INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 3, Pages 273–282

SUPERCONVERGENCE OF TETRAHEDRAL LINEAR FINITE ELEMENTS

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(Communicated by Zhimin Zhang)

Abstract. In this paper, we show that the piecewise linear finite element solution u_h and the linear interpolation u_I have superclose gradient for tetrahedral meshes, where most elements are obtained by dividing approximate parallelepiped into six tetrahedra. We then analyze a post-processing gradient recovery scheme, showing that the global L^2 projection of ∇u_h is a superconvergent gradient approximation to ∇u.

Key Words. superconvergence, finite element methods, tetrahedral elements, post-processing

1. Introduction

Superconvergence of the gradient for the finite element approximation for second order elliptic boundary value problems and gradient recovery schemes have been an active research topic; see, for example, Babuška and Strouboulis [1], Chen and Huang [8], Lin and Yan [12], Wahlbin [13] and Lakhany, Marek, and Whiteman [11] for overviews of this field. Recently Bank and Xu [2, 3] have developed some new techniques and obtained some new superconvergence results for linear finite element elements on two dimensional triangular meshes. The goal of this paper is to extend their results to three dimensions, namely to linear tetrahedral finite element.

The model problem that we study in this paper is

$$-\nabla\cdot(\mathcal{D}(x)\nabla u) = f,\ x\in\Omega$$
$$u = 0,\ x\in\partial\Omega.$$

Here $\mathcal{D}(x)$ is a 3×3 symmetric matrix function in $(L^\infty(\Omega))^{3\times 3}$ and uniformly positive definite. For simplicity, we assume f is smooth enough and Ω is a polyedr in $\mathbb{R}^3$ partitioned into a quasiuniform triangulation $\mathcal{T}_h$ with mesh size $h\in(0,1)$. Let $\mathcal{V}_h\subset H_0^1(\Omega)$ be the corresponding finite element space of continuous piecewise linear functions associated with $\mathcal{T}_h$, and let $u_h\in\mathcal{V}_h$ be the finite element solution of the above second order elliptic boundary value problem.

Unlike in the two dimensional case, superconvergence results in three dimensions are relatively rare [7, 9, 10, 5]. The difficulty is partially due to the loss of symmetry in three dimensions [4]. In this paper, we only deal with a special triangulation of which most elements are obtained by dividing each $O(h^2)$ parallelepiped into six tetrahedra (see Section 3 for details). For this kind of triangulation, we numerically

Received by the editors July 1, 2004 and, in revised form, October 22, 2004.
2000 *Mathematics Subject Classification.* 65N30.
This work was supported in part by NSF DMS-0074299, NSF DMS-0209497,NSF DMS-0215392 and the Center for Computational Mathematics and Application at Penn State.

observed that superconvergence occurs for linear elements, due to the cancellation of the lowest order terms in some asymptotic expansion of the local error. It is, however, difficult to combine elementwise error estimates together, since the normal component of the gradients of the test functions is discontinuous. Thus we follow the new approach in [2] to derive some expressions for the element error that involve only the tangential derivative of the test function on the edges.

Our first result is that the gradient of the finite element approximation u_h is superclose to the gradient of the piecewise linear interpolant u_I of the solution u. More precisely, we have

$$|u_h - u_I|_{1,\Omega} \lesssim h^{1+\min(\sigma,1)} \|u\|_{3,\infty,\Omega}. \tag{1}$$

Estimate (1) holds on quasi-uniform meshes, where most elements are obtained by dividing each $O(h^2)$ parallelepiped into six tetrahedra except for a region of size $O(h^{2\sigma})$; see Section 3 for details.

The estimate (1) is known in the literature for the special case $\sigma = \infty$ [7, 9, 10]. Recently Brandts and Křížek [5] extend the results of [7, 9, 10] for tetrahedra into arbitrary n- simplex. Our new estimate (1) is a significant generalization, since firstly, our analysis is based on local identities for each element and thus, it is straightforward to extend our results to meshes in which an $O(h^{1+\alpha})$ (instead of $O(h^2)$) approximate local symmetry property holds for most patches of edges. Second, the relaxation parameter σ makes our analysis to work for more general meshes, especially for domains with unstructured boundaries.

Based on the superconvergence results, one can construct schemes to get better approximations of ∇u; see for example, [16, 17, 14, 15] and [6]. The second major component of this work is a superconvergent approximation to ∇u by a gradient recovery procedure. In Section 4, we show that

$$\|\nabla u - Q_h \nabla u_h\|_{0,\Omega} \lesssim h^{1+\min(\sigma,1/2)} \|u\|_{3,\infty,\Omega}, \tag{2}$$

where Q_h is the L^2 projection to $\mathcal{V}_h^3$. As remarked in [2], both the superconvergence and gradient recovery results can be generalized to a more general non-self-adjoint and possibly indefinite problem.

The rest of this paper is organized as follows. We introduce some notation and technical identities for our analysis in Section 2. We prove the estimate (1) and (2) in Section 3 and Section 4 respectively.

