U))\)

\[
C^{-1}h_T^2 \leq |T| \leq Ch_T^2, \quad C^{-1}h_{T_U}^2 \leq |T_U| \leq Ch_{T_U}^2
\]

where \(|T|\) \((|T_U|)\) is the area of \( T \) \((T_U)\) and \( h_T \) \((h_{T_U})\) is the diameter of \( T \) \((T_U)\). Let \( h = \max h_T \) \((h_U = \max h_{T_U})\).

Let \( V_h \times W_h \subset V \times W \) denote the Raviart-Thomas space [28] of the lowest order associated with the triangulations or rectangulations \( T^h \) of \( \Omega \). Let \( P_k \) denote polynomials of total degree at most \( k \), and \( Q_{m,n}(K) \) indicate the space of polynomials of degree no more than \( m \) and \( n \) in \( x \) and \( y \) variables respectively. If \( T \) is a triangle, \( V(T) = \{v \in P^2_0 + x \cdot P_0\} \) and if \( T \) is a rectangle, \( V(T) = \{v \in Q_{1,0} \times Q_{0,1}\} \). We define

\[
V_h := \{v_h \in V : \forall T \in T_h, \ v_h|_T \in V(T)\},
\]

\[
W_h := \{w_h \in W : \forall T \in T_h, \ w_h|_T = \text{constant}\}.
\]

Associated with \( T_h(\Omega_U) \) is another finite dimensional subspace \( U_h \) of \( U \):

\[
U_h := \{\bar{u}_h \in U : \forall T_U \in T_h(\Omega_U), \ \bar{u}_h|_{T_U} = \text{constant}\}.
\]

The mixed finite element approximation of (CCP) (2.4)-(2.6) is as follows: (CCP)

\[
\begin{align*}
\min_{u_h \in K_h \subset U_h} & \frac{1}{2} \left\{ \int_{\Omega} |p_h - p_0|^2 + \int_{\Omega} (y_h - y_0)^2 + \int_{\Omega_U} u_h^2 \right\} \\
(A^{-1}p_h, v_h) - (y_h, \text{div}v_h) &= 0, \quad \forall \ v_h \in V_h, \\
(\text{div}p_h, w_h) &= (f + Bu_h, w_h), \quad \forall \ w_h \in W_h.
\end{align*}
\]

where \( K_h \) is a closed convex set in \( U_h \). This control problem (CCP) (2.16)-(2.18) again has a unique solution \((p_h, y_h, u_h)\), and a triplet \((p_h, y_h, u_h) \in V_h \times W_h \times U_h \) is the solution of (CCP) (2.16)-(2.18) if and only if there is a co-state \((q_h, z_h) \in \)
\( V_h \times W_h \) such that \((p_h, y_h, q_h, z_h, u_h)\) satisfies the following optimal conditions: (CCP-OPT)\(_h\)

\[
\begin{align*}
(2.19) & \quad (A^{-1}p_h, v_h) - (y_h, \text{div} v_h) = 0, \\
(2.20) & \quad (\text{div}p_h, w_h) = (f + Bu_h, w_h), \\
(2.21) & \quad (A^{-1}q_h, v_h) - (z_h, \text{div} v_h) = -(p_h - p_0, v_h), \\
(2.22) & \quad (\text{div}q_h, w_h) = (y_h - y_0, w_h), \\
(2.23) & \quad (u_h + B^* z_h, u_h - u_h) \geq 0,
\end{align*}
\]

for all \(v_h \in V_h, w_h \in W_h\), and \(\tilde{u}_h \subset K_h\), where \(B^*\) is the adjoint operator of \(B\).

