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SUPERCONVERGENCE OF LEAST-SQUARES MIXED FINITE ELEMENTS

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Abstract. In this paper we consider superconvergence and supercloseness in the least-squares mixed finite element method for elliptic problems. The supercloseness is with respect to the standard and mixed finite element approximations of the same elliptic problem, and does not depend on the properties of the mesh. As an application, we will derive more precise a priori bounds for the least squares mixed method. The superconvergence may be used to define a posteriori error estimators in the usual way. As a by-product of the analysis, a strengthened Cauchy-Buniakowskii-Schwarz inequality is used to prove the coercivity of the least-squares mixed bilinear form in a straight-forward manner. Using the same inequality, it can moreover be shown that the least-squares mixed finite element linear system of equations can basically be solved with one single iteration step of the Block Jacobi method.

Key Words. least squares mixed elements, supercloseness, superconvergence

1. Introduction

Superconvergence in finite element methods is an important topic in current research, as is reflected in the references in the classical overview paper [19] but also in the proceedings [20] and of course this issue of this journal. In the past decade, much progress has been made. On the one hand, the so-called Chinese school [12, 29, 30] has made progress in developing suitable interpolants of the exact solution of a PDE to which its finite element approximation is *superclose*. This strategy became necessary since results in [21] (and earlier work by the same author) showed that the nodal interpolant often lacks this property, in particular for n-simplicial elements in dimension $n \geq 2$ of degree d with $d > n$. On the other hand, the so-called *patch-recovery technique* [28, 26, 27] allows for superconvergence on irregular meshes at the cost of additional computations on a patch of elements surrounding an element. Finally, progress has also been made in proving (and, in fact, disproving) localized bounds [14, 15, 23, 24, 25] for standard and mixed finite element methods.

1.1. Least squares mixed finite elements. In this paper we turn our attention to supercloseness and superconvergence in least-squares mixed finite element methods [11, 22] for elliptic equations. These methods aim to provide approximations for the potential and the flux separately, just as mixed finite element methods. The difference is that instead of posing a Ritz-Galerkin condition to select approximations from the subspaces, which results in a saddle-point problem that is not trivial

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[9, 10] to discretize, it employs a least-squares approach. Just as in the standard Galerkin method, this leads to a symmetric coercive bilinear form and straightforward discretization.

A drawback of the least-squares mixed finite element method is that the errors in both the potential and the flux influence one another; as a result, the well-known Lemma by Céa is only able to yield a bound for the largest error of the two. It may however be the case that one of the two errors is of higher order than the other. To prove that in some situations this is indeed the case, one needs to rely on other techniques. In [5] it was proposed to use supercloseness of the least-squares mixed finite element approximations to well-known and well-defined reference functions from the approximating spaces in order to give separate results for both the potential and the flux.

1.2. Outline of this paper. We start in Section 2 with defining our model problem and fix our notations for Sobolev spaces and norms, in particular for some weighted norms on product spaces. In Section 2.2, we recall the strengthened Cauchy-Buniakowskii-Schwarz (CBS) inequality from [6] and put it in a slightly more general context. In Section 2.3 we describe the least-squares mixed finite element method for our model problem and give a one-line proof of the coercivity of the associated bilinear form. Due to the strengthened CBS inequality, block-diagonal preconditioning of the linear system results in a condition number of the preconditioned matrix that is bounded uniformly in the stepsize; as an illustration, we prove separately that the block-Jaboci method (which is equivalent to the block-diagonally preconditioned Richardson iteration) has convergence factor γ when measured in the appropriate norm. Then, in Section 3, we turn to the application of supercloseness to derive a priori bounds for the separate variables that improve the standard bounds by Céa's Lemma in case both approximating spaces have different approximation quality. Finally, we briefly discuss superconvergence by post-processing as a consequence of the supercloseness.