2. Local Error Expansion

In this section we shall derive some useful identities for our analysis. The key identity is contained in Lemma 2.4, which is a generalization of the integral formulas of rectangular elements [12] and triangular elements [2] in two dimensions to tetrahedral elements in three dimensions.

Let τ be a tetrahedron in $\mathbb{R}^3$, with vertices $\{\mathbf{p}_k\}_{k=1}^4$ and the corresponding nodal basis functions (barycentric coordinates) $\{\varphi_k\}_{k=1}^4$. We assume that $\mathbb{R}^3$ has the orientation given by the right-hand rule and τ has the induced orientation. Let F_k denote the surface opposite vertex $\mathbf{p}_k$ with the induced orientation and $\mathbf{n}_k$ the unit outward normal vectors of F_k. We also use the symbol $\triangle_{klm}$ to denote the face with vertices $\mathbf{p}_k, \mathbf{p}_l$, and $\mathbf{p}_m$. If the orientation of $\triangle_{klm}$, given by the order of k, l, m, coincides with the induced orientation from τ, we say $\triangle_{klm}$ has the consistent orientation with τ. Let $\mathbf{e}_{ij}$ denote the oriented edges of element τ from $\mathbf{p}_i$ to $\mathbf{p}_j$ and $\mathbf{t}_{ij}$, d_{ij} the corresponding unit tangent vectors and length, respectively (see Fig 1). Let θ_{kl} be the angle between $\mathbf{t}_{kl}$ and the supporting plane of F_l. In

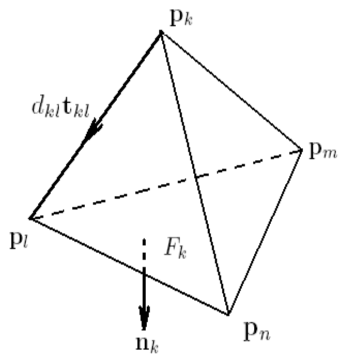

FIGURE 1. A tetrahedron

general, $\theta_{kl} \neq \theta_{lk}$. Let $\mathcal{D}_\tau$ be a constant symmetric 3×3 matrix defined on τ. We define $\xi_{ij} = \mathbf{n}_i \cdot \mathcal{D}_\tau \mathbf{n}_j$. Since $\mathcal{D}_\tau$ is symmetric, $\xi_{ij} = \xi_{ji}$.

Lemma 2.1. *Under the above assumptions we have*

$$\nabla u \cdot \mathcal{D}_\tau \mathbf{n}_k = \sum_{l=1, l\neq k}^{4} \frac{\xi_{kl}}{\cos\theta_{kl}} \frac{\partial u}{\partial \mathbf{t}_{kl}}.$$

Proof. It is an immediate consequence of

$$\mathcal{D}_\tau \mathbf{n}_k = \sum_{l=1,l\neq k}^{4} \frac{\mathbf{n}_l \cdot \mathcal{D}_\tau \mathbf{n}_k}{\mathbf{n}_l \cdot \mathbf{t}_{kl}} \mathbf{t}_{kl} = \sum_{l=1,l\neq k}^{4} \frac{\xi_{lk}}{\cos\theta_{kl}} \mathbf{t}_{kl}.$$

□

Lemma 2.2. *Let u_q be a quadratic polynomial and u_I the continuous piecewise linear interpolant for u_q on τ. Then for a constant vector* $\mathbf{t}$,

$$\int_\tau \nabla(u_I - u_q) \cdot \mathbf{t} = \frac{1}{2} \sum_{k=1}^{4} \mathbf{n}_k \cdot \mathbf{t} \int_{F_k} \sum_{i,j=1,i<j}^{4} d_{ij}^2 \varphi_i \varphi_j \frac{\partial^2 u_q}{\partial \mathbf{t}_{ij}^2}.$$

Proof. By the Taylor expansion,

$$(u_I - u_q)(x) = \frac{1}{2} \sum_{k=1}^{4} \varphi_k (\mathbf{p}_k - x) \cdot \nabla^2 u_q (\mathbf{p}_k - x). \tag{3}$$

Noting that $\mathbf{p}_k - x = \sum_i \varphi_i \mathbf{e}_{ki}$, we get

$$(u_I - u_q)(x) = \frac{1}{2} \sum_{i,j=1,i<j}^{4} d_{ij}^2 \varphi_i \varphi_j \frac{\partial^2 u}{\partial \mathbf{t}_{ij}^2}.$$

The desired result follows from Green's formula. □

Lemma 2.3. *For a function $u \in W^{1,1}(\tau)$ we have*

$$\frac{1}{\cos\theta_{lk}} \int_{F_l} \varphi_k \varphi_m u - \frac{1}{\cos\theta_{kl}} \int_{F_k} \varphi_l \varphi_m u = \frac{1}{\cos\theta_{lk}} \frac{1}{\cos\theta_{kl}} \int_\tau (\varphi_k + \varphi_l) \varphi_m \frac{\partial u}{\partial \mathbf{t}_{kl}},$$

$$\textit{and} \quad \frac{1}{\cos\theta_{lk}} \int_{F_l} u - \frac{1}{\cos\theta_{kl}} \int_{F_k} u = \frac{1}{\cos\theta_{lk}} \frac{1}{\cos\theta_{kl}} \int_\tau \frac{\partial u}{\partial \mathbf{t}_{kl}}.$$

