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GLOBAL SUPERCONVERGENCE FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY STOKES EQUATIONS

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Abstract. In this paper, the global superconvergence analysis for the finite element approximation of the distributed optimal control governed by Stokes equations is discussed. For the control, a global superconvergence result is derived by applying patch recovery technique. For the state and the co-state, the global superconvergence results are derived by applying some postprocessing techniques for the bilinear-constant scheme over the uniform rectangular meshes. Based on the global superconvergence analysis, recovery type a posteriori error estimates are derived. It is shown that the recovery type a posteriori error estimators provided in this paper are asymptotically exact if the conditions for the superconvergence are satisfied.

Key Words. optimal control, Stokes equations, finite element approximation, global superconvergence, recovery, a posteriori error estimate.

1. Introduction

Flow control problems are crucial to many engineering applications. Extensive research has been carried out on various theoretical aspects of flow control problems, see, for example, [1], [6], [8], [10], [21], [22], [27], [30]. It is obvious that efficient numerical methods are essential to successful applications of flow control. It is well known that the finite element method is undoubtedly the most widely used numerical method in computing optimal control problems, including flow control problems. Systematic introductions to the finite element method for PDEs and optimal control problems can be found in, for example, [3], [11], [28], and [30]. There have been extensive theoretical studies of finite element approximation for various optimal control problems. For instance, a priori error estimates of finite element approximation were established long ago for the optimal control problems governed by linear elliptic and parabolic state equations; see, for example, [5], [13], and [26]. Furthermore, finite element approximation of some flow control has been studied, and a priori error estimates have been established; see [8], [9], [10], and [12]. A posteriori error estimates of finite element approximation were derived for the optimal control problems governed by Stokes equations and for convex boundary control problems, see, i.e., [2], [23], [24], [25].

In recent years, superconvergence for finite element solutions has been an active research area in numerical analysis. The main objective for superconvergence is

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to improve the existing approximation accuracy by applying certain postprocessing techniques which are easy to implement. For the stationary Stokes problem, many finite element schemes which satisfy the Babuška-Brezzi condition have been introduced in [7]. It has been found that there exits a potential for high accuracy or superconvergence for several finite element schemes when the exact solution is smooth enough and the mesh is sufficiently regular; see, for example, [16], [17], [18], [29]. A principle technique for the proof of the global technique is the integral identity technique which has proven to be an efficient tool for the superconvergence analysis of rectangular finite elements (cf. [15]). For the distributed convex optimal control problems governed by elliptic equations, some superconvergence results have been established by applying recovery operators (see, i.e., in [4], [14]).

In this paper, by means of the techniques used in [14] and [29], the global superconvergence for the control problems governed by Stokes equations is discussed. It is shown that if the solution is smooth enough, the mesh for the state and the co-state is the uniform rectangular mesh and the bilinear-constant scheme is adopted for the state and co-state equations, the global superconvergence for the control, state and co-state can be proved. Based on the supercovergence analysis, recovery type a posteriori error estimators are provided.

The outline of this paper is as follows: In Section 2, we provide a weak form for the distributed control problem governed by Stokes equation and its finite element approximation scheme. In Section 3, a global superconvergence result for the control $\mathbf{u}$ is derived by applying recovery operator and the supercovergence analysis technique. Moreover, the global superconvergence results for the state $\mathbf{y}$ and the co-state $\mathbf{p}$ (also r and s) are derived by applying the integral identity technique in Section 4. In the Section 5, based on the global superconvergence analysis provided in Sections 3 and 4, the recovery type a posteriori error estimate is discussed. In the last section, we discuss briefly some possible future work.

Let Ω and $\Omega_{\mathbf{U}}$ be two bounded open sets in R^2 with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_{\mathbf{U}}$, respectively. In this paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|.\|_{m,q,\Omega}$ and seminorm $|.|_{m,q,\Omega}$. We shall extend these (semi) norms to vector functions whose components belong to $W^{m,q}(\Omega)$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)(H_0^m(\Omega))$ with the norm $\|.\|_{m,\Omega}$ and the seminorm $|.|_{m,\Omega}$. In addition, c or C denotes a general positive constant independent of h.

2. Finite element approximation of optimal control problems

In this section, we discuss the finite element approximation of distributed convex optimal control problems governed by the Stokes equations. Let $\mathbf{Y} = (H_0^1(\Omega))^2, \mathbf{U} = (L^2(\Omega_{\mathbf{U}}))^2, \mathbf{H} = (L^2(\Omega))^2$, and $Q = L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q = 0\}$. The state space and the control space will be $\mathbf{Y} \times Q$ and $\mathbf{U}$, respectively. Let B be a linear continuous operator from $\mathbf{U}$ to $\mathbf{H}$, let g be a strictly convex functional which is continuously differentiable on $\mathbf{H}$, and let $\mathbf{K}$ be a closed convex set in the control space $\mathbf{U}$ such that

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{U} : \mathbf{v} \geq 0\}.$$

We further assume that the functional $g(\cdot)$ is bounded below.

We are interested in the following optimal control problem: find $(\mathbf{y}, r, \mathbf{u}) \in \mathbf{Y} \times Q \times \mathbf{U}$ such that

$$\begin{aligned}
&\min_{\mathbf{u}\in K\subset U}\{g(\mathbf{y})+\frac{\alpha}{2}\|\mathbf{u}\|^2_{0,\Omega_{\mathbf{U}}}\},\\
&-\triangle\mathbf{y}+\nabla r=\mathbf{f}+B\mathbf{u} \quad \text{in } \Omega,\\
&\operatorname{div}\mathbf{y}=0 \quad \text{in } \Omega,\\
&\mathbf{y}=0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{2.1}$$

where $\mathbf{f} \in \mathbf{L} = (L^2(\Omega))^2$, α is a positive constant. To consider the finite element approximation of the above optimal control problem, we have to give a weak formulation for the state equations. Let

$$a(\mathbf{y},\mathbf{w})=\int_\Omega \nabla\mathbf{y}\cdot\nabla\mathbf{w} \qquad \forall \mathbf{y},\mathbf{w}\in\mathbf{Y},$$

$$b(\mathbf{v},r)=\int_\Omega r\operatorname{div}\mathbf{v} \qquad \forall(\mathbf{v},r)\in\mathbf{Y}\times Q,$$

$$(\mathbf{f}+B\mathbf{u},\mathbf{w})=\int_\Omega(\mathbf{f}+B\mathbf{u})\cdot\mathbf{w} \qquad \forall \mathbf{f},\mathbf{u},\mathbf{w}\in\mathbf{L}\times\mathbf{U}\times\mathbf{Y}.$$

Then the standard weak formulation for the state equations reads as follows: Given $\mathbf{f} \in \mathbf{L}$, find $(\mathbf{y}(\mathbf{u}), r(\mathbf{u})) \in \mathbf{Y} \times Q$ such that

$$\begin{aligned}
a(\mathbf{y}(\mathbf{u}),\mathbf{w})-b(\mathbf{w},r(\mathbf{u}))&=(\mathbf{f}+B\mathbf{u},\mathbf{w}) \qquad \forall\mathbf{w}\in\mathbf{Y},\\
b(\mathbf{y}(\mathbf{u}),\phi)&=0 \qquad \forall\phi\in Q.
\end{aligned} \tag{2.2}$$

For the above problem, it is well known that the following Babuška-Brezzi condition holds (see [7], for example).

Lemma 2.1 *Let* $ß = (H^1_0(\Omega))^2 \times L^2_0(\Omega)$*, and define a bilinear form* L *on* $ß \times ß$ *by* $L([\mathbf{u}, p]; [\mathbf{v}, q]) \equiv a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + b(\mathbf{u}, q)$*; then*

$$\sup_{(0,0)\neq(\mathbf{v},q)\in ß}\frac{L([\mathbf{u},p];[\mathbf{v},q])}{\|\mathbf{v}\|_1+\|q\|_0}\geq C(\|\mathbf{u}\|_1+\|p\|_0) \quad \forall(\mathbf{u},p)\in ß, \tag{2.3}$$

where C *is a constant independent of* $\mathbf{u}, \mathbf{v}, p$ *and* q*.*

Using the weak formulation, our control problem can be restated as the following (SCP):

$$\begin{aligned}
&\min_{\mathbf{u}\in\mathbf{K}\subset\mathbf{U}}\{g(\mathbf{y})+\frac{\alpha}{2}\|\mathbf{u}\|^2_{0,\Omega_{\mathbf{U}}}\},\\
&a(\mathbf{y}(\mathbf{u}),\mathbf{w})-b(\mathbf{w},r(\mathbf{u}))=(\mathbf{f}+B\mathbf{u},\mathbf{w}) \qquad \forall\mathbf{w}\in\mathbf{Y},\\
&b(\mathbf{y}(\mathbf{u}),\phi)=0 \qquad \forall\phi\in Q.
\end{aligned} \tag{2.4}$$

It is well known (see, e.g., [21]) that the control problem (SCP) has a unique solution $(\mathbf{y}, r, \mathbf{u})$ and that $(\mathbf{y}, r, \mathbf{u})$ is the solution of (SCP) if and only if there is a co-state $(\mathbf{p}, s) \in \mathbf{Y} \times Q$ such that $(\mathbf{y}, r, \mathbf{p}, s, \mathbf{u})$ satisfies the following optimality conditions (SCP-OPT):

$$\begin{aligned}
a(\mathbf{y},\mathbf{w})-b(\mathbf{w},r)&=(\mathbf{f}+B\mathbf{u},\mathbf{w}) && \forall\mathbf{w}\in\mathbf{Y},\\
b(\mathbf{y},\phi)&=0 && \forall\phi\in Q,\\
a(\mathbf{q},\mathbf{p})+b(\mathbf{q},s)&=(g'(\mathbf{y}),\mathbf{q}) && \forall\mathbf{q}\in\mathbf{Y},\\
b(\mathbf{p},\psi)&=0 && \forall\psi\in Q,\\
(\alpha\mathbf{u}+B^*\mathbf{p},\mathbf{v}-\mathbf{u})_{\mathbf{U}}&\geq 0 && \forall\mathbf{v}\in\mathbf{K}\subset\mathbf{U},
\end{aligned} \tag{2.5}$$

where B^* is the adjoint operator of B, and $(\cdot, \cdot)_{\mathbf{U}}$ is the inner product of $\mathbf{U}$.

We note that for any $\mathbf{y} \in \mathbf{Y}$, $g'(\mathbf{y})$ is in $\mathbf{H} = \mathbf{H}' = (L^2(\Omega))^2$. Therefore, it can be viewed as a function in $(L^2(\Omega))^2$ by the well-known representation theorem in a Hilbert space.

Let us consider the finite element approximation of the control problem (SCP). Here we consider only the bilinear-constant scheme of finite elements which are conforming elements for the state and the co-state equations.