3. Some Preliminaries and interpolation operators

Now, we define the standard \(L^2(\Omega)\)-orthogonal projection \([6]\) \(P_h\): \(W \rightarrow W_h\), which satisfies: for any \(\phi \in W\)

\[
(3.1) \quad (\phi - P_h\phi, w_h) = 0, \quad \forall \, w_h \in W_h.
\]

Next, let us recall the Fortin projection (see [4] and [6]) \(\Pi_h : V \rightarrow V_h\), which satisfies: for any \(q \in V\)

\[
(3.2) \quad (\text{div}(q - \Pi_h q), w_h) = 0, \quad \forall \, w_h \in W_h.
\]

We have that the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\text{div}} & W \\
\Pi_h \downarrow & & \downarrow P_h \\
V_h & \xrightarrow{\text{div}} & W_h
\end{array}
\]

i.e., \(\text{div}\Pi_h = P_h\text{div}: V \xrightarrow{\text{onto}} W_h\), and the following approximation properties:

\[
\begin{align*}
(3.4) & \quad ||q - \Pi_h q|| \leq C||q||_h, \\
(3.5) & \quad ||\text{div}(q - \Pi_h q)||_{-s} \leq C||q||_{1+s}, \quad s = 0, 1 \\
(3.6) & \quad ||\phi - P_h\phi||_{-s} \leq C||\phi||_{1+s}, \quad s = 0, 1.
\end{align*}
\]

Furthermore, we also define the standard \(L^2(\Omega_U)\)-orthogonal projection \(Q_h\): \(U \rightarrow U_h\), which satisfies: for any \(\tilde{u} \in U\)

\[
(3.7) \quad (\tilde{u} - Q_h\tilde{u}, \tilde{u}_h) = 0, \quad \forall \, \tilde{u}_h \in U_h.
\]

Similar to (3.6), we have the approximation property:

\[
(3.8) \quad ||\tilde{u} - Q_h\tilde{u}||_{-s} \leq C||\tilde{u}||_{1+s}, \quad s = 0, 1.
\]

4. Error estimates for the intermediate error

We assume that \(\Omega\) is 2-regular, i.e. the Dirichlet problem

\[
\begin{align*}
(4.1) & \quad -\text{div}(A(x)\nabla \phi) = \psi, \quad x \in \Omega, \\
& \quad \phi = 0, \quad x \in \partial \Omega,
\end{align*}
\]

has a unique solution \(\phi \in H^2(\Omega)\) for \(\psi \in L^2(\Omega)\) and \(||\phi||_2 \leq C||\psi||_0\) for all \(\psi \in L^2(\Omega)\).

For any \(\tilde{u} \in U\), let \((\tilde{p}(\tilde{u}), \tilde{y}(\tilde{u})) \in V \times W\) be the solution of the following equations:

\[
\begin{align*}
(4.2) & \quad (A^{-1}p(\tilde{u}), v) - (\tilde{y}(\tilde{u}), \text{div} v) = 0, \quad \forall \, v \in V, \\
(4.3) & \quad (\text{div}p(\tilde{u}), w) = (f + B\tilde{u}, w), \quad \forall \, w \in W,
\end{align*}
\]
Let \((p, y, u) \in V \times W \times U\) and \((p_h, y_h, u_h) \in V_h \times W_h \times U_h\) be the solutions of (CCP\((2.4)-(2.6)\)) and (CCP\(_h\) \((2.16)-(2.18)\)) respectively. By the regularity of \((4.1)\) and the uniqueness of the solution for \((4.2)-(4.3)\), we can get that \(y(u_h) \in H^2(\Omega)\) satisfies

\[
-\text{div}(A\nabla y(u_h)) = f + Bu_h,\]

and

\[
||y(u_h)||_2 \leq C||f + Bu_h||.
\]

**Remark:** Obviously, we can see that \((p_h, y_h)\) is the mixed finite element approximation of the elliptic problem \((4.4)\).