Proof. By Green's formula

$$\begin{aligned}\int_\tau \nabla(fu)\cdot \mathbf{t}_{kl} &= \mathbf{n}_k\cdot\mathbf{t}_{kl}\int_{F_k} fu + \mathbf{n}_l\cdot\mathbf{t}_{kl}\int_{F_l} fu\\ &= -\mathbf{n}_k\cdot\mathbf{t}_{lk}\int_{F_k} fu + \mathbf{n}_l\cdot\mathbf{t}_{kl}\int_{F_l} fu.\end{aligned}$$

We then set $f = (\varphi_k + \varphi_l)\varphi_m$ to get the first identity, where we use facts $f|_{F_k} = \varphi_l\varphi_m, f|_{F_l} = \varphi_k\varphi_m$ and f is a constant along lines parallel to $\mathbf{t}_{kl}$, since $f = (1-\varphi_m-\varphi_n)\varphi_m$. The second identity is obtained by setting $f = 1$. □

To prove the next lemma, we need the following identity for the triangle $\triangle_{klm}$.

$$d_{lm}^2\frac{\partial^2 u}{\partial \mathbf{t}_{lm}^2} - d_{km}^2\frac{\partial^2 u}{\partial \mathbf{t}_{km}^2} = (d_{lm}^2 - d_{km}^2)\frac{\partial^2 u}{\partial \mathbf{t}_{kl}^2} + 4|\triangle_{klm}|\frac{\partial^2 u}{\partial \mathbf{t}_{kl}\partial \mathbf{n}_{kl,m}}, \tag{4}$$

where $\mathbf{n}_{kl,m}$ is the unit outward normal vector of edge $\mathbf{t}_{kl}$ on the supporting plane of triangle $\triangle_{klm}$. The proof can be found in [2].

The next identity is a fundamental one in our analysis.

Lemma 2.4. *Let u_q be a quadratic polynomial and $v_h \in \mathcal{V}_h$. Then we have*

$$\begin{aligned}&\int_\tau \nabla(u_I - u_q)\cdot \mathcal{D}_\tau \nabla v_h\\ &= \sum_{k,l=1,k\neq l}^{4} \frac{\partial v_h}{\partial \mathbf{t}_{kl}}\frac{\xi_{kl}}{4\cos\theta_{kl}}\Big[(d_{lm}^2 - d_{km}^2)\int_{F_k}\varphi_l\varphi_m\frac{\partial^2 u_q}{\partial \mathbf{t}_{kl}^2}\\ &+4|\triangle_{klm}|\int_{F_k}\varphi_l\varphi_m\frac{\partial^2 u_q}{\partial \mathbf{t}_{kl}\partial \mathbf{n}_{kl,m}}\Big],\end{aligned}$$

where we choose m such that $\triangle_{klm}$ has the consistent orientation with τ.

Proof. By Lemmas 2.1-2.3, we have:

$$\begin{aligned}\int_\tau \nabla(u_I - u_q)\cdot\mathcal{D}_\tau\nabla v_h &= \frac{1}{2}\sum_{k=1}^{4}\nabla v_h\cdot\mathcal{D}_\tau\mathbf{n}_k\int_{F_k}\sum_{i,j=1,i<j}^{4} d_{ij}^2\varphi_i\varphi_j\frac{\partial^2 u_q}{\partial t_{ij}^2}\\ &= \sum_{k,l=1,k\neq l}^{4}\int_{F_k}\frac{\partial v_h}{\partial \mathbf{t}_{kl}}\frac{\xi_{kl}}{2\cos\theta_{kl}}\sum_{i,j=1,i<j}^{4} d_{ij}^2\varphi_i\varphi_j\frac{\partial^2 u_q}{\partial \mathbf{t}_{ij}^2}\\ &= \sum_{k,l=1,k\neq l}^{4}\Big[\frac{\partial v_h}{\partial \mathbf{t}_{kl}}\frac{\xi_{kl}}{4\cos\theta_{kl}}\int_{F_k}\sum_{i,j=1,i<j}^{4} d_{ij}^2\varphi_i\varphi_j\frac{\partial^2 u_q}{\partial \mathbf{t}_{ij}^2}\\ &\quad+\frac{\partial v_h}{\partial \mathbf{t}_{lk}}\frac{\xi_{kl}}{4\cos\theta_{lk}}\int_{F_l}\sum_{i,j=1,i<j}^{4} d_{ij}^2\varphi_i\varphi_j\frac{\partial^2 u}{\partial \mathbf{t}_{ij}^2}\Big]\\ &= \sum_{k,l=1,k\neq l}^{4}(I_1+I_2+I_3),\end{aligned}$$