Assume that Ω is a polygon with boundaries parallel to the axes and let $T^h = \{e\}$ be a uniform rectangular mesh for Ω with mesh size h. Then, associated with T^h is a finite dimensional subspace $\mathbf{Y}^h \times \mathbf{Q}^h$ of $(H_0^1(\Omega))^2 \times L_0^2(\Omega)$. Here, we choose $\mathbf{Y}^h$ as the general conforming bilinear finite element space. For Q^h, we assume that the subdivision T^h has been obtained from $T^{2h} = \{\tau\}$ by dividing each element of T^{2h} into four small congruent rectangles. Let $\tilde{Q}^h$ consist of piecewise constant function with respect to T^h and the local basis functions for $\tilde{Q}^h$ on a 2×2-patch of τ shown in FIG.1. The finite element space Q^h is defined by $\tilde{Q}^h \cap L_0^2(\Omega)$, and thus $\mathbf{Y}^h$ and Q^h for the bilinear-constant scheme of Stokes equations are described by (see, [7])

$$(2.6) \qquad \begin{cases} \mathbf{Y}^h = \left\{ \mathbf{y} \in (C(\overline{\Omega}))^2 : \ \mathbf{y}|_e \in (Q_1(e))^2, \ \mathbf{y}|_{\partial\Omega} = 0, \ e \in T^h \right\}, \\ Q^h = \left\{ q \in L_0^2(\Omega) : \ q|_\tau = \sum_{i=1}^3 \lambda_i^\tau \varphi_i^\tau, \ \sum_{\tau \in T_{2h}} \lambda_1^\tau = 0, \ \tau \in T^{2h} \right\}, \end{cases}$$

where $Q_1(e) = \left\{ y : \ y = \sum_{i=0,j=0}^1 a_{ij} x_1^i x_2^j, \ (x_1, x_2) \in e \right\}$.

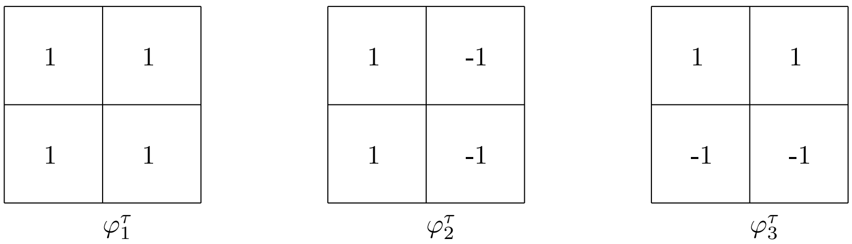

FIG. 1. Local basis functions of $\tilde{Q}^h$

Then the discrete week form of the state equations reads as

$$(2.7) \qquad \begin{aligned} a(\mathbf{y}_h, \mathbf{w}_h) - b(\mathbf{w}_h, r_h) &= (\mathbf{f} + B\mathbf{u}_h, \mathbf{w}_h) \qquad && \forall \mathbf{w}_h \in \mathbf{Y}^h \subset Y, \\ b(\mathbf{y}_h, \phi_h) &= 0 && \forall \phi_h \in Q^h \subset Q. \end{aligned}$$

It can be seen from [7] that for the bilinear-constant mixed finite element space, the Babuška-Brezzi condition follows. That is,

$$(2.8) \qquad \sup_{(\mathbf{w},\phi) \in \mathbf{Y}^h \times Q^h} \frac{(\nabla \mathbf{y}, \nabla \mathbf{w}) - (r, \mathrm{div}\mathbf{w}) + (\phi, \mathrm{div}\mathbf{y})}{\|\mathbf{w}\|_1 + \|\phi\|_0} \geq c(\|\mathbf{y}\|_1 + \|r\|_0)$$

$$\forall (\mathbf{y}, r) \in \mathbf{Y}^h \times Q^h.$$

Furthermore, for the bilinear-constant scheme, the following a priori error estimate is well known (see [7]).

Lemma 2.2 *Assume that Ω is convex, let $(\mathbf{\Psi}, \rho)$ be the solution of the following equations:*

$$\begin{aligned} a(\mathbf{\Psi}, \mathbf{w}) \pm b(\mathbf{w}, \rho) &= (\mathbf{\Phi}, \mathbf{w}) \qquad && \forall \mathbf{w} \in (H_0^1(\Omega))^2, \\ b(\mathbf{\Psi}, q) &= 0 && \forall q \in L_0^2(\Omega). \end{aligned}$$

Let $\mathbf{\Psi} \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, $\rho \in H^1(\Omega) \cap L_0^2(\Omega)$. Then,

$$\|\mathbf{\Psi} - \mathbf{\Psi}_h\|_{0,\Omega} \leq Ch^2 \{ |\mathbf{\Psi}|_{2,\Omega} + |\rho|_{1,\Omega} \},$$

where $\mathbf{\Psi}_h$ is the mixed finite element approximation of $\mathbf{\Psi}$ by the bilinear-constant scheme.

Let $\Omega^h_{\mathbf{U}}$ be a polygonal approximation to $\Omega_{\mathbf{U}}$ with a boundary $\partial\Omega^h_{\mathbf{U}}$. For simplicity, we assume that $\Omega^h_{\mathbf{U}} = \Omega_{\mathbf{U}}$ in the paper. Let $T^h_{\mathbf{U}}$ be a partitioning of $\Omega^h_{\mathbf{U}}$ into disjoint regular triangular $\tau_{\mathbf{U}}$, so that $\overline{\Omega}^h_{\mathbf{U}} = \cup_{\tau_{\mathbf{U}}\in T^h_{\mathbf{U}}}\overline{\tau}_{\mathbf{U}}$. $\overline{\tau}_{\mathbf{U}}$ and $\overline{\tau}'_{\mathbf{U}}$ have either only one common vertex or a whole face or are disjoint if $\tau_{\mathbf{U}}$ and $\tau'_{\mathbf{U}} \in T^h_{\mathbf{U}}$.

Associated with $T^h_{\mathbf{U}}$ is another finite dimensional subspace $W^h_{\mathbf{U}}$ of $L^2(\Omega^h_{\mathbf{U}})$, such that $\chi|_{\tau_{\mathbf{U}}}$ are polynomials of order m $(m \geq 0)$ $\forall\chi \in W^h_{\mathbf{U}}$ and $\tau_{\mathbf{U}} \in T^h_{\mathbf{U}}$. Here there is no requirement for the continuity. Let $\mathbf{U}^h = (W^h_{\mathbf{U}})^2$, it is easy to see that $\mathbf{U}^h \subset \mathbf{U}$.

In this paper, we will only consider the simplest finite element spaces, i.e., m=0 for $\mathbf{U}^h$. Let $h_{\tau_{\mathbf{U}}}$ denote the maximum diameter of the element $\tau_{\mathbf{U}}$ in $T^h_{\mathbf{U}}$, Let $h_{\mathbf{U}} = \max_{\tau\in T^h_{\mathbf{U}}}\{h_{\tau_{\mathbf{U}}}\}$.

Then a possible finite element approximation of (SCP) is the control problem $(\mathrm{SCP})^h$:

$$\begin{aligned} &\min_{\mathbf{u}_h\in\mathbf{K}^h\subset\mathbf{U}^h}\{g(\mathbf{y}_h) + \frac{\alpha}{2}\|\mathbf{u}_h\|^2_{0,\Omega}\} \\ (2.9)\qquad &a(\mathbf{y}_h,\mathbf{w}_h) - b(\mathbf{w}_h,r_h) = (\mathbf{f} + B\mathbf{u}_h,\mathbf{w}_h) \qquad \forall\mathbf{w}_h \in \mathbf{Y}^h \subset \mathbf{Y}, \\ &b(\mathbf{y}_h,\phi_h) = 0 \qquad \forall\phi_h \in Q^h \subset Q, \end{aligned}$$

where $\mathbf{K}^h$ is a closed convex set in $\mathbf{U}^h$, an approximation of $\mathbf{K}$. In this paper, let $\mathbf{K}^h = \mathbf{U}^h \cap \mathbf{K}$.

It follows that the control problem $(\mathrm{SCP})^h$ has a unique solution $(\mathbf{y}_h, r_h, \mathbf{u}_h)$ and that $(\mathbf{y}_h, r_h, \mathbf{u}_h) \in \mathbf{Y}^h \times Q^h \times \mathbf{U}^h$ is the solution of $(\mathrm{SCP})^h$ if and only if there is a co-state $(\mathbf{p}_h, s_h) \in \mathbf{Y}^h \times Q^h$ such that $(\mathbf{y}_h, r_h, \mathbf{p}_h, s_h, \mathbf{u}_h) \in \mathbf{Y}^h\times Q^h\times\mathbf{Y}^h\times Q^h\times\mathbf{U}^h$ satisfies the following optimality conditions $(\mathrm{SCP\text{-}OPT})^h$:

$$\begin{aligned} a(\mathbf{y}_h,\mathbf{w}_h) - b(\mathbf{w}_h,r_h) &= (\mathbf{f} + B\mathbf{u}_h,\mathbf{w}_h) && \forall\mathbf{w}_h \in \mathbf{Y}^h \subset \mathbf{Y}, \\ b(\mathbf{y}_h,\phi_h) &= 0 && \forall\phi_h \in Q^h \subset Q, \\ a(\mathbf{q}_h,\mathbf{p}_h) + b(\mathbf{q}_h,s_h) &= (g'(\mathbf{y}_h),\mathbf{q}_h) && \forall\mathbf{q}_h \in \mathbf{Y}^h \subset Y, \qquad (2.10) \\ b(\mathbf{p}_h,\psi_h) &= 0 && \forall\psi_h \in Q^h \subset Q, \\ (\alpha\mathbf{u}_h + B^*\mathbf{p}_h, \mathbf{v}_h - \mathbf{u}_h)_{\mathbf{U}} &\geq 0 && \forall\mathbf{v}_h \in \mathbf{K}^h \subset \mathbf{U}^h \subset \mathbf{U}. \end{aligned}$$

It is well known that for the problem (2.5) and its finite element approximation (2.10), the following error estimate holds:

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}} + \|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega} + \|\mathbf{p} - \mathbf{p}_h\|_{1,\Omega} \leq C(h + h_{\mathbf{U}}),$$

if $\mathbf{y}, \mathbf{p} \in (H^2(\Omega))^2$, $\mathbf{u} \in (H^1(\Omega_{\mathbf{U}}))^2$, where $(\mathbf{y}, \mathbf{p}, \mathbf{u})$ and $(\mathbf{y}_h, \mathbf{p}_h, \mathbf{u}_h)$ are the solutions of (2.5) and (2.10), respectively.

To derive a superconvergence result, we need some stable interpolators from $\mathbf{Y}$ to $\mathbf{Y}^h$, and from Q to Q^h, respectively. They are given in the following lemmas, which are important in deriving superconvergence results. The following lemmas are well known, and their proofs can be found in [3] and [7].

Lemma 2.3 *Let $\bar{\pi}_h$ be such that*

$$\bar{\pi}_h p|_\tau = \int_\tau p/|\tau| \qquad \forall p \in L^2(\tau), \tau \in T^h,$$

where $|\tau|$ is the measure of τ. Then for $1 \leq q \leq \infty$ and $p \in W^{1,q}(\tau)$,

$$\|p - \bar{\pi}_h p\|_{0,q,\tau} \leq Ch_\tau |p|_{1,q,\tau}.$$

Lemma 2.4 *Let i_h be the standard piecewise linear Lagrange interpolation operator. Then for $1 \leq q \leq \infty$ and $v \in W^{2,q}(\tau)$,*

$$|v - i_h v|_{m,q,\tau} \leq C h_\tau^{2-m} |v|_{2,q,\tau}, \qquad m = 0, 1.$$

3. Superconvergence analysis and recovery for the control u

In this section, we shall establish the superconvergence result for the control $\mathbf{u}$. Let

$$\Omega_{\mathbf{U}}^+ = \{\cup_{\tau_{\mathbf{U}}} : \tau_{\mathbf{U}} \subset \Omega_{\mathbf{U}}, \mathbf{u}|_{\tau_{\mathbf{U}}} > 0\},$$

$$\Omega_{\mathbf{U}}^0 = \{\cup_{\tau_{\mathbf{U}}} : \tau_{\mathbf{U}} \subset \Omega_{\mathbf{U}}, \mathbf{u}|_{\tau_{\mathbf{U}}} = 0\},$$

$$\Omega_{\mathbf{U}}^b = \Omega_{\mathbf{U}} \backslash (\Omega_{\mathbf{U}}^+ \cup \Omega_{\mathbf{U}}^0).$$

We will assume that $\mathbf{u}$ and $T_{\mathbf{U}}^h$ are regular such that $meas(\Omega_{\mathbf{U}}^b) \leq C h_{\mathbf{U}}$.