Set some intermediate errors:

\[
\varepsilon_1 := p(u_h) - p_h \quad \text{and} \quad \varepsilon_1 := y(u_h) - y_h.
\]

To analyze the intermediate errors, let us first note the following error equations from \((2.17)-(2.18)\) and \((4.2)-(4.3)\) with the choice \(\tilde{u} = u_h:\)

\[
(A^{-1}\varepsilon_1, v_h) - (\varepsilon_1, \text{div}v_h) = 0, \quad \forall \ v_h \in V_h, \tag{4.7}
\]

\[
(\text{div}\varepsilon_1, w_h) = 0, \quad \forall \ w_h \in W_h. \tag{4.8}
\]

From Theorem 4.1 in [6], we first establish the following error estimates:

**Lemma 4.1.** For \(h\) sufficiently small, there exists a positive constant \(C\) which only depends on \(A\) and \(\Omega\), such that

\[
||p(u_h) - p_h||_{\text{div}} + ||y(u_h) - y_h|| \leq Ch||f + Bu_h|| \leq Ch\left(||f|| + ||u_h||\right). \tag{4.9}
\]

For any \(\tilde{u} \in U\), let \((q(\tilde{u}), z(\tilde{u})) \in V \times W\) be the solution of the following equations:

\[
(A^{-1}q(\tilde{u}), v) - (z(\tilde{u}), \text{div}v) = -(p(\tilde{u}) - p_0, v), \tag{4.10}
\]

\[
(\text{div}q(\tilde{u}), w) = (y(\tilde{u}) - y_0, w), \tag{4.11}
\]

for all \(v \in V\) and \(w \in W\).

Let \((p, y, q, z, u) \in (V \times W)^2 \times U\) and \((p_h, y_h, q_h, z_h, u_h) \in (V_h \times W_h)^2 \times U_h\) be the solutions of (CCP-OPT \((2.7)-(2.11)\)) and (CCP-OPT\(_h\) \((2.19)-(2.23)\)) respectively. Set some other intermediate errors again:

\[
\varepsilon_2 := q(u_h) - q_h \quad \text{and} \quad \varepsilon_2 := z(u_h) - z_h. \tag{4.12}
\]

To obtain error estimates for \(\varepsilon_2\) and \(\varepsilon_2\), we need the following error equations which come from \((2.21)-(2.22)\) and \((4.10)-(4.11)\) with the choice \(\tilde{u} = u_h:\)

\[
(A^{-1}\varepsilon_2, v_h) - (\varepsilon_2, \text{div}v_h) = (p_h - p(u_h), v_h), \tag{4.13}
\]

\[
(\text{div}\varepsilon_2, w_h) = (y(u_h) - y_h, w_h),
\]

for all \(v_h \in V_h\) and \(w_h \in W_h\).

Using the stability result [4] of the standard mixed finite elements and Lemma 4.1, we then obtain the following result:

**Lemma 4.2.** For \(h\) sufficiently small, there exists a positive constant \(C\) which only depends on \(A\) and \(\Omega\), such that

\[
||q(u_h) - q_h||_{\text{div}} + ||z(u_h) - z_h|| \leq Ch\left(||f|| + ||u_h||\right). \tag{4.14}
\]
5. Error estimates for optimal control problems

In this paper, we deal with general cases where the convex set $K$ may not be the whole control space $U$ and the discretized constraint set $K_h$ may not be a subset of $K$. For a specific choice, we define

$$K = \{ \tilde{u} \in U : \alpha(x) \leq \tilde{u}(x) \leq \beta(x) \quad \text{ a.e. } x \in \Omega \},$$

$$K_h = \{ \tilde{u}_h \in U_h : \tilde{u}_h|_{T_U} \in ((\alpha)_{T_U}, (\beta)_{T_U}), \ T_U \in T_h(\Omega_U) \},$$

where $\alpha(x)$ and $\beta(x)$ are given functions in $L^\infty(\Omega)$,

$$(\alpha)_{T_U} = \frac{1}{|T_U|} \int_{T_U} \alpha(x) dx \quad \text{and} \quad (\beta)_{T_U} = \frac{1}{|T_U|} \int_{T_U} \beta(x) dx.$$

From Lemma 4-5 in [8], we have the following result:

**Lemma 5.1.** For any $\tilde{u}_h \in K_h$, there exists a function $\tilde{u}^* \in K$ such that

$$(5.1) \quad (\tilde{u}^*)_{T_U} = \tilde{u}_h|_{T_U} \quad \text{for all } T_U \in T_h(\Omega_U),$$

and moreover, the following estimates holds

$$\| \tilde{u}^* \|_{1, T_U} \leq (\|\alpha\|_{1, T_U}^2 + \|\beta\|_{1, T_U}^2)^{1/2} \quad \forall \ T_U \in T_h(\Omega_U),$$

$$\| \tilde{u}_h - \tilde{u}^* \|_{-1} \leq Ch_h^2 (\|\alpha\|_1^2 + \|\beta\|_1^2)^{1/2}. \quad (5.3)$$

Let $(p, y, q, z, u) \in (V \times W)^2 \times U$ and $(p_h, y_h, q_h, z_h, u_h) \in (V_h \times W_h)^2 \times U_h$ be the solutions of (CCP-OPT) (2.7)-(2.11) and (CCP-OPT) \(h\) (2.19)-(2.23) respectively. With the intermediate errors, we can decompose the errors as following

$$(5.4) \quad \begin{align*}
p - p_h &= p - p(u_h) + p(u_h) - p_h := e_1 + e_1, \\
y - y_h &= y - y(u_h(y_h - y_h) := r_1 + e_1, \\
q - q_h &= q - q(u_h) + q(u_h) - q_h := e_2 + e_2, \\
z - z_h &= z - z(u_h) + z(u_h) - z_h := r_2 + e_2.
\end{align*}$$

From (2.7)-(2.8) and (4.2)-(4.3), (2.9)-(2.10) and (4.10)-(4.11), we derive the following error equations:

$$(5.5) \quad (A^{-1}e_1, v) - (r_1, \div v) = 0,$$

$$(5.6) \quad (\div e_1, w) = (B(u - u_h), w),$$

$$(5.7) \quad (A^{-1}e_2, v) - (r_2, \div v) = (p(u_h) - p, v),$$

$$(5.8) \quad (\div e_2, w) = (y - y(u_h), w),$$

for all $v \in V$ and $w \in W$. The stability result [4] of the standard mixed finite elements implies that

$$(5.9) \quad \|e_1\|_{\div} + \|r_1\| \leq C\|u - u_h\|_U,$$

and

$$(5.10) \quad \|e_2\|_{\div} + \|r_2\| \leq C\left(\|p - p(u_h)\| + \|y - y(u_h)\|\right) \leq C\|u - u_h\|_U.$$

In the following we estimate $\|u - u_h\|_U$ and then obtain the results:

**Theorem 5.1.** Let $(p, y, q, z, u) \in (V \times W)^2 \times U$ and $(p_h, y_h, q_h, z_h, u_h) \in (V_h \times W_h)^2 \times U_h$ be the solutions of (CCP) (2.7)-(2.11) and (CCP) \(h\) (2.19)-(2.23) respectively. Then, we have

$$\|p - p_h\|_{\div} + \|y - y_h\| \leq C(h + h_U),$$

$$\|q - q_h\|_{\div} + \|z - z_h\| \leq C(h + h_U),$$

where $\alpha(x)$ and $\beta(x)$ are given functions in $L^\infty(\Omega)$,

$$(\alpha)_{T_U} = \frac{1}{|T_U|} \int_{T_U} \alpha(x) dx \quad \text{and} \quad (\beta)_{T_U} = \frac{1}{|T_U|} \int_{T_U} \beta(x) dx.$$
and
\begin{equation}
\|u - u_h\|_U \leq C(h + h_U).
\end{equation}