where

$$
\begin{aligned}
I_1 &= \frac{\partial v_h}{\partial \mathbf{t}_{kl}} \frac{\xi_{kl}}{4} \left(\frac{1}{\cos\theta_{kl}} \int_{F_k} d^2_{mn} \varphi_m \varphi_n \frac{\partial^2 u_q}{\partial \mathbf{t}^2_{mn}} - \frac{1}{\cos\theta_{lk}} \int_{F_l} d^2_{mn} \varphi_m \varphi_n \frac{\partial^2 u_q}{\partial \mathbf{t}^2_{mn}} \right), \\
I_2 &= \frac{\partial v_h}{\partial \mathbf{t}_{kl}} \frac{\xi_{kl}}{4} \left(\frac{1}{\cos\theta_{kl}} \int_{F_k} d^2_{lm} \varphi_l \varphi_m \frac{\partial^2 u_q}{\partial \mathbf{t}^2_{lm}} - \frac{1}{\cos\theta_{lk}} \int_{F_l} d^2_{km} \varphi_k \varphi_m \frac{\partial^2 u_q}{\partial \mathbf{t}^2_{km}} \right), \text{ and} \\
I_3 &= \frac{\partial v_h}{\partial \mathbf{t}_{lk}} \frac{\xi_{lk}}{4} \left(\frac{1}{\cos\theta_{lk}} \int_{F_l} d^2_{kn} \varphi_k \varphi_n \frac{\partial^2 u_q}{\partial \mathbf{t}^2_{kn}} - \frac{1}{\cos\theta_{kl}} \int_{F_k} d^2_{ln} \varphi_l \varphi_n \frac{\partial^2 u_q}{\partial \mathbf{t}^2_{ln}} \right).
\end{aligned}
$$

Here we choose m, n such that $\triangle_{klm}$ and $\triangle_{lkn}$ have the consistent orientation with τ. By Lemma 2.3 and identity (4), we have:

$$
\begin{aligned}
I_1 &= 0, \\
I_2 &= \frac{\xi_{kl}}{\cos\theta_{kl}} \int_{F_k} \varphi_l \varphi_m \Big[(d^2_{lm} - d^2_{km}) \frac{\partial^2 u_q}{\partial \mathbf{t}^2_{kl}} + 4|\triangle_{klm}| \frac{\partial^2 u}{\partial \mathbf{t}_{kl} \partial \mathbf{n}_{kl,m}} \Big] \frac{\partial v_h}{\partial \mathbf{t}_{kl}}, \\
I_3 &= \frac{\xi_{lk}}{\cos\theta_{lk}} \int_{F_l} \varphi_k \varphi_n \Big[(d^2_{kn} - d^2_{ln}) \frac{\partial^2 u_q}{\partial \mathbf{t}^2_{lk}} + 4|\triangle_{lkn}| \frac{\partial^2 u}{\partial \mathbf{t}_{lk} \partial \mathbf{n}_{lk,n}} \Big] \frac{\partial v_h}{\partial \mathbf{t}_{lk}}.
\end{aligned}
$$

Hence, the lemma follows. □

From Lemma 2.4, using the standard scaling argument and the Bramble-Hilbert Lemma on the reference element, we obtain the following result.

Lemma 2.5. *Let $u \in H^3(\tau)$. Then we have*

$$
\begin{aligned}
&\int_\tau \nabla(u_I - u) \cdot \mathcal{D}_\tau \nabla v_h \\
&= \sum_{k,l=1,k\neq l}^{4} \frac{\partial v_h}{\partial \mathbf{t}_{kl}} \frac{\xi_{kl}}{4\cos\theta_{kl}} \Big[(d^2_{lm} - d^2_{km}) \int_{F_k} \varphi_l \varphi_m \frac{\partial^2 u}{\partial \mathbf{t}^2_{kl}} \\
&+4|\triangle_{klm}| \int_{F_k} \varphi_l \varphi_m \frac{\partial^2 u}{\partial \mathbf{t}_{kl} \partial \mathbf{n}_{kl,m}} \Big] + O(h^3) \|u\|_{3,\tau} \|v\|_{1,\tau},
\end{aligned}
$$

where m is defined as in Lemma 2.4.

3. A Superconvergence result

In this section, we shall derive a superconvergence result for the linear finite element approximation for a model second order elliptic problem based on the identities we derived in the previous section.

We begin with the property of the triangulation $\mathcal{T}_h$. We say four points $\{\mathbf{p}_k\}_{k=1}^4$ in space form an $O(h^2)$ parallelogram, if $\|\mathbf{e}_{21} + \mathbf{e}_{41} - \mathbf{e}_{31}\| = O(h^2)$. A hexadron is called $O(h^2)$ approximated parallelepiped if each face is an $O(h^2)$ parallelogram. Let e be an interior edge in the triangulation $\mathcal{T}_h$. The patch of e, which is the union of tetrahedra sharing e, is denoted by Ω_e.

