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A BLOCK MONOTONE DOMAIN DECOMPOSITION ALGORITHM FOR A NONLINEAR SINGULARLY PERTURBED PARABOLIC PROBLEM

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Abstract. This paper deals with discrete monotone iterative algorithms for solving a nonlinear singularly perturbed parabolic problem. A block monotone domain decomposition algorithm based on a Schwarz alternating method and on a block iterative scheme is constructed. This monotone algorithm solves only linear discrete systems at each time level and converges monotonically to the exact solution of the nonlinear problem. The rate of convergence of the block monotone domain decomposition algorithm is estimated. Numerical experiments are presented.

Key Words. singularly perturbed parabolic problem, block monotone method, nonoverlapping domain decomposition, monotone Schwarz alternating algorithm.

1. Introduction

We are interested in monotone Schwarz alternating algorithms for solving the nonlinear reaction-diffusion problem

$$-\mu^2 (u_{xx} + u_{yy}) + u_t = -f(P,t,u), \tag{1}$$

$$P = (x,y), \quad (P,t) \in Q = \Omega \times (0,T], \quad \Omega = \{0 < x < 1, 0 < y < 1\},$$

$$f_u(P,t,u) \geq 0, \quad (P,t,u) \in \overline{Q} \times (-\infty,\infty), \quad (f_u \equiv \partial f/\partial u),$$

where μ is a small positive parameter. The initial-boundary conditions are defined by

$$u(P,t) = g(P,t),\ (P,t) \in \partial\Omega \times (0,T], \quad u(P,0) = u^0(P),\ P \in \overline{\Omega},$$

where $\partial\Omega$ is the boundary of Ω. The functions $f(P,t,u)$, $g(P,t)$ and $u^0(P)$ are sufficiently smooth. Under suitable continuity and compatibility conditions on the data, a unique solution $u(P,t)$ of (1) exists (see [6] for details). For $\mu \ll 1$, problem (1) is singularly perturbed and characterized by the boundary layers of width $O(\mu|\ln\mu|)$ at the boundary $\partial\Omega$ (see [1] for details).

In the study of numerical solutions of nonlinear singularly perturbed problems by the finite difference method, the corresponding discrete problem is usually formulated as a system of nonlinear algebraic equations. A major point about this system is to obtain reliable and efficient computational algorithms for computing the solution. In the case of the parabolic problem (1), the implicit method is usually in use. On each time level, this method leads to a nonlinear system which requires some kind of iterative scheme for the computation of numerical solutions.

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A fruitful method for the treatment of these nonlinear systems is the method of upper and lower solutions and its associated monotone iterations (in the case of "unperturbed" problems see [8], [9] and references therein). Since the initial iteration in the monotone iterative method is either an upper or a lower solution, which can be constructed directly from the difference equation without any knowledge of the exact solution (see [2] for details), this method eliminates the search for the initial iteration as is often needed in Newton's method. This elimination gives a practical advantage in the computation of numerical solutions.

In [3], we proposed a discrete iterative algorithm which combines the monotone approach and the iterative domain decomposition method based on the Schwarz alternating procedure. In the case of small values of the perturbation parameter μ, the convergence factor $\tilde{\rho}$ of the monotone domain decomposition algorithm is estimated by

$$\tilde{\rho} = \rho + \mathcal{O}(\tau),$$

where ρ is the convergence factor of the monotone (undecomposed) method and τ is the step size in the t-direction.

The purpose of this paper is to extend the monotone domain decomposition algorithm from [3] in a such way that computation of the discrete linear subsystems on subdomains which are located outside the boundary layers is implemented by the block iterative scheme (see [12] for details of the block iterative scheme). A basic advantage of the block iterative scheme is that the Thomas algorithm can be used for each linear subsystem defined on these subdomains in the same manner as for one-dimensional problems, and the scheme is stable and is suitable for parallel computing.

For solving nonlinear discrete elliptic problems without domain decomposition, the block monotone iterative methods were constructed and studied in [10]. In [10], the convergence analysis does not contain any estimates on a convergence rate of the proposed iterative methods, and the numerical experiments show that these algorithms applied to some model problems converge very slowly. In contrast, a numerical algorithm based on a combination of the domain decomposition approach and the block iterative method applied on subdomains outside the boundary layers converges more quickly than the original block iterative method.

The structure of the paper is as follows. In Section 2, we consider a monotone iterative method for solving the implicit difference scheme which approximates the nonlinear problem (1). In Section 3, we construct and investigate a block monotone domain decomposition algorithm. The rate of convergence of the block monotone domain decomposition algorithm is estimated in Section 4. The final Section 5 presents results of numerical experiments for the proposed algorithm.

2. Monotone iterative method

On $\overline{Q}$ introduce a rectangular mesh $\overline{\Omega}^h \times \overline{\Omega}^\tau$, $\overline{\Omega}^h = \overline{\Omega}^{hx} \times \overline{\Omega}^{hy}$:

$$\overline{\Omega}^{hx} = \{x_i,\ 0 \le i \le N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i\},$$

$$\overline{\Omega}^{hy} = \left\{y_j,\ 0 \le j \le N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j\right\},$$

$$\overline{\Omega}^\tau = \{t_k = k\tau,\ 0 \le k \le N_\tau,\ N_\tau \tau = T\}.$$

For a mesh function $U(P,t)$, we use the implicit difference scheme

$$\mathcal{L}^h U(P,t) + \frac{1}{\tau}\left[U(P,t) - U(P,t-\tau)\right] = -f(P,t,U),\ (P,t) \in \Omega^h \times \Omega^\tau, \tag{2}$$

$$U(P,t) = g(P,t),\ (P,t) \in \partial\Omega^h \times \Omega^\tau, \quad U(P,0) = u^0(P),\ P \in \overline{\Omega}^h,$$

where $\mathcal{L}^h U(P,t)$ is defined by

$$\mathcal{L}^h U = -\mu^2 \left(D_+^x D_-^x + D_+^y D_-^y \right) U.$$

$D_+^x D_-^x U(P,t)$, $D_+^y D_-^y U(P,t)$ are the central difference approximations to the second derivatives

$$D_+^x D_-^x U_{ij}^k = (\hbar_{xi})^{-1} \left[\left(U_{i+1,j}^k - U_{ij}^k \right) (h_{xi})^{-1} - \left(U_{ij}^k - U_{i-1,j}^k \right) (h_{xi-1})^{-1} \right],$$

$$D_+^y D_-^y U_{ij}^k = (\hbar_{yj})^{-1} \left[\left(U_{i,j+1}^k - U_{ij}^k \right) (h_{yj})^{-1} - \left(U_{ij}^k - U_{i,j-1}^k \right) (h_{yj-1})^{-1} \right],$$

$$\hbar_{xi} = 2^{-1} \left(h_{xi-1} + h_{xi} \right), \quad \hbar_{yj} = 2^{-1} \left(h_{yj-1} + h_{yj} \right),$$

where $U_{ij}^k \equiv U\left(x_i, y_j, t_k\right)$.