2. Preliminaries

As our model serves the following second order elliptic problem. Given $f \in H^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a convex polytope, find $u \in H_0^1(\Omega)$ such that

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where A is uniformly symmetric positive definite with Lipschitz continuous coefficients and with eigenvalues in the interval $[\beta^2, \beta^{-2}]$ for some $\beta \in (0, 1]$. The formulation of (1) as a system of first-order equations lies at the basis of the least-squares mixed finite element method. This formulation is to find functions $u \in H_0^1(\Omega)$ and $\mathbf{p} \in \mathbf{H}(\operatorname{div}; \Omega)$ such

$$\mathbf{p} = -A\nabla u \text{ in } \Omega, \quad \operatorname{div} \mathbf{p} = f \text{ in } \Omega. \tag{2}$$

Since the spaces $H_0^1(\Omega)$ and $\mathbf{H}(\operatorname{div}; \Omega)$ play a central part in the analysis, we will derive a useful but nevertheless simple result that involves both of them. First however some notations.

2.1. Weighted Sobolev norms and other notations. We use standard notations for Sobolev spaces and their norms and semi-norms; the L_2-norm and inner product we denote by $|\cdot|_0$ and $(\cdot,\cdot)_0$. Additional to the usual norms on $\mathbf{H}(\operatorname{div}; \Omega)$

and $H_0^1(\Omega)$ we will define norms by means of A-weighted inner products. Firstly, let

$$(\mathbf{q},\mathbf{r})_{\mathrm{div},A} = (A^{-1}\mathbf{q},\mathbf{r}) + (\operatorname{div}\mathbf{q},\operatorname{div}\mathbf{r}) \quad \text{and} \quad \|\mathbf{q}\|^2_{\mathrm{div},A} = (\mathbf{q},\mathbf{q})_{\mathrm{div},A}. \tag{3}$$

Secondly, we set

$$(v,w)_{1,A} = (A\nabla v,\nabla w) \quad \text{and} \quad |v|^2_{1,A} = (v,v)_{1,A}. \tag{4}$$

Notice that (4) defines the usual energy norm on $H_0^1(\Omega)$. To conclude, the product space $H_0^1(\Omega)\times\mathbf{H}(\operatorname{div};\Omega)$ will be equipped with the canonical product norm

$$\|(v,\mathbf{q})\|^2_{1\times\mathrm{div},A} = |v|^2_{1,A} + \|\mathbf{q}\|^2_{\mathrm{div},A}. \tag{5}$$

Notice that if $A=I$, the weighted norms above reduce to the usual norms on those spaces. It is not difficult to prove that they are equivalent to the usual norms if $A\neq I$.

By the Poincaré-Friedrichs inequality and the assumption on the eigenvalues of A we have that

$$\sup_{0\neq v\in H_0^1(\Omega)} \frac{|v|_0}{|v|_{1,A}} = d_A < \infty, \quad \text{or equivalently,} \quad \forall v\in H_0^1(\Omega), \quad |v|_0 \le d_A |v|_{1,A}. \tag{6}$$

The constant d_A depends only on the diameter of Ω and on β.

2.2. A strengthened Cauchy-Buniakowskii-Schwarz inequality. In [6] it was proved that $\nabla H_0^1(\Omega)\subset[L^2(\Omega)]^n$ and $\mathbf{H}(\operatorname{div};\Omega)\subset[L^2(\Omega)]^n$ satisfy a strengthened Cauchy-Buniakowskii-Schwarz type inequality [18] in the sense that for all $v\in H_0^1(\Omega)$ and $\mathbf{q}\in\mathbf{H}(\operatorname{div};\Omega)$,

$$(\nabla v,\mathbf{q})_0 \le \gamma |v|_{1,A}\|\mathbf{q}\|_{\mathrm{div},A}, \quad \text{where } \gamma = \sqrt{\frac{d_A^2}{d_A^2+1}} < 1. \tag{7}$$

Since $(\nabla v,\mathbf{r})_0 = -(v,\operatorname{div}\mathbf{r})_0 = 0$ for solenoidal vector fields $\mathbf{r}$, this bound is not always sharp. This can be fixed by stating that for all $\mathbf{r}\in\mathbf{H}(\operatorname{div};\Omega)$ with $\operatorname{div}\mathbf{r}=0$ we have that

$$(\nabla v,\mathbf{q})_0 = (\nabla v,\mathbf{q}-\mathbf{r})_0 \le \gamma|v|_{1,A}\|\mathbf{q}-\mathbf{r}\|_{\mathrm{div},A}. \tag{8}$$

In fact, the constant γ can be strictly less than one because the norm $\|\mathbf{q}\|_{\mathrm{div},A}$ is stronger than the norm $|\mathbf{q}|_0$ that the direct application of the Cauchy-Buniakowskii-Schwarz inequality would yield.