Definition 3.1. *Assume that the triangulation $\mathcal{T}_h = \mathcal{T}_{1,h} \cup \mathcal{T}_{2,h}$ and $\mathcal{T}_{1,h}$ is obtained by dividing $O(h^2)$ parallelepipeds into six tetradra (see Fig. 2.). Let $\mathcal{E} = \mathcal{E}_1 \bigoplus \mathcal{E}_2$ denote the set of interior edges in $\mathcal{T}_h$ and $\Omega_{i,h} = \bigcup_{e\in\mathcal{E}_i} \Omega_e$. The triangulation $\mathcal{T}_h$ is $O(h^{2\sigma})$ irregular if for each $e \in \mathcal{E}_1$, $\Omega_e \subset \mathcal{T}_{1,h}$, while $|\Omega_{2,h}| = O(h^{2\sigma})$.*

Lemma 3.2. *Let the triangulation $\mathcal{T}_h$ be $O(h^{2\sigma})$ irregular. Let $\mathcal{D}_\tau$ be a piecewise constant matrix function defined on $\mathcal{T}_h$, whose entries $\mathcal{D}_{\tau,ij}$ satisfy*

$$
|\mathcal{D}_{\tau,ij}| \lesssim 1, \ |\mathcal{D}_{\tau,ij} - \mathcal{D}_{\tau',ij}| \lesssim h,
$$

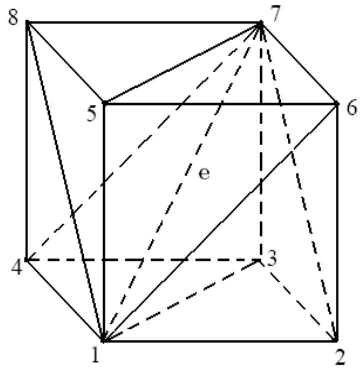

FIGURE 2. An $O(h^2)$ parallelepiped is triangulated into six tetrahedra

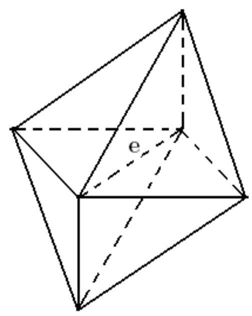

FIGURE 3. Another type of the patch Ω_e for $e \in \mathcal{E}_1$

for $i, j = 1, 2, 3$. Here τ and τ' are tetrahedra having a common edge. Then

$$|\sum_{\tau \in \mathcal{T}_h} \int_\tau \nabla(u - u_I) \cdot \mathcal{D}_\tau \nabla v_h| \lesssim h^{1+\min(\sigma,1)} \|u\|_{3,\infty,\Omega} |v|_{1,\Omega}.$$

Proof. Let $\mathcal{E} = \mathcal{E}_1 \bigoplus \mathcal{E}_2 \bigoplus \mathcal{E}_3$. $\mathcal{E}_1, \mathcal{E}_2$ are defined in the Definition 3.1 and $\mathcal{E}_3$ denotes the set of boundary edges. For each τ, we denote

$$\alpha_{klm} = \frac{\xi_{kl}}{4\cos\theta_{kl}}(d_{km}^2 - d_{lm}^2), \ \beta_{klm} = \frac{\xi_{kl}}{\cos\theta_{kl}} |\triangle_{klm}|.$$

Applying Lemma 2.5, we only need to estimate

$$\begin{aligned} I &= \sum_{\tau \in \mathcal{T}_h} \sum_{k,l=1,k\neq l}^{4} \frac{\partial v_h}{\partial \mathbf{t}_{kl}} \left[\alpha_{klm} \int_{F_k} \varphi_l \varphi_m \frac{\partial^2 u}{\partial \mathbf{t}_{kl}^2} + \beta_{klm} \int_{F_k} \varphi_l \varphi_m \frac{\partial^2 u}{\partial \mathbf{t}_{kl} \partial \mathbf{n}_{kl,m}} \right], \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_i = \sum_{e_{kl} \in \mathcal{E}_i} \sum_{\tau \in \Omega_{e_{kl}}} \frac{\partial v_h}{\partial \mathbf{t}_{kl}} \left[\alpha_{klm} \int_{F_k} \varphi_l \varphi_m \frac{\partial^2 u}{\partial \mathbf{t}_{kl}^2} + \beta_{klm} \int_{F_k} \varphi_l \varphi_m \frac{\partial^2 u}{\partial \mathbf{t}_{kl} \partial \mathbf{n}_{kl,m}} \right],$$

for $i = 1, 2, 3$. In above formulas, F_k, α_{klm} and β_{klm} are different for different tetrahedra. To simplify the notation, we omit the index τ.