Remark 1. *In this paper, we use the standard backward Euler approximation to the first derivative u_t. Our analysis can be extended to higher order schemes in time* [4].

Now, we construct an iterative method for solving the nonlinear difference scheme (2) which possesses the monotone convergence. Represent the difference equation from (2) in the equivalent form

$$\mathcal{L} U(P,t) = -f(P,t,U) + \tau^{-1} U(P,t-\tau), \tag{3}$$
$$\mathcal{L} U(P,t) \equiv \mathcal{L}^h U(P,t) + \tau^{-1} U(P,t).$$

We say that on a time level $t \in \Omega^\tau$, $\overline{V}(P,t)$ is an upper solution of the difference problem

$$\mathcal{L} V(P,t) + f(P,t,V) - \tau^{-1} V(P,t-\tau) = 0, \ P \in \Omega^h, \tag{4}$$
$$V(P,t) = g(P,t), \ P \in \partial\Omega^h,$$

if it satisfies

$$\mathcal{L} \overline{V}(P,t) + f\left(P,t,\overline{V}\right) - \tau^{-1} V(P,t-\tau) \geq 0, \ P \in \Omega^h,$$
$$\overline{V}(P,t) = g(P,t), \ P \in \partial\Omega^h.$$

Similarly, $\underline{V}(P,t)$ is called a lower solution on a time level $t \in \Omega^\tau$ with a given function $V(P,t-\tau)$, if it satisfies the reversed inequality and the boundary condition.

Additionally, we assume that $f(P,t,u)$ from (1) satisfies the two-sided constraints

$$0 \leq f_u \leq c^*, \quad c^* = \text{const}. \tag{5}$$

The iterative solution $V(P,t)$ to (2) is constructed in the following way. On each time level $t \in \Omega^\tau$, we calculate n_* iterates $V^{(n)}(P,t)$, $P \in \overline{\Omega}^h$, $n = 1, \ldots, n_*$ using the recurrence formulas

$$\begin{aligned} \mathcal{L} Z^{(n+1)}(P,t) + c^* Z^{(n+1)}(P,t) = - & \left[\mathcal{L} V^{(n)}(P,t) + f\left(P,t,V^{(n)}\right) \right. \\ & \left. -\tau^{-1} V(P,t-\tau) \right], \quad P \in \Omega^h, \end{aligned} \tag{6}$$

$$Z^{(n+1)}(P,t) = 0, \ P \in \partial\Omega^h, \quad n = 0, \ldots, n_* - 1,$$
$$V^{(n+1)}(P,t) = V^{(n)}(P,t) + Z^{(n+1)}(P,t), \ P \in \overline{\Omega}^h,$$
$$V(P,t) \equiv V^{(n_*)}(P,t), \ P \in \overline{\Omega}^h, \quad V(P,0) = u^0(P), \ P \in \overline{\Omega}^h,$$

where an initial guess $V^{(0)}(P,t)$ satisfies the boundary condition

$$V^{(0)}(P,t) = g(P,t), \ P \in \partial\Omega^h.$$

The following theorem gives the properties of the iterative algorithm (6).

Theorem 1. *Let $V^{(0)}(P,t)$ be an upper or lower solution in the iterative method (6), and let $f(P,t,u)$ satisfy (5). Suppose that on each time level the number of iterates n_* satisfies $n_* \geq 2$. Then the following estimate on convergence rate holds*

$$\max_{t\in\Omega^\tau} \|V(t)-U(t)\|_{\overline{\Omega}^h} \leq C(\rho)^{n_*-1},$$

where $U(P,t)$ is the solution to (2), $\rho = c^*/\left(c^*+\tau^{-1}\right)$, constant C is independent of τ and $\|W(t)\|_{\overline{\Omega}^h} \equiv \max_{P\in\overline{\Omega}^h} |W(P,t)|$. Furthermore, on each time level the sequence $\left\{V^{(n)}(P,t)\right\}$ converges monotonically.

The proof of this theorem can be found in [3].

Remark 2. *Consider the following approach for constructing initial upper and lower solutions $\overline{V}^{(0)}(P,t)$ and $\underline{V}^{(0)}(P,t)$. Suppose that for t fixed, a mesh function $R(P,t)$ is defined on $\overline{\Omega}^h$ and satisfies the boundary condition $R(P,t)=g(P,t)$ on $\partial\Omega^h$. Introduce the following difference problems*

$$\mathcal{L}Z_q^{(0)}(P,t) = q\left|\mathcal{L}R(P,t)+f(P,t,R)-\tau^{-1}V(P,t-\tau)\right|,\ P\in\Omega^h, \tag{7}$$
$$Z_q^{(0)}(P,t)=0,\ P\in\partial\Omega^h, \quad q=1,-1.$$

Then the functions $\overline{V}^{(0)}(P,t)=R(P,t)+Z_1^{(0)}(P,t)$, $\underline{V}^{(0)}(P,t)=R(P,t)+Z_{-1}^{(0)}(P,t)$ are upper and lower solutions, respectively. The proof of this result can be found in [3].