Remark 2.1. *Inequality (7) can be written down in a more symmetric manner. For this we also recall the space* $\mathbf{H}(\operatorname{curl};\Omega)$ *with norm* $\|\cdot\|_{\mathrm{curl}}$*. Since* $\nabla H_0^1(\Omega)\subset\mathbf{H}(\operatorname{curl};\Omega)$ *and* $\operatorname{curl}\nabla v = 0$ *we may restate (7) as follows, where we set* $A=I$ *for simplicity: for all* $\mathbf{q}\in\mathbf{H}(\operatorname{div};\Omega)$ *and all irrotational* $\boldsymbol{\sigma}\in\mathbf{H}(\operatorname{curl};\Omega)$*,*

$$(\boldsymbol{\sigma},\mathbf{q})_0 \le \gamma\|\boldsymbol{\sigma}\|_{\mathrm{curl}}\|\mathbf{q}\|_{\mathrm{div}} \tag{9}$$

Similarly, by reversing the role of both spaces, we find that (9) also holds for all $\boldsymbol{\sigma}\in\mathbf{H}(\operatorname{curl};\Omega)$ *and all solenoidal* $\mathbf{q}\in\mathbf{H}(\operatorname{div};\Omega)$*.*

2.3. Least-squares mixed finite elements. The least-squares mixed finite element method applied to the elliptic problem (2) consists of minimizing the quadratic functional $J : H^1_0(\Omega) \times \mathbf{H}(\mathrm{div}\,;\Omega) \to \mathbb{R}$ defined by

$$J(v, \mathbf{q}) = (f - \mathrm{div}\,\mathbf{q}, f - \mathrm{div}\,\mathbf{q})_0 + (\mathbf{q} + A\nabla v, A^{-1}(\mathbf{q} + A\nabla v))_0, \tag{10}$$

over suitable subspaces $V_h \times \mathbf{\Gamma}_h \subset H^1_0(\Omega) \times \mathbf{H}(\mathrm{div}\,;\Omega)$. Setting the first variation in (10) to zero boils down to the discrete problem to find $(u_h, \mathbf{p}_h) \in V_h \times \mathbf{\Gamma}_h$ such that

$$(\mathbf{p}_h, \mathbf{q}_h)_{\mathrm{div},A} - (u_h, \mathrm{div}\,\mathbf{q}_h)_0 = (f, \mathrm{div}\,\mathbf{q}_h)_0, \tag{11}$$

$$-(v_h, \mathrm{div}\,\mathbf{p}_h)_0 + (u_h, v_h)_{1,A} = 0, \tag{12}$$

for all $v_h \in V_h$ and $\mathbf{q}_h \in \mathbf{\Gamma}_h$. Adding both equations together, the left-hand side can be seen to be derived from the continuous and coercive bilinear form

$$B(w, \mathbf{r}; v, \mathbf{q}) = (\mathbf{r}, \mathbf{q})_{\mathrm{div},A} + (w, v)_{1,A} - (w, \mathrm{div}\,\mathbf{q})_0 - (v, \mathrm{div}\,\mathbf{r})_0 \tag{13}$$

on the product space $H^1_0(\Omega) \times \mathbf{H}(\mathrm{div}\,;\Omega)$. The proof for coercivity stems from [22] but it was observed in [6] that it could be simplified considerably. Indeed, choosing $\mathbf{q} = \mathbf{r}$ and $v = w$ and using Green's formula in combination with (7) we get that

$$B(w, \mathbf{r}; w, \mathbf{r}) = \|(w, \mathbf{r})\|^2_{1\times\mathrm{div},A} - 2(w, \mathrm{div}\,\mathbf{q}) \geq (1 - \gamma)\|(w, \mathbf{r})\|^2_{1\times\mathrm{div},A}. \tag{14}$$

It can easily be verified that the continuity constant is not larger than $1 + \gamma$. The continuity and coercivity gives unique solvability of the discrete system according to the Lax-Milgram Lemma and quasi-optimal convergence in $H^1_0(\Omega) \times \mathbf{H}(\mathrm{div}\,;\Omega)$ according to Céa's Lemma.