It is easy to get

$$|\alpha_{klm}| \lesssim h^2, |\beta_{klm}| \lesssim h^2, \tag{5}$$

and

$$|\int_{F_k} \varphi_l \varphi_m \frac{\partial^2 u}{\partial \mathbf{t}_{kl}^2} \frac{\partial v_h}{\partial \mathbf{t}_{kl}}| \lesssim h^{-1} |u|_{2,\infty,\Omega} \int_\tau |\nabla v_h|. \tag{6}$$

Since we only consider the homogeneous Dirichlet boundary conditions, $I_3 = 0$. Now we estimate I_1. For $e \in \mathcal{E}_1$, the patch Ω_e is one of two types in Fig. 2 and Fig. 3. We first consider the case that Ω_e is an $O(h^2)$ parallelepiped. Without loss of generality, we index the vertices in Fig. 2. We will group the coefficients of $\frac{\partial v_h}{\partial \mathbf{t}_{17}}$

and $\frac{\partial v_h}{\partial \mathbf{t}_{71}}$ in different tetrahedra together to see the cancellation. In $\tau = \tau_{1567}$, we have term

$$\frac{\partial v_h}{\partial \mathbf{t}_{17}}\left[\alpha_{175}\int_{F_{567}}\varphi_7\varphi_5\frac{\partial^2 u}{\partial \mathbf{t}_{17}^2}+\beta_{175}\int_{F_{567}}\varphi_7\varphi_5\frac{\partial^2 u}{\partial \mathbf{t_{17}}\partial \mathbf{n}_{17,5}}\right],$$

while in $\tau' = \tau_{1347}$ we have

$$\frac{\partial v_h}{\partial \mathbf{t}_{71}}\left[\alpha'_{713}\int_{F_{134}}\varphi_1\varphi_3\frac{\partial^2 u}{\partial \mathbf{t}_{71}^2}+\beta'_{713}\int_{F_{134}}\varphi_1\varphi_3\frac{\partial^2 u}{\partial \mathbf{t_{71}}\partial \mathbf{n}'_{71,3}}\right].$$

Since Ω_e is an $O(h^2)$ parallelepiped, it is easy to see that

$$\begin{aligned}
&|\alpha_{175}-\alpha'_{713}| \lesssim h^3,\ |\beta_{175}-\beta'_{713}| \lesssim h^3,\\
&\Big|\int_{F_{567}}\varphi_7\varphi_5\frac{\partial^2 u}{\partial \mathbf{t}_{17}^2}-\int_{F_{134}}\varphi_1\varphi_3\frac{\partial^2 u}{\partial \mathbf{t}_{17}^2}\Big| \lesssim \|u\|_{3,\infty,\Omega}h^3,\ \text{and}\\
&\Big|\int_{F_{567}}\varphi_7\varphi_5\frac{\partial^2 u}{\partial \mathbf{t}_{17}\partial \mathbf{n}_{17,3}}-\int_{F_{134}}\varphi_1\varphi_3\frac{\partial^2 u}{\partial \mathbf{t}_{71}\partial \mathbf{n}'_{71,5}}\Big| \lesssim \|u\|_{3,\infty,\Omega}h^3.
\end{aligned}$$

By paring all coefficients of $\frac{\partial v_h}{\partial \mathbf{t}_{17}}$ and $\frac{\partial v_h}{\partial \mathbf{t}_{71}}$ in the similar way and using (5) and (6), we have

$$|I_1| \lesssim h^2\|u\|_{3,\infty,\Omega}|v_h|_{1,\Omega}.$$

For edges in $\mathcal{E}_2$, we simply use (5) and (6) to get

$$|I_2| \lesssim h^2\int_{\Omega_{2,h}}|\nabla^2 u||\nabla v_h| \lesssim h(\int_{\Omega_{2,h}}|\nabla^2 u|^2)^{1/2}|v_h|_{1,\Omega} \lesssim h^{1+\sigma}|u|_{2,\infty,\Omega}|\nabla v_h|_{1,\Omega}.$$

Combing them together, we get the desired estimate. For the patch in Fig. 3, we can also pair the coefficients in the similar way to get the desired estimate. □

With Lemma 3.2 we can prove a superconvergence result for the following elliptic equation

$$-\nabla\cdot(\mathcal{D}(x)\nabla u) = f,\ x\in\Omega, \tag{7}$$

$$u = 0,\ x\in\partial\Omega. \tag{8}$$

Here $\mathcal{D}(x)$ is a 3×3 symmetric matrix function in $(L^\infty(\Omega))^{3\times 3}$ and uniformly positive definite. The weak form of (7) and (8) is to find $u \in H_0^1(\Omega)$ such that

$$B(u,v) = \int_\Omega \nabla v\cdot(\mathcal{D}\nabla u) = f(v) \tag{9}$$

for all $v \in H_0^1(\Omega)$.

In order to insure that (9) has a unique solution, we assume that the eigenvalues of $\mathcal{D}$ satisfy $0 < \mu \le \lambda_{\min} \le \lambda_{\max} \le \nu$ uniformly in Ω. Let $\mathcal{V}_h \subset H_0^1(\Omega)$ be the space of continuous piecewise linear polynomials associated with a quasi-uniform triangulation $\mathcal{T}_h$, and consider the approximate problem: find $u_h \in \mathcal{V}_h$ such that

$$B(u_h, v_h) = f(v_h) \tag{10}$$

for all $v_h \in \mathcal{V}_h$. The following result is standard in FEM

$$\|u-u_h\|_{1,\Omega} \le \frac{\nu}{\mu}\inf_{v_h\in V_h}\|u-v_h\|_{1,\Omega}.$$

We define the piecewise constant matrix function $\mathcal{D}_\tau$ in terms of the diffusion matrix $\mathcal{D}$ as follows:

$$\mathcal{D}_{\tau ij} = \frac{1}{|\tau|}\int_\tau \mathcal{D}_{ij}dx.$$

Note that $\mathcal{D}_\tau$ is symmetric and positive definite.