Remark 3. *Since the initial iteration in the monotone iterative method (6) is either an upper or a lower solution, which can be constructed directly from the difference equation without any knowledge of the solution as we have suggested in the previous remark, this algorithm eliminates the search for the initial iteration as is often needed in Newton's method. This elimination gives a practical advantage in the computation of numerical solutions.*

Remark 4. *In fact, it is sufficient that the condition (5) holds true only between the upper and lower solutions defined by (7).*

Remark 5. *We can modify the iterative method (6) in the following way. The monotone convergent property of the upper and lower sequences in Theorem 1 still holds true if the coefficient c^* in the difference equation from (6) is replaced by*

$$c^{(n)}(P,t)=\max f_u(P,t,U),\ \underline{V}^{(n)}(P,t)\leq U(P)\leq \overline{V}^{(n)}(P,t),\ t=\mathit{fixed}.$$

To perform the modified algorithm we have to compute two sequences of upper and lower solutions simultaneously. But, on the other hand, this modification increases the rate of the convergence of the iterative method.

Remark 6. *The implicit two-level difference scheme (2) is of the first order with respect to τ. From here and since $\rho\leq c^*\tau$, one may choose $n_*=2$ to keep the global error of the monotone iterative method (6) consistent with the global error of the difference scheme (2).*

3. Monotone domain decomposition algorithms

We consider decomposition of the domain $\overline{\Omega}$ into M nonoverlapping subdomains (vertical strips) $\overline{\Omega}_m, m=1,\dots,M$:

$$\Omega_m=\Omega_m^x\times(0,1), \quad \Omega_m^x=\left(x_{m-1},x_m\right), \quad x_0=0,\ x_M=1,$$
$$\Gamma_m^b=\left\{x=x_{m-1},0\leq y\leq 1\right\}, \quad \Gamma_m^e=\left\{x=x_m,0\leq y\leq 1\right\},$$

$$\overline{\Omega}_{m-1} \cap \overline{\Omega}_m = \Gamma_m^b, \quad \Gamma_m^b = \Gamma_{m-1}^e, \ m = 2, \dots, M.$$

Thus, we can write down the boundary of Ω_m as

$$\partial\Omega_m = \Gamma_m^b \cup \Gamma_m^e \cup \Gamma_m^0, \quad \Gamma_m^0 = \partial\Omega \cap \partial\Omega_m.$$

Additionally, introduce $(M-1)$ interfacial subdomains $\omega_m, m = 1, \dots, M-1$:

$$\omega_m = \omega_m^x \times (0,1), \quad \omega_m^x = \left(x_m^b, x_m^e\right),$$

$$\omega_{m-1} \cap \omega_m = \emptyset, \quad x_m^b < x_m < x_m^e, \ m = 1, \dots, M-1.$$

The boundaries of ω_m are denoted by

$$\gamma_m^b = \left\{x = x_m^b, 0 \le y \le 1\right\}, \ \gamma_m^e = \left\{x = x_m^e, 0 \le y \le 1\right\}, \ \gamma_m^0 = \partial\Omega \cap \partial\omega_m.$$

Figure 1 illustrates the x-section of the multidomain decomposition.

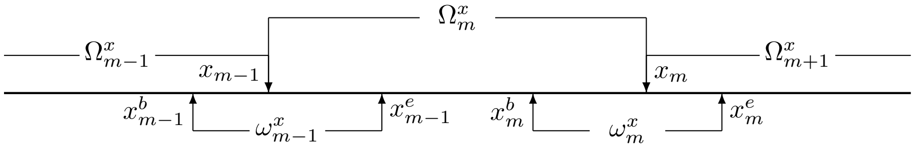

Figure 1.

We now introduce meshes on $\overline{\Omega}_m, \ m = 1, \dots, M$ and on $\overline{\omega}_m, \ m = 1, \dots, M-1$. Suppose that the following set of mesh points belongs to mesh $\overline{\Omega}^h$

$$\{x_m^b, x_m, x_m^e\}_{m=1}^{M-1} \subset \Omega^{hx},$$

then

$$\overline{\Omega}_m^h = \overline{\Omega}_m \cap \overline{\Omega}^h, \quad \overline{\omega}_m^h = \overline{\omega}_m \cap \overline{\Omega}^h,$$

$$\Gamma_m^{hb,e,0} = \Gamma_m^{b,e,0} \cap \overline{\Omega}_m^h, \quad \gamma_m^{hb,e,0} = \gamma_m^{b,e,0} \cap \overline{\omega}_m^h.$$

3.1. Statement and convergence of monotone domain decomposition algorithm. Consider a parallel domain decomposition algorithm for solving problem (2). On each time level $t \in \Omega^\tau$, we calculate n_* iterates $V^{(n)}(P,t), \ P \in \overline{\Omega}^h, \ n = 1, \dots, n_*$. To find $V^{(n)}$, firstly, we solve problems on the nonoverlapping subdomains $\overline{\Omega}_m^h, m = 1, \dots, M$ with Dirichlet boundary conditions passed from the previous iterate. Then Dirichlet data are passed from these subdomains to the interfacial subdomains $\overline{\omega}_m^h, m = 1, \dots, M-1$, and problems on the interfacial subdomains are computed. Finally, we piece together the solutions on the subdomains.

Step 0. Initialization: On the mesh $\overline{\Omega}^h$, choose an upper or lower solution $V^{(0)}(P,t), P \in \overline{\Omega}^h$ satisfying the boundary condition $V^{(0)}(P,t) = g(P,t)$ on $\partial\Omega^h$.

For $n = 1$ to n_* do Steps 1-3

Step 1. *For $m = 1$ to M do*

On the subdomain $\overline{\Omega}_m^h$, compute the mesh function $Z_m^{(n)}(P,t)$, satisfying the difference scheme

$$\begin{aligned} \mathcal{L}Z_m^{(n)}(P,t) + c^* Z_m^{(n)}(P,t) = - & \Big[\mathcal{L}V^{(n-1)}(P,t) + f\left(P,t,V^{(n-1)}\right) \\ & -\tau^{-1}V(P,t-\tau)\Big], \quad P \in \Omega_m^h, \end{aligned} \tag{8}$$

$$Z_m^{(n)}(P,t) = 0, \quad P \in \partial\Omega_m^h,$$

and denote

$$V_m^{(n)}(P,t) = V^{(n-1)}(P,t) + Z_m^{(n)}(P,t), \ P \in \overline{\Omega}_m^h.$$