Lemma 3.1 *Let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of (2.5) and (2.10), respectively. Let $\mathbf{u}_I \equiv \bar{\pi}_h \mathbf{u} \in \mathbf{K}^h$ be the L^2-projection of $\mathbf{u}$. Assume that $\mathbf{u} \in (W^{1,\infty}(\Omega_{\mathbf{U}}))^2$, $\mathbf{p} \in (W^{1,\infty}(\Omega))^2$, and g' is Lipschitz continuous. Then,*

$$\|\mathbf{u}_h - \mathbf{u}_I\|_{0,\Omega_{\mathbf{U}}} \leq C(h_{\mathbf{U}}^{1.5} + h^2). \tag{3.1}$$

Proof. Noting that $\mathbf{u}_h, \mathbf{u}_I \in \mathbf{K}^h \subset \mathbf{K}$, it follows from (2.5) and(2.10) that

$$(\alpha \mathbf{u} + B^* \mathbf{p}, \mathbf{u} - \mathbf{u}_h) \leq 0,$$

and

$$(\alpha \mathbf{u}_h + B^* \mathbf{p}_h, \mathbf{u}_h - \mathbf{u}_I) \leq 0.$$

Hence,

$$\begin{aligned}
& \alpha \|\mathbf{u}_h - \mathbf{u}_I\|_{0,\Omega_{\mathbf{U}}}^2 = \alpha(\mathbf{u}_h - \mathbf{u}_I, \mathbf{u}_h - \mathbf{u}_I)_{\mathbf{U}} \\
\leq\; & -(B^* \mathbf{p}_h, \mathbf{u}_h - \mathbf{u}_I)_{\mathbf{U}} - (\alpha \mathbf{u}_I, \mathbf{u}_h - \mathbf{u}_I)_{\mathbf{U}} \\
=\; & (B^* \mathbf{p}, \mathbf{u} - \mathbf{u}_h)_{\mathbf{U}} + (B^* \mathbf{p}, \mathbf{u}_I - \mathbf{u})_{\mathbf{U}} + (\alpha \mathbf{u}_I, \mathbf{u}_I - \mathbf{u}_h)_{\mathbf{U}} \\
& + (B^*(\mathbf{p} - \mathbf{p}_h), \mathbf{u}_h - \mathbf{u}_I)_{\mathbf{U}} \\
\leq\; & -\alpha(\mathbf{u}, \mathbf{u} - \mathbf{u}_h)_{\mathbf{U}} + (B^* \mathbf{p}, \mathbf{u}_I - \mathbf{u})_{\mathbf{U}} + \alpha(\mathbf{u}_I, \mathbf{u}_I - \mathbf{u}_h)_{\mathbf{U}} \\
& + (B^*(\mathbf{p} - \mathbf{p}_h), \mathbf{u}_h - \mathbf{u}_I)_{\mathbf{U}} \\
=\; & \alpha(\mathbf{u}_I - \mathbf{u}, \mathbf{u}_I - \mathbf{u}_h)_{\mathbf{U}} + (\alpha \mathbf{u} + B^* \mathbf{p}, \mathbf{u}_I - \mathbf{u})_{\mathbf{U}} \\
& + (B^*(\mathbf{p} - \mathbf{p}(\mathbf{u}_h)), \mathbf{u} - \mathbf{u}_I)_{\mathbf{U}} + (B^*(\mathbf{p} - \mathbf{p}(\mathbf{u}_h)), \mathbf{u}_h - \mathbf{u})_{\mathbf{U}} \\
& + (B^*(\mathbf{p}(\mathbf{u}_h) - \mathbf{p}_h), \mathbf{u}_h - \mathbf{u}_I)_{\mathbf{U}},
\end{aligned} \tag{3.2}$$

where $\mathbf{p}(\mathbf{u}_h)$ is the solution of the auxiliary equation:

$$\begin{aligned}
a(\mathbf{y}(\mathbf{u}_h), \mathbf{w}) - b(\mathbf{w}, r(\mathbf{u}_h)) &= (\mathbf{f} + B\mathbf{u}_h, \mathbf{w}) && \forall \mathbf{w} \in \mathbf{Y}, \\
b(\mathbf{y}(\mathbf{u}_h), \phi) &= 0 && \forall \phi \in Q, \\
a(\mathbf{q}, \mathbf{p}(\mathbf{u}_h)) + b(\mathbf{q}, s(\mathbf{u}_h)) &= (g'(\mathbf{y}(\mathbf{u}_h)), \mathbf{q}) && \forall \mathbf{q} \in \mathbf{Y}, \\
b(\mathbf{p}(\mathbf{u}_h), \psi) &= 0 && \forall \psi \in Q.
\end{aligned} \tag{3.3}$$

It follows from the definition of $\mathbf{u}_I$ that

$$(\mathbf{u}_I-\mathbf{u},\mathbf{u}_I-\mathbf{u}_h)_{\mathbf{U}}=\sum_{\tau_{\mathbf{U}}}(\mathbf{u}_I-\mathbf{u}_h)\int_{\tau_{\mathbf{U}}}(\bar{\pi}_h\mathbf{u}-\mathbf{u})$$

$$=\sum_{\tau_{\mathbf{U}}}(\mathbf{u}_I-\mathbf{u}_h)\int_{\tau_{\mathbf{U}}}(\frac{\int_{\tau_{\mathbf{U}}}\mathbf{u}}{|\tau_{\mathbf{U}}|}-\mathbf{u})=\sum_{\tau_{\mathbf{U}}}(\mathbf{u}_I-\mathbf{u}_h)(\int_{\tau_{\mathbf{U}}}\frac{\int_{\tau_{\mathbf{U}}}\mathbf{u}}{|\tau_{\mathbf{U}}|}-\int_{\tau_{\mathbf{U}}}\mathbf{u}) \tag{3.4}$$

$$=\sum_{\tau_{\mathbf{U}}}(\mathbf{u}_I-\mathbf{u}_h)(\int_{\tau_{\mathbf{U}}}\mathbf{u}-\int_{\tau_{\mathbf{U}}}\mathbf{u})=0.$$

Lemma 2.3 implies that

$$(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h)),\mathbf{u}-\mathbf{u}_I)_{\mathbf{U}}$$

$$=\sum_{\tau_{\mathbf{U}}}\int_{\tau_{\mathbf{U}}}\Big(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h))-\bar{\pi}_h(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h)))\Big)(\mathbf{u}-\bar{\pi}_h\mathbf{u})$$

$$\leq\sum_{\tau_{\mathbf{U}}}\|B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h))-\bar{\pi}_h(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h)))\|_{L^2(\Omega_{\mathbf{U}})}\|\mathbf{u}-\bar{\pi}_h\mathbf{u}\|_{L^2(\Omega_{\mathbf{U}})}$$

$$\leq C\sum_{\tau_{\mathbf{U}}}h^2_{\tau_{\mathbf{U}}}|B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h))|_{1,\tau_{\mathbf{U}}}|\mathbf{u}|_{1,\tau_{\mathbf{U}}} \tag{3.5}$$

$$\leq Ch^2_{\mathbf{U}}\|\mathbf{p}-\mathbf{p}(\mathbf{u}_h)\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega_{\mathbf{U}}}.$$

It follows from (2.5), (3.3) and B-B condition (2.3) that

$$\begin{aligned}\|\mathbf{p}-\mathbf{p}(\mathbf{u}_h)\|_{1,\Omega}&\leq C\|\mathbf{y}-\mathbf{y}(\mathbf{u}_h)\|_{0,\Omega}\leq C\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}\\&\leq C\|\mathbf{u}-\mathbf{u}_I\|_{0,\Omega_{\mathbf{U}}}+C\|\mathbf{u}_I-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}\\&\leq Ch_{\mathbf{U}}|\mathbf{u}|_{1,\Omega_{\mathbf{U}}}+C\|\mathbf{u}_I-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}},\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}(B^*(\mathbf{p}-\mathbf{p}(\mathbf{u}_h)),\mathbf{u}_h-\mathbf{u})_{\mathbf{U}}&=(\mathbf{p}-\mathbf{p}(\mathbf{u}_h),B(\mathbf{u}_h-\mathbf{u}))\\&=(g'(\mathbf{y})-g'(\mathbf{y}(\mathbf{u}_h)),\mathbf{y}(\mathbf{u}_h)-\mathbf{y})\leq 0.\end{aligned} \tag{3.7}$$

The Schwarz inequality then yields

$$\begin{aligned}(B^*(\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h),\mathbf{u}_h-\mathbf{u}_I)_{\mathbf{U}}&\leq C\|B^*(\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h)\|_{0,\Omega_{\mathbf{U}}}\|\mathbf{u}_h-\mathbf{u}_I\|_{0,\Omega_{\mathbf{U}}}\\&\leq C\|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h\|^2_{0,\Omega}+C\delta\|\mathbf{u}_h-\mathbf{u}_I\|^2_{0,\Omega_{\mathbf{U}}},\end{aligned} \tag{3.8}$$

where δ is an arbitrary small positive constant. Then, by (3.2)-(3.8),

$$\begin{aligned}\alpha\|\mathbf{u}_h-\mathbf{u}_I\|^2_{0,\Omega_{\mathbf{U}}}&\leq(\alpha\mathbf{u}+B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}}+Ch^2_{\mathbf{U}}\|\mathbf{u}\|_{1,\Omega_{\mathbf{U}}}(h_{\mathbf{U}}\|\mathbf{u}\|_{1,\Omega_{\mathbf{U}}}\\&\quad+\|\mathbf{u}_I-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}})+C\|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h\|^2_{0,\Omega}+C\delta\|\mathbf{u}_h-\mathbf{u}_I\|^2_{0,\Omega_{\mathbf{U}}}\\&\leq(\alpha\mathbf{u}+B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}}+Ch^3_{\mathbf{U}}\|\mathbf{u}\|^2_{1,\Omega_{\mathbf{U}}}+C\|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h\|^2_{0,\Omega}\\&\quad+C\delta\|\mathbf{u}_h-\mathbf{u}_I\|^2_{0,\Omega_{\mathbf{U}}}.\end{aligned}$$

Hence,

$$\|\mathbf{u}_h-\mathbf{u}_I\|^2_{0,\Omega_{\mathbf{U}}}\leq C(\alpha\mathbf{u}+B^*\mathbf{p},\mathbf{u}_I-\mathbf{u})_{\mathbf{U}}+Ch^3_{\mathbf{U}}+C\|\mathbf{p}(\mathbf{u}_h)-\mathbf{p}_h\|^2_{0,\Omega}. \tag{3.9}$$

Let $\mathbf{p}(\mathbf{y}_h)\in(H^1_0(\Omega))^2$ be the solution of the equation:

$$\begin{aligned}a(\mathbf{q},\mathbf{p}(\mathbf{y}_h))+b(\mathbf{q},s(\mathbf{y}_h))&=(g'(\mathbf{y}_h),\mathbf{q})\quad\forall\mathbf{q}\in\mathbf{Y},\\b(\mathbf{p}(\mathbf{y}_h),\psi)&=0\quad\forall\psi\in Q.\end{aligned}$$

