INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 2, Pages 232–254

WAVEFORM RELAXATION METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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(Communicated by Edward J. Allen)

Abstract. L^p-convergence of waveform relaxation methods (WRMs) for numerical solving of systems of ordinary stochastic differential equations (SDEs) is studied. For this purpose, we convert the problem to an operator equation $X = \mathbb{T}X + G$ in a Banach space $\mathcal{E}$ of $\mathcal{F}_t$-adapted random elements describing the initial- or boundary value problem related to SDEs with weakly coupled, Lipschitz-continuous subsystems. The main convergence result of WRMs for SDEs depends on the spectral radius of a matrix associated to a decomposition of $\mathbb{T}$. A generalization to one-sided Lipschitz continuous coefficients and a discussion on the example of singularly perturbed SDEs complete this paper.

Key Words. waveform relaxation methods, stochastic differential equations, stochastic-numerical methods, iteration methods, large scale systems

1. Introduction

The solution of complex and large scale systems plays a crucial role in recent scientific computations. In particular, large scale stochastic dynamical systems represent very complex systems incorporating the random appearances of physical processes in nature. The development of efficient numerical methods to study such large scale systems, which can be characterized as weakly coupled subsystems with quite different behavior, is an important challenge. Under some conditions, block-iterative methods are very efficient. One of these methods to solve large scale systems is given by the *waveform relaxation method*. This method was first proposed by Lelarasmee, Ruehli and Sangiovanni–Vincentelli [27] for the time-domain analysis of large scale integrated circuits. For the waveform algorithm concerning deterministic processes and related aspects, many research papers can be found, e.g. Bremer and Schneider [4], Bremer [5], Burrage [6], in't Hout [12], Jackiewicz and Kwapisz [16], Jansen et al. [17], Jansen and Vandewalle [18], Leimkuhler [25, 26], Miekkala and Nevanlinna [30], Nevanlinna and Odeh [32], Sand and Burrage [36], Schneider [37, 38, 39], Ta'asan and Zhang [44], Zennaro [48], Zubik–Koval and Vandewalle [50], among many others.

In what follows we present a theoretical foundation for the construction and convergence of waveform iterations applied to systems of ordinary stochastic differential equations (SDEs) which are decomposable into weakly coupled subsystems. The attention is restricted to Itô-interpreted SDEs and L^p-solutions (i.e. strong solutions in the Banach space of $L^p(\Omega, \mathcal{F}, \mathbb{P})$-integrable random processes). For

Received by the editors October 29, 2004.

2000 *Mathematics Subject Classification.* 65C30, 65L20, 65D30, 34F05, 37H10, 60H10.

This research was supported by Weierstrass-Institute, University of Minnesota, Universidad de Los Andes, Southern Illinois University and Texas Tech University.

original works on stochastic integration, see Itô [13, 14, 15]. For basic aspects on the theory of SDEs in the spirit of Itô [13], see e.g. Arnold [1, 2], Dynkin [8], Gard [10], Khas'minskij [19], Krylov [22], Mao [28], Protter [34] and Revuz and Yor [35].

We see our main contribution in deriving precise bounds for the Lipschitz-constants of the related stochastic integral operator and in describing their influence on the L^p-convergence of waveform iteration methods depending on the splitting into subsystems. However, a qualitative comparison with other numerical techniques for stochastic differential equations (SDEs) is left to the interested reader.

The paper is organized as follows. In Section 2 we describe the key idea of waveform relaxation method. Section 3 presents a proof for the existence and uniqueness of an initial value problem for Itô-type stochastic differential equations (SDEs) using fixed point techniques on appropriate Banach spaces in order to derive conditions for the L^p-convergence of waveform relaxation methods with $p \geq 2$. Section 4 generalizes this idea to the case of one-sided Lipschitz-continuity of the drift part, restricted to drift coefficients satisfying an angle condition. An illustrative example is given in Section 5. Section 6 closes this paper with some concluding remarks.

2. The general idea of waveform relaxation methods

At first we convert the initial-value problem problem related to Itô-interpreted stochastic differential equations into a fixed point problem. Therefore, we can consider

$$x \;=\; \top\!\!\top x \,+\, g \tag{1}$$

where $\top\!\!\top$ maps the function space $\mathcal{U}$ into itself, and $g \in \mathcal{U}$. There are several techniques to find appropriate conditions on the operator $\top\!\!\top$ guaranteeing a unique solution $x^* \in \mathcal{U}$ of system (1) and resulting in an efficient algorithm to approximate x^*. In the case that (1) represents a network of weakly connected subsystems with quite different behavior, i.e. (1) carries the feature of a large scale system, the *waveform relaxation method* is an efficient approach to approximate x^*, formulated as follows:

(i) *Decomposition step*: Find a suitable representation of the space $\mathcal{U}$ as a product of subspaces $\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_n$, i.e.

$$\mathcal{U} \;=\; \mathcal{U}_1 \times \mathcal{U}_2 \times ... \times \mathcal{U}_n\,, \tag{2}$$

and a corresponding splitting of $\top\!\!\top$ into $\top\!\!\top_1, \ldots, \top\!\!\top_n$ and g into $g_1, ..., g_n$ such that the fixed point problem (1) is equivalent to the system

$$\begin{array}{rcl} x^{(1)} & = & \top\!\!\top_1(x^{(1)}, ..., x^{(n)}) + g_1, \\ x^{(2)} & = & \top\!\!\top_2(x^{(1)}, ..., x^{(n)}) + g_2, \\ \ldots\ldots & .. & \ldots\ldots\ldots\ldots\ldots\ldots \\ x^{(n)} & = & \top\!\!\top_n(x^{(1)}, ..., x^{(n)}) + g_n \end{array} \tag{3}$$

where $x^{(k)}, g_k \in \mathcal{U}_k$, and $\top\!\!\top_k$ maps $\mathcal{U}$ into the subspace $\mathcal{U}_k$ for $k = 1, 2, ..., n$.

(ii) *Solution step*: By an appropriate procedure, solve the k-th subsystem

$$x^{(k)} \;=\; \top\!\!\top_k(x^{(1)}, ..., x^{(k-1)}, x^{(k)}, x^{(k+1)}, ..., x^{(n)}) \,+\, g_k\,. \tag{4}$$

Here, $x^{(j)}, j = 1, 2, ..., n$ with $j \neq k$ are the inputs from other subsystems.

(iii) *Relaxation step*: Derive conditions such that the successive solution of subsystems (4) leads to the unique solution of the large scale system (of SDEs, specified later)

An alternative approach based on monotone iteration techniques is described in Ladde, Lakshmikantham and Vatsala [23] or Zhao [49]. We follow the fairly general idea of waveform iterations as originally suggested by Lelarasmee, Ruehli and Sangiovanni–Vincentelli [27], summarized by steps (i) - (iii) and specified for systems of SDEs. As commonly known, the work out of (iii) is strongly connected with efficient proofs for the existence and uniqueness of solutions of related operator equations and heavily depends on their specific structure. Moreover, the steps (ii) and (iii) can be also combined to some "diagonalized" iteration scheme (see Schneider [39] for details). In the case of the Gauss–Jacobi procedure

$$\begin{aligned} x_i^{(1)} &= \top\!\!\top_1(x_i^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}, ..., x_{i-1}^{(n-1)}, x_{i-1}^{(n)}) + g_1, \\ x_i^{(2)} &= \top\!\!\top_2(x_{i-1}^{(1)}, x_i^{(2)}, x_{i-1}^{(3)}, ..., x_{i-1}^{(n-1)}, x_{i-1}^{(n)}) + g_2, \\ \ldots\ldots & \;..\; \ldots\ldots\ldots\ldots\ldots\ldots \\ x_i^{(n)} &= \top\!\!\top_n(x_{i-1}^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}, ..., x_{i-1}^{(n-1)}, x_i^{(n)}) + g_n \end{aligned} \tag{5}$$

we get the diagonalized iteration scheme

$$\begin{aligned} x_i^{(1)} &= \top\!\!\top_1(x_{i-1}^{(1)}, x_{i-1}^{(2)}, \ldots, x_{i-1}^{(n)}) + g_1, \\ x_i^{(2)} &= \top\!\!\top_2(x_{i-1}^{(1)}, x_{i-1}^{(2)}, \ldots, x_{i-1}^{(n)}) + g_2, \\ \ldots\ldots & \;..\; \ldots\ldots\ldots\ldots\ldots\ldots \\ x_i^{(n)} &= \top\!\!\top_n(x_{i-1}^{(1)}, x_{i-1}^{(2)}, \ldots, x_{i-1}^{(n)}) + g_n \end{aligned} \tag{6}$$

which represents a block Picard iteration. To prove the convergence of (6) we assume

- (H_1) For $k = 1, ..., n$, $\mathcal{U}_k$ is a complete metric space with norm $||.||_k$.
- (H_2) For $k = 1, \ldots, n$, $\top\!\!\top_k : \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n \to \mathcal{U}_k$ is a globally Lipschitz continuous, nonlinear mapping, i.e.

$$\begin{aligned} &||\top\!\!\top_k(x^{(1)}, \ldots, x^{(n)}) - \top\!\!\top_k(\bar{x}^{(1)}, \ldots, \bar{x}^{(n)})||_k \\ &\quad \le\; l_{k1}||x^{(1)} - \bar{x}^{(1)}||_1 + \cdots + l_{kn}||x^{(n)} - \bar{x}^{(n)}||_n \end{aligned} \tag{7}$$

for all $x^{(1)}, \bar{x}^{(1)} \in \mathcal{U}_1, \ldots, x^{(n)}, \bar{x}^{(n)} \in \mathcal{U}_n$.

Let $L := (l_{kj})_{1 \le k,j \le n}$, be the matrix of Lipschitz constants l_{kj} of operators $\top\!\!\top_k$.