Step 2. *For $m = 1$ to $M - 1$ do*
On the interfacial subdomain $\overline{\omega}_m^h$, compute the difference problem

$$\mathcal{L}\tilde{Z}_m^{(n)}(P,t) + c^*\tilde{Z}_m^{(n)}(P,t) = - \Big[\mathcal{L}V^{(n-1)}(P,t) + f\left(P,t,V^{(n-1)}\right) - \tau^{-1}V(P,t-\tau)\Big], \quad P \in \omega_m^h, \tag{9}$$

$$\tilde{Z}_m^{(n)}(P,t) = \begin{cases} 0, & P \in \gamma_m^{h0}; \\ Z_m^{(n)}(P,t), & P \in \gamma_m^{hb}; \\ Z_{m+1}^{(n)}(P,t), & P \in \gamma_m^{he}, \end{cases}$$

and denote

$$\tilde{V}_m^{(n)}(P,t) = V^{(n-1)}(P,t) + \tilde{Z}_m^{(n)}(P,t), \ P \in \overline{\omega}_m^h.$$

Step 3. Compute the solution $V^{(n)}(P,t), P \in \overline{\Omega}^h$ by piecing the solutions on the subdomains

$$V^{(n)}(P,t) = \begin{cases} V_m^{(n)}(P,t), & P \in \overline{\Omega_m^h \setminus (\omega_{m-1}^h \cup \omega_m^h)}, m = 1, \dots, M; \\ \tilde{V}_m^{(n)}(P,t), & P \in \overline{\omega}_m^h, m = 1, \dots, M-1. \end{cases} \tag{10}$$

Step 4. Set up

$$V(P,t) = V^{(n_*)}(P,t), \quad P \in \overline{\Omega}^h. \tag{11}$$

Remark 7. *We note that the original Schwarz alternating algorithm with overlapping subdomains is a purely sequential algorithm. To obtain parallelism, one needs a subdomain coloring strategy, so that a set of independent subproblems can be introduced. The proposed modification of the Schwarz algorithm is very suitable for parallel computing. Algorithm (8)-(11) can be carried out by parallel processing, since the M problems (8) for $V_m^{(n)}(P,t), m = 1, \dots, M$ and the $(M-1)$ problems (9) for $\tilde{V}_m^{(n)}(P,t), m = 1, \dots, M-1$ can be implemented concurrently.*

Remark 8. *Remark 2 on the initial iteration holds true for algorithm (8)-(11), i.e. an upper or lower solution can be constructed directly from the difference equation without any knowledge of the solution.*

On mesh $\overline{\Omega}_*^h = \overline{\Omega}_*^{hx} \times \overline{\Omega}^{hy}$:

$$\overline{\Omega}_*^{hx} = \{x_i, i = 0, 1, \dots, N_x^*; x_0 = x_a, x_{N_x^*} = x_b\},$$

where $x_a < x_b$, consider the following difference problems:

$$\mathcal{L}\Phi^s(P) + c^*\Phi^s(P) = 0, \ P \in \Omega_*^h, \tag{12}$$

$$\Phi^s(P) = 1, \ P \in \Gamma^{hs}, \quad \Phi^s(P) = 0, \ P \in \partial\Omega_*^h \backslash \Gamma^{hs}, \ s = 1, 2, 3, 4,$$

where $\mathcal{L}$ from (3) and Γ^{hs} is the s-th side of the rectangular mesh $\overline{\Omega}_*^h$. We suppose that

$$\Gamma^{h1} = \{x = x_a; \ y = y_j, 0 \le j \le N_y\}, \ \Gamma^{h2} = \{x = x_b; \ y = y_j, 0 \le j \le N_y\},$$

$$\Gamma^{h3} = \{x = x_i, 0 \le i \le N_x^*; \ y = 0\}, \ \Gamma^{h4} = \{x = x_i, 0 \le i \le N_x^*; \ y = 1\}.$$

Introduce the notation

$$\hbar_m^b = 2^{-1}\left(h_m^{b-} + h_m^{b+}\right), \quad \hbar_m^e = 2^{-1}\left(h_m^{e-} + h_m^{e+}\right),$$

where h_m^{b-}, h_m^{b+} are the mesh step sizes on the left and on the right from point x_m^b, respectively, and h_m^{e-}, h_m^{e+} are the mesh step sizes on the left and on the right from point x_m^e, respectively,

$$\kappa_m^b = \frac{\mu^2}{(c^* + \tau^{-1})\,\hbar_m^b h_m^{b+}}, \quad \kappa_m^e = \frac{\mu^2}{(c^* + \tau^{-1})\,\hbar_m^e h_m^{e-}},$$

$$q_m^b = \left\|\Phi_m^{II}\right\|_{\gamma_m^{hb+}}, \quad q_m^e = \left\|\Phi_m^{I}\right\|_{\gamma_m^{he-}}, \quad \|W\|_{\gamma^{hb+,he-}} \equiv \max_{P\in\gamma^{hb+,he-}} |W(P)|,$$

$$\gamma_m^{hb\pm} = \left\{x = x_m^b \pm h_m^{b\pm}, 0 \le y \le 1\right\}, \quad \gamma_m^{he\pm} = \left\{x = x_m^e \pm h_m^{e\pm}, 0 \le y \le 1\right\},$$

where $\Phi_m^{II}(P)$ is the solution to (12) on $\overline{\vartheta}_m^{hb} = \overline{\Omega_m^h \cap \omega_m^h}$ with $s = 2$ and $\Phi_m^I(P)$ is the solution to (12) on $\overline{\vartheta}_m^{he} = \overline{\Omega_{m+1}^h \cap \omega_m^h}$ with $s = 1$.

The following theorem contains the properties of algorithm (8)-(11), and the proof of this theorem can be found in [3].

Theorem 2. *Let $V^{(0)}(P,t)$ be an upper or lower solution in the domain decomposition algorithm (8)-(11), and let $f(P,t,u)$ satisfy (5). Suppose that on each time level, the number of iterates n_* satisfies $n_* \ge 2$. Then the following estimate on convergence rate holds*

$$\max_{1\le k\le N_\tau} \|V(t_k) - U(t_k)\|_{\overline{\Omega}^h} \le C\,(c^* + \nu)\,(\rho + \lambda)^{n_*-1}, \tag{13}$$

$$\lambda = \max_{1\le m\le M-1} \left\{\kappa_m^b q_m^b; \kappa_m^e q_m^e\right\}, \quad \nu = \left(c^* + \tau^{-1}\right)\lambda,$$

where $U(P,t)$ is the solution to (2), $\rho = c^*/(c^* + \tau^{-1})$ and constant C is independent of τ. Furthermore, on each time level the sequence $\left\{V^{(n)}(P,t)\right\}$ converges monotonically.