Then,

$$\|\mathbf{p}(\mathbf{y}_h)-\mathbf{p}(\mathbf{u}_h)\|_{0,\Omega}\leq C\|g'(\mathbf{y}_h)-g'(\mathbf{y}(\mathbf{u}_h))\|_{0,\Omega}\leq C\|\mathbf{y}_h-\mathbf{y}(\mathbf{u}_h)\|_{0,\Omega}. \tag{3.10}$$

Note that $\mathbf{y}_h$ and $\mathbf{p}_h$ are the standard finite element approximations of $\mathbf{y}(\mathbf{u}_h)$ and $\mathbf{p}(\mathbf{y}_h)$, respectively. Using Lemma 2.2, we obtain

$$\|\mathbf{y}_h - \mathbf{y}(\mathbf{u}_h)\|_{0,\Omega} \leq Ch^2\{|\mathbf{y}(\mathbf{u}_h)|_{2,\Omega} + |r(\mathbf{u}_h)|_{1,\Omega}\} \leq Ch^2 \tag{3.11}$$

and

$$\|\mathbf{p}_h - \mathbf{p}(\mathbf{y}_h)\|_{0,\Omega} \leq Ch^2\{|\mathbf{p}(\mathbf{y}_h)|_{2,\Omega} + |s(\mathbf{y}_h)|_{1,\Omega}\} \leq Ch^2. \tag{3.12}$$

Therefore, it follows from (3.10)-(3.12) that

$$\|\mathbf{p}(\mathbf{u}_h) - \mathbf{p}_h\|_{0,\Omega} \leq \|\mathbf{p}(\mathbf{u}_h) - \mathbf{p}(\mathbf{y}_h)\|_{0,\Omega} + \|\mathbf{p}(\mathbf{y}_h) - \mathbf{p}_h\|_{0,\Omega} \leq Ch^2. \tag{3.13}$$

Note that

$$\begin{aligned}(\alpha\mathbf{u} + B^*\mathbf{p}, \mathbf{u}_I - \mathbf{u})_{\mathbf{U}} &= \int_{\Omega^+_{\mathbf{U}}} (\alpha\mathbf{u} + B^*\mathbf{p})(\mathbf{u}_I - \mathbf{u}) + \int_{\Omega^0_{\mathbf{U}}} (\alpha\mathbf{u} + B^*\mathbf{p})(\mathbf{u}_I - \mathbf{u}) \\ &\quad + \int_{\Omega^b_{\mathbf{U}}} (\alpha\mathbf{u} + B^*\mathbf{p})(\mathbf{u}_I - \mathbf{u}),\end{aligned}$$

and

$$(\alpha\mathbf{u} + B^*\mathbf{p})|_{\Omega^+_{\mathbf{U}}} = 0, \quad (\mathbf{u}_I - \mathbf{u})|_{\Omega^0_{\mathbf{U}}} = 0.$$

Then,

$$\begin{aligned}(\alpha\mathbf{u} + B^*\mathbf{p}, \mathbf{u}_I - \mathbf{u})_{\mathbf{U}} &= \int_{\Omega^b_{\mathbf{U}}} (\alpha\mathbf{u} + B^*\mathbf{p})(\mathbf{u}_I - \mathbf{u}) \\ &= \sum_{\tau_{\mathbf{U}} \subset \Omega^b_{\mathbf{U}}} \int_{\tau_{\mathbf{U}}} \Big(\alpha\mathbf{u} + B^*\mathbf{p} - \bar{\pi}_h(\alpha\mathbf{u} + B^*\mathbf{p})\Big)(\bar{\pi}_h\mathbf{u} - \mathbf{u}) \\ &\leq C \sum_{\tau_{\mathbf{U}} \subset \Omega^b_{\mathbf{U}}} h^2_{\tau_{\mathbf{U}}} |\alpha\mathbf{u} + B^*\mathbf{p}|_{1,\tau_{\mathbf{U}}} |\mathbf{u}|_{1,\tau_{\mathbf{U}}} \\ &\leq Ch^2_{\mathbf{U}}(\|\mathbf{u}\|^2_{1,\infty,\Omega_{\mathbf{U}}} + \|\mathbf{p}\|^2_{1,\infty,\Omega}) meas(\Omega^b_{\mathbf{U}}) \leq Ch^3_{\mathbf{U}}.\end{aligned} \tag{3.14}$$

Therefore, (3.1) follows from (3.9), (3.13) and (3.14). □

Lemma 3.1 shows that the error order of $\|\mathbf{u}_h - \mathbf{u}_I\|_{0,\Omega_{\mathbf{U}}}$ is a half order higher than the optimal error order for the piecewise constant finite element space. This property is called superclose (see [32]). In order to provide the global superconvergence for the control $\mathbf{u}$, we construct the recovery operator R_h on triangular meshes. Let $R_h v$ be a continuous piecewise linear function. The values of $R_h v$ on the nodes are defined by least-squares argument on an element patches surrounding the nodes, similar as ZZ patch recovery (see, e.g., [36], [37]), as follows. Let z be a node, $\omega_z = \cup_{\tau_{\mathbf{U}} \in T^h_{\mathbf{U}}, z \in \bar{\tau}_U} \tau_U$, V_z be the linear function space on ω_z. Set $R_h v(z) = \sigma_z(z)$ where

$$E(\sigma_z) = \min_{w \in V_z} E(w),$$

and

$$E(w) = \sum_{\tau_U \subset \omega_z} (\int_{\tau_U} w - \int_{\tau_U} v)^2.$$

When $z \in \partial\Omega$, we should add a few extra neighboring elements to ω_z such that ω_z contains more than three elements. For the regular mesh and suitable choice of ω_z, we can conclude that for any $v \in L^2(\Omega)$, $R_h v$ exists. Moreover, for any domain $D \in \Omega$, $R_h v = v$ on D if v is a linear function on $\hat{D}$, where $\hat{D} = \{\cup_{\tau_{\mathbf{U}}} : \bar{\tau}_{\mathbf{U}} \cap \bar{D} \neq 0\}$.

Remark 3.1 *For a triangular partition, $R_h v(z)$, $z = (x_0, y_0)$, can be calculated as follows. Let $\omega_z = \sum_{i=1}^m \tau^i_{\mathbf{U}}$, the three nodes of $\tau^i_{\mathbf{U}}$ are (x_0, y_0), (x_i, y_i) and (x_{i+1}, y_{i+1}), $i = 1, 2, \cdots, m-1$, the three nodes of $\tau^m_{\mathbf{U}}$ are (x_0, y_0), (x_m, y_m) and*

(x_1, y_1). *Then,* $R_h v(z) = a + bx_0 + dy_0$*, where* (a, b, d) *is the solution of the following linear system:*

$$\begin{bmatrix} \sum_{i=1}^{m}|k_i|^2 & \frac{1}{3}\sum_{i=1}^{m}|k_i|^2 \times(x_0+x_i+x_{i+1}) & \frac{1}{3}\sum_{i=1}^{m}|k_i|^2 \times(y_0+y_i+y_{i+1}) \\ \frac{1}{3}\sum_{i=1}^{m}|k_i|^2 \times(x_0+x_i+x_{i+1}) & \frac{1}{9}\sum_{i=1}^{m}|k_i|^2 \times(x_0+x_i+x_{i+1})^2 & \frac{1}{9}\sum_{i=1}^{m}|k_i|^2 \times(x_0+x_i+x_{i+1}) \times(y_0+y_i+y_{i+1}) \\ \frac{1}{3}\sum_{i=1}^{m}|k_i|^2 \times(y_0+y_i+y_{i+1}) & \frac{1}{9}\sum_{i=1}^{m}|k_i|^2 \times(x_0+x_i+x_{i+1}) \times(y_0+y_i+y_{i+1}) & \frac{1}{9}\sum_{i=1}^{m}|k_i|^2 \times(y_0+y_i+y_{i+1})^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m}|k_i|\int_{\tau_U^i} v \\ \frac{1}{3}\sum_{i=1}^{m}|k_i|\int_{\tau_U^i} v \times(x_0+x_i +x_{i+1}) \\ \frac{1}{3}\sum_{i=1}^{m}|k_i|\int_{\tau_U^i} v \times(y_0+y_i +y_{i+1}) \end{bmatrix},$$

where $|k_i|$ *is the area of the element* $\tau_{\mathbf{U}}^i$*, and* $(x_{m+1}, y_{m+1}) = (x_1, y_1)$.

Lemma 3.2 *Assume that* $\mathbf{u} \in (W^{1,\infty}(\Omega_{\mathbf{U}}))^2$ *and* $u_i|_{\Omega_j} \in H^2(\Omega_j)$*, where* u_i *is the component of* $\mathbf{u}$*,* $\bar{\Omega}_{\mathbf{U}} = \cup_j \bar{\Omega}_j$*, and* $\bar{\Omega}_j \cap \bar{\Omega}_k$*,* $j \neq k$*, is the free boundary of* u_i*. Then,*

$$\|R_h\mathbf{u} - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}} \leq Ch_{\mathbf{U}}^{1.5}, \tag{3.15}$$

where R_h *is the recovery operator defined above.*

Proof. Note that $\mathbf{u} \in (W^{1,\infty}(\Omega_{\mathbf{U}}))^2$, and $\mathbf{u} \in (H^2(\Omega_{\mathbf{U}}^+ \cup \Omega_{\mathbf{U}}^0))^2$. Let

$$\Omega_{\mathbf{U}}^{++} = \{\cup_{\tau_{\mathbf{U}}} : \omega_z \subset \Omega_{\mathbf{U}}^+, \forall z \in \bar{\tau}_{\mathbf{U}}\},$$

$$\Omega_{\mathbf{U}}^{00} = \{\cup_{\tau_{\mathbf{U}}} : \omega_z \subset \Omega_{\mathbf{U}}^0, \forall z \in \bar{\tau}_{\mathbf{U}}\}.$$

and

$$\Omega_{\mathbf{U}}^{bb} = \Omega_{\mathbf{U}} \setminus (\Omega_{\mathbf{U}}^{++} \cup \Omega_{\mathbf{U}}^{00})$$

Then,

$$R_h\mathbf{u}(x) = \mathbf{u}(x) = 0 \quad \forall x \in \Omega_{\mathbf{U}}^{00}. \tag{3.16}$$

Note that $\mathbf{u} \in (H^2(\Omega_{\mathbf{U}}^+))^2$. It can be proved by the standard technique (see, e.g., [3]) that

$$\|R_h\mathbf{u} - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}^{++}} \leq Ch_{\mathbf{U}}^2\|\mathbf{u}\|_{2,\Omega_{\mathbf{U}}^+}. \tag{3.17}$$

It follows from $\mathbf{u} \in (W^{1,\infty}(\Omega_{\mathbf{U}}))^2$ that

$$\|R_h\mathbf{u} - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}^{bb}}^2 \leq Ch_{\mathbf{U}}^2\|\mathbf{u}\|_{1,\Omega_{\mathbf{U}}^{bb}}^2 \leq Ch_{\mathbf{U}}^2\|\mathbf{u}\|_{1,\infty,\Omega_{\mathbf{U}}}^2 meas(\Omega_{\mathbf{U}}^{bb}),$$

Note that $meas(\Omega_{\mathbf{U}}^b) = O(h_{\mathbf{U}})$ and hence $meas(\Omega_{\mathbf{U}}^{bb}) = O(h_{\mathbf{U}})$. We have that

$$\|R_h\mathbf{u} - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}^{bb}} \leq Ch_{\mathbf{U}}^{1.5}. \tag{3.18}$$