2.4. Optimal preconditioning of the linear system. The discrete problem (11) can, after a suitable choice of a basis for $V_h \times \mathbf{\Gamma}_h$, be written as a two-by-two block linear system of equations of the form,

$$\begin{pmatrix} D & C \\ C^* & S \end{pmatrix} \begin{pmatrix} \mathbf{p}_N \\ u_M \end{pmatrix} \begin{pmatrix} f_N \\ 0 \end{pmatrix}. \tag{15}$$

Here, D corresponds to the term $(\mathbf{p}_h, \mathbf{q}_h)_{\mathrm{div},A}$ and S to $(u_h, v_h)_{1,A}$. The matrix C contains the interaction between the spaces V_h and $\mathbf{\Gamma}_h$ represented by the terms $-(u_h, \mathrm{div}\,\mathbf{q}_h)_0$ and $-(v_h, \mathrm{div}\,\mathbf{p}_h)_0$. A standard consequence of the strengthened Cauchy-Buniakowskii-Schwarz inequality (7) and its discrete counterpart, which is that for all vectors v and z of the appropriate lengths,

$$|v^* C z| \leq \gamma \sqrt{v^* D v}\sqrt{z^* S z}, \tag{16}$$

is that block diagonal preconditioning of a linear system with the matrix from (15) results in a preconditioned matrix with condition number (see [2])

$$\kappa = \frac{1 + \gamma}{1 - \gamma}. \tag{17}$$

Since γ does not depend on the discretization at all, we see that the number of iterations of the preconditioned conjugate gradient method that is needed to solve (15), is independent of problem parameters. In fact, convergence independent of problem parameters of the block Jacobi method can be established as well, by considering the following equivalent formulation of the block-diagonally preconditioned iteration,

$$\text{given } u^0_h \text{ and } \mathbf{p}^0_h, \text{ iterate} \quad \begin{cases} (\mathbf{p}^{j+1}_h, \mathbf{q}_h)_{\mathrm{div},A} = (f, \mathrm{div}\,\mathbf{q}_h)_0 + (u^j_h, \mathrm{div}\,\mathbf{q}_h)_0, \\ (u^{j+1}_h, v_h)_{1,A} = -(v_h, \mathrm{div}\,\mathbf{p}^j_h)_0. \end{cases} \tag{18}$$

Proposition 2.2. *The iterates* $(u_h^j, \mathbf{p}_h^j)$ *defined in* (18) *satisfy*

$$\|(u_h - u_h^{j+1}, \mathbf{p}_h - \mathbf{p}_h^{j+1})\|_{1\times\mathrm{div},A} \le \gamma \|(u_h - u_h^j, \mathbf{p}_h - \mathbf{p}_h^j)\|_{1\times\mathrm{div},A}. \tag{19}$$

Proof. Subtract (18) from the least-squares mixed discrete equations (11) and substitute $\mathbf{q}_h = \mathbf{p}_h - \mathbf{p}_h^j$ and $v_h = u_h - u_h^{j+1}$. Adding the resulting two equations gives

$$\|(u_h - u_h^{j+1}, \mathbf{p}_h - \mathbf{p}_h^{j+1})\|^2_{1\times\mathrm{div},A} = (u_h - u_h^j, \mathrm{div}\,(\mathbf{p}_h - \mathbf{p}_h^{j+1})) - (u_h - u_h^{j+1}, \mathrm{div}\,(\mathbf{p}_h - \mathbf{p}_h^j)). \tag{20}$$

Applying Green's formula and (7) shows that

$$\|(u_h - u_h^{j+1}, \mathbf{p}_h - \mathbf{p}_h^{j+1})\|^2_{1\times\mathrm{div},A} \le \gamma \left(|u_h - u_h^j|_{1,A} \|\mathbf{p}_h - \mathbf{p}_h^{j+1}\|_{\mathrm{div},A} + |u_h - u_h^{j+1}|_{1,A} \|\mathbf{p}_h - \mathbf{p}_h^j\|_{\mathrm{div},A} \right). \tag{21}$$

Applying the Cauchy-Schwarz inequality for 2-vectors proves the statement. □

Of course, each preconditioning step involves solving two linear systems: one with D and one with S. For these systems there exist, however, optimal complexity multigrid methods. See in particular [1] for solving sytems with D.