Remark 9. *We mention here that the monotone domain decomposition algorithm (8)-(11) is a particular case of the block monotone domain decomposition which will be constructed in the next section. In Remark 15 and Section 4.2 below, we present sufficient conditions to guarantee the inequality $\rho + \lambda < 1$ required in Theorem 2, and in Section 4.2 we study how constant C depends on the small parameter μ and the space mesh.*

Remark 10. *Remarks 4 and 5 to Theorem 1 hold true for algorithm (8)-(11) in Theorem 2.*

3.2. Block monotone domain decomposition algorithm. On time level t_k, write down the difference scheme (2) at an interior mesh point $(x_i, y_j) \in \Omega^h$ in the form

$$\begin{aligned} d_{ij}U_{ij}^k - l_{ij}U_{i-1,j}^k - r_{ij}U_{i+1,j}^k - b_{ij}U_{i,j-1}^k - t_{ij}U_{i,j+1}^k &= -f\left(x_i, y_j, t_k, U_{ij}^k\right) \\ &\quad + \tau^{-1}U_{ij}^{k-1} + G_{ij}^k, \end{aligned}$$

$$d_{ij} = l_{ij} + r_{ij} + b_{ij} + t_{ij} + \tau^{-1},\ l_{ij} = \mu^2\left(\hbar_{xi}h_{x,i-1}\right)^{-1},\ r_{ij} = \mu^2\left(\hbar_{xi}h_{xi}\right)^{-1},$$
$$b_{ij} = \mu^2\left(\hbar_{yj}h_{y,j-1}\right)^{-1},\ t_{ij} = \mu^2\left(\hbar_{yj}h_{yj}\right)^{-1},$$

where G_{ij}^k is associated with the boundary function $g(P, t_k)$. Define vectors and diagonal matrices by

$$U_i^k = \left(U_{i1}^k, \ldots, U_{i,N_y-1}^k\right)', \quad G_i^k = \left(G_{i1}^k, \ldots, G_{i,N_y-1}^k\right)',$$

$$F_i^k\left(U_i^k\right) = \left(f_{i1}^k\left(U_{i1}^k\right), \ldots, f_{i,N_y-1}^k\left(U_{i,N_y-1}^k\right)\right)',$$

$$L_i = \operatorname{diag}\left(l_{i1}, \ldots, l_{i,N_y-1}\right), \quad R_i = \operatorname{diag}\left(r_{i1}, \ldots, r_{i,N_y-1}\right).$$

Then the difference scheme (2) may be written in the form

$$A_i U_i^k - \left(L_i U_{i-1}^k + R_i U_{i+1}^k\right) = -F_i^k\left(U_i^k\right) + \tau^{-1} U_i^{k-1} + G_i^k,\ i = 1, \ldots, N_x - 1,$$

with the tridiagonal matrix A_i

$$A_i = \begin{bmatrix} d_{i1} & -t_{i1} & & 0 \\ -b_{i2} & d_{i2} & -t_{i2} & \\ \ddots & \ddots & \ddots & \\ & -b_{i,N_y-2} & d_{i,N_y-2} & -t_{i,N_y-2} \\ 0 & & -b_{i,N_y-1} & d_{i,N_y-1} \end{bmatrix}.$$

Matrices L_i and R_i contain the coupling coefficients of a mesh point respectively to the mesh point of the left line and the mesh point of the right line.

Since $d_{ij}, b_{ij}, t_{ij} > 0$ and A_i is strictly diagonally dominant, then A_i is an M-matrix and $A_i^{-1} \geq 0$ (cf. [12]).

Introduce two nonoverlapping ordered sets of indices

$$\mathcal{M}_\alpha = \{m_{k_\alpha} |\ m_{1_\alpha}, \ldots, m_{M_\alpha}\},\ \alpha = 1, 2, \quad M_1 + M_2 = M,$$

$$\mathcal{M}_1 \neq \emptyset, \quad \mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset, \quad \mathcal{M}_1 \cup \mathcal{M}_2 = \{1, \ldots, M\}.$$

Now, we modify Step 1 in algorithm (8)-(11) in the following way.

Step $1'$**.** On subdomain $\overline{\Omega}_m^h, m \in \mathcal{M}_1$, compute

$$V_m^{(n)}(P, t_k) = \left\{V_{m,i}^{(n)}(P, t_k),\ 0 \leq i \leq i_m\right\}, \quad m \in \mathcal{M}_1,$$

satisfying the difference scheme

$$\begin{aligned} A_{m,i} V_{m,i}^{(n)}(t_k) + c^* V_{m,i}^{(n)}(t_k) &= L_{m,i} V_{m,i-1}^{(n-1)}(t_k) + R_{m,i} V_{m,i+1}^{(n-1)}(t_k) \\ &\quad + c^* V_{m,i}^{(n-1)}(t_k) - F_{m,i}^k\left(V_{m,i}^{(n-1)}(t_k)\right) \\ &\quad + \tau^{-1} V_{m,i}^{k-1} + G_{m,i}^k,\ 1 \leq i \leq i_m - 1, \end{aligned} \tag{14}$$

$$V_{m,i}^{(n)}(t_k) = V_{m,i}^{(n-1)}(t_k),\ i = 0, i_m,$$

where $i = 0$ and $i = i_m$ are the boundary vertical lines, and $G_m^k = \{G_{m,i}^k, 1 \leq i \leq i_m - 1\}$, $V_m^{k-1} = \{V_{m,i}^{k-1}, 1 \leq i \leq i_m - 1\}$, $V_m^{(n-1)}(t_k) = \{V_{m,i}^{(n-1)}(t_k), 0 \leq i \leq i_m\}$ are the parts of $g(P, t_k)$, $V(P, t_{k-1})$ and $V^{(n-1)}(P, t_k)$, respectively, which correspond to subdomain $\overline{\Omega}_m^h$.