Theorem 2.1. *We assume the hypotheses (H_1) and (H_2) to be satisfied. Under the additional assumption that the spectral radius $\varrho(L)$ of matrix L is lesser than one, the iteration scheme (6) converges in $\mathcal{U}$ with respect to an appropriate norm $|||.|||$ (for its definition, see (9) in the proof below).*

Proof. Without loss of generality we may assume that all entries of L are strictly positive. Then, by a theorem of Perron (see [9]), the fact $\varrho(L) < 1$ implies that $\varrho(L)$ is an eigenvalue of L to which an eigenfunction e with strictly positive components $e_1, \ldots, e_n$ exists. From (6) and (7), for $k = 1, \ldots, n$, we get to

$$||x_i^{(k)} - \bar{x}_{i-1}^{(k)}||_k \le l_{k1}||x_{i-1}^{(1)} - \bar{x}_{i-2}^{(1)}||_1 + \cdots + l_{kn}||x_{i-1}^{(n)} - \bar{x}_{i-2}^{(n)}||_n.$$

Hence, we have

$$\begin{aligned} &e_1||x_i^{(1)} - x_{i-1}^{(1)}||_1 + \cdots + e_n||x_i^{(n)} - x_{i-1}^{(n)}||_n \\ &\quad\le\; (e_1 l_{11} + e_2 l_{21} + \cdots + e_n l_{n1})||x_{i-1}^{(1)} - x_{i-2}^{(1)}||_1 + \ldots \\ &\qquad +(e_1 l_{n1} + e_2 l_{n2} + \cdots + e_n l_{nn})||x_{i-1}^{(n)} - x_{i-2}^{(n)}||_n \\ &\quad=\; \varrho(L)\left(e_1||x_{i-1}^{(1)} - x_{i-2}^{(1)}||_1 + \cdots + e_n||x_{i-1}^{(n)} - x_{i-2}^{(n)}||_n\right). \end{aligned} \tag{8}$$

Now we introduce a norm $|||.|||$ on $\mathcal{U} := \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$ by

$$|||x||| := e_1||x^{(1)}||_1 + \cdots + e_n||x^{(n)}||_n. \tag{9}$$

Using this norm we obtain from (8) that $|||x_i - x_{i-1}||| \le \varrho(L)\, |||x_{i-1} - x_{i-2}|||$. Thus, the iteration scheme (6) is convergent in $\mathcal{U}$ with respect to the norm (9), provided that $\varrho(L) < 1$. $\diamond$ □

Similar convergence results can be derived for modified schemes. The iterative methods to solve the subsystems can be applied in form of Gauss–Jacobi, Gauss–Seidel, successive overrelaxation (SOR) or Picard iterations in general, where the related spectral radii control the convergence of these algorithms in appropriate Banach spaces. For example, if we replace the Gauss–Jacobi procedure (6) by the Gauss–Seidel iteration

$$\begin{array}{rcl} x_i^{(1)} & = & \Pi_1(x_{i-1}^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}, ..., x_{i-1}^{(n-1)}, x_{i-1}^{(n)}) + g_1, \\ x_i^{(2)} & = & \Pi_2(x_i^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}, ..., x_{i-1}^{(n-1)}, x_{i-1}^{(n)}) + g_2, \\ .... & .. & .... \\ x_i^{(n)} & = & \Pi_n(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, ..., x_i^{(n-1)}, x_{i-1}^{(n)}) + g_n, \end{array} \tag{10}$$

then the corresponding matrix $\tilde{L} = (\tilde{l}_{k,j})$ of Lipschitz-constants can be determined from the estimates

$$\begin{array}{rcl} \Delta x_i^{(1)} & \le & l_{11}\Delta x_{i-1}^{(1)} + l_{12}\Delta x_{i-1}^{(2)} + \cdots + l_{1n}\Delta x_{i-1}^{(n)}, \\ \Delta x_i^{(2)} & \le & l_{21}l_{11}\Delta x_i^{(1)} + (l_{21}l_{12} + l_{22})\Delta x_{i-1}^{(2)} + \cdots + (l_{21}l_{1n} + l_{2n})\Delta x_{i-1}^{(n)}, \\ .... & .. & .... \end{array} \tag{11}$$

where $\Delta x_i^{(k)} = ||x_i^{(k)} - x_{i-1}^{(k)}||_k$. For example, for the Gauss–Seidel iteration

$$\begin{array}{rcl} x_i^{(1)} & = & \Pi_1(x_{i-1}^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}) + g_1, \\ x_i^{(2)} & = & \Pi_2(x_i^{(1)}, x_{i-1}^{(2)}, x_{i-1}^{(3)}) + g_2, \\ x_i^{(3)} & = & \Pi_n(x_i^{(1)}, x_i^{(2)}, x_{i-1}^{(3)}) + g_3 \end{array} \tag{12}$$

in the case $n = 3$, we obtain that the matrix $\tilde{L}$ is equal to

$$\begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21}l_{11} & l_{21}l_{12} + l_{22} & l_{21}l_{13} + l_{23} \\ l_{31}l_{11} + l_{32}l_{21}l_{11} & l_{31}l_{12} + l_{32}(l_{21}l_{12} + l_{22}) & l_{31}l_{13} + l_{33} + l_{32}(l_{21}l_{13} + l_{23}) \end{pmatrix}.$$

Thus, $\varrho(\tilde{L}) < 1$ implies the convergence of the iteration scheme (12). Consequently, Theorem 2.1 can be modified for this iteration scheme as well. General convergence theorems for iteration methods are also found in standard references, e.g. Zeidler [47].

Remark 2.1. *Theorem 2.1 is applicable to operators describing deterministic as well as stochastic processes. The main problem to be tackled in applying the waveform relaxation method to stochastic systems consists of estimating the influence of stochastic terms on the Lipschitz-constants. A first approach is presented in the next section.*

Remark 2.2. *It is worth noting that system (5) permits the application of multi-processor computers (parallel computing) – a fact which renders the waveform algorithm to be very attractive for numerical solving of large scale systems.*

3. Waveform relaxation methods for SDEs

3.1. Notation and main assumptions. Let $< .,. >_d$ denote the Euclidean scalar product defined by $< x,y >_d = \sum_{i=1}^d x_i y_i$ for vectors x,y in $\mathbb{R}^d$, $d \geq 1$ the current dimension, and $\|.\|_d$ the Euclidean vector norm in $\mathbb{R}^d$. Throughout this paper $\mathcal{B}^d$ represents the set of all Borel-measurable sets of $\mathbb{R}^d$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space, and $\mathcal{T} = [0,T]$ a fixed finite time interval. Suppose that $(\mathcal{F}_t)_{t\in\mathcal{T}}$ performs a filtration such that $(\Omega, \mathcal{F}, \mathcal{F}_{t\in\mathcal{T}}, \mathbb{P})$ presents a complete stochastic basis. In the following we consider only $\mathcal{F}_t$-adapted stochastic processes $(X_t)_{t\in\mathcal{T}}$ defined on $(\Omega, \mathcal{F}, \mathcal{F}_{t\in\mathcal{T}}, \mathbb{P})$, with finite p-th absolute moments for all times $t \in \mathcal{T}$, where $p \geq 1$. Recall that a stochastic process is called "cadlag (a.s.)" if and only if all trajectories are continuous from the right side, and left hand limits exist almost surely (almost surely with respect to probability measure $\mathbb{P}$).

Definition 3.1. *The space $\mathcal{E}_{p,d}$ is defined to be*

(13)
$$\mathcal{E}_{p,d} := \left\{ (X_t)_{0\leq t\leq T} : \begin{array}{l} X_t = X_t(\omega) \text{ is a cadlag (a.s.) stochastic process,} \\ X_t(\omega) : [0,T] \times (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}) \longrightarrow (\mathbb{R}^d, \mathcal{B}^d), \\ X_t \text{ is } \mathcal{F}_t\text{-adapted, } \mathbb{E} \sup_{0\leq t\leq T} \|X_t\|_d^p < +\infty \end{array} \right\}$$

and the space $\mathcal{E}^0_{p,d}$

(14)
$$\mathcal{E}^0_{p,d} := \left\{ (X_t)_{0\leq t\leq T} : \begin{array}{l} X_t = X_t(\omega) \text{ is a continuous (a.s.) stochastic process,} \\ X_t(\omega) : [0,T] \times (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}) \longrightarrow (\mathbb{R}^d, \mathcal{B}^d), \\ X_t \text{ is } \mathcal{F}_t\text{-adapted, } \mathbb{E} \max_{0\leq t\leq T} \|X_t\|_d^p < +\infty \end{array} \right\}.$$

Proposition 3.1. *The spaces $\mathcal{E}_{p,d}, \mathcal{E}^0_{p,d}$ are Banach spaces with respect to the norm*

$$\|X\|_{\mathcal{E}_{p,d}} = \left(\mathbb{E} \sup_{0\leq t\leq T} \|X_t\|_d^p \right)^{1/p} \tag{15}$$

for $X \in \mathcal{E}_{p,d}$ or $X \in \mathcal{E}^0_{p,d}$, respectively.

Proof. The proofs of this assertion for $\mathcal{E}_{p,d}$ and $\mathcal{E}^0_{p,d}$ are similar, hence we restrict ourselves to the case of $\mathcal{E}^0_{p,d}$. The fact that $\mathcal{E}^0_{p,d}$ is a normed linear space follows from the linearity of $\mathbb{E}$-operation and properties of vector norm $\|.\|_d$ in $\mathbb{R}^d$. It remains to show the completeness of $\mathcal{E}^0_{p,d}$. Let $(X^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{E}^0_{p,d}$. That is that, we know that $\forall\varepsilon > 0\ \exists n_0(\varepsilon) \in \mathbb{N}\ \forall n,m \geq n_0(\varepsilon)\ :\ \|X^{(n)} - X^{(m)}\|_{\mathcal{E}^0_{p,d}} < \varepsilon$. Let $X^{(n)}$ converge to $\hat{X}$. Then, for all $n,m \geq n_0(\varepsilon)$, it follows that

$$\begin{aligned} \|\hat{X} - X^{(m)}\|^p_{\mathcal{E}^0_{p,d}} &= \mathbb{E} \sup_{0\leq t\leq T} \|\hat{X}_t - X_t^{(m)}\|_d^p \leq \sup_{n\geq m} \left(\mathbb{E} \sup_{0\leq t\leq T} \|X_t^{(n)} - X_t^{(m)}\|_d^p \right) \\ &\leq \varepsilon^p. \end{aligned}$$

Hence, by the Lemma of Fatou (see Bauer [5], p. 92), we get $\hat{X} - X^{(m)} \in \mathcal{E}^0_{p,d}$ for all $m \geq n_0(\varepsilon)$. Therefore, the proof is completed by

$$\hat{X} = \hat{X} - X^{(m)} + X^{(m)} \in \mathcal{E}^0_{p,d}\,.\diamond$$

□

Remark 3.1. *For $p = 2$, the function spaces $\mathcal{E}_{p,d}, \mathcal{E}^0_{p,d}$ form Hilbert spaces endowed with the naturally induced scalar product. For fixed parameters p, d, one finds the natural inclusion $\mathcal{E}^0_{p,d} \subset \mathcal{E}_{p,d}$.*

Our goal is to study the class of Itô-interpreted stochastic differential equations (SDEs) in conjunction with convergence of waveform relaxation methods. Let W^1_t, W^2_t, ..., W^m_t be m given independent, one-dimensional Wiener processes adapted to the filtration $\mathcal{F}_t$. Define $W^0_t = t$ for all $t \in [0, T]$. Consider the initial value problem

$$\begin{aligned} dX_t &= \sum_{j=0}^{m} f_j(t, X_t)\, dW^j_t \\ X_0 &= x_0(\omega) \text{ fixed and } \mathcal{F}_0\text{−measurable},\ 0 \le t \le T, \end{aligned} \tag{16}$$

driven by the Wiener process $W_t = (W^1_t, W^2_t, ..., W^m_t)$. We need to derive conditions on the functions f_j in order to guarantee the convergence of approximations based on waveform relaxation methods to the unique solution of (16) within the space $\mathcal{E}^0_{p,d}$. For this purpose, we take into account the following splitting of the d-dimensional system (16) into n interacting subsystems of dimension d_k

$$\begin{aligned} dX^{(1)}_t &= \sum_{j=0}^{m} f_{1,j}(t, X^{(1)}_t, X^{(2)}_t, ..., X^{(n)}_t)\, dW^j_t, \\ dX^{(2)}_t &= \sum_{j=0}^{m} f_{2,j}(t, X^{(1)}_t, X^{(2)}_t, ..., X^{(n)}_t)\, dW^j_t, \\ .... &\ . \ .... \\ dX^{(n)}_t &= \sum_{j=0}^{m} f_{n,j}(t, X^{(1)}_t, X^{(2)}_t, ..., X^{(n)}_t)\, dW^j_t, \\ (X^{(1)}_0, X^{(2)}_0, ..., X^{(n)}_0) &= (x^{(1)}_0(\omega), x^{(2)}_0(\omega), ..., x^{(n)}_0(\omega)),\ 0 \le t \le T, \end{aligned} \tag{17}$$

where $f_j = (f_{1j}, f_{2j}, ..., f_{nj})^T, j = 0, 1, ..., m$, with $f_{kj} : [0, T] \times \mathrm{I\!R}^d \longrightarrow \mathrm{I\!R}^{d_k}$, $d = \sum_{k=1}^{n} d_k$, $k = 1, 2, ..., n$. Concerning the functions f_j we assume that

(A_0) $f_j, j = 0, 1, ..., m$ are Lebesgue-measurable.