**Proof.** For any \( \tilde{u} \in K \) and \( \tilde{u}_h \in K_h \), it follows from (2.8) and (4.3) that
\begin{equation}
(\text{div}(p - p(\tilde{u})), w) = (B(u - \tilde{u}), w), \quad \forall \ w \in W,
\end{equation}
and from (2.20) and (4.3) with the choice \( \tilde{u} = \tilde{u}_h \) that
\begin{equation}
(\text{div}(p_h - p(\tilde{u}_h)), w_h) = (B(u_h - \tilde{u}_h), w_h), \quad \forall \ w_h \in W_h.
\end{equation}
It then follows from (2.11) that
\begin{equation}
0 \geq (u + B^* z, u - \tilde{u}) = (u, u - \tilde{u}) + (B(u - \tilde{u}), z).
\end{equation}
From (5.14) and (2.9), we have that
\begin{equation}
(B(u - \tilde{u}), z) = (\text{div}(p - p(\tilde{u})), z) = (A^{-1} q, p - p(\tilde{u})) + (p - p_0, p - p(\tilde{u})).
\end{equation}
Using (2.7), (4.2), and (2.10), we deduce that
\begin{equation}
(A^{-1} q, p - p(\tilde{u})) = (A^{-1}(p - p(\tilde{u})), q) = (y - y(\tilde{u}), \text{div} q) = (y - y_0, y - y(\tilde{u})).
\end{equation}
Thus, we combine (5.16)-(5.18) to get that,
\begin{equation}
(u, u - \tilde{u}) + (p - p_0, p - p(\tilde{u})) + (y - y_0, y - y(\tilde{u})) \leq 0, \quad \forall \ \tilde{u} \in K.
\end{equation}
Similar to (5.19), from (2.23), (5.15), (2.21), (2.19), (4.3) with the choice \( \tilde{u} = \tilde{u}_h \), and (2.22), we can prove that
\begin{equation}
(u_h, u_h - \tilde{u}_h) + (p_h - p_0, p_h - p(\tilde{u}_h)) + (y_h - y_0, y_h - y(\tilde{u}_h)) \leq 0,
\end{equation}
for all \( \tilde{u}_h \in K_h \).

Next, the relations (2.9)-(2.10) and (4.2)-(4.3) imply that for any \( \tilde{u} \in U \)
\begin{equation}
(p - p_0, p(\tilde{u}) - p(u_h)) + (y - y_0, y(\tilde{u}) - y(u_h)) = (f + B\tilde{u}, z),
\end{equation}
so that with the choice \( \tilde{u} = u_h \) in (5.21) we find that
\begin{equation}
(p - p_0, p(\tilde{u}) - p(u_h)) + (y - y_0, y(\tilde{u}) - y(u_h)) = (B^* z, \tilde{u} - u_h),
\end{equation}
for all \( \tilde{u} \in K \). Similarly,
\begin{equation}
(p(u_h) - p_0, p - p(u_h)) + (y(u_h) - y_0, y - y(u_h)) = (B^* z(u_h), u - u_h),
\end{equation}
for all \( \tilde{u}_h \in K_h \).

Finally, for any \( \tilde{u} \in K \) and \( \tilde{u}_h \in K_h \), we observe that
\begin{equation}
\begin{align*}
c\|u - u_h\|_U^2 &\leq \|u - u_h\|_U^2 + \|p - p(u_h)\|_U^2 + \|y - y(u_h)\|_U^2 \\
&= (u, u - u_h) + (p - p_0, p - p(u_h)) + (y - y_0, y - y(u_h)) \\
&\quad - (u_h, u - u_h) - (p(u_h) - p_0, p - p(u_h)) \\
&\quad - (y(u_h) - y_0, y - y(u_h)) \\
&= (u, u - \tilde{u}) + (p - p_0, p - p(\tilde{u})) + (y - y_0, y - y(\tilde{u})) \\
&\quad + (u, \tilde{u} - u_h) + (p - p_0, p(\tilde{u}) - p(u_h)) \\
&\quad + (y - y_0, y(\tilde{u}) - y(u_h)) - (u_h, u - u_h) \\
&\quad - (p(u_h) - p_0, p - p(u_h)) - (y(u_h) - y_0, y - y(u_h)).
\end{align*}
\end{equation}
Thus the estimates (5.11)-(5.12) are proved.