On subdomain $\overline{\Omega}_m^h, m \in \mathcal{M}_2$, compute mesh function $V_m^{(n)}(P, t_k)$, $m \in \mathcal{M}_2$ satisfying (8).

Remark 11. *Algorithm (14) may be considered as the line Jacobi method or the block Jacobi method for solving the five-point difference scheme (8) on subdomain* $\Omega_m^h, m \in \mathcal{M}_1$ *(cf.* [12]*). Basic advantages of the block iterative scheme (14) are that the Thomas algorithm can be used for each subsystem* $i, i = 1, \ldots, i_m - 1$ *and all the subsystems can be computed in parallel.*

On $\overline{\Omega}_*^h = \overline{\Omega}_*^{hx} \times \overline{\Omega}^{hy}$, we represent a difference scheme in the following canonical form

$$d(P)W(P) = \sum_{P' \in S(P)} e(P, P')W(P') + F(P),\ P \in \Omega_*^h, \tag{15}$$

$$W(P) = W^0(P),\ P \in \partial\Omega_*^h,$$

and suppose that

$$d(P) > 0,\ e(P, P') \geq 0,\ c(P) = d(P) - \sum_{P' \in S'(P)} e(P, P') > 0,\ P \in \Omega_*^h,$$

where $S'(P) = S(P) \setminus \{P\}$, $S(P)$ is a stencil of the difference scheme. Now, we formulate a discrete maximum principle and give an estimate on the solution to (15).

Lemma 1. *Let the positive property of the coefficients of the difference scheme (15) be satisfied.*

(i) If $W(P)$ satisfies the conditions

$$d(P)W(P) - \sum_{P' \in S(P)} e(P, P')W(P') - F(P) \geq 0 (\leq 0),\ P \in \Omega_*^h,$$

$$W(P) \geq 0 (\leq 0),\ P \in \partial\Omega_*^h,$$

then $W(P) \geq 0 (\leq 0),\ P \in \overline{\Omega}_*^h$.

(ii) The following estimate on the solution to (15) holds true

$$\|W\|_{\overline{\Omega}_*^h} \leq \max\left[\left\|W^0\right\|_{\partial\Omega_*^h}; \|F/c\|_{\Omega_*^h}\right]. \tag{16}$$

The proof of the lemma can be found in [11].

Theorem 3. *Let $V(P, t-\tau)$ be given and $\overline{V}^{(0)}(P,t)$, $\underline{V}^{(0)}(P,t)$ be upper and lower solutions corresponding to $V(P, t-\tau)$. Suppose that $f(P,t,u)$ satisfies (5). Then the upper sequence $\left\{\overline{V}^{(n)}(P,t)\right\}$ generated by (8)-(10), (14) converges monotonically from above to the unique solution $\mathcal{V}(P,t)$ of the problem (4), and the lower sequence $\left\{\underline{V}^{(n)}(P,t)\right\}$ generated by (8)-(10), (14) converges monotonically from below to $\mathcal{V}(P,t)$:*

$$\underline{V}^{(0)}(P,t) \leq \underline{V}^{(n)}(P,t) \leq \underline{V}^{(n+1)}(P,t) \leq \mathcal{V}(P,t), \quad P \in \overline{\Omega}^h,$$

$$\mathcal{V}(P,t) \leq \overline{V}^{(n+1)}(P,t) \leq \overline{V}^{(n)}(P,t) \leq \overline{V}^{(0)}(P,t),\ P \in \overline{\Omega}^h.$$

Proof. Introduce the notation

$$Z_m^{(n)}(P,t_k) = V_m^{(n)}(P)(t_k) - V^{(n-1)}(P,t_k),\ P \in \Omega_m^h,\ m \in \mathcal{M}_1.$$

Consider the case of the upper sequence, i.e. $\overline{V}^{(0)}(P,t_k)$ is an upper solution. For $n = 1$ and $m \in \mathcal{M}_1$, by (14)

$$\begin{aligned} A_{m,i}Z_{m,i}^{(1)}(t_k) + c^* Z_{m,i}^{(1)}(t_k) &= -\Big[A_{m,i}\overline{V}_{m,i}^{(0)}(t_k) - \Big(L_{m,i}\overline{V}_{m,i-1}^{(0)}(t_k) \\ &\quad + R_{m,i}\overline{V}_{m,i+1}^{(0)}(t_k)\Big) + F_{m,i}^k\left(\overline{V}_{m,i}^{(0)}(t_k)\right) \\ &\quad -\tau^{-1}V_{m,i}^{k-1} - G_{m,i}^k\Big] \leq 0, \end{aligned} \tag{17}$$

$$1 \leq i \leq i_m - 1,$$

where we have taken into account that $\overline{V}^{(0)}(P,t_k)$ is the upper solution. Since $A_{m,i}^{-1} \geq 0$ then $(A_{m,i} + c^* I_m)^{-1} \geq 0$, where I_m is the $(i_m - 1) \times (i_m - 1)$ identity matrix. Thus, we conclude that $Z_m^{(1)}(P,t_k) \leq 0, P \in \overline{\Omega}_m^h$, $m \in \mathcal{M}_1$. By (8)

$$\begin{aligned} \mathcal{L}Z_m^{(1)}(P,t_k) + c^* Z_m^{(1)}(P,t_k) = - &\Big[\mathcal{L}\overline{V}^{(0)}(P,t_k) + f\left(P,t_k,\overline{V}^{(0)}\right) \\ &-\tau^{-1}V(P,t_{k-1})\Big] \leq 0, \end{aligned} \tag{18}$$

$$P \in \Omega_m^h, \quad Z_m^{(1)}(P,t) = 0,\ P \in \partial\Omega_m^h, \quad m \in \mathcal{M}_2.$$

By the maximum principle in Lemma 1, we conclude that $Z_m^{(1)}(P,t_k) \leq 0, P \in \overline{\Omega}_m^h$, $m \in \mathcal{M}_2$. Thus

$$Z_m^{(1)}(P,t_k) \leq 0,\ P \in \overline{\Omega}_m^h,\ m = 1,\dots,M. \tag{19}$$