Therefore, it follows from(3.16)-(3.18) that

$$\begin{aligned} \|R_h\mathbf{u} - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}}^2 &= \|R_h\mathbf{u} - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}^{++}}^2 + \|R_h\mathbf{u} - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}^{00}}^2 + \|R_h\mathbf{u} - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}^{bb}}^2 \\ &\leq Ch_{\mathbf{U}}^4 + 0 + Ch_{\mathbf{U}}^3 \leq Ch_{\mathbf{U}}^3. \end{aligned}$$

This proves (3.15). □

Remark 3.2 *The assumption of Lemma 3.2 is reasonable. We can't assume that* $\mathbf{u} \in (H^2(\Omega_{\mathbf{U}}))^2$ *on the whole domain, because the derivative of* $\mathbf{u}$ *is discontinuous over the free boundary. Hence, we only can assume that* $\mathbf{u} \in (W^{1,\infty}(\Omega_{\mathbf{U}}))^2$ *on the whole domain, and* $u_i|_{\Omega_j} \in H^2(\Omega_j)$ *on piecewise subdomains.*
Theorem 3.1 *Suppose all conditions of Lemma 3.1 and 3.2 are valid. Then,*

$$\|R_h\mathbf{u}_h - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}} \leq C(h_{\mathbf{U}}^{1.5} + h^2). \tag{3.19}$$

Proof. Let $\mathbf{u}_I \equiv \bar{\pi}_h\mathbf{u}$, which is defined in Lemma 2.3. Then,

$$\begin{aligned} \|R_h\mathbf{u}_h - \mathbf{u}\|_{0,\Omega_{\mathbf{U}}} \leq & \|\mathbf{u} - R_h\mathbf{u}\|_{0,\Omega_{\mathbf{U}}} + \|R_h\mathbf{u} - R_h\mathbf{u}_I\|_{0,\Omega_{\mathbf{U}}} \\ & + \|R_h\mathbf{u}_I - R_h\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}. \end{aligned} \tag{3.20}$$

It follows from Lemma 3.2 that

$$\|\mathbf{u} - R_h\mathbf{u}\|_{0,\Omega_{\mathbf{U}}} \leq Ch_{\mathbf{U}}^{1.5}. \tag{3.21}$$

Noting the definition of R_h, we have that

$$R_h\mathbf{u} = R_h\mathbf{u}_I, \tag{3.22}$$

and

$$\|R_h\mathbf{u}_I - R_h\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}} \leq C\|\mathbf{u}_I - \mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}. \tag{3.23}$$

It has been proved in Lemma 3.1 that

$$\|\mathbf{u}_I - \mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}} \leq C(h_{\mathbf{U}}^{\frac{3}{2}} + h^2). \tag{3.24}$$

Therefore, (3.19) follows from (3.20)-(3.24). □

Corollary 3.1 *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solutions of (2.5) and (2.10), respectively. Suppose all conditions of Lemma 3.1 and 3.2 are valid. Then,*

$$\|\mathbf{u} - \mathbf{u}_h\|_{-1,\Omega_{\mathbf{U}}} \leq C(h_{\mathbf{U}}^{\frac{3}{2}} + h^2). \tag{3.25}$$

Proof. For any function $\mathbf{\Phi} \in (H^1(\Omega_{\mathbf{U}}))^2$, let $\mathbf{\Phi}_I = \bar{\pi}_h\mathbf{\Phi} \in \mathbf{U}^h$ be the L^2-projection of $\mathbf{\Phi}$, such that

$$\mathbf{\Phi}_I|_{\tau_U} = \frac{\int_{\tau_{\mathbf{U}}} \mathbf{\Phi}}{|\tau_{\mathbf{U}}|}.$$

Then,

$$(\mathbf{u} - \mathbf{u}_h, \mathbf{\Phi})_{\mathbf{U}} = (\mathbf{u} - \mathbf{u}_h, \mathbf{\Phi} - \mathbf{\Phi}_I)_{\mathbf{U}} + (\mathbf{u} - \mathbf{u}_h, \mathbf{\Phi}_I)_{\mathbf{U}}. \tag{3.26}$$

Note that

$$\begin{aligned} (\mathbf{u} - \mathbf{u}_h, \mathbf{\Phi} - \mathbf{\Phi}_I)_{\mathbf{U}} & \leq \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}\|\mathbf{\Phi} - \mathbf{\Phi}_I\|_{0,\Omega_{\mathbf{U}}} \\ & \leq C(h_{\mathbf{U}} + h)h_{\mathbf{U}}\|\mathbf{\Phi}\|_{1,\Omega_{\mathbf{U}}} \leq C(h^2 + h_{\mathbf{U}}^2)\|\mathbf{\Phi}\|_{1,\Omega_{\mathbf{U}}}, \end{aligned} \tag{3.27}$$

and it follows from Lemma 3.1 that

$$\begin{aligned} (\mathbf{u} - \mathbf{u}_h, \mathbf{\Phi}_I)_{\mathbf{U}} & = (\mathbf{u}_I - \mathbf{u}_h, \mathbf{\Phi})_{\mathbf{U}} \\ & \leq \|\mathbf{u}_I - \mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}\|\mathbf{\Phi}\|_{0,\Omega_{\mathbf{U}}} \leq C(h_{\mathbf{U}}^{\frac{3}{2}} + h^2)\|\mathbf{\Phi}\|_{1,\Omega_{\mathbf{U}}}. \end{aligned} \tag{3.28}$$

Therefore, (3.26)-(3.28) imply

$$\|\mathbf{u} - \mathbf{u}_h\|_{-1,\Omega_{\mathbf{U}}} = \sup_{\mathbf{\Phi} \in H^1(\Omega_{\mathbf{U}})} \frac{(\mathbf{u} - \mathbf{u}_h, \mathbf{\Phi})_{\mathbf{U}}}{\|\mathbf{\Phi}\|_{1,\Omega_{\mathbf{U}}}} \leq C(h_{\mathbf{U}}^{\frac{3}{2}} + h^2).$$

This proves (3.25). □

4. Superconvergence analysis for the state y and the co-state p

In this section, we shall consider the superconvergence analysis for the state $\mathbf{y}$ and the co-state $\mathbf{p}$. In the following discussion we always assume that $\tau = \cup_{i=1}^{4} e_i \in T_{2h}$ with $e_i \in T_h (1 \le i \le 4)$ (see FIG.2), and $e \in T_h$.

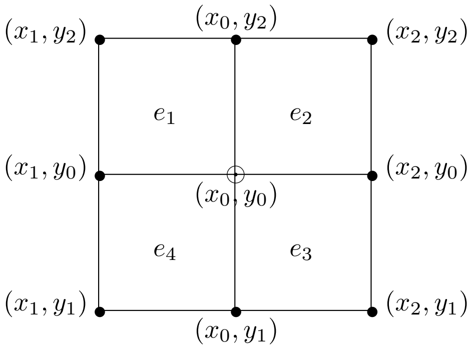

FIG. 2. recovery of node z

Let $i_h((C(\bar{\Omega}))^2 \to \mathbf{Y}^h)$ be the standard Lagrange interpolation operator for the state $\mathbf{y}$ and the co-state $\mathbf{p}$. For r and s, we use the local L^2-projection $\bar{\pi}_h$ defined by Lemma 2.3, such that

$$\bar{\pi}_h r|_e = \frac{1}{|e|} \int_e r \qquad \forall e \in T_h,$$

and then define the interpolation operator π_h on τ by

$$\pi_h r|_{e_i} = \begin{cases} \bar{\pi}_h r - \frac{1}{4}\alpha_\tau & i = 1, 4, \\ \bar{\pi}_h r + \frac{1}{4}\alpha_\tau & i = 2, 3, \end{cases} \tag{4.1}$$

where $\alpha_\tau = r_1^\tau - r_2^\tau - r_3^\tau + r_4^\tau$, and $r_i^\tau = \bar{\pi}_h r|_{e_i} (i = 1, 2, 3, 4)$. A direct calculation shows that

$$\pi_h r|_\tau = \frac{1}{4}\left[\left(\sum_{i=1}^{4} r_i^\tau\right)\varphi_1^\tau + (r_1^\tau - r_2^\tau + r_3^\tau - r_4^\tau)\varphi_2^\tau + (r_1^\tau + r_2^\tau - r_3^\tau - r_4^\tau)\varphi_3^\tau\right],$$

which implies that $\pi_h r \in Q^h$ for $r \in L_0^2(\Omega)$.

Using the integral identity technique, we introduce the following lemmas, which are important in deriving superconvergence results. These lemmas can be found in [29].

Lemma 4.1 *Let* $\mathbf{y} \in (H^3(\Omega))^2$. *Then*

$$a(\mathbf{y} - i_h\mathbf{y}, \mathbf{w}) = (\nabla(\mathbf{y} - i_h\mathbf{y}), \nabla\mathbf{w}) \le Ch^2\|\mathbf{y}\|_{3,\Omega}\|\mathbf{w}\|_{1,\Omega} \quad \forall \mathbf{w} \in \mathbf{Y}^h.$$

Lemma 4.2 *Let* $\mathbf{y} \in (H^3(\Omega))^2$. *Then*

$$b(\mathbf{y} - i_h\mathbf{y}, \phi) = (\phi, div(\mathbf{y} - i_h\mathbf{y})) \le Ch^2\|\mathbf{y}\|_{3,\Omega}\|\phi\|_{0,\Omega} \quad \forall \phi \in Q^h.$$

Lemma 4.3 *Let* $r \in H^2(\Omega)$, *Then*

$$b(\mathbf{w}, r - \pi_h r) = (r - \pi_h r, div\mathbf{w}) \le Ch^2\|r\|_{2,\Omega}\|\mathbf{w}\|_{1,\Omega} \quad \forall \mathbf{w} \in \mathbf{Y}^h.$$

Then, we shall consider the superconvergence for the state $\mathbf{y}$ *and the co-state* $\mathbf{p}$.

Lemma 4.4 *Suppose that T^h is a uniform rectangular partitioning. Let $(\mathbf{y}, r, \mathbf{p}, s,)$ be the solution of the equations (2.5) with $\mathbf{y}, \mathbf{p} \in (H^3(\Omega))^2$ and $r, s \in H^2(\Omega)$, and $(\mathbf{y}_h, r_h, \mathbf{p}_h, s_h)$ be the solution of the equations (2.10). Assume that the conditions in Lemma 3.1, and Lemma 4.1-4.3 are valid. Then*

$$\|\mathbf{y}_h - i_h\mathbf{y}\|_{1,\Omega} + \|r_h - \pi_h r\|_{0,\Omega} \le C(h_{\mathbf{U}}^{1.5} + h^2), \tag{4.2}$$

$$\|\mathbf{p}_h - i_h\mathbf{p}\|_{1,\Omega} + \|s_h - \pi_h s\|_{0,\Omega} \le C(h_{\mathbf{U}}^{1.5} + h^2). \tag{4.3}$$

Proof. It follows from replacing $\mathbf{y}$ by $\mathbf{y}_h - i_h\mathbf{y}$ and r by $r_h - \pi_h r$ in (2.8) that we have the following result:

$$\begin{aligned} &\|\mathbf{y}_h - i_h\mathbf{y}\|_{1,\Omega} + \|r_h - \pi_h r\|_{0,\Omega} \\ \le\ & C \sup_{(\mathbf{w},\phi)\in\mathbf{Y}^h\times Q^h} \frac{a(\mathbf{y}_h - i_h\mathbf{y}, \mathbf{w}) - b(\mathbf{w}, r_h - \pi_h r) + b(\mathbf{y}_h - i_h\mathbf{y}, \phi)}{\|\mathbf{w}\|_{1,\Omega} + \|\phi\|_{0,\Omega}}. \end{aligned}$$