(A_1) $f_{kj}(t, x), j = 0, 1, ..., m; k = 1, 2, ..., n$ are globally Lipschitz continuous in x, uniformly with respect to time $t \in [0, T]$, i.e. there are constants $L^{(i)}_{k,j} \in \mathrm{I\!R}^1$ $(i = 1, 2, ..., n)$ such that

$$\forall t \in [0, T] \quad \forall (x^{(1)}, ..., x^{(n)}), (y^{(1)}, ..., y^{(n)}) \in \mathrm{I\!R}^{d_1} \times ... \times \mathrm{I\!R}^{d_n}$$

$$\|f_{k,j}(t, x^{(1)}, ..., x^{(n)}) - f_{k,j}(t, y^{(1)}, ..., y^{(n)})\|_{d_k} \le \sum_{i=1}^{n} L^{(k)}_{i,j} \|x^{(i)} - y^{(i)}\|_{d_i}.$$

(A_2) For $k = 1, 2, ..., n$, and $j = 0, 1, ..., m$ we have

$$\sup_{0 \le t \le T} \inf_{y \in \mathrm{I\!R}^d} \|f_{k,j}(t, y)\| \le K_B < +\infty\,.$$

3.2. On existence and uniqueness of the solution of (16) using contractive operators. In this subsection we present a constructive proof for the existence and uniqueness of the solution of the Cauchy problem (16) taking into account the splitting (17). The goal of this procedure is to extract conditions for the convergence of waveform relaxation methods applied to SDEs. For this purpose, we make use of the representation of the Banach space $\mathcal{E}^0_{p,d}$ as the product space

$$\mathcal{E}^0_{p,d} \ = \ \mathcal{E}^0_{p,d_1} \times \mathcal{E}^0_{p,d_2} \times ... \times \mathcal{E}^0_{p,d_n}$$

with $d = \sum_{k=1}^n d_k$. The spaces $\mathcal{E}^0_{p,d_k}$ are equipped with the norm $\|.\|_{\mathcal{E}^0_{p,d_k}}$, hence they are Banach spaces according to the Proposition 3.1. Later we shall introduce an appropriate norm in $\mathcal{E}^0_{p,d}$ which renders $\mathcal{E}^0_{p,d}$ to be a Banach space (This new norm is equivalent to the norm given by (15)). Now, define the random operators $\mathbb{T}_k$ by

(18)
$$[\mathbb{T}_k \left(X^{(1)}, X^{(2)}, ..., X^{(n)} \right)]_t \ = \ X_0^{(k)} + \sum_{j=0}^{m} \int_0^t f_{k,j}(s, X_s^{(1)}, X_s^{(2)}, ..., X_s^{(n)})\, dW_s^j$$

for all $X^{(k)} \in \mathcal{E}^0_{p,d_k}$, mapping $\mathcal{E}^0_{p,d}$ into $\mathcal{E}^0_{p,d_k}$ $(k = 1, 2, ..., n)$. Then a solution of the initial value problem (17) is understood as a solution of the integral equations

$$[\mathbb{T}_k \left(X^{(1)}, X^{(2)}, ..., X^{(n)} \right)]_t \ = \ X_t^{(k)}, \ k = 1, 2, ..., n. \tag{19}$$

Introducing the operator $\mathbb{T} = (\mathbb{T}_1, ..., \mathbb{T}_n)$, a solution of (19) corresponds to a fixed point of the operator $\mathbb{T}$. The proof of the following theorem relies on the contractivity of operator $\mathbb{T}$ in the product Banach space $\mathcal{E}^0_{p,d}$.

Theorem 3.1. *Let $p \geq 1$. Assume that the given functions $f_{k,j}$ satisfy the conditions (A_0) - (A_2), and that $\mathbb{E}\,\|X_0^{(k)}\|^p_{d_k} < +\infty$ for all $k = 1, 2, ..., n; j = 0, 1, ..., m$. Then the initial value problem (17) has an unique, $\mathcal{F}_t$-adapted and continuous (a.s.) solution in the space $\mathcal{E}^0_{p,d}$.*

Proof. The proof is carried out in two main steps. First, we shall show that the decomposed operator $\mathbb{T}$ is a mapping from the Banach space $\mathcal{E}^0_{p,d}$ into itself. Second, the operator $\mathbb{T}$ forms a contraction in $\mathcal{E}^0_{p,d}$ with respect to appropriately constructed norm. Then Banach's fixed point theorem provides the conclusion of Theorem 3.1.
Step 1: We prove that $\|\mathbb{T}_k(X)\|_{\mathcal{E}^0_{p,d_k}} < +\infty$ for $X \in \mathcal{E}^0_{p,d}$ whenever the functions $f_{k,j}(t,x)$ fulfill assumptions (A_0) - (A_2). It is well-known that the linear-polynomial boundedness of Lipschitz-continuous functions f_{kj} can be verified under (A_0) - (A_2), i.e. there exist corresponding constants $c_0(f_{kj})$ and $c_1(f_{kj})$ such that

$$\forall t \in [0,T] \ \forall x \in \mathbb{R}^d \ : \ \|f(t,x)\|_d \ \leq \ c_0(f) + c_1(f)\|x\|_d.$$

Using the latter fact, the norm of images of operators $\top\!\!\top_k$ is estimated by

$$\begin{aligned}
\|[\top\!\!\top_k(X)]_t\|_{d_k}^p &\leq \Big(\|X_0^{(k)}\|_{d_k} + \|\int_0^t f_{k,0}(s,X_s)ds\|_{d_k} + \sum_{j=1}^m \|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}\Big)^p \\
&\leq (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + \|\int_0^t f_{k,0}(s,X_s)ds\|_{d_k}^p + \sum_{j=1}^m \|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big) \\
&\leq (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + t^{p-1}\int_0^t \|f_{k,0}(s,X_s)\|_{d_k}^p ds + \sum_{j=1}^m \|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big) \\
&\leq (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + 2^{p-1}(t^p c_0^p(f_{k,0}) + t^{p-1}c_1^p(f_{k,0})\int_0^t \|X_s\|_d^p ds)\Big) \\
&\quad + (m+2)^{p-1}\Big(\sum_{j=1}^m \|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big) \\
&\leq (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + 2^{p-1}t^p(c_0^p(f_{k,0}) + c_1^p(f_{k,0})\max_{0\leq u\leq t}\|X_u\|_d^p)\Big) \\
&\quad + (m+2)^{p-1}\Big(\sum_{j=1}^m \sup_{0\leq u\leq t} \|\int_0^u f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big) \\
&\leq (m+2)^{p-1}\Big(\|X_0^{(k)}\|_{d_k}^p + 2^{p-1}T^p(c_0^p(f_{k,0}) + c_1^p(f_{k,0})\max_{0\leq t\leq T}\|X_t\|_d^p)\Big) \\
&\quad + (m+2)^{p-1}\Big(\sum_{j=1}^m \sup_{0\leq t\leq T} \|\int_0^t f_{k,j}(s,X_s)dW_s^j\|_{d_k}^p\Big)
\end{aligned}$$

with appropriate constants $c_0(f_{k,0})$ and $c_1(f_{k,0})$ as mentioned above (Remember also $X_t = (X_t^{(1)}, ..., X_t^{(k)}, ..., X_t^{(n)})$). Using the Burkholder–Davis–Gundy inequality and basic properties of quadratic variation of Itô integrals with respect to Brownian motions W_s^j (see Revuz and Yor [35]), there are constants $c_{p,k,j}$ such that

$$\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq T}\left\|\int_0^t f_{k,j}(s,X_s)dW_s^j\right\|_{d_k}^p &\leq c_{p,k,j}\,\mathbb{E}\left(\int_0^T \|f_{k,j}(s,X_s)\|_{d_k}^2 d < W^j, W^j >_s\right)^{p/2} \\
&= c_{p,k,j}\,\mathbb{E}\left(\int_0^T \|f_{k,j}(s,X_s)\|_{d_k}^2 ds\right)^{p/2}
\end{aligned}$$

where $< M, M >_s$ denotes the total quadratic variation of inscribed martingale M on $[0,s]$. In fact, applying the Burkholder inequality as stated in Protter [34, p. 174–175] to continuous time, local martingales (here represented by stochastic Itô integrals) and the constants $c_{p,k,j}$ can be chosen universally, e.g.

$$c_{p,k,j} \leq \left(\left(\frac{p}{p-1}\right)^p \left(\frac{p(p-1)}{2}\right)\right)^{\frac{p}{2}}$$

for $p \geq 2$, see also Krylov [22, p. 160–163] for an alternative estimate with $p \in (0,+\infty)$. Note that a deterministic T naturally is a $\mathcal{F}_t$-stopping time and that here $f_{k,j}(s,X_s)$ are bounded in the sense of norm $\|.\|_{\mathcal{E}_{p,d_k}}$, thus one has the right to apply the Burkholder–Davis–Gundy inequality. Using this fact, returning to the estimation of $\|[\top\!\!\top_k(X)]_t\|_{d_k}^p$, taking supremum and expectation $\mathbb{E}$, one arrives at

$$
\begin{aligned}
\|\top_k(X)\|^p_{\mathcal{E}_{p,d_k}} &= \mathbb{E} \sup_{0\le t\le T} \|[\top_k(X)]_t\|^p_{d_k} \\
&\le (m+2)^{p-1}\Big(\mathbb{E}\,\|X_0^{(k)}\|^p_{d_k} + 2^{p-1}T^p(c_0^p(f_{k,0}) + c_1^p(f_{k,0})\|X\|^p_{\mathcal{E}_{p,d}})\Big) \\
&\quad +(m+2)^{p-1}2^{p/2-1}\Big(\sum_{j=1}^m c_{p,k,j}\,\mathbb{E}\left(\int_0^T (c_0^2(f_{k,j}) + c_1^2(f_{k,j})\|X_s\|^2_d)ds\right)^{p/2}\Big) \\
&\le (m+2)^{p-1}\Big(\mathbb{E}\,\|X_0^{(k)}\|^p_{d_k} + 2^{p-1}T^p(c_0^p(f_{k,0}) + c_1^p(f_{k,0})\|X\|^p_{\mathcal{E}^0_{p,d}})\Big) \\
&\quad +(m+2)^{p-1}2^{p/2-1}T^{p/2}\Big(\sum_{j=1}^m c_{p,k,j}(c_0^p(f_{k,j}) + c_1^p(f_{k,j})\|X\|^p_{\mathcal{E}^0_{p,d}})\Big) \;<\; +\infty\,,
\end{aligned}
$$