By (9)

$$\begin{aligned} \mathcal{L}\tilde{Z}_m^{(1)}(P,t_k) + c^*\tilde{Z}_m^{(1)}(P,t_k) = - \quad & \Big[\mathcal{L}\overline{V}^{(0)}(P,t_k) + f\left(P,t_k,\overline{V}^{(0)}\right) \\ & -\tau^{-1}V(P,t_{k-1})\Big] \leq 0,\ P \in \omega_m^h, \end{aligned} \tag{20}$$

$$\tilde{Z}_m^{(1)}(P,t_k) = \begin{cases} 0, & P \in \gamma_m^{h0}; \\ Z_m^{(1)}(P,t_k), & P \in \gamma_m^{hb}; \\ Z_{m+1}^{(1)}(P,t_k), & P \in \gamma_m^{he}. \end{cases}$$

Using the nonpositive property of $Z_m^{(1)}(P,t_k)$, by the maximum principle in Lemma 1, we conclude that

$$\tilde{Z}_m^{(1)}(P,t_k) \leq 0,\ P \in \overline{\omega}_m^h,\ m = 1,\dots,M-1. \tag{21}$$

(19) and (21) show that $\overline{V}^{(1)}(P,t_k) \leq \overline{V}^{(0)}(P,t_k), P \in \overline{\Omega}^h$. By induction, we prove that $\overline{V}^{(n)}(P,t_k) \leq \overline{V}^{(n-1)}(P,t_k), P \in \overline{\Omega}^h$ for each $n \geq 1$.

Now we verify that $\overline{V}^{(n)}(P,t_k)$ is an upper solution for each n. From the boundary conditions for $V_m^{(n)}(P,t_k)$ and $\tilde{V}_m^{(n)}(P,t_k)$, it follows that $\overline{V}^{(n)}(P,t_k)$ satisfies the boundary condition in (2). Represent (14) in the form

$$\begin{aligned} \left[\Lambda V_m^{(n)}(t_k)\right]_i \ = \ & -L_{m,i}Z_{m,i-1}^{(n)}(t_k) - R_{m,i}Z_{m,i+1}^{(n)}(t_k) \\ & + \left[c^*\overline{V}_{m,i}^{(n-1)}(t_k) - F_{m,i}^k\left(\overline{V}_{m,i}^{(n-1)}(t_k)\right)\right] \\ & - \left[c^*V_{m,i}^{(n)}(t_k) - F_{m,i}^k\left(V_{m,i}^{(n)}(t_k)\right)\right], \end{aligned} \tag{22}$$

where we have introduced the notation

$$\Lambda V_m^{(n)}(P,t_k) = \mathcal{L}V_m^{(n)}(P,t_k) + f\left(P,t_k,V_m^{(n)}\right) - \tau^{-1}V(P,t_{k-1}),$$

$$\begin{aligned} \left[\Lambda V_m^{(n)}(P,t_k)\right]_i \ = \ & A_{m,i}V_{m,i}^{(n)}(t_k) - \left(L_{m,i}V_{m,i-1}^{(n)}(t_k) + R_{m,i}V_{m,i+1}^{(n)}(t_k)\right) \\ & + F_{m,i}^k\left(V_{m,i}^{(n)}(t_k)\right) - \tau^{-1}V_{m,i}^{k-1} - G_{m,i}^k. \end{aligned}$$

By the mean-value theorem and (5)

$$[c^*W - f(W)] - [c^*Z - f(Z)] = c^*(W-Z) - F_u^{(n)}(W-Z) \geq 0,$$

whenever $W \geq Z$. Using this property, (19) and $L_{m,i} \geq 0$, $R_{m,i} \geq 0$, we conclude

$$\Lambda V_m^{(n)}(P,t_k) \geq 0,\ P \in \Omega_m^h, \quad m \in \mathcal{M}_1.$$

From (8) for $m \in \mathcal{M}_2$, by the mean-value theorem, (5) and (19), we have

$$\Lambda V_m^{(n)}(P,t_k) = -\left(c^* - f_u^{(n)}\right)Z_m^{(n)}(P,t_k) \geq 0, \quad P \in \Omega_m^h,\ m \in \mathcal{M}_2. \tag{23}$$

Similarly, we prove

$$\Lambda \tilde{V}_m^{(n)}(P,t_k) = -\left(c^* - f_u^{(n)}\right)\tilde{Z}_m^{(n)}(P,t_k) \geq 0, \quad P \in \omega_m^h. \tag{24}$$

Thus, by the definition of $\overline{V}^{(n)}$ in (10), we conclude that

$$\mathcal{L}\overline{V}^{(n)}(P,t_k) + f\left(P,t_k,\overline{V}^{(n)}\right) - \tau^{-1}V(P,t_{k-1}) \geq 0, \ P \in \Omega^h \setminus \gamma^h,$$

$$\gamma^h = \bigcup_{m=1}^{M-1} \gamma_m^{hb,e}.$$

To prove that $\overline{V}^{(n)}(P,t_k)$ is an upper solution of (4), we have to verify only that the last inequality holds true on the interfacial boundaries $\gamma_m^{hb}, \gamma_m^{he}$, $m = 1, \dots M-1$. We check this inequality in the case of the left interfacial boundary γ_m^{hb}, since the second case is checked in a similar way. Introduce the notation $W_m^{(n)}(P,t_k) = V_m^{(n)}(P,t_k) - \tilde{V}_m^{(n)}(P,t_k)$. From (14) and (9), we have

$$A_{m,i}W_{m,i}^{(n)}(t_k) + c^* W_{m,i}^{(n)}(t_k) = -L_{m,i}Z_{m,i-1}^{(n)}(t_k) - R_{m,i}Z_{m,i+1}^{(n)}(t_k), \tag{25}$$

$$P \in \vartheta_m^{hb}, \quad \vartheta_m^{hb} = \Omega_m^h \cap \omega_m^h, \quad m \in \mathcal{M}_1.$$

In view of $(A_{m,i} + c^* I_m)^{-1} \geq 0$, $L_{m,i} \geq 0$, $R_{m,i} \geq 0$ and (19) which holds true for each $n \geq 1$,

$$W_m^{(n)}(P,t_k) \geq 0, \ P \in \overline{\vartheta}_m^{hb}, \quad m \in \mathcal{M}_1.$$