Theorem 3.3. *Assume that the solution of (9) satisfies $u \in W^{3,\infty}(\Omega)$. Further, assume the hypotheses of Lemma 3.2, with $\mathcal{D}_\tau$ defined as above. Then*

$$\|u_h - u_I\|_{1,\Omega} \lesssim h^{1+\min(\sigma,1)}\|u\|_{3,\infty,\Omega}.$$

Proof. We begin with the identity

$$\begin{aligned} B(u - u_I, v_h) &= \sum_{\tau\in\mathcal{T}_h}\int_\tau \nabla(u-u_I)\cdot\mathcal{D}_\tau\nabla v_h dx \\ &\quad + \sum_{\tau\in\mathcal{T}_h}\int_\tau \nabla(u-u_I)\cdot(\mathcal{D}-\mathcal{D}_\tau)\nabla v_h dx \\ &= I_1 + I_2. \end{aligned}$$

The first term I_1 can be estimated using Lemma 3.2. The second term I_2 can be easily estimated by

$$|I_2| \lesssim h^2\|u\|_{2,\Omega}\|v\|_{1,\Omega}.$$

Thus

$$|B(u-u_I, v_h)| \lesssim h^{1+\min(\sigma,1)}\|u\|_{3,\infty,\Omega}\|v_h\|_{1,\Omega}.$$

We complete the proof by choosing $v_h = u_h - u_I$ and using the fact that

$$\mu\|u_h-u_I\|_{1,\Omega}^2 \le B(u_h-u_I, u_h-u_I) = B(u-u_I, u_h-u_I) \le \nu\|u-u_I\|_{1,2,\Omega}\|u_h-u_I\|_{1,2,\Omega}.$$

□

4. A Gradient Recovery Algorithm

Once we get the supercloseness between ∇u_h and ∇u_I, we can develop post-processing schemes to improve the approximation order of ∇u_h. In this section, we show that $Q_h\nabla u_I$ is superconvergent to ∇u for $O(h^2)$ irregular meshes, where Q_h is the L^2 projection to $\mathcal{V}_h^3$, namely $(Q_h\mathbf{u}, \mathbf{v}_h) = (\mathbf{u}, \mathbf{v}_h)$ for all $\mathbf{u} \in (L^2(\Omega))^3$ and $\mathbf{v}_h \in \mathcal{V}_h^3$.

Lemma 4.1.

$$\int_\tau (u_I - u) = \frac{1}{40}\int_\tau \sum_{i,j=1,i<j}^{4} d_{ij}^2\frac{\partial^2 u_q}{\partial \mathbf{t}_{ij}^2} + \int_\tau (u_q - u),$$

where u_q is the quadratic interpolant of u.

Proof. The proof is similar to that in Lemma 2.2. Here we use the fact that

$$\int_\tau \varphi_i\varphi_j = \frac{|\tau|}{20}.$$

□

Lemma 4.2. *Let $u \in W^{3,\infty}(\Omega)$ and assume the hypotheses of Lemma 3.2 hold. Then*

$$\|\nabla u - Q_h\nabla u_I\|_{0,\Omega} \lesssim h^{1+\min(\sigma,1/2)}\|u\|_{3,\infty,\Omega}.$$

Proof. Given $\mathbf{v}_h \in \mathcal{V}_h^3$, we have

$$(Q_h\nabla(u-u_I), \mathbf{v}_h) = (\nabla(u-u_I), \mathbf{v}_h) = (u_I - u, \nabla\cdot\mathbf{v}_h) + \int_{\partial\Omega}(u-u_I)\mathbf{v}_h\cdot\mathbf{n}. \tag{11}$$

We estimate the two terms on the right-hand side of (11). First,

$$|\int_{\partial\Omega}(u-u_I)\mathbf{v}_h\cdot\mathbf{n}| \lesssim h^{3/2}|u|_{2,\infty,\Omega}\|v\|_{0,\Omega}. \tag{12}$$