with appropriate constants $c_0(f_{k,j})$ and $c_1(f_{k,j})$ (see above), since $X \in \mathcal{E}^0_{p,d}$. That is, the images of operators $\top_k$ cannot blow up (a.s.) at finite times $t \in [0,T]$. Therefore, and thanks to integral construction of operators $\top_k$, the non-blowing up (a.s.) images of operators $\top_k$ are continuous (a.s.) and $\mathcal{F}_t$-adapted stochastic processes $\top_k(X) \in \mathcal{E}^0_{p,d_k}$ whenever the domain element X to which the operator $\top_k$ is applied lies in the space $\mathcal{E}^0_{p,d}$, and the functions $f_{k,j}$ are globally Lipschitz continuous (A_1). As a consequence, the decomposed operator $\top = (\top_1, ..., \top_n)$ represents a mapping from the closed space $\mathcal{E}^0_{p,d}$ into itself.
Step 2: It remains to show the property of contractivity of the operator $\top$ with respect to an appropriate norm of the product space $\mathcal{E}^0_{p,d}$. Assume that $X_0^{(k)} = Y_0^{(k)}$ (a.s.), $k = 1, 2, ..., n$. Set $\Delta\top_k(t) := [\top_k(X^{(1)}, ..., X^{(n)}) - \top_k(Y^{(1)}, ..., Y^{(n)})](t)$ for all $t \in [0,T]$, and $\Delta f_{k,j}(s) := f_{k,j}(s, X_s^{(1)}, ..., X_s^{(n)}) - f_{k,j}(s, Y_s^{(1)}, ..., Y_s^{(n)})$ for all $s \in [0,T]$. For any fixed $(X^{(1)}, ..., X^{(n)}), (Y^{(1)}, ..., Y^{(n)}) \in \mathcal{E}^0_{p,d}$ one has

$$
\begin{aligned}
\|\Delta\top_k(t)\|^p_{d_k} &\le \left(\left\|\int_0^t \Delta f_{k,0}(s)ds\right\|_{d_k} + \sum_{j=1}^m \left\|\int_0^t \Delta f_{k,j}(s)dW_s^j\right\|_{d_k}\right)^p \\
&\le (m+1)^{p-1}\left(\left\|\int_0^t \Delta f_{k,0}(s)ds\right\|^p_{d_k} + \sum_{j=1}^m \left\|\int_0^t \Delta f_{k,j}(s)dW_s^j\right\|^p_{d_k}\right) \\
&\le (m+1)^{p-1}\left(t^{p-1}\int_0^t \|\Delta f_{k,0}(s)\|^p_{d_k}\,ds + \sum_{j=1}^m \left\|\int_0^t \Delta f_{k,j}(s)dW_s^j\right\|^p_{d_k}\right)
\end{aligned}
$$

using the triangle inequality and using the Hölder inequality several times. We may estimate $\|\Delta f_{k,j}(s)\|^p_{d_k} \le n^{p-1}\sum_{i=1}^n (L_{i,j}^{(k)})^p \|X_s^{(i)} - Y_s^{(i)}\|^p_{d_i}$ under global Lipschitz-continuity of $f_{k,j}$ for $p \ge 1$. Therefore it follows that

$$
\begin{aligned}
\|\Delta\top_k(t)\|^p_{d_k} &\le (m+1)^{p-1}n^{p-1}t^p \sum_{i=1}^n (L_{i,0}^{(k)})^p \max_{0\le s\le t} \|X_s^{(i)} - Y_s^{(i)}\|^p_{d_i} \\
&\quad +(m+1)^{p-1}\sum_{j=1}^m \|\int_0^t \Delta f_{k,j}(s)dW_s^j\|^p_{d_k},
\end{aligned}
$$

hence

$$\begin{aligned}\|\Delta \Pi_k(t)\|_{d_k}^p &\le (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^{n} (L_{i,0}^{(k)})^p \max_{0\le t\le T} \|X_t^{(i)} - Y_t^{(i)}\|_{d_i}^p \\ &\quad + (m+1)^{p-1} \sum_{j=1}^{m} \sup_{0\le t\le T} \| \int_0^t \Delta f_{k,j}(s) dW_s^j \|_{d_k}^p .\end{aligned}$$

Now, by taking the operation of expectation $\mathbb{E}$ on both sides, this implies

$$\begin{aligned}\|\Delta \Pi_k\|_{\mathcal{E}_{p,d_k}}^p &= \mathbb{E} \sup_{0\le t\le T} \|\Delta \Pi_k(t)\|_{d_k}^p \\ &\le (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^{n} (L_{i,0}^{(k)})^p \|X^{(i)} - Y^{(i)}\|_{\mathcal{E}_{p,d_i}^0}^p \\ &\quad + (m+1)^{p-1} \sum_{j=1}^{m} \mathbb{E} \sup_{0\le t\le T} \| \int_0^t \Delta f_{k,j}(s) dW_s^j \|_{d_k}^p .\end{aligned}$$

The herein occurring terms $\int_0^t \Delta f_{k,j}(s) dW_s^j$ form continuous and $\mathcal{F}_t$-adapted martingales started at initial value 0 under the global Lipschitz-continuity (A_1) of functions $f_{k,j}$ and for $X^{(k)} \in \mathcal{E}_{p,d_k}^0$, where $k = 1, 2, ..., n; j = 1, 2, ..., m$. This can be shown in the same way as in step 1. Using the Burkholder–Davis–Gundy inequality and basic properties of quadratic variation of Itô integrals with respect to Brownian motions W_s^j (see Revuz and Yor [35, p. 153]), there are constants $C_{p,k,j}$ such that

$$\begin{aligned}\mathbb{E} \sup_{0\le t\le T} \left\| \int_0^t \Delta f_{k,j}(s) dW_s^j \right\|_{d_k}^p &\le C_{p,k,j} \mathbb{E} \left(\int_0^T \|\Delta f_{k,j}(s)\|_{d_k}^2 d < W^j, W^j >_s \right)^{p/2} \\ &= C_{p,k,j}\, \mathbb{E} \left(\int_0^T \|\Delta f_{k,j}(s)\|_{d_k}^2 ds \right)^{p/2}\end{aligned}$$

where $< M, M >_s$ denotes the total quadratic variation of inscribed martingale M on $[0, s]$. As already stated, we can find an universal estimate of $C_{p,k,j}$ arising from the Burkholder inequality (see Protter [34, p. 174–175], as before), e.g. with

$$C_{p,k,j} \le \left(\left(\frac{p}{p-1} \right)^p \left(\frac{p(p-1)}{2} \right) \right)^{\frac{p}{2}}$$

for $p \ge 2$, which still depends on p. Note that a deterministic T naturally is a $\mathcal{F}_t$-stopping time, and $\Delta f_{k,j}(s)$ are bounded in the sense of norm $\|.\|_{\mathcal{E}_{p,d_k}}$, thus one has the right to apply the Burkholder–Davis–Gundy inequality. Using the last observations and returning to the estimation of $\|\Delta \Pi_k\|^p_{\mathcal{E}_{p,d_k}}$, we have

$$\begin{aligned}
&\|\Delta \Pi_k\|^p_{\mathcal{E}_{p,d_k}} \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^{n} (L_{i,0}^{(k)})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}} \\
&\quad + (m+1)^{p-1} \sum_{j=1}^{m} C_{p,k,j}\, \mathbb{E} \left(\int_0^T \|\Delta f_{k,j}(s)\|^2_{d_k} ds \right)^{p/2} \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^{n} (L_{i,0}^{(k)})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}} \\
&\quad + (m+1)^{p-1} n^{p/2} \sum_{j=1}^{m} C_{p,k,j}\, \mathbb{E} \left(\sum_{i=1}^{n} (L_{i,j}^{(k)})^2 \int_0^T \|X_s^{(i)} - Y_s^{(i)}\|^2_{d_i} ds \right)^{p/2} \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^{n} (L_{i,0}^{(k)})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}} \\
&\quad + (m+1)^{p-1} (nT)^{p/2} \sum_{j=1}^{m} C_{p,k,j}\, \mathbb{E} \left(\sum_{i=1}^{n} (L_{i,j}^{(k)})^2 \sup_{0\leq t\leq T} \|X_t^{(i)} - Y_t^{(i)}\|^2_{d_i} \right)^{p/2} \\
&\leq (m+1)^{p-1} n^{p-1} T^p \sum_{i=1}^{n} (L_{i,0}^{(k)})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}} \\
&\quad + (m+1)^{p-1} n^{p-1} T^{p/2} \sum_{j=1}^{m} C_{p,k,j} \sum_{i=1}^{n} (L_{i,j}^{(k)})^p \|X^{(i)} - Y^{(i)}\|^p_{\mathcal{E}^0_{p,d_i}}
\end{aligned}$$

under Lipschitz-continuity of $f_{k,j}$. Hence, by taking the p-th root, we have

$$\|\Delta \Pi_k\|_{\mathcal{E}_{p,d_k}} \leq (m+1)^{(p-1)/p} n^{(p-1)/p} \sqrt{T} \sum_{i=1}^{n} \mathbf{k}_{i,k} \|X^{(i)} - Y^{(i)}\|_{\mathcal{E}^0_{p,d_i}}$$

where the coefficients $\mathbf{k}_{i,k}$ are given by $\mathbf{k}_{i,k} = \sqrt{T} L_{i,0}^{(k)} + \sum_{j=1}^{m} (C_{p,k,j})^{1/p} L_{i,j}^{(k)}$. Summarizing, we have the relation

$$\begin{pmatrix} \|\Delta \Pi_1\|_{\mathcal{E}^0_{p,d_1}} \\ \|\Delta \Pi_2\|_{\mathcal{E}^0_{p,d_2}} \\ \cdots\cdots \\ \|\Delta \Pi_n\|_{\mathcal{E}^0_{p,d_n}} \end{pmatrix} \leq (m+1)^{(p-1)/p} n^{(p-1)/p} \sqrt{T}\, \mathbf{K} \begin{pmatrix} \|X^{(1)} - Y^{(1)}\|_{\mathcal{E}^0_{p,d_1}} \\ \|X^{(2)} - Y^{(2)}\|_{\mathcal{E}^0_{p,d_2}} \\ \cdots\cdots \\ \|X^{(n)} - Y^{(n)}\|_{\mathcal{E}^0_{p,d_n}} \end{pmatrix},$$

for any $X^{(k)}, Y^{(k)} \in \mathcal{E}^0_{p,d_k}$ with $X_0^{(k)} = Y_0^{(k)}$ (a.s.), where the inequality sign $\leq$ is understood componentwise, and where $\mathbf{K}$ is the $n \times n$-matrix defined by $\mathbf{K} = (\mathbf{k}_{i,l})_{1\leq i,l\leq n}$. Under the assumption that T is sufficiently small we can conclude that the spectral radius $\varrho(L)$ of the matrix $L := (m+1)^{(p-1)/p} n^{(p-1)/p} \sqrt{T}\, \mathbf{K}$ is lesser than one. Thus, $\varrho(L)$ is an eigenvalue of L to which an eigenvector $(e_1, ..., e_n)$ with strictly positive components e_i corresponds. Now we introduce the norm

$$|||X|||_{\mathcal{E}^0_{p,d}} := \left(\sum_{k=1}^{n} e_k \|X^{(k)}\|^p_{\mathcal{E}_{p,d_k}} \right)^{1/p} \tag{20}$$

in the Banach space $\mathcal{E}^0_{p,d}$. Then the vector-valued operator $\top\!\!\top$ mapping the closed set $\mathcal{E}^0$ into itself is strictly contractive with the contraction constant $\varrho(L)$. Consequently, the sequence generated by iterative application of operator $\top\!\!\top$ converges with respect to norm $|||.|||_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$ to an unique element of $\mathcal{E}^0_{p,d}$ which is a solution of original system (17). Since the norm $|||.|||_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$ is equivalent to the norm $\|.\|_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$, we know that the solution of system (17) also lies in the original Banach space $\mathcal{E}^0_{p,d}$.