3. Supercloseness and applications

Notice that although Céa's Lemma gives quasi-optimal convergence in the weighted product norm on $H_0^1(\Omega) \times \mathbf{H}(\mathrm{div}\,;\Omega)$ in the sense that for all $(v_h, \mathbf{q}_h) \in H_0^1(\Omega) \times \mathbf{H}(\mathrm{div}\,;\Omega)$,

$$\|(u - u_h, \mathbf{p} - \mathbf{p}_h)\|_{1\times\mathrm{div},A} \le \frac{1+\gamma}{1-\gamma} \|(u - v_h, \mathbf{p} - \mathbf{q}_h)\|_{1\times\mathrm{div},A}, \tag{22}$$

it does not give bounds for norms of the individual errors $\mathbf{p} - \mathbf{p}_h$ and $u - u_h$ other than that each one of them is bounded by the right-hand side of (22). In fact, the product norm kills the approximation quality of the best of V_h and $\mathbf{\Gamma}_h$, as is depicted in Figure 1 below.

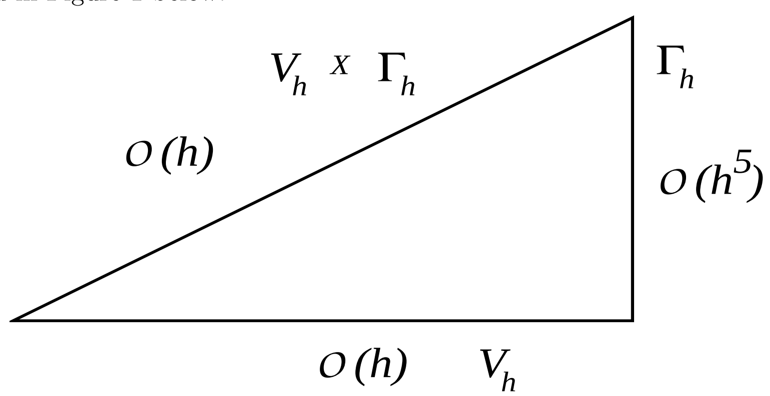

Figure 1. Illustration of Céa's Lemma in the weighted product norm.

In this figure, as an illustration, the space V_h has approximation order $\mathcal{O}(h)$, and $\mathbf{\Gamma}_h$ has approximation order $\mathcal{O}(h^5)$. Then from Céa's Lemma (22) it can only be concluded that

$$\|\mathbf{p} - \mathbf{p}_h\|_{\mathrm{div},A} \le \frac{1+\gamma}{1-\gamma} \mathcal{O}(h). \tag{23}$$

There is good reason to believe that this result can be improved, since we know that the coupling between the diagonal blocks in (15) is weak in some sense; after all, the weakest possible coupling $C = 0$ would result in both approximations u_h and $\mathbf{p}_h$ getting the approximation quality of the space in which it is approximated.

3.1. Improved a priori bounds due to supercloseness. Let us for the moment assume that $\pi_h u$ is an interpolant of u such that for all $v_h \in V_h$,

$$|u - \pi_h u|_{1,A} \leq C|u - v_h|_{1,A}.$$

Moreover, let $\mathbf{\Pi}_h \mathbf{p}$ be an interpolant of the vector field $\mathbf{p}$ such that for all $\mathbf{q}_h \in \mathbf{\Gamma}_h$,

$$\|\mathbf{p} - \mathbf{\Pi}_h \mathbf{p}\|_{\mathrm{div},A} \leq C\|\mathbf{p} - \mathbf{q}_h\|_{\mathrm{div},A}.$$