From Lemma 5.1, there exists a function $u^* \in K$ such that for all $T_U \in T_h(\Omega_U)$

\begin{equation}
(u^*)_{T_U} = u_h|_{T_U}
\end{equation}

and

\begin{equation}
||u_h - u^*||_{-1} \leq C h_U^2(||\alpha||^2 + ||\beta||^2)^{1/2}.
\end{equation}

In (5.25), we choose that

\begin{equation}
\tilde{u} = u^*, \quad \tilde{u}_h = Q_h u,
\end{equation}

and use (5.27), (3.8), (5.10) and Lemma 4.2 to obtain that

\begin{equation}
||u - u_h||^2_{T_U} \leq ||u + B^*z||_1 \cdot \left[ ||u^* - u_h||_{-1} + ||u - Q_h u||_{-1} \right] + \left[ ||u - u_h||_{T_U} + C||z - z(u_h)|| \right] \cdot ||u - Q_h u||_{T_U} + C||z_h - z(u_h)|| \cdot ||u - Q_h u||_{T_U} + ||u - u_h||_{T_U}
\end{equation}

\begin{equation}
\leq C h_U^2 ||u + B^*z||_1 + C||u - u_h||_{T_U} \cdot ||u - Q_h u||_{T_U} + Ch \left[ ||f|| + ||u_h|| \right] \cdot ||u - Q_h u||_{T_U} + ||u - u_h||_{T_U}
\end{equation}

\begin{equation}
\leq C(h^2 + h_U^2) + \delta ||u - u_h||^2_{T_U},
\end{equation}

for any small $\delta > 0$, where we applied the $\varepsilon$-Cauchy’s inequality. Thus, (5.29) implies the result (5.13).

By using (5.4), Lemma 4.1-4.2, (5.9)-(5.10), and (5.29), we obtain that

\begin{equation}
||p - p_h||_{\text{div}} + ||y - y_h||
\end{equation}

\begin{equation}
\leq ||p - p(u_h)||_{\text{div}} + ||y - y(u_h)|| + ||p(u_h) - p_h||_{\text{div}} + ||y(u_h) - y_h||
\end{equation}

\begin{equation}
\leq C||u - u_h||_{T_U} + Ch(||f|| + ||u_h||) \leq C(h + h_U),
\end{equation}

\begin{equation}
||q - q_h||_{\text{div}} + ||z - z_h||
\end{equation}

\begin{equation}
\leq ||q - q(u_h)||_{\text{div}} + ||z - z(u_h)|| + ||q(u_h) - q_h||_{\text{div}} + ||z(u_h) - z_h||
\end{equation}

\begin{equation}
\leq C||u - u_h||_{T_U} + Ch(||f|| + ||u_h||) \leq C(h + h_U),
\end{equation}

Thus the estimates (5.11)-(5.12) are proved. \hfill \Box
6. Superconvergence result for optimal control problems

Recently, the superconvergence property of optimal control problems is discussed in R. Li, W.B. Liu and N.N. Yan [17] and C. Meyer, A. Rösch [25], by using standard finite element methods. In their work, the state and the adjoint state are approximated by linear finite elements and the control is approximated by piecewise constant functions. In this paper, our aim is to establish superconvergence results for optimal control problems by using mixed finite element methods.

Of particular interest in this section, we consider the Raviart-Thomas finite elements on the uniform partition $\mathcal{T}_h$. Moreover, we set

$$K = \{ \tilde{u} \in U : a \leq \tilde{u}(x) \leq b \quad \text{a.e. } x \in \Omega_U \},$$

and $K_h = K \cap U_h \subset K$, where $a$ and $b$ are real numbers. For simplicity, we make some assumptions:

- The control and the state has the same domain and mesh, namely, $\Omega_U = \Omega$ and $\mathcal{T}_h = \mathcal{T}_h(\Omega_U)$;
- The continue linear operator $B$ can be expressed as $B = a(x) \in W^{1,\infty}(\Omega)$;
- Set $K = \{ \tilde{u} \in U : \tilde{u}(x) \geq 0 \}$ and $K_h \subset K$. Let $\Omega^+ = \{ T : u|_T > 0 \}$, $\Omega^- = \{ T : u|_T = 0 \}$, and $\Omega^0 = \Omega \setminus (\Omega^+ \cup \Omega^0)$. We assume that $u$ and $\mathcal{T}_h$ are regular such that $\text{meas}(\Omega^0) \leq C h$.