We have seen that $\top\!\!\top$ is contractive in $\mathcal{E}^0_{p,d}$ for sufficiently small T. To get the result for any T we divide $[0,T]$ in a finite number of sufficiently small subintervals and repeat the prior proof-steps successively. This completes the proof. ⋄ □

Remark 3.2. *For $p = 2$, thanks to Doob's maximum inequality (see Revuz and Yor* [35]*), we can choose $c_{2,k,j} = C_{2,k,j} = 4$ in the estimation above. Following Protter* [34, p. 174–175] *we may apply the Burkholder inequality to continuous time, local martingales (here represented by stochastic Itô integrals), and the universal estimation*

$$\max(c_{p,k,j}, C_{p,k,j}) \le \left(\left(\frac{p}{p-1}\right)^p \left(\frac{p(p-1)}{2}\right)\right)^{\frac{p}{2}} \tag{21}$$

is established for $p \ge 2$. Krylov [22] *and Mao* [28] *have also proved some estimates for $p \in (0, +\infty)$.*

Remark 3.3. *To get rid of dividing the interval $[0,T]$ in sufficiently small subintervals one may take weighted random norms on Banach spaces. One easily verifies that the appropriately weighted random norms are equivalent to the original norm (note that we make use of deterministic weights!).*

Remark 3.4. *In the case $m = 0$ (i.e. no stochastic terms) with $p = 1$, Theorem 3.1 yields a convergence criterion for the case of ordinary differential equations (here there is no dependence on the splitting parameter n).*

3.3. Convergence of waveform relaxation methods. The proof of Theorem 3.1 is based on general contraction principles and can be used to derive a sufficient condition for the convergence of the waveform relaxation method. If we consider the block Picard iteration as a special waveform relaxation technique for the fixed point problem (18), then we get the following sufficient condition for its convergence from the proof of Theorem 3.1.

Theorem 3.2. *Assume the hypotheses of Theorem 3.1 hold. Define $L = (l_{ik})$ by*

$$l_{ik} := (m+1)^{(p-1)/p} n^{(p-1)/p} \sqrt{T}\Big(\sqrt{T} L^{(k)}_{i,0} + \sum_{j=1}^{m} (C_p)^{1/p} L^{(k)}_{i,j}\Big)$$

with corresponding universal constants C_p occurring at the right hand side of the Burkholder–Davis–Gundy inequality (or substituted by estimates as in (21)).
Then $\varrho(L) < 0$ implies the convergence of the waveform relaxation algorithm based on the block Picard iterations (6) for the initial value problem (16) in the Banach space $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$ with norm $|||\cdot|||$ defined by (20), where $\mathcal{U}_k = \mathcal{E}^0_{p,d_k}$. If we modify this algorithm with Gauss–Seidel iterations (10) applied to the initial value problem (16), then the condition $\varrho(\tilde{L}) < 0$ implies its convergence with respect to corresponding norm $|||\cdot|||$.

Proof. For the completion of the proof, it only remains to determine the matrix of Lipschitz-constants L. These constants can be extracted from the last steps of the proof of previous Theorem 3.1 directly. Finally, one applies Theorem 2.1 to establish the claimed convergence with respect to the specifically constructed norm of $\mathcal{U}$. $\diamond$ $\square$

4. The case of one-sided Lipschitz continuous and anticoercive drift

The conditions for convergence of waveform relaxation methods can be relaxed as follows. The global Lipschitz-continuity of drift coefficients of SDEs is replaced by local one, but, additionally, the one-sided Lipschitz-continuity and anticoercivity (latter also originally called geometric or angle condition) of the drift is required. We shall combine the idea of monotonicity of coefficients of SDEs, as indicated by Krylov [21, 22] for the analytical solution, and as used by Bremer [5] for the convergence of waveform relaxation methods for ODEs.

Definition 4.1. *A function $f_0 : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is said to be* (uniformly) one-sided Lipschitz continuous *if for the splitting $f_0 = (f_{1,0}, ..., f_{k,0}, ..., f_{n,0})^T$ there are constants $\hat{L}^{(k)}_{i,0} \in \mathbb{R}^1 (i,k = 1,2,...,n)$ such that*

$$(A_3) \qquad \forall x = (x^{(1)}, ..., x^{(n)}), y = (y^{(1)}, ..., y^{(n)}) \in \mathbb{R}^{d_1} \times ... \times \mathbb{R}^{d_n}$$

$$< f_{k,0}(t, x^{(1)}, ..., x^{(n)}) - f_{k,0}(t, y^{(1)}, ..., y^{(n)}), x^{(k)} - y^{(k)} >_{d_k} \leq \sum_{i=1}^{n} \hat{L}^{(k)}_{i,0} \|x^{(i)} - y^{(i)}\|^2_{d_i}$$

for all $t \in [0,T]$. A function $f : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is called (uniformly) anticoercive *(or* to satisfy the autonomous angle condition*) if*

$$(A_4) \qquad \exists c_a \in \mathbb{R}^1 \ \forall t \in [0,T] \ \forall x \in \mathbb{R}^d \ : < f(t,x), x >_d \leq \ c_a \Big(1 + \|x\|^2_d\Big) .$$

4.1. On existence and uniqueness of the solution of (16). One encounters the following result. Assume measurebility (A_0) of all coefficients f_j.

Theorem 4.1. *Fix an exponent $p \geq 2$. Let the drift function $f_0 = f_0(t,x)$ be local and uniformly one-sided Lipschitz continuous (i.e. (A_3) holds), and the diffusion functions $f_{k,j} = f_{k,j}(t,x), j = 1,2,...,m; k = 1,2,...,n$ satisfy the conditions (A_1) of global Lipschitz-continuity and boundedness (A_2). Additionally, assume that f_0 possesses the property (A_4) of uniform anticoercivity, and $\mathbb{E}\, \|X^{(k)}_0\|^p_{d_k} < +\infty, k = 1,2,...,n$.*
Then the initial value problem (16) has an unique, $\mathcal{F}_t$-adapted and continuous (a.s.) solution in the space $\mathcal{E}^0_{p,d}$.

Proof. Again, the proof is carried out in two main steps. First, we shall show that the decomposed operator $\mathbb{T}$ is a mapping from the Banach space $\mathcal{E}^0_{p,d}$ into itself. Second, $\mathbb{T}$ forms a contraction in $\mathcal{E}_{p,d}$ with respect to an appropriately constructed norm. Then standard fixed point principles provide the conclusion of Theorem 4.1.
Step 1: Obviously, the existence of the unique solution of system (17) in any ball of $\mathbb{R}^d$ with finite radius $r > 0$ follows from the proof of Theorem 3.1 while assuming local Lipschitz-continuity of the components of f_0. That is that we can justify the unique solvability of the stopped system

$$(22) \qquad dX^r_t \ = \ \chi_{\{\sup_{0 \leq s \leq t} \|X^r_s\|_d < r\}}(t) \sum_{j=0}^{m} f_j(t, X^r_t)\, dW^j_t$$

in the space $\mathcal{E}^0_{p,d}$, where $\chi_{\{.\}}(t)$ represents the characteristic function of the subscribed set $\{.\}$ evaluated at time t. Here X^r_t denotes the solution of the system

(22) truncating the system (16) such that the solutions X_t^r of (22) and X_t of (16) coincide up to the first exit time from the ball of radius r. It remains to show an aposterori estimate of the sequence $(X^r)_{r>0}$ of local and continuous (a.s.) solutions X^r of truncated system (22) such that its uniform limit uniquely exists in $\mathcal{E}^0_{p,d}$ as the radius r tends to infinity. Using the well-known Itô formula, the local Lipschitz-continuity and anticoercivity (A_4) of drift coefficient f_0 and the Lipschitz-continuity (A_1) of diffusion coefficients $f_{k,j}$ of the considered system of SDEs (17), one recognizes that the stopped solution processes X_t^r must satisfy

$$\|X_t^r\|_d^p \;=\; \|X_0^r\|_d^p + \sum_{j=0}^{m} \int_0^t \mathcal{L}^j\left(\|X_s^r\|_d^p\right) dW_s^j$$

with the operators $\mathcal{L}^j$ originating from the Itô formula. Thus, we have

$$\begin{aligned}
\mathcal{L}^0\left(\|x\|_d^p\right) &= p\, g(x)\|x\|_d^{p-2}\,,\\
g(x) &= < f_0(t,x), x >_d + \frac{1}{2}\sum_{j=1}^{m}\|f_j(t,x)\|_d^2 + \frac{p-2}{2}\sum_{j=1}^{m}\frac{< f_j(t,x), x >_d^2}{\|x\|_d^2}\\
&\le < f_0(t,x), x >_d + \frac{p-1}{2}\sum_{j=1}^{m}\|f_j(t,x)\|_d^2\,,\\
\mathcal{L}^j\left(\|x\|_d^p\right) &= p < f_j(t,x), x >_d \|x\|_d^{p-2} \;\le\; p\,\|f_j(t,x)\|_d\|x\|_d^{p-1}
\end{aligned}$$

where $x \in \mathbb{R}^d$ and $j = 1, 2, ..., m$. For technical reasons, at first assume that we have $\mathbb{E}\,||X_0^r||^{2p}_{\mathcal{E}_{p,d}} < +\infty$. Taking the supremum, taking into account the uniform anticoercivity (A_4) of drift f_0 and the linear-polynomial boundedness of globally Lipschitz continuous diffusion functions $f_j (j = 1, 2, ..., m)$ under condition (A_2), and using the elementary inequality

$$(c_0 + c_1\|x\|^2)\|x\|^{p-2} \le c_0 + (c_0 + c_1)\|x\|^p$$

(a slightly more efficient estimate by application of the Hölder inequality would also be applicable here with $(c_0 + c_1\|x\|^2)\|x\|^{p-2} \le c_0\frac{2}{p} + (c_0\frac{p-2}{p} + c_1)\|x\|^p$) implies that

$$\begin{aligned}
\|X^r\|^p_{\mathcal{E}_{p,d}} \;\le\; & \mathbb{E}\,\|X_0^r\|_d^p + p\,\mathbb{E}\sup_{0\le t\le T}\int_0^t \Big(c_a(f_0)(1 + \|X_s^r\|_d^2) + \\
&+ \frac{p-1}{2}\sum_{j=1}^{m}(c_0(f_j) + c_1(f_j)\|X_s^r\|_d)^2\Big)\|X_s^r\|_d^{p-2}ds\\
&+ \sum_{j=1}^{m}\mathbb{E}\sup_{0\le t\le T}\int_0^t \mathcal{L}^j(\|X_s^r\|_d^p)dW_s^j
\end{aligned}$$