Then $(u_h, \mathbf{p}_h)$ is called *superclose* to $(\pi_h u, \mathbf{\Pi}_h \mathbf{p})$ in the product norm if

$$\|(u_h - \pi_h u, \mathbf{p}_h - \mathbf{\Pi}_h \mathbf{p})\|_{1\times\mathrm{div},A} \leq Ch\|(u - \pi_h u, \mathbf{p} - \mathbf{\Pi}_h \mathbf{p})\|_{1\times\mathrm{div},A}. \tag{24}$$

The supercloseness helps us to improve the a priori bounds for the variable that is approximated in the space of highest order approximation quality; here we think of the usual type of finite element spaces in which the quality of approximation is expressed as a power of the meshsize h. First, observe that by a simple triangle inequality we have that

$$|u - u_h|_{1,A} \leq |u - \pi_h u|_{1,A} + Ch\|(u - \pi_h u, \mathbf{p} - \mathbf{\Pi}_h \mathbf{p})\|_{1\times\mathrm{div},A}.$$

Now there are two options:

(A) $\mathbf{\Gamma}_h$ has a higher approximation order then V_h, which results in

$$|u - u_h|_{1,A} \leq |u - \pi_h u|_{1,A} + Ch|u - \pi_h u|_{1,A}.$$

Clearly, this gives no improvement over Céa's Lemma, and this is not surprising, since Céa's Lemma is sharp for the variable that is approximated in the space of lowest approximation order. The second option is however of more interest:

(B) V_h has a higher approximation order then $\mathbf{\Gamma}_h$, which results in

$$|u - u_h|_{1,A} \leq |u - \pi_h u|_{1,A} + Ch|\mathbf{p} - \mathbf{\Pi}_h \mathbf{p}|_{1,A}.$$

This improves the bound that results from Céa's Lemma, because the influence of the second term in the right-hand side is diminished by the supercloseness factor h. Of course, similar observations hold for $\|\mathbf{p} - \mathbf{p}_h\|_{\mathrm{div},A}$.

In [5] it was proved that by choosing for $\pi_h u$ the standard finite element approximation $u_h^s \in V_h$ of the Poisson problem, or in other words, the elliptic projection, and by choosing for $\mathbf{\Pi}_h \mathbf{p}$ the mixed finite element approximation $\mathbf{p}_h^m \in \mathbf{\Gamma}_h$ of $\mathbf{p}$, the supercloseness is indeed present as given in (24). Sufficient conditions for this result are:

- Elliptic regularity of the model problem
- BBL stability for the pair $\mathbf{\Gamma}_h, \mathrm{div}\,(\mathbf{\Gamma}_h)$
- The space $\mathbf{\Gamma}_h$ satisfies the property that for all $\mathbf{r} \in H^1(\Omega)]^2$,

$$\inf_{\mathbf{q}_h \in \mathbf{\Gamma}_h} |\mathbf{r} - \mathbf{q}_h|_0 \leq Ch|\mathbf{r}|_1.$$

- The piecewise constants are in $\mathrm{div}\,(\mathbf{\Gamma}_h)$ and the continuous piecewise linears are in V_h.

The proof of (24) in [5] contains duality arguments of both the standard [13] and the mixed [16] finite element method. Notice that the result does not require uniform meshes.

3.2. Superconvergence after post-processing. As mentioned above, the supercloseness does not require uniform meshes. However, it is well known that if the meshes are uniform, like for example the three-directional mesh in two space dimensions, or uniform simplicial meshes in higher dimensions as defined in [7], the standard finite element solution u_h^s and the mixed finite element solution $\mathbf{p}_h^m$ can be superclose to local interpolants of the exact solution. See for instance [3] and [4] for supercloseness of Raviart-Thomas approximations $\mathbf{p}_h^m$ to the Fortin interpolant $\mathbf{\Pi}_h\mathbf{p}$ of $\mathbf{p}$ in case of three-directional planar meshes or [17] in case of rectangular meshes, and [8] for supercloseness of u_h^s to the quadratic nodal interpolant $Q_h u$ in the context of tetrahedral quadratic standard elements. In the papers just mentioned, it is explained how the discrete solutions can be post-processed into higher order (or in other words *superconvergent*) approximations.

The same techniques can be used to post-process the least-squares mixed finite element approximations, since a simple triangle inequality shows that they too are superclose to the local interpolants, in case the standard and mixed finite element approximations are superclose to those local interpolants.

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