**Theorem 6.1.** Let $(p, q, z, u) \in (V \times W)^2 \times U$ and $(p_h, q_h, z_h, u_h) \in (V_h \times W_h)^2 \times U_h$ be the solutions of (CCP) (2.7)-(2.11) and (CCP)$_h$ (2.19)-(2.23) respectively. We assume that the exact state and control solution satisfy

$$p(u), q(u) \in [H^2(\Omega)]^2, \quad y(u), z(u) \in H^1(\Omega),$$

and

$$u, z \in W^{1,\infty}(\Omega), \quad y \in W^{1.4}(\Omega).$$

Then, we have

(6.1) $||\Pi_h p - p_h||_{\text{div}} + ||P_h y - y_h||_{L^2(\Omega)} \leq Ch^{3/2}$;

(6.2) $||\Pi_h q - q_h||_{\text{div}} + ||P_h z - z_h||_{L^2(\Omega)} \leq Ch^{3/2}$;

(6.3) $||Q_h u - u_h|| \leq Ch^{3/2}$.

**Proof.** For the sake of brevity, our argument is merely outlined here.

First, we define some mixed finite element solutions to the intermediate variables. From [7], we can derive the superconvergence result for the intermediate variables by fixing the control approximation $u_h$,

(6.4) $||\Pi_h p(\tilde{u}) - p_h(\tilde{u})||_{\text{div}} + ||P_h y(\tilde{u}) - y_h(\tilde{u})||_{L^2(\Omega)} \leq Ch^2$;

(6.5) $||\Pi_h q(\tilde{u}) - q_h(\tilde{u})||_{\text{div}} + ||P_h z(\tilde{u}) - z_h(\tilde{u})||_{L^2(\Omega)} \leq Ch^2$.

Next, we establish a variational inequality for the function $Q_h u$ by the continuity of $u$.

(6.6) $(u_h + B^* z_h - (u + B^* z), Q_h u - u_h) + (u + B^* z, Q_h u - u) \geq 0$.

Hence

$$c||Q_h u - u_h||^2 \leq (Q_h u - u, Q_h u - u_h) + (B^* z_h - B^* z, Q_h u - u_h) + (u + B^* z, Q_h u - u).$$
Then, we derive the superconvergence result for the control by using an average operator $\pi_c$ as in [17], and $L^2$-projection. Note that
\[(u + B^* z, Q_h u - u) = \int_{\Omega^+} + \int_{\Omega^0} + \int_{\Omega^b} (u + B^* z)(Q_h u - u)dx,
\]
and $(Q_h u - u)|_{\Omega^b} = 0$. From (2.11), we have pointwise a.e. $(u + B^* z) \geq 0$. In (2.11), we choose $\bar{u}|_{\Omega^+} = 0$ and $\bar{u}|_{\Omega \setminus \Omega^+} = u$, so that $(u + B^* z, u)|_{\Omega^+} \leq 0$. Therefore $(u + B^* z)|_{\Omega^+} = 0$.

Then,
\[
(u + B^* z, Q_h u - u) = (u + B^* z, Q_h u - u)|_{\Omega^b} \\
= \left( u + B^* z - \pi_c(u + B^* z), Q_h u - u \right)_{\Omega^b} \\
\leq Ch^2 \cdot ||u + B^* z||_{1, \Omega^b} \cdot ||u||_{1, \Omega^b} \\
\leq Ch^2 \cdot ||u + B^* z||_{1, \infty} \cdot ||u||_{1, \infty} \cdot \text{meas}(\Omega^b) \leq Ch^3.
\]

Finally, by applying the standard stability argument and the properties of some projections, we obtain the superconvergence result for the state and co-state. \hfill \Box

7. Conclusion

We have derived optimal a priori error bounds for mixed finite element approximation of quadratic optimal control problems. We further studied superconvergence properties of these schemes. We shall study error estimates and superconvergence of mixed finite element methods for general convex optimal control problems, give the detailed proof for superconvergence results and study a posteriori error estimates of the schemes in future work.

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References


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