Then, there exists $(\mathbf{w}, \phi) \in \mathbf{Y}^h \times Q^h$ such that

$$\begin{aligned} & c(\|\mathbf{y}_h - i_h\mathbf{y}\|_{1,\Omega} + \|r_h - \pi_h r\|_{0,\Omega})(\|\mathbf{w}\|_{1,\Omega} + \|\phi\|_{0,\Omega}) \\ \le\ & a(\mathbf{y}_h - i_h\mathbf{y}, \mathbf{w}) - b(\mathbf{w}, r_h - \pi_h r) + b(\mathbf{y}_h - i_h\mathbf{y}, \phi). \end{aligned} \tag{4.4}$$

Using Lemma 4.1, Lemma 4.2 and Lemma 4.3, we obtain

$$\begin{aligned} & a(\mathbf{y}_h - i_h\mathbf{y}, \mathbf{w}) - b(\mathbf{w}, r_h - \pi_h r) + b(\mathbf{y}_h - i_h\mathbf{y}, \phi) \\ =\ & a(\mathbf{y}_h - \mathbf{y} + \mathbf{y} - i_h\mathbf{y}, \mathbf{w}) - b(\mathbf{w}, r_h - r + r - \pi_h r) \\ & + b(\mathbf{y}_h - \mathbf{y} + \mathbf{y} - i_h\mathbf{y}, \phi) \\ =\ & a(\mathbf{y} - i_h\mathbf{y}, \mathbf{w}) - b(\mathbf{w}, r - \pi_h r) + b(\mathbf{y} - i_h\mathbf{y}, \phi) \\ & + a(\mathbf{y}_h - \mathbf{y}, \mathbf{w}) - b(\mathbf{w}, r_h - r) + b(\mathbf{y}_h - \mathbf{y}, \phi) \\ \le\ & Ch^2\|\mathbf{y}\|_{3,\Omega}\|\mathbf{w}\|_{1,\Omega} + Ch^2\|r\|_{2,\Omega}\|\mathbf{w}\|_{1,\Omega} \\ & + Ch^2\|\mathbf{y}\|_{3,\Omega}\|\phi\|_{0,\Omega} + (B(\mathbf{u}_h - \mathbf{u}), \mathbf{w}). \end{aligned} \tag{4.5}$$

It follows from Corollary 3.1 that

$$\begin{aligned} |(B(\mathbf{u}_h - \mathbf{u}), \mathbf{w})| &= |(\mathbf{u}_h - \mathbf{u}, B^*\mathbf{w})_{\mathbf{U}}| \le C\|\mathbf{u}_h - \mathbf{u}\|_{-1,\Omega_{\mathbf{U}}}\|B^*\mathbf{w}\|_{1,\Omega_{\mathbf{U}}} \\ &\le C(h_{\mathbf{U}}^{1.5} + h^2)|\mathbf{w}|_{1,\Omega} \le C(h_{\mathbf{U}}^{1.5} + h^2)\|\mathbf{w}\|_{1,\Omega}, \end{aligned} \tag{4.6}$$

and (4.5) and (4.6) imply

$$\begin{aligned} & a(\mathbf{y}_h - i_h\mathbf{y}, \mathbf{w}) - b(\mathbf{w}, r_h - \pi_h r) + b(\mathbf{y}_h - i_h\mathbf{y}, \phi) \\ \le\ & Ch^2\|\mathbf{y}\|_{3,\Omega}\|\mathbf{w}\|_{1,\Omega} + Ch^2\|r\|_{2,\Omega}\|\mathbf{w}\|_{1,\Omega} \\ & + Ch^2\|\mathbf{y}\|_{3,\Omega}\|\phi\|_{0,\Omega} + C(h_{\mathbf{U}}^{1.5} + h^2)\|\mathbf{w}\|_{1,\Omega} \\ \le\ & C(h_{\mathbf{U}}^{1.5} + h^2)(\|\mathbf{w}\|_{1,\Omega} + \|\phi\|_{0,\Omega}). \end{aligned} \tag{4.7}$$

Therefore, (4.2) follows from (4.4) and (4.7), and we find

$$\|\mathbf{y}_h - i_h\mathbf{y}\|_{1,\Omega} \le C(h_{\mathbf{U}}^{1.5} + h^2). \tag{4.8}$$

Similarly, the B-B condition (2.8) leads to

$$\begin{aligned} &\|\mathbf{p}_h - i_h\mathbf{p}\|_{1,\Omega} + \|s_h - \pi_h s\|_{0,\Omega} \\ \le\ & C \sup_{(\mathbf{w},\phi)\in\mathbf{Y}^h\times Q^h} \frac{a(\mathbf{p}_h - i_h\mathbf{p}, \mathbf{w}) + b(\mathbf{w}, s_h - \pi_h s) - b(\mathbf{p}_h - i_h\mathbf{p}, \phi)}{\|\mathbf{w}\|_{1,\Omega} + \|\phi\|_{0,\Omega}}. \end{aligned}$$

Then, there exists $(\mathbf{w}, \phi) \in \mathbf{Y}^h \times Q^h$ such that

$$\begin{aligned} & c(\|\mathbf{p}_h - i_h\mathbf{p}\|_{1,\Omega} + \|s_h - \pi_h s\|_{0,\Omega})(\|\mathbf{w}\|_{1,\Omega} + \|\phi\|_{0,\Omega}) \\ \leq\ & a(\mathbf{p}_h - i_h\mathbf{p}, \mathbf{w}) + b(\mathbf{w}, s_h - \pi_h s) - b(\mathbf{p}_h - i_h\mathbf{p}, \phi). \end{aligned} \tag{4.9}$$

Using Lemma 4.1, Lemma 4.2 and Lemma 4.3, we obtain

$$\begin{aligned} & a(\mathbf{p}_h - i_h\mathbf{p}, \mathbf{w}) + b(\mathbf{w}, s_h - \pi_h s) - b(\mathbf{p}_h - i_h\mathbf{p}, \phi) \\ =\ & a(\mathbf{p} - i_h\mathbf{p}, \mathbf{w}) + b(\mathbf{w}, s - \pi_h s) - b(\mathbf{p} - i_h\mathbf{p}, \phi) \\ & + a(\mathbf{p}_h - \mathbf{p}, \mathbf{w}) + b(\mathbf{w}, s_h - s) - b(\mathbf{p}_h - \mathbf{p}, \phi) \\ \leq\ & Ch^2\|\mathbf{p}\|_{3,\Omega}\|\mathbf{w}\|_{1,\Omega} + Ch^2\|s\|_{2,\Omega}\|\mathbf{w}\|_{1,\Omega} \\ & + Ch^2\|\mathbf{p}\|_{3,\Omega}\|\phi\|_{0,\Omega} + (g'(\mathbf{y}_h) - g'(\mathbf{y}), \mathbf{w}). \end{aligned} \tag{4.10}$$

It follows from (4.8), the Poincare inequality and Lemma 2.4 that

$$\begin{aligned} & |(g'(\mathbf{y}_h) - g'(\mathbf{y}), \mathbf{w})| \leq C\|\mathbf{y}_h - \mathbf{y}\|_{0,\Omega}\|\mathbf{w}\|_{0,\Omega} \\ \leq\ & C(\|\mathbf{y}_h - i_h\mathbf{y}\|_{1,\Omega} + \|i_h\mathbf{y} - \mathbf{y}\|_{0.\Omega})|\mathbf{w}|_{1,\Omega} \\ \leq\ & C(h_{\mathbf{U}}^{1.5} + h^2 + h^2|\mathbf{y}|_{2,\Omega})|\mathbf{w}|_{1,\Omega} \leq C(h_{\mathbf{U}}^{1.5} + h^2)\|\mathbf{w}\|_{1,\Omega}. \end{aligned} \tag{4.11}$$

Hence, by (4.10) and (4.11),

$$\begin{aligned} & a(\mathbf{p}_h - i_h\mathbf{p}, \mathbf{w}) + b(\mathbf{w}, s_h - \pi_h s) - b(\mathbf{p}_h - i_h\mathbf{p}, \phi) \\ \leq\ & Ch^2\|\mathbf{p}\|_{3,\Omega}\|\mathbf{w}\|_{1,\Omega} + Ch^2\|s\|_{2,\Omega}\|\mathbf{w}\|_{1,\Omega} \\ & + Ch^2\|\mathbf{p}\|_{3,\Omega}\|\phi\|_{0,\Omega} + C(h_{\mathbf{U}}^{1.5} + h^2)\|\mathbf{w}\|_{1,\Omega} \\ \leq\ & C(h_{\mathbf{U}}^{1.5} + h^2)(\|\mathbf{w}\|_{1,\Omega} + \|\phi\|_{0,\Omega}). \end{aligned} \tag{4.12}$$

Therefore, (4.3) follows from (4.9) and (4.12) . □

Again, Lemma 4.4 shows the superclose property for the state $\mathbf{y}$ and co-state $\mathbf{p}$. In order to get the global superconvergence result for the state $\mathbf{y}$ and $\mathbf{p}$, we need to define two postprocessing operators. A similar idea has been used in, e.g., [15], [19], [20] and [29]. Let I_{2h} be the postprocessing interpolation operator for the state $\mathbf{y}$ and the co-state $\mathbf{p}$ associated with T^{2h} such that

$$\begin{cases} I_{2h}\mathbf{y} \in (Q_2(\tau))^2 \\ I_{2h}\mathbf{y}(z_i) = \mathbf{y}(z_i), \quad i = 1, 2, \cdots, 9, \end{cases}$$

where Q_2 is the space of the biquadratic functions, z_i, $i = 1, 2, \cdots, 9$, are nodes of T^h on the large element τ, and $\tau \in T^{2h}$ consists of the four small element e_i, $i = 1, 2, 3, 4$, in T^h (see Fig 2). For r and s, we define Π_{2h} as

$$\begin{cases} \Pi_{2h}r \in Q_1(\tau) \\ \int_{e_i}(\Pi_{2h}r - r) = 0, \quad i = 1, 2, 3, 4, \end{cases}$$

where Q_1 is the space of the bilinear functions, τ and e_i, $i = 1, 2, 3, 4$, are small elements defined as above. Then, the following properties can be checked:

$$\begin{cases} I_{2h}i_h = I_{2h}, \\ \|I_{2h}\mathbf{v}\|_1 \leq C\|\mathbf{v}\|_1, \forall \mathbf{v} \in \mathbf{Y}^h, \\ \|I_{2h}\mathbf{y} - \mathbf{y}\|_1 \leq Ch^2\|\mathbf{y}\|_3, \end{cases} \qquad \begin{cases} \Pi_{2h}\bar{\pi}_h = \Pi_{2h}, \\ \|\Pi_{2h}q\|_0 \leq C\|q\|_0, \forall q \in Q^h, \\ \|\Pi_{2h}r - r\|_0 \leq Ch^2\|r\|_2. \end{cases} \tag{4.13}$$

For the postprocessing interpolation Π_{2h} and the interpolation π_h defined by (4.1), there is an important property as follows. The proof can be found in [29].

Lemma 4.5 *Let $r \in H^2(\Omega)$. then*

$$\|\Pi_{2h}\pi_h r - r\|_{0,\Omega} \leq Ch^2\|r\|_{2,\Omega}.$$

Now, we are going to prove the global superconvergence for the state $\mathbf{y}$ and the co-state $\mathbf{p}$ (also r and s). Based on uniform rectangular meshes, we have the following theorem.