For the other, we use Lemma 4.1 to get

$$\begin{aligned}
\int_\tau (u_I - u)\nabla\cdot\mathbf{v}_h &= \frac{1}{40}\int_\tau \sum_{i,j=1,i\neq j}^4 d_{ij}^2 \frac{\partial^2 u_q}{\partial \mathbf{t}_{ij}^2}\nabla\cdot\mathbf{v}_h + \int_\tau (u-u_q)\nabla\cdot\mathbf{v}_h \\
&= \frac{1}{40}\int_\tau \sum_{i,j=1,i\neq j}^4 d_{ij}^2 \frac{\partial^2 u}{\partial \mathbf{t}_{ij}^2}\nabla\cdot\mathbf{v}_h \\
&\quad + \frac{1}{40}\int_\tau \sum_{i,j=1,i\neq j}^4 d_{ij}^2 \frac{\partial^2 (u_q-u)}{\partial \mathbf{t}_{ij}^2}\nabla\cdot\mathbf{v}_h + \int_\tau (u-u_q)\nabla\cdot\mathbf{v}_h \\
&= \frac{1}{40}\sum_{k=1}^4 \sum_{i,j=1,i\neq j}^4 \int_{F_k} d_{ij}^2 \frac{\partial^2 u}{\partial \mathbf{t}_{ij}^2}\mathbf{v}_h\cdot\mathbf{n}_k + \frac{1}{40}\int_\tau \sum_{i,j=1,i\neq j}^4 d_{ij}^2 \nabla\frac{\partial^2 u}{\partial \mathbf{t}_{ij}^2}\mathbf{v}_h \\
&\quad \frac{1}{40}\int_\tau \sum_{i,j=1,i\neq j}^4 d_{ij}^2 \frac{\partial^2 (u_q-u)}{\partial \mathbf{t}_{ij}^2}\nabla\cdot\mathbf{v}_h + \int_\tau (u-u_q)\nabla\cdot\mathbf{v}_h \\
&= I_1 + I_2 + I_3 + I_4
\end{aligned}$$

Easy estimates show

$$|I_3| + |I_4| \lesssim h^3\|u\|_{3,\tau}|\mathbf{v}_h|_{1,\tau} \lesssim h^2\|u\|_{3,\tau}\|\mathbf{v}_h\|_{0,\tau}, \text{ and } |I_2| \lesssim h^2\|u\|_{3,\tau}\|\mathbf{v}_h\|_{0,\tau}.$$

Now we estimate

$$\begin{aligned}
\sum_{\tau\subset\mathcal{T}_h} I_1 &= \frac{1}{40}\Big[\sum_{F_k\subset\partial\Omega}\sum_{\tau\supset F_k}\int_{F_k}\sum_{i,j=1,i<j}^4 d_{ij}^2\frac{\partial^2 u}{\partial \mathbf{t}_{ij}^2}\mathbf{v}_h\cdot\mathbf{n}_k \\
&\quad + \sum_{F_k\subset\Omega}\sum_{\tau\supset F_k}\int_{F_k}\sum_{i,j=1,i<j}^4 d_{ij}^2\frac{\partial^2 u}{\partial \mathbf{t}_{ij}^2}\mathbf{v}_h\cdot\mathbf{n}_k\Big] \\
&= I_{11} + I_{12.}
\end{aligned}$$

Since we only consider the homogeneous Dirichlet boundary condition, $I_{11} = 0$. For I_{12}, we first notice that for each interior triangle, there are two tetrahedra containing it with opposite normal direction. Thus I_{12} can be simplified in this way

$$I_{12} = \sum_{i=1}^2 \sum_{e_{kl}\in\mathcal{E}_i}\sum_{\tau\in\Omega_{e_{kl}}}\int_{F_k} d_{kl}^2\frac{\partial^2 u}{\partial \mathbf{t}_{kl}^2}\mathbf{v}_h\cdot\mathbf{n}_k.$$

Then following the pattern of the proof in Lemma 3.2, we get

$$|(Q_h\nabla(u-u_I),\mathbf{v}_h)| \lesssim h^{1+\min(\sigma,1/2)}\|u\|_{3,\infty,\Omega}\|\mathbf{v}_h\|_{0,\Omega.}$$

Taking $\mathbf{v}_h = Q_h\nabla(u-u_I)$, we find that

$$\|Q_h\nabla(u-u_I)\|_{0,\Omega} \lesssim h^{1+\min(\sigma,1/2)}\|u\|_{3,\infty,\Omega}.$$

Lemma 4.2 now follows from the triangle inequality

$$\|\nabla u - Q_h\nabla u_I\|_{0,\Omega} \le \|\nabla u - Q_h\nabla u\|_{0,\Omega} + \|Q_h\nabla(u-u_I)\|_{0,\Omega}.$$

□

An immediate consequence of Theorems 3.3 and Lemma 4.2 is

Theorem 4.3. *Let $u\in W^{3,\infty}(\Omega)$ and assume the hypotheses of Theorem 3.3 and Lemma 4.2 are valid. Then*

$$\|\nabla u - Q_h\nabla u_h\|_{0,\Omega} \lesssim h^{1+\min(\sigma,1/2)}\|u\|_{3,\infty,\Omega}.$$

Proof. Using the triangle inequality

$$\|\nabla u - Q_h \nabla u_h\|_{0,\Omega} \le \|\nabla u - Q_h \nabla u_I\|_{0,\Omega} + \|Q_h \nabla(u_I - u_h)\|_{0,\Omega}, \tag{13}$$

estimate the two terms on the right-hand side of (13) by Theorem 3.3 and Lemma 4.2. □

Remark 4.4. *The superconvergence rate in Theorem 4.3 is weaker than that in Theorem 3.3 due to boundary constrain (12).*

Acknowledgments

The author is grateful to Prof. Jinchao Xu and Dr. Pengtao Sun for numerous discussions and to Prof. Jinchao Xu, Prof. J. Brandts and the referee for kind edit of the English.

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