and hence

$$
\begin{aligned}
\|X^r\|^p_{\mathcal{E}_{p,d}} \leq\;& \mathbb{E}\,\|X^r_0\|^p_d + pT\Big(c_a(f_0) + (p-1)\sum_{j=1}^m c_0^2(f_j)\Big)\\
&+p\Big(c_a(f_0) + (p-1)\sum_{j=1}^m (c_0^2(f_j) + c_1^2(f_j))\Big)\int_0^T \mathbb{E}\,\|X^r_t\|^p_d dt\\
&+p2\sqrt{2}\sum_{j=1}^m \left(\mathbb{E}\int_0^T (c_0^2(f_j) + c_1^2(f_j)\|X^r_t\|^2_d)\|X^r_t\|^{2p-2}dt\right)^{1/2}\\
\leq\;& \mathbb{E}\,\|X^r_0\|^p_d + pT\Big(c_a(f_0) + (p-1)\sum_{j=1}^m c_0^2(f_j)\Big)\\
&+p\Big(c_a(f_0) + (p-1)\sum_{j=1}^m (c_0^2(f_j) + c_1^2(f_j))\Big)\int_0^T \mathbb{E}\,\|X^r_t\|^p_d dt\\
&+p2\sqrt{2}\sum_{j=1}^m \left(\sqrt{T}c_0(f_j) + (c_0(f_j) + c_1(f_j))\Big(\int_0^T \mathbb{E}\,\|X^r_t\|^{2p}dt\Big)^{1/2}\right)
\end{aligned}
$$

for all radii $r > 0$, where we have applied Doob's maximum inequality to the occurring integrals (as in proof above). Note that $c_a(f_0)$ represents the constant of anticoercivity (A_4) of drift f_0 and $c_0(f_j), c_1(f_j)$ the constants of linear-polynomial growth of globally Lipschitz continuous diffusion functions f_j, respectively. Now, one can show that

$$
\begin{aligned}
\int_0^T \mathbb{E}\,\|X^r_t\|^p dt &\leq T\sup_{r>0}\sup_{0\leq t\leq T} \mathbb{E}\,\|X^r_t\|^p < +\infty \quad \text{and}\\
\int_0^T \mathbb{E}\,\|X^r_t\|^{2p} dt &\leq T\sup_{r>0}\sup_{0\leq t\leq T} \mathbb{E}\,\|X^r_t\|^{2p} < +\infty
\end{aligned}
$$

by applying Dynkin's formula (see Dynkin [9] or Khas'minskij [15]) to the functionals $\mathbb{E}\,\|X^r_t\|^p_d$ and $\mathbb{E}\,\|X^r_t\|^{2p}_d$, respectively, while $\sup_{r>0}\mathbb{E}\,\|X^r_0\|^{2p}_d < +\infty$. After that step and using Gronwall–Bellman inequality, one finds that

$$
\lim_{r\to+\infty}\|X^r\|_{\mathcal{E}_{p,d}} \leq \sup_{r>0}\|X^r\|_{\mathcal{E}_{p,d}} < +\infty\,.
$$

Now, by use of standard localization procedures, one may relax the assumption $\mathbb{E}\,||X^r_0||^{2p} < +\infty$ to the weaker requirement $\mathbb{E}\,||X^r_0||^p < +\infty$.

Thus, from uniform anticoercivity (A_4) of functions f_j and $\mathbb{E}\,\|X^r_0\|^p_d < +\infty$, we know that the uniform limit of continuous (a.s.) stochastic processes X^r as the radius r tends to $+\infty$ must exist with finite norm $\|.\|_{\mathcal{E}_{p,d}}$. Therefore, by the completeness of space $\mathcal{E}^0_{p,d}$, the limit process $\lim_{r\to+\infty}X^r$ which also solves the original system (16) must exist, be continuous (a.s.), be $\mathcal{F}_t$-adapted and have a finite norm $\|.\|_{\mathcal{E}_{p,d}}$. Consequently, the decomposed operator Π is a mapping from $\mathcal{E}^0_{p,d}$ into itself.

Step 2: Contractivity of operator Π on the space $\mathcal{E}^0_{p,d}$. Assume that $X^{(k)}_0 = Y^{(k)}_0$ (a.s.). Take $\Delta X^{(k)}_s = X^{(k)}_s - Y^{(k)}_s$ for $k = 1, 2, ..., n$, and $\Delta X_s = X_s - Y_s$. Set

$$
\Delta\Pi_k(t) := [\Pi_k(X^{(1)}, ..., X^{(n)}) - \Pi_k(Y^{(1)}, ..., Y^{(n)})](t)
$$

for all $t \in [0, T]$, and

$$
\Delta f_{k,j}(s) := f_{k,j}(s, X^{(1)}_s, ..., X^{(n)}_s) - f_{k,j}(s, Y^{(1)}_s, ..., Y^{(n)}_s)
$$

for all $s \in [0,T]$. Fix any $(X^{(1)},...,X^{(n)}),(Y^{(1)},...,Y^{(n)}) \in \mathcal{E}^0_{p,d}$, where $X^{(k)} \neq Y^{(k)}$ (a.s.). Define

$$g_k(x,y) := < f_{k,0}(t,x) - f_{k,0}(t,y), x^{(k)} - y^{(k)} >_{d_k} + \frac{1}{2}\sum_{j=1}^{m} \|f_{k,j}(t,x) - f_{k,j}(t,y)\|^2_{d_k}$$
$$+ \frac{p-2}{2}\sum_{j=1}^{m} \frac{< f_{k,j}(t,x) - f_{k,j}(t,y), x^{(k)} - y^{(k)} >^2_{d_k}}{\|x-y\|^2_{d_k}}$$

and estimate $g_k = g_k(x,y)$ by

$$g_k \leq < f_{k,0}(t,x) - f_{k,0}(t,y), x^{(k)} - y^{(k)} >_{d_k} + \frac{p-1}{2}\sum_{j=1}^{m} \|f_{k,j}(t,x) - f_{k,j}(t,y)\|^2_{d_k}$$

where $x = (x^{(1)},...,x^{(k)},...,x^{(n)})^T, y = (y^{(1)},...,y^{(k)},...,y^{(n)})^T \in \mathbb{R}^d$. In the following let $[.]_+$ denote the nonnegative part of the inscribed expression. Then one has

$$\|\Delta \mathbb{T}_k(t)\|^p_{d_k} = \int_0^t \mathcal{L}^0\Big(\|\Delta X^{(k)}_s\|^p_{d_k}\Big)ds + \sum_{j=1}^{m}\int_0^t \mathcal{L}^j\Big(\|\Delta X^{(k)}_s\|^p_{d_k}\Big)dW^j_s$$
$$= p\int_0^t g_k(X_s,Y_s)\|\Delta X^{(k)}_s\|^{p-2}_{d_k}ds$$
$$+p\sum_{j=1}^{m}\int_0^t < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} \|\Delta X^{(k)}_s\|^{p-2}_{d_k}dW^j_s$$
$$\leq p\int_0^t \Big(\sum_{i=1}^{n}\hat{L}^{(k)}_{i,0}\|\Delta X^{(i)}_s\|^2_{d_i} + \frac{p-1}{2}\sum_{j=1}^{m}(\sum_{l=1}^{n} L^{(k)}_{l,j}\|\Delta X^{(l)}_s\|)^2\Big)\|\Delta X^{(k)}_s\|^{p-2}_{d_k}ds$$
$$+p\sum_{j=1}^{m}\Big|\int_0^t < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} \|\Delta X^{(k)}_s\|^{p-2}_{d_k}dW^j_s\Big|$$
$$\leq p\int_0^t \Big(\sum_{i=1}^{n}(\hat{L}^{(k)}_{i,0} + n\frac{p-1}{2}\sum_{j=1}^{m}(L^{(k)}_{i,j})^2)\|\Delta X^{(i)}_s\|^2_{d_i}\Big)\|\Delta X^{(k)}_s\|^{p-2}_{d_k}ds \tag{23}$$
$$+p\sum_{j=1}^{m}\Big|\int_0^t < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} \|\Delta X^{(k)}_s\|^{p-2}_{d_k}dW^j_s\Big|$$
$$\leq p\int_0^t \Big(\sum_{l=1}^{n}[\hat{L}^{(k)}_{l,0} + n\frac{p-1}{2}\sum_{j=1}^{m}(L^{(k)}_{l,j})^2]_+\Big)\sum_{i=1}^{n}\|\Delta X^{(i)}_s\|^p_{d_i}ds$$
$$+p\sum_{j=1}^{m}\Big|\int_0^t < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} \|\Delta X^{(k)}_s\|^{p-2}_{d_k}dW^j_s\Big|$$
$$\leq pt\Big(\sum_{l=1}^{n}[\hat{L}^{(k)}_{l,0} + n\frac{p-1}{2}\sum_{j=1}^{m}(L^{(k)}_{l,j})^2]_+\Big)\sum_{i=1}^{n}\sup_{0\leq s\leq t}\|\Delta X^{(i)}_s\|^p_{d_i}$$
$$+p\sum_{j=1}^{m}\Big|\int_0^t < \Delta f_{k,j}(s), \Delta X^{(k)}_s >_{d_k} \|\Delta X^{(k)}_s\|^{p-2}_{d_k}dW^j_s\Big|.$$

Therefore

$$\begin{aligned}\|\Delta \Pi_k(t)\|_{d_k}^p &\leq pT\Big(\sum_{l=1}^{n}[\hat{L}_{l,0}^{(k)} + n\frac{p-1}{2}\sum_{j=1}^{m}(L_{l,j}^{(k)})^2]_+\Big)\sum_{i=1}^{n}\sup_{0\leq t\leq T}\|\Delta X_t^{(i)}\|_{d_i}^p \\ &+p\sum_{j=1}^{m}\sup_{0\leq t\leq T}|\int_0^t <\Delta f_{k,j}(s),\Delta X_s^{(k)}>_{d_k}\|\Delta X_s^{(k)}\|_{d_k}^{p-2}dW_s^j|\end{aligned}$$

using the Itô lemma applied to $\|\Delta X_s\|_{d_k}^p$, triangle inequality, the Hölder inequality, and the Lipschitz conditions (A_1) and (A_3), respectively. Note that the operators $\mathcal{L}^0$ and $\mathcal{L}^j$ are those operators arising at the application of Itô formula. Now, by taking the operation of expectation $\mathbb{E}$ on both sides, this implies

$$\begin{aligned}\|\Delta \Pi_k\|_{\mathcal{E}_{p,d_k}}^p &= \mathbb{E}\max_{0\leq t\leq T}\|\Delta \Pi_k(t)\|_{d_k}^p \\ &\leq pT\Big(\sum_{l=1}^{n}[\hat{L}_{l,0}^{(k)} + n\frac{p-1}{2}\sum_{j=1}^{m}(L_{l,j}^{(k)})^2]_+\Big)\sum_{i=1}^{n}\|\Delta X^{(i)}\|_{\mathcal{E}_{p,d_i}}^p \\ &+p\left(\sum_{j=1}^{m}\mathbb{E}\max_{0\leq t\leq T}|\int_0^t <\Delta f_{k,j}(s),\Delta X_s^{(k)}>_{d_k}\|\Delta X_s^{(k)}\|_{d_k}^{p-2}dW_s^j|\right).\end{aligned}$$