Theorem 4.1 *Suppose all conditions of Lemma 4.4 are valid. Then*

$$\|I_{2h}\mathbf{y}_h - \mathbf{y}\|_{1,\Omega} + \|\Pi_{2h}r - r\|_{0.\Omega} \leq C(h_{\mathbf{U}}^{1.5} + h^2), \tag{4.14}$$

$$\|I_{2h}\mathbf{p}_h - \mathbf{p}\|_{1,\Omega} + \|\Pi_{2h}s - s\|_{0.\Omega} \leq C(h_{\mathbf{U}}^{1.5} + h^2). \tag{4.15}$$

Proof. We only need to prove (4.14). It follows from the property (4.13) and Lemma 4.4, 4.5 that

$$\begin{aligned}
&\|I_{2h}\mathbf{y}_h - \mathbf{y}\|_{1,\Omega} + \|\Pi_{2h}r - r\|_{0.\Omega} \\
\leq\ & \|I_{2h}(\mathbf{y}_h - i_h\mathbf{y})\|_{1,\Omega} + \|I_{2h}\mathbf{y} - \mathbf{y}\|_{1,\Omega} + \|\Pi_{2h}(r_h - \pi_h r)\|_{0,\Omega} + \|\Pi_{2h}\pi_h r - r\|_{0,\Omega} \\
\leq\ & C\|\mathbf{y}_h - i_h\mathbf{y}\|_{1,\Omega} + Ch^2\|\mathbf{y}\|_{3,\Omega} + C\|r_h - \pi_h r\|_{0,\Omega} + Ch^2\|r\|_{2,\Omega} \\
\leq\ & C(h_{\mathbf{U}}^{1.5} + h^2) + Ch^2\|\mathbf{y}\|_{3,\Omega} + Ch^2\|r\|_{2,\Omega} \\
\leq\ & C(h_{\mathbf{U}}^{1.5} + h^2).
\end{aligned}$$

Then, (4.14) is proved. (4.15) can be verified similarly. □

Remark 4.1 *For arbitrary quadrilateral meshes, the bilinear-constant element does not satisfy the Babuška-Brezzi condition (see* [31]*). Only on the almost uniform rectangular mesh, it can be proved that the bilinear-constant element satisfies the Babuška-Brezzi condition and is superconvergent. Unfortunately, the uniform rectangular mesh can not be adapted to a general domain. In the following, we will extend our superconvergence estimates to the general bounded convex domains. For an arbitrary bounded convex domain, we first make a rectangle $D \supset \Omega$. Then use an uniform rectangular partitioning $\bar{T}^h$ covering D and make some modification near the boundary of Ω, so that the boundary of Ω can be approximated by the edges of the elements (see Fig. 3). The partitioning T^h of Ω is composed of two parts: i) $e_1 \in \bar{T}^h$, and $e_1 \subset \Omega$. Let $\Omega_1 = \cup e_1$. ii) $e_2 \in \bar{T}^h$, and $\bar{e}_2 \cap \partial\Omega \neq \emptyset$. Let $\Omega_2 = \cup e_2$. From the above definition, it is easy to see that the partitioning of Ω_1 is uniform rectangular partitioning and the partitioning of Ω_2 is irrectangular partitioning (see,* Fig.3*). It is well known that for the general domain Ω, $meas(\Omega_2) = O(h)$, i.e., the area of the irrectangular partitioning domain Ω_2 is very small. Thus we get the mostly uniform rectangular meshes with a boundary layer of irrectangular elements. We use the bilinear-constant element introduced in Section 2 on the domain Ω_1, and use the mini element on Ω_2. It is easy to see that the above finite element space is conforming on the inner boundary $\overline{\Omega}_1 \cap \overline{\Omega}_2$. Based on above mostly uniform rectangular mesh and finite element space, it follows from Lemma 4.1 that*

$$\begin{aligned}
a_\Omega(\mathbf{y} - i_h\mathbf{y}, \mathbf{w}) &= a_{\Omega_1}(\mathbf{y} - i_h\mathbf{y}, \mathbf{w}) + a_{\Omega_2}(\mathbf{y} - i_h\mathbf{y}, \mathbf{w}) \\
&\leq Ch^2\|\mathbf{y}\|_{3,\Omega_1}\|\mathbf{w}\|_{1,\Omega_1} + C\|\mathbf{y} - i_h\mathbf{y}\|_{1,\Omega_2}\|\mathbf{w}\|_{1,\Omega_2} \\
&\leq Ch^2\|\mathbf{y}\|_{3,\Omega_1}\|\mathbf{w}\|_{1,\Omega_1} + Ch\|\mathbf{y}\|_{2,\Omega_2}\|\mathbf{w}\|_{1,\Omega_2} \\
&\leq Ch^2\|\mathbf{y}\|_{3,\Omega_1}\|\mathbf{w}\|_{1,\Omega_1} + Ch(meas(\Omega_2))^{\frac{1}{2}}\|\mathbf{y}\|_{2,\infty,\Omega_2}\|\mathbf{w}\|_{1,\Omega_2} \\
&\leq Ch^{1.5}\|\mathbf{y}\|_{2,\infty,\Omega}\|\mathbf{w}\|_{1,\Omega}, \quad \forall \mathbf{w} \in \mathbf{Y}^h.
\end{aligned}$$

In above estimate, $a_\Omega(\mathbf{w}, \mathbf{v}) = \int_\Omega \nabla\mathbf{w}\nabla\mathbf{v}$. For simplicity, we omit the effect of curved boundary, and assume that $\Omega = \Omega_1 \cup \Omega_2$. Similarly, we can prove the *following results:*

$$b(\mathbf{y} - i_h\mathbf{y}, \phi) = Ch^{1.5}\|\mathbf{y}\|_{2,\infty,\Omega}\|\phi\|_{0,\Omega} \quad \forall \phi \in Q^h,$$

For subdomain Ω_1, no zero boundary condition imposed on the interior boundary. So the error of Lemma 4.3 loses a half order. Therefore, we have that

$$b(\mathbf{w}, r - \pi_h r) \leq Ch^{1.5}(\|r\|_{2,\Omega} + \|r\|_{1,\infty,\Omega})\|\mathbf{w}\|_{1,\Omega} \quad \forall \mathbf{w} \in \mathbf{Y}^h.$$

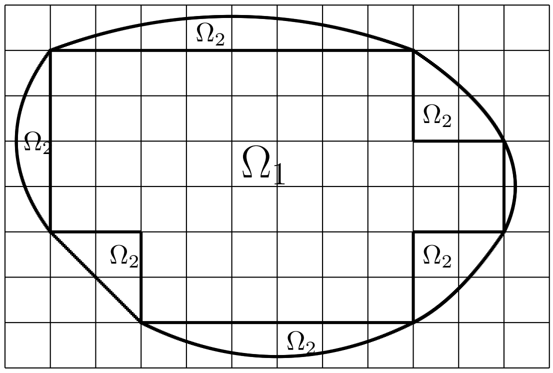

FIG. *3. the partitioning of* Ω

Remark 4.2 *For arbitrary quadrilateral of T^h, the higher order element $Q_l - P_{l-1}$ schemes ($l \geq 2$) satisfy the Babuška-Brezzi condition. In order to get the superconvergence, $\mathbf{y} \in (H^{l+2}(\Omega))^2$ is required for the $Q_l - P_{l-1}$ finite elements. But we only can assume that $\mathbf{u} \in H^1(\Omega_{\mathbf{U}})$ and $\mathbf{y} \in H^3(\Omega)$ for the control problem (2.1), because the derivative of $\mathbf{u}$ should be jumping over the free boundary. So we did not discuss the high order element in this paper, even though it can be used on polygon directly.*

Similar to the proof of Lemma 4.4, we can prove the following lemma on the mostly uniform rectangular meshes.

Lemma 4.6 *Suppose that T^h is mostly uniform rectangular meshes with a boundary layer of irrectangular elements. Let $(\mathbf{y}, r, \mathbf{p}, s,)$ be the solution of the equations (2.5) with $\mathbf{y}, \mathbf{p} \in (H^3(\Omega))^2 \cap (W^{2,\infty}(\Omega))^2$, $r, s \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, and $(\mathbf{y}_h, r_h, \mathbf{p}_h, s_h)$ be the solution of the equations (2.10). Assume that the conditions in Lemma 3.1 and Remark 4.1 are valid. Then*

$$\|\mathbf{y}_h - i_h\mathbf{y}\|_{1,\Omega} + \|r_h - \pi_h r\|_{0,\Omega} \leq C(h_{\mathbf{U}}^{1.5} + h^{1.5}), \tag{4.16}$$

$$\|\mathbf{p}_h - i_h\mathbf{p}\|_{1,\Omega} + \|s_h - \pi_h s\|_{0,\Omega} \leq C(h_{\mathbf{U}}^{1.5} + h^{1.5}). \tag{4.17}$$

Similar to the proof of Theorem 4.1, using the mostly uniform rectangular meshes with a boundary layer of irrectangular elements, we have the following theorem.

Theorem 4.2 *Suppose all conditions of Lemma 4.6 are valid. Then*

$$\|I_{2h}\mathbf{y}_h - \mathbf{y}\|_{1,\Omega} + \|\Pi_{2h}r - r\|_{0.\Omega} \leq C(h_{\mathbf{U}}^{1.5} + h^{1.5}), \tag{4.18}$$

$$\|I_{2h}\mathbf{p}_h - \mathbf{p}\|_{1,\Omega} + \|\Pi_{2h}s - s\|_{0.\Omega} \leq C(h_{\mathbf{U}}^{1.5} + h^{1.5}), \tag{4.19}$$

where I_{2h}, Π_{2h} are defined before Lemma 4.5 on Ω_1, and defined to be identity operators on the domain Ω_2.

5. Recovery type a posteriori error estimate

Based on the results of the global superconvergence presented in Sections 3 and 4, we will discuss the recovery type a posteriori error estimates for the control problems discussed in this paper.

Theorem 5.1 *Assume that all the conditions in Theorems 3.1 , 4.1 and 4.2 are valid. Then,*

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}} = \|R_h\mathbf{u}_h-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}+\epsilon_1, \tag{5.1}$$

$$\|\mathbf{y}-\mathbf{y}_h\|_{1,\Omega} = \|I_{2h}\mathbf{y}_h-\mathbf{y}_h\|_{1,\Omega}+\epsilon_2, \tag{5.2}$$

$$\|\mathbf{p}-\mathbf{p}_h\|_{1,\Omega} = \|I_{2h}\mathbf{p}_h-\mathbf{p}_h\|_{1,\Omega}+\epsilon_2, \tag{5.3}$$

$$\|r-r_h\|_{0,\Omega} = \|\Pi_{2h}r_h-r_h\|_{0,\Omega}+\epsilon_2, \tag{5.4}$$

$$\|s-s_h\|_{0,\Omega} = \|\Pi_{2h}s_h-s_h\|_{0,\Omega}+\epsilon_2, \tag{5.5}$$

where

$$\epsilon_1 = O(h_{\mathbf{U}}^{\frac{3}{2}}+h^2).$$

$$\epsilon_2 = \begin{cases} O(h_{\mathbf{U}}^{1.5}+h^2) & T^h \text{ be uniform rectangular meshes,} \\ O(h_{\mathbf{U}}^{1.5}+h^{1.5}) & T^h \text{ be mostly uniform rectangular meshes.} \end{cases}$$

Proof. Note that

$$\left|\, \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}} - \|R_h\mathbf{u}_h-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}} \,\right| \le \|\mathbf{u}-R_h\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}. \tag{5.6}$$

It has been proved in Theorem 3.1 that

$$\|\mathbf{u}-R_h\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}} \le \epsilon_1. \tag{5.7}$$

Then (5.1) follows from (5.6) and (5.7). (5.2)-(5.5) can be proved similarly from Theorem 4.1 and Theorem 4.2. ☐

Theorem 5.1 provides a recovery type a posteriori estimator:

$$\begin{aligned}\eta_g^2 &= \|R_h\mathbf{u}_h-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}^2+\|I_{2h}\mathbf{y}_h-\mathbf{y}_h\|_{1,\Omega}^2+\|I_{2h}\mathbf{p}_h-\mathbf{p}_h\|_{1,\Omega}^2 \\ &\quad +\|\Pi_{2h}r_h-r_h\|_{0,\Omega}^2+\|\Pi_{2h}s_h-s_h\|_{0,\Omega}^2,\end{aligned}$$

which is a good approximation of the exact error:

$$e^2 = \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}^2+\|\mathbf{y}-\mathbf{y}_h\|_{1,\Omega}^2+\|\mathbf{p}-\mathbf{p}_h\|_{1,\Omega}^2+\|r-r_h\|_{0,\Omega}^2+\|s-s_h\|_{0,\Omega}^2.$$

Note that

$$\lim_{h,h_{\mathbf{U}}\to 0}\frac{\eta_g}{e}=1,$$

because ϵ_i is a higher order term. Then, the recovery type a posteriori estimator η_g defined above is asymptotically exact if the conditions for the superconvergence are valid.