The occurring terms $\int_0^t <\Delta f_{k,j}(s),\Delta X_s^{(k)}>_{d_k}\|\Delta X_s^{(k)}\|_{d_k}^{p-2}dW_s^j$ form continuous and $\mathcal{F}_t$-adapted martingales started at initial value 0 under the global Lipschitz-continuity (A_1) of diffusion functions $f_{k,j}$ and for $X^{(k)} \in \mathcal{E}_{p,d_k}$, where $k = 1,2,...,n; j = 1,2,...,m$. As in proof of Theorem 3.1, using the Burkholder–Davis–Gundy inequality and basic properties of quadratic variation of Itô integrals with respect to Brownian motions W_s^j, there are constants $\hat{C}_{p,k,j}$ such that

$$\begin{aligned}&\mathbb{E}\max_{0\leq t\leq T}|\int_0^t <\Delta f_{k,j}(s),\Delta X_s^{(k)}>_{d_k}\|\Delta X_s^{(k)}\|_{d_k}^{p-2}dW_s^j| \\ &\leq \hat{C}_{p,k,j}\,\mathbb{E}\left(\int_0^T |<\Delta f_{k,j}(s),\Delta X_s^{(k)}>_{d_k}|^2\Delta X_s^{(k)}\|_{d_k}^{2(p-2)}d<W^j,W^j>_s\right)^{1/2} \\ &= \hat{C}_{p,k,j}\,\mathbb{E}\left(\int_0^T |<\Delta f_{k,j}(s),\Delta X_s^{(k)}>_{d_k}|^2\|\Delta X_s^{(k)}\|_{d_k}^{2(p-2)}ds\right)^{1/2} \\ &\leq \hat{C}_{p,k,j}\,\mathbb{E}\left(\int_0^T \|\Delta f_{k,j}(s)\|^2\|\Delta X_s^{(k)}\|_{d_k}^{2p-2}ds\right)^{1/2} \\ &\leq \hat{C}_{p,k,j}\,\mathbb{E}\left(\int_0^T (\sum_{i=1}^{n}L_{i,j}^{(k)}\|\Delta X_s^{(i)}\|_{d_i})^2\|\Delta X_s^{(k)}\|_{d_k}^{2p-2}ds\right)^{1/2}\end{aligned}$$

$$\leq \hat{C}_{p,k,j}\sqrt{n}\,\mathbb{E}\left(\int_0^T\sum_{i=1}^{n}(L_{i,j}^{(k)})^2\|\Delta X_s^{(i)}\|_{d_i}^2\|\Delta X_s^{(k)}\|_{d_k}^{2p-2}ds\right)^{1/2} \tag{24}$$

$$\leq \hat{C}_{p,k,j}\sqrt{n(\sum_{i=1}^{n}(L_{i,j}^{(k)})^2)}\,\mathbb{E}\left(\int_0^T\sum_{i=1}^{n}\|\Delta X_s^{(i)}\|_{d_i}^{2p}ds\right)^{1/2}.$$

Hence

$$
\begin{aligned}
&\mathbb{E} \max_{0\le t\le T} \Big| \int_0^t < \Delta f_{k,j}(s), \Delta X_s^{(k)} >_{d_k} \|\Delta X_s^{(k)}\|_{d_k}^{p-2} dW_s^j \Big| \\
\le\ & \hat{C}_{p,k,j} \sqrt{nT(\sum_{i=1}^n (L_{i,j}^{(k)})^2)}\ \mathbb{E}\left(\sum_{i=1}^n \max_{0\le t\le T} \|\Delta X_t^{(i)}\|_{d_i}^{2p} \right)^{1/2} \\
\le\ & \hat{C}_{p,k,j} \sqrt{nT(\sum_{i=1}^n (L_{i,j}^{(k)})^2)} \sum_{i=1}^n \mathbb{E} \max_{0\le t\le T} \|\Delta X_t^{(i)}\|_{d_i}^{p} \\
=\ & \hat{C}_{p,k,j} \sqrt{nT(\sum_{i=1}^n (L_{i,j}^{(k)})^2)} \sum_{i=1}^n \|\Delta X^{(i)}\|_{\mathcal{E}_{p,d_i}}^p .
\end{aligned}
$$

Using the last estimate and returning to the estimation of $\|\Delta \top\!\top_k\|_{\mathcal{E}_{p,d_k}}^p$, we have

$$
\begin{aligned}
\|\Delta \top\!\top_k\|_{\mathcal{E}_{p,d_k}}^p \le\ & pT\Big(\sum_{l=1}^n [\hat{L}_{l,0}^{(k)} + n\frac{p-1}{2} \sum_{j=1}^m (L_{l,j}^{(k)})^2]_+ \Big) \sum_{i=1}^n \|\Delta X^{(i)}\|_{\mathcal{E}_{p,d_i}}^p \\
& + p \left(\sum_{j=1}^m \hat{C}_{p,k,j} \sqrt{nT(\sum_{i=1}^n (L_{i,j}^{(k)})^2)} \sum_{i=1}^n \|\Delta X^{(i)}\|_{\mathcal{E}_{p,d_i}}^p \right)
\end{aligned}
$$

under one-sided Lipschitz-continuity (A_3) of $f_{k,0}$. Hence, one finds

$$
\|\Delta \top\!\top_k\|_{\mathcal{E}_{p,d_k}} \le \sqrt[p]{p}\ \sqrt[2p]{T} \sum_{i=1}^n \hat{\mathbf{k}}_{i,k} \|\Delta X^{(i)}\|_{\mathcal{E}_{p,d_i}}
$$

by taking the p-th root, where the coefficients $\hat{\mathbf{k}}_{i,k}$ are given by

$$
\hat{\mathbf{k}}_{i,k} = \sqrt[2p]{T} \sum_{l=1}^n [\hat{L}_{l,0}^{(k)} + n\frac{p-1}{2} \sum_{j=1}^m (L_{l,j}^{(k)})^2]_+^{1/p} + \sum_{j=1}^m (\hat{C}_{p,k,j})^{1/p} \sqrt[2p]{n(\sum_{l=1}^n (L_{l,j}^{(k)})^2)} .
$$

Summarizing the main result, we have shown the relation

$$
\begin{pmatrix} \|\Delta \top\!\top_1\|_{\mathcal{E}_{p,d_1}} \\ \|\Delta \top\!\top_2\|_{\mathcal{E}_{p,d_2}} \\ \cdots\cdots \\ \|\Delta \top\!\top_n\|_{\mathcal{E}_{p,d_n}} \end{pmatrix} \le \sqrt[p]{p}\ \sqrt[2p]{T}\, \hat{\mathbf{K}} \begin{pmatrix} \|X^{(1)} - Y^{(1)}\|_{\mathcal{E}_{p,d_1}} \\ \|X^{(2)} - Y^{(2)}\|_{\mathcal{E}_{p,d_2}} \\ \cdots\cdots \\ \|X^{(n)} - Y^{(n)}\|_{\mathcal{E}_{p,d_n}} \end{pmatrix},
$$

for all $X^{(k)}, Y^{(k)} \in \mathcal{E}_{p,d_k}$ with $X_0^{(k)} = Y_0^{(k)}$ (a.s.), where the inequality sign $\le$ is understood componentwise, and where the $n \times n$-matrix $\hat{\mathbf{K}}$ is given by $\hat{\mathbf{K}} = (\hat{\mathbf{k}}_{i,l})_{1\le i,l\le n}$. For sufficiently small T, we can conclude that the spectral radius $\varrho(\hat{L})$ of $\hat{L} := \sqrt[p]{p}\ \sqrt[2p]{T}\, \hat{\mathbf{K}}$ is less than one. Thus, $\varrho(\hat{L})$ is an eigenvalue of $\hat{L}$ to which an eigenvector with strictly positive components $(e_1, ..., e_n)$ corresponds. Now, we introduce the norm

$$
|||X|||_{\mathcal{E}_{p,d}^0} := \left(\sum_{k=1}^n e_k \|X^{(k)}\|_{\mathcal{E}_{p,d_k}}^p \right)^{1/p} \tag{25}
$$

in the Banach space $\mathcal{E}_{p,d}^0$. Then the vector-valued operator $\top\!\top$ mapping the closed set $\mathcal{E}_{p,d}^0$ into itself is strictly contractive with the contraction constant $\varrho(\hat{L})$. Consequently, the sequence generated by iterative application of operator $\top\!\top$ converges with respect to norm $|||.|||_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$ to an unique element of $\mathcal{E}^0_{p,d}$ which is a solution of the original system (17). Since the norm $|||.|||_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$ is equivalent to the norm $\|.\|_{\mathcal{E}^0_{p,d}}$ of $\mathcal{E}^0_{p,d}$, we know that the solution of system (17) also lies in the original Banach space $\mathcal{E}^0_{p,d}$.

We have seen that $\mathbb{T}$ is contractive in $\mathcal{E}^0_{p,d}$ for sufficiently small T. To get the result for any T we divide $[0,T]$ in a finite number of sufficiently small subintervals and repeat the prior proof-steps successively. This completes the proof. $\diamond$ □

4.2. Convergence of waveform relaxation methods. The contractivity of operator $\mathbb{T}$ can be used to establish a theorem on the convergence of waveform relaxation methods. Analogous to Theorem 3.2 we have

Theorem 4.2. *Assume the hypotheses of Theorem 4.1 are valid. Define* $\hat{L} = (\hat{l}_{ik})$ *by*

$$\hat{l}_{ik} := \left(p\sqrt{T} \left[\sqrt{T} \sum_{l=1}^{n} [\hat{L}^{(k)}_{l,0} + n\frac{p-1}{2} \sum_{j=1}^{m} (L^{(k)}_{l,j})^2]_+ + \sum_{j=1}^{m} \hat{C}_p \sqrt{n (\sum_{l=1}^{n} (L^{(k)}_{l,j})^2)} \right] \right)^{1/p}$$

with corresponding universal constants C_p occurring at the right hand side of the Burkholder–Davis–Gundy inequality (or substituted by estimates as in (21)). Then $\varrho(\hat{L}) < 0$ implies the convergence of the waveform relaxation algorithm based on the block Picard iterations (6) for the initial value problem (16) in the Banach space $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$ with norm $|||\cdot|||$ defined by (25), where $\mathcal{U}_k = \mathcal{E}^0_{p,d_k}$. If we modify this algorithm with Gauss–Seidel iterations (10) applied to the initial value problem (16), then the condition $\varrho(\tilde{\hat{L}}) < 0$ implies its convergence with respect to corresponding norm $|||\cdot|||$.

The proof of Theorem 4.2 is omitted as an immediate conclusion of Theorem 4.1.

4.3. Further remarks. One could think of slight improvements in the estimation of the coefficients $\hat{k}_{ik}$ from the proof of Theorem 4.1 and $\hat{l}_{ik}$ from the Theorem 4.2. For this purpose one returns to inequalities (23) and (24), respectively. Now, make use of the inequalities

$$\sum_{i=1}^{n} c_{ik}\, x_i\, x_k^{p-1} \;\leq\; \frac{1}{p} \sum_{i=1:i\neq k}^{n} c_{ik}\, x_i^p \;+\; \left(\frac{p-1}{p} \sum_{i=1:i\neq k}^{n} c_{ik} + c_{kk} \right) x_k^p, p \geq 1, \quad \text{and}$$

$$\sum_{i=1}^{n} c_{ik}\, x_i^2\, x_k^{p-2} \;\leq\; \frac{2}{p} \sum_{i=1:i\neq k}^{n} c_{ik}\, x_i^p \;+\; \left(\frac{p-2}{p} \sum_{i=1:i\neq k}^{n} c_{ik} + c_{kk} \right) x_k^p, p \geq 2,$$

where c_{ik}, x_i, x_k are nonnegative numbers. In passing note that these inequalities are obtained by the application of well-known Young's inequality. Let $[.]_+$ denote the nonnegative part of the inscribed expression. So one would arrive at coefficients

$$(\hat{\mathbf{k}}_{ik})^p = \sqrt{T} \Big[\Big(\frac{2}{p}\Big)^{1-\delta_{i,k}} [L^{(k)}_{i,0} + n\frac{p-1}{2} \sum_{j=1}^{m} (L^{(k)}_{i,j})^2]_+ +$$

$$+ \delta_{i,k} \frac{p-2}{2} \sum_{l=1:l\neq k}^{n} [L^{(k)}_{l,0} + n\frac{p-1}{2} \sum_{j=1}^{m} (L^{(k)}_{l,j})^2]_+ \Big] +$$

$$+ \sqrt{n} \Big[(1-\delta_{i,k}) \sqrt{\frac{1}{p}} \sum_{j=1}^{m} \hat{C}_{p,k,j} L^{(k)}_{i,j} + \delta_{i,k} \sum_{j=1}^{m} \hat{C}_{p,k,j} \Big(\sqrt{\frac{p-1}{p}} \sum_{l=1:l\neq k}^{n} L^{(k)}_{l,j} + L^{(k)}_{k,j} \Big) \Big]$$

occurring at $\|\Delta \top\!\!\top_k\|_{\mathcal{E}_{p,d_k}} \leq (p\sqrt{T})^{1/p} \sum_{i=1}^{n} \hat{\mathbf{k}}_{i,k} \|\Delta X^{(i)}\|_{\mathcal{E}_{p,d_i}}$, where $\delta_{i,k}$ represents the Kronecker symbol. However, the evaluation of this result leads to more complex expressions for the spectral radius of the matrix $\hat{L} = (\hat{l}_{ik})$ with $\hat{l}_{ik} = (p\sqrt{T})^{1/p} \hat{\mathbf{k}}_{i,k}$ controlling the convergence of the waveform iterations for SDEs with one-sided Lipschitz continuous drift part. This is the reason why we preferred to use the more elementary estimates $\sum_{i=1}^{n} c_{ik}\, x_i^2\, x_k^{p-2} \leq \sum_{l=1}^{n} c_{lk} \cdot \sum_{i=1}^{n} x_i^p$ with $p \geq 2$ after the inequality (23), and $\sum_{i=1}^{n} c_{ik}\, x_i\, x_k^{p-1} \leq \sum_{l=1}^{n} c_{lk} \cdot \sum_{i=1}^{n} x_i^p$ with $p \geq 1$ after the inequality (24), where $c_{ik}, x_i, x_k \geq 0$.

The assertions of Theorems 4.1, 4.2 remain valid in case $1 \leq p < 2$. In that case one needs slight modifications in some estimations of corresponding proof-steps.

The crucial point in all generalizations with locally Lipschitz continuous coefficients is to find an appropriate aposteriori estimation such that the limit process $\lim_{r\to+\infty} X^r$, where $X^r = (X^{r,(1)}, ..., X^{r,(n)})^T$ represents the solution of the corresponding truncated system (22), cannot blow up (a.s) at finite times. However, generically, the solutions do not lie in the original Banach space $\mathcal{E}_{p,d}$ anymore.

As a by-product, we have shown that solutions of (16) posess finite moments $\sup_{0\leq t\leq T} \mathbb{E}\|X_t\|_d^{2p} < +\infty$ under assumptions of Theorem 4.1 and $\mathbb{E}\,\|X_0\|^{2p} < +\infty$.

5. An illustrative example with different time scales

There are a lot of real-life processes containing several time scales. For example, a rich class is given by biochemical processes. The presence of fast and slow variables can be expressed by *singularly perturbed differential equations* of the type

$$\frac{dx}{ds} = f(x,y,s), \varepsilon \frac{dy}{ds} = g(x,y,s). \tag{26}$$

By introducing the fast time $t = s/\varepsilon$ we get the system

$$\frac{dx}{dt} = \varepsilon\, f(x,y,\varepsilon t), \frac{dy}{dt} = g(x,y,\varepsilon t). \tag{27}$$

Now, suppose that system (27) is randomly perturbed in its x-component by a stochastic term $\sqrt{\varepsilon} h(x,y,\varepsilon t) dW_t$ where $W = (W_t)_{t\in[0,T/\varepsilon]}$ is a standard Brownian motion. The system we obtain, which is to be understood in integral sense, is

$$\begin{aligned} dX_t &= \varepsilon f(X_t, Y_t, \varepsilon t)\, dt + \sqrt{\varepsilon} h(X_t, Y_t, \varepsilon t)\, dW_t, \\ dY_t &= g(X_t, Y_t, \varepsilon t)\, dt\,. \end{aligned} \tag{28}$$

Stochastic singularly perturbed systems have been considered by many authors. For example, a qualitative theory is found by [29] and a block diagonalization procedure is exploited in [24]. In contrast to analytical techniques, it is much lesser known on their numerical approximations. Golec and Ladde [11] have studied Euler-type approximations in the mean square sense. It is worth stressing that singularly perturbed differential equations (28) with their naturally inherited splitting into slowly and fastly varying components form a suitable class for an application of waveform iteration techniques as a further numerical method. The waveform iteration technique can be applied to approximate the solution of the initial value problem to (28) as follows. First, fix some initial guess $X_t^{(0)}$ for X_t, e.g. $X_t^{(0)} = X_0$. Second, compute an approximation for $Y = (Y_t)_{t\in[0,T/\varepsilon]}$ satisfying the initial value problem for

$$dY_t^{(k)} = g(X_t^{(k-1)}, Y_t^{(k)}, \varepsilon t)\, dt$$

while freezing the first component, for example, pathwise by deterministic numerical methods. Afterwards, by plugging $Y_t^{(k)}$ into the first equation one solves the system

$$dX_t^{(k)} = \varepsilon f(X_t^{(k)}, Y_t^{(k)}, \varepsilon t)\, dt + \sqrt{\varepsilon} h(X_t^{(k)}, Y_t^{(k)}, \varepsilon t)\, dW_t$$

by stochastic-numerical methods. This procedure will be repeated iteratively until a required accuracy has been reached.

To guarantee the convergence of the waveform algorithm applied to systems (28) one has to check the spectral radius criterion of corresponding matrix of Lipschitz-coefficients. Concerning the functions f, g, h, we assume that they are continuous and globally Lipschitz continuous in x and y uniformly with respect to t, i.e.

$$\begin{aligned} \|f(x,y,t) - f(\bar{x},\bar{y},t)\|_1 &\leq L^1_{1,0}\|x-\bar{x}\|_1 + L^1_{2,0}\|y-\bar{y}\|_2, \\ \|g(x,y,t) - g(\bar{x},\bar{y},t)\|_2 &\leq L^2_{1,0}\|x-\bar{x}\|_1 + L^2_{2,0}\|y-\bar{y}\|_2, \\ \|h(x,y,t) - h(\bar{x},\bar{y},t)\|_1 &\leq L^1_{1,1}\|x-\bar{x}\|_1 + L^1_{2,1}\|y-\bar{y}\|_2 \end{aligned} \tag{29}$$

for all $x, \bar{x} \in \mathbb{R}^{d_1}, y, \bar{y} \in \mathbb{R}^{d_2}, t \in [0,T]$, where $\|.\|_i$ represents the Euclidean norm in $\mathbb{R}^{d_i}$. Taking into account $L^2_{1,1} = L^2_{2,1} = 0$ we arrive at 2×2 matrix $L = (l_{i,k})$ with (30)

$$L \;=\; 4^{(p-1)/p}\sqrt{T}\begin{pmatrix} (\varepsilon\sqrt{T}L^1_{1,0} + \sqrt{\varepsilon}\,C_p^{1/p}L^1_{1,1}) & (\varepsilon\sqrt{T}L^1_{2,0} + \sqrt{\varepsilon}\,C_p^{1/p}L^1_{2,1}) \\ \sqrt{T}L^2_{1,0} & \sqrt{T}L^2_{2,0} \end{pmatrix}$$

as found at the end of the proof of Theorem 3.1. Recall that the constant C_p arises as the constant on the right side of the well-known Burkholder–Davis–Gundy inequality and can be replaced by any of their majorants, e.g.

$$\tilde{C}_p = C_p{}^{\frac{1}{p}} \leq \sqrt{\left(\frac{p}{p-1}\right)^p \left(\frac{p(p-1)}{2}\right)}$$

where $p \geq 1$. Finally, the condition $\varrho(L) < 1$ on the spectral radius $\varrho(L)$ controls the convergence of corresponding Picard iterations. Correspondingly, the condition $\varrho(\tilde{L}) < 1$ ($\tilde{L}$ belonging to (12)) on the spectral radius $\varrho(\tilde{L})$ of matrix

$$\tilde{L} \;=\; \begin{pmatrix} l_{11} & l_{12} \\ l_{21}l_{11} & l_{21}l_{12} + l_{22} \end{pmatrix}$$

guarantees the convergence of waveform methods based on Gauss–Seidel iteration.

6. Conclusions, summary and remarks

This paper is an continuation of [37] - [43] concerning the approximation of the solution of initial value problems for systems of explicit stochastic differential equations. Here, we extended the standard idea of waveform iteration method to nonlinear ordinary stochastic differential equations (SDEs) driven by Wiener processes. It turns out that the Lipschitz-continuity of the coefficients of SDEs and the form of its splitting into subsystems are crucial to establish the convergence of waveform relaxation methods. In particular, the Lipschitz-coefficients determine the length of integration intervals to which the waveform iterations are applied (windowing techniques). We have shown its convergence with respect to the metric on the Banach space of $L^p(\Omega, \mathcal{F}, \mathbb{P})$-integrable, adapted, cadlag processes ($p \geq 2$).

Waveform iteration methods provide an alternative approach to approximating the solution of a system of stochastic differential equations. Compared with the traditional time-incremental methods as described in [20], [31] or [45], the waveform relaxation technique forms a global iteration scheme on a given time interval. Its efficiency depends on an appropriate decomposition of the large original system into weakly interacting subsystems. These methods are particularly designed to treat very large scale systems by parallel computations.

It is worth emphasizing that there are other attempts to treat stochastic large-scale systems in the literature. However, a systematic comparison study of the performance of waveform iteration techniques compared to those attempts exceeds the intention and length of this paper. Therefore this is omitted here.

Acknowledgments

The authors like to express their gratitude to the Weierstrass Institute for Applied Analysis and Stochastics at Berlin, University of Los Andes at Bogotá, University of Minnesota at Minneapolis, Southern Illinois University at Carbondale and Texas Tech University at Lubbock for their support of this research project. This work initiated in 1998 is the written version of an invited talk of the first author presented at SciCADE01 meeting (Vancouver, July 29 – August 3, 2001).

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