Note that $\nabla(I_{2h}\mathbf{y}_h)$, $\nabla(I_{2h}\mathbf{p}_h)$ and $\Pi_{2h}r_h$, $\Pi_{2h}s_h$ are not continuous on the edges of the large element τ. We are going to introduce some new recovery postprocessing operators, which are continuous on the whole domain, and can easily be extended to the more complicated meshes .

For a uniform rectangular partition, we construct the recovery operator $\bar{R}_h$ similar to the recovery operator R_h defined in Section 3. Let $\omega_z = \sum_{i=1}^4 e_i$ which is shown in FIG.2. Then, $\bar{R}_h v(z)$, $z = (x_0, y_0)$, can be calculated as follows: $\bar{R}_h v(z) = a + bx_0 + dy_0$, where (a, b, d) is the solution of the following linear system:

$$\begin{bmatrix} 4 & \begin{matrix} x_0+x_1 \\ +x_0+x_2 \end{matrix} & \begin{matrix} y_0+y_1 \\ +y_0+y_2 \end{matrix} \\ \begin{matrix} x_0+x_1 \\ +x_0+x_2 \end{matrix} & \begin{matrix} \frac{1}{2}[(x_0+x_1)^2 \\ +(x_0+x_2)^2] \end{matrix} & \begin{matrix} \frac{1}{4}(2x_0+x_1+x_2) \\ \times(2y_0+y_1+y_2) \end{matrix} \\ \begin{matrix} y_0+y_1 \\ +y_0+y_2 \end{matrix} & \begin{matrix} \frac{1}{4}(2x_0+x_1+x_2) \\ \times(2y_0+y_1+y_2) \end{matrix} & \begin{matrix} \frac{1}{2}[(y_0+y_1)^2 \\ +(y_0+y_2)^2] \end{matrix} \end{bmatrix} \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{|k|} \sum_{i=1}^{4} \int_{e_i} v \\ \frac{1}{2|k|} \left[(x_0+x_1) \int_{e_1 \cup e_4} v + (x_0+x_2) \int_{e_2 \cup e_3} v \right] \\ \frac{1}{2|k|} \left[(y_0+y_2) \int_{e_1 \cup e_2} v + (y_0+y_1) \int_{e_3 \cup e_4} v \right] \end{bmatrix},$$

where $|k|$ is the common area of the element e_i.

For the gradient of $\mathbf{y}$ and $\mathbf{p}$, we construct the gradient recovery operator G_h such that

$$G_h \mathbf{v} = (\bar{R}_h \mathbf{v}_x, \bar{R}_h \mathbf{v}_y),$$

where $\bar{R}_h$ is defined as above, $\mathbf{v}_x = \frac{\partial \mathbf{v}}{\partial x}$ and $\mathbf{v}_y = \frac{\partial \mathbf{v}}{\partial y}$. It should be noted that G_h is similar to the Z-Z gradient recovery (see e.g., [36], [37]) in the piecewise bilinear case. And there are more other kinds of recovery operators (see, e.g., [34]). For r and s, we use the recovery operator $\bar{R}_{2h}$. Again, $\bar{R}_{2h}$ is defined as above, but it is applied on the mesh T^{2h} instead of T^h. For the recovery operator G_h and $\bar{R}_{2h}$ defined above, the following global superconvergence property still follows.

Lemma 5.1 *Let $\mathbf{y}$, $\mathbf{p}$, r, s be the solutions of the equations (2.5), and $\mathbf{y}_h$, $\mathbf{p}_h$, r_h, s_h be the solutions of the equations (2.10). Assume that the conditions in Lemma 4.4 and 4.5 are valid. Then*

$$\|G_h \mathbf{y}_h - \nabla \mathbf{y}\|_{0,\Omega} + \|G_h \mathbf{p}_h - \nabla \mathbf{p}\|_{0,\Omega} + \|\bar{R}_{2h} r_h - r\|_{0,\Omega} + \|\bar{R}_{2h} s_h - s\|_{0,\Omega} \le \epsilon_2, \tag{5.8}$$

where ϵ_2 is defined as Theorem 5.1.

Proof. Note that

$$\|G_h \mathbf{y}_h - \nabla y\|_{0,\Omega} \le \|G_h \mathbf{y}_h - G_h \mathbf{y}_I\|_{0,\Omega} + \|G_h \mathbf{y}_I - \nabla \mathbf{y}\|_{0,\Omega}, \tag{5.9}$$

where $\mathbf{y}_I = i_h \mathbf{y}$. Note also that for any uniform mesh, G_h is a bounded linear operator with an upper bound independent of h. Then, it follows from Lemma 4.4 and 4.5 that

$$\|G_h \mathbf{y}_h - G_h \mathbf{y}_I\|_{0,\Omega} \le \|G_h\| \, \|\nabla \mathbf{y}_h - \nabla \mathbf{y}_I\|_{0,\Omega} \le \epsilon_2. \tag{5.10}$$

It has been proved in [35] that $G_h v_I = \nabla v$ on e if v is a quadratic function on the neighborhood of e $(\cup_{\bar{e}' \cap \bar{e} \neq \emptyset} \{e'\})$. Then, it follows from the standard interpolation error estimate technique (see, [3]) that

$$\|G_h \mathbf{y}_I - \nabla \mathbf{y}\|_{0,\Omega} \le Ch^2 |\mathbf{y}|_{3,\Omega}. \tag{5.11}$$

Therefore, (5.9)-(5.11) imply that

$$\|G_h \mathbf{y}_h - \nabla \mathbf{y}\|_{0,\Omega} \le \epsilon_2. \tag{5.12}$$

Similarly, it can be proved that

$$\|G_h \mathbf{p}_h - \nabla \mathbf{p}\|_{0,\Omega} \le \epsilon_2. \tag{5.13}$$

Moreover, note that $\int_\tau (v-\pi_h v)=0$, and hence $\bar{R}_{2h}v=\bar{R}_{2h}\pi_h v$. We then obtain

$$\begin{aligned} \|\bar{R}_{2h}r_h - r\|_{0,\Omega} &\le \|\bar{R}_{2h}r_h - \bar{R}_{2h}\pi_h r\|_{0,\Omega} + \|\bar{R}_{2h}\pi_h r - r\|_{0,\Omega} \\ &= \|\bar{R}_{2h}(r_h - \pi_h r)\|_{0,\Omega} + \|\bar{R}_{2h} r - r\|_{0,\Omega} \\ &\le C\|r_h - \pi_h r\|_{0,\Omega} + Ch^2\|r\|_{2,\Omega} \le \epsilon_2. \end{aligned} \tag{5.14}$$

Similarly, it can be proved that

$$\|\bar{R}_{2h}s_h - s\|_{0,\Omega} \le \epsilon_2. \tag{5.15}$$

Therefore, (5.8) follows from (5.12)-(5.15). □

Based on the superconvergence analysis presented by Lemma 5.1, we have the following results for the recovery type a posteriori error estimate.

Theorem 5.2 *Suppose that all conditions of Theorem 3.1 and 4.1 are valid. Then,*

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}} = \|R_h\mathbf{u}_h - \mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}} + \epsilon_1, \tag{5.16}$$

$$|\mathbf{y}-\mathbf{y}_h|_{1,\Omega} = \|G_h\mathbf{y}_h - \nabla\mathbf{y}_h\|_{0,\Omega} + \epsilon_2, \tag{5.17}$$

$$|\mathbf{p}-\mathbf{p}_h|_{1,\Omega} = \|G_h\mathbf{p}_h - \nabla\mathbf{p}_h\|_{0,\Omega} + \epsilon_2, \tag{5.18}$$

$$\|r-r_h\|_{0,\Omega} = \|\bar{R}_{2h}r_h - r_h\|_{0,\Omega} + \epsilon_2, \tag{5.19}$$

$$\|s-s_h\|_{0,\Omega} = \|\bar{R}_{2h}s_h - s_h\|_{0,\Omega} + \epsilon_2, \tag{5.20}$$

where ϵ_1 and ϵ_2 are defined as Theorem 5.1.

Proof. (5.16)-(5.20) are direct consequences of Lemma 5.1. The proof is similar to the one for Theorem 5.1. □

Let

$$\begin{aligned} \tilde{\eta}_g^2 &= \|R_h\mathbf{u}_h - \mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}^2 + \|G_h\mathbf{y}_h - \nabla\mathbf{y}_h\|_{0,\Omega}^2 + \|G_h\mathbf{p}_h - \nabla\mathbf{p}_h\|_{0,\Omega}^2 \\ &\quad + \|\bar{R}_{2h}r_h - r_h\|_{0,\Omega}^2 + \|\bar{R}_{2h}s_h - s_h\|_{0,\Omega}^2, \end{aligned}$$

and again let

$$e^2 = \|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_{\mathbf{U}}}^2 + |\mathbf{y}-\mathbf{y}_h|_{1,\Omega}^2 + |\mathbf{p}-\mathbf{p}_h|_{1,\Omega}^2 + \|r-r_h\|_{0,\Omega}^2 + \|s-s_h\|_{0,\Omega}^2.$$

It follows from Theorem 5.2 that a recovery type a posteriori estimator $\tilde{\eta}_g$ is still a good approximation of the exact error e, and is asymptotically exact if the conditions for the superconvergence are valid. Moreover, $R_h\mathbf{u}_h$, $G_h\mathbf{y}_h$, $G_h\mathbf{p}_h$, $\bar{R}_{2h}r_h$ and $\bar{R}_{2h}s_h$ are all continuous on whole domain. It is predictable that this recovery type a posteriori error estimator may be applied on the more complicated meshes (see, e.g., [14], [33]).

6. Discussions

In this paper, we discussed the global superconvergence for the control problems governed by the Stokes equations. It is shown that if the solution is smooth enough, the mesh for the state and the co-state is the uniform rectangular mesh and the bilinear-constant scheme is adopted for the state and co-state equations, the global superconvergence for the control, the state and the co-state can be proved. Based on the supercovergence analysis, the recovery type a posteriori error estimators are provided.

There are many important issues still to be addressed in this area, for example, deriving the global superconvergence analysis and a recovery type a posteriori error estimate for more complicated control problems and finite element schemes. It is also interesting and very important to investigate the more complicated constrained optimal control problems and more general finite element meshes, i.e., the closed convex set K is more complicated or the meshes are general regular meshes instead of the rectangular meshes. Finally, many computational issues have to be studied, and we will give numerical examples in the coming paper.

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