From (8), (9) and (21), we conclude

$$\mathcal{L}W_m^{(n)}(P,t_k) + c^* W_m^{(n)}(P,t_k) = 0, \ P \in \vartheta_m^{hb}, \tag{26}$$

$$W_m^{(n)}(P,t_k) = 0, \ P \in \partial\vartheta_m^{hb} \setminus \Gamma_m^{he}, \quad W_m^{(n)}(P,t_k) \geq 0, \ P \in \Gamma_m^{he}.$$

In view of the maximum principle in Lemma 1, $W_m^{(n)}(P,t_k) \geq 0$, $P \in \overline{\vartheta}_m^{hb}$, $m \in \mathcal{M}_2$. Thus

$$V_m^{(n)}(P,t_k) - \tilde{V}_m^{(n)}(P,t_k) \geq 0, \ P \in \overline{\vartheta}_m^{hb}, \ m = 1, \dots, M-1. \tag{27}$$

From (2), (9), (10) and $\tilde{V}_m^{(n)}(P,t_k) = V_m^{(n)}(P,t_k), P \in \gamma_m^{hb}$,

$$-\mu^2 D_+^y D_-^y V_m^{(n)}(P,t_k) = -\mu^2 D_+^y D_-^y \overline{V}^{(n)}(P,t_k), \ P \in \gamma_m^{hb}.$$

From (2), (10) and (27), we obtain

$$-\mu^2 D_+^x D_-^x V_m^{(n)}(P,t_k) \leq -\mu^2 D_+^x D_-^x \overline{V}^{(n)}(P,t_k), \ P \in \gamma_m^{hb}.$$

In the notation from (22), we conclude

$$\Lambda\overline{V}^{(n)}(P,t_k) \geq \Lambda V_m^{(n)}(P,t_k) \geq 0, \ P \in \gamma_m^{hb}.$$

This leads to the fact that $\overline{V}^{(n)}(P,t_k)$ is an upper solution of problem (4).

By (19), (21), sequence $\{\overline{V}^{(n)}\}$ is monotone decreasing and bounded by a lower solution. Indeed, if $\underline{V}$ is a lower solution, then by the definitions of lower and upper solutions and the mean-value theorem, for $\delta^{(n)} = \overline{V}^{(n)} - \underline{V}$, we have

$$\mathcal{L}\delta^{(n)}(P,t_k) + f_u \delta^{(n)}(P,t_k) \geq 0, \ P \in \Omega^h,$$

$$\delta^{(n)}(P,t_k) = 0, \ P \in \partial\Omega^h.$$

In view of the maximum principle in Lemma 1, it follows that $\underline{V} \leq \overline{V}^{(n)}, n \geq 0$. Thus, $\lim \overline{V}^{(n)} = \overline{V}$ as $n \to \infty$ exists and satisfies the relation

$$\overline{V}(P,t_k) \leq \overline{V}^{(n+1)}(P,t_k) \leq \overline{V}^{(n)}(P,t_k) \leq \overline{V}^{(0)}(P,t_k), \quad P \in \overline{\Omega}^h.$$

Now we prove the last point of this theorem that the limiting function $\overline{V}$ is the solution to (4). Letting $n \to \infty$ in (8), (9) and (14) shows that $\overline{V}$ is the solution of (4) on $\Omega^h \setminus \gamma^h$. Now we verify that $\overline{V}$ satisfies (4) on the interfacial boundaries

$\gamma_m^{hb}, \gamma_m^{he}$, $m = 1, \ldots, M-1$. Since $V_m^{(n)}(P,t_k) - \tilde{V}_m^{(n)}(P,t_k) = \overline{V}^{(n-1)}(P,t_k) - \overline{V}^{(n)}(P,t_k), P \in \Gamma_m^{he}$, we conclude that

$$\lim_{n\to\infty} V_m^{(n)}(P,t_k) = \lim_{n\to\infty} \tilde{V}_m^{(n)}(P,t_k) = \overline{V}(P,t_k),\ P \in \overline{\vartheta}_m^{hb}.$$

From here it follows that

$$\lim_{n\to\infty} \Lambda \overline{V}^{(n)}(P,t_k) = \lim_{n\to\infty} \Lambda V_m^{(n)}(P,t_k) = 0,\ P \in \gamma_m^{hb},$$

and hence, $\overline{V}(P,t_k)$ solves (4) on γ_m^{hb}. In a similar way, we can prove the last result on γ_m^{he}. Under the condition (5), problem (4) has the unique solution $\mathcal{V}$ (see [3] for details), hence $\overline{V} = \mathcal{V}$. This proves the theorem. □

Remark 12. *The proposed algorithm (8)-(10), (14) can be applied for solving "unperturbed" problems of form (1), i.e. in the case of* $\mu = \mathcal{O}(1)$. *Furthermore, the block iterative scheme (14) can be applied on the whole computational domain* $\overline{\Omega}^h$. *However, as we show below, this algorithm can be most efficiently used at small values of* μ *and in the case of the location of subdomains* $\overline{\Omega}_m^h$, $m \in \mathcal{M}_1$, *outside the boundary layers.*

Remark 13. *Remarks 4 and 5 to Theorem 1 hold true for algorithm (8)-(10), (14) in Theorem 3.*

4. Convergence of algorithm (8)-(11), (14)

We now establish convergence properties of algorithm (8)-(11), (14).

4.1. Convergence analysis of algorithm (8)-(11), (14). If we denote

$$Z^{(n)}(P,t_k) = V^{(n)}(P,t_k) - V^{(n-1)}(P,t_k),\ P \in \overline{\Omega}^h,$$

then from (8)-(11), (14), it follows that on $\overline{\Omega}_m^h, m = 1, \ldots, M$, $Z^{(n)}$ can be written in the form

$$Z^{(n)}(P,t_k) = \begin{cases} Z_{m-1}^{(n)}(P,t_k), & x_{m-1} \le x \le x_{m-1}^e; \\ \tilde{Z}_m^{(n)}(P,t_k), & x_{m-1}^e \le x \le x_m^b; \\ Z_m^{(n)}(P,t_k), & x_m^b \le x \le x_m, \end{cases}$$

where for simplicity, we indicate the discrete domains only in the x-variable, i.e. $x_{m-1} \le x \le x_{m-1}^e$ means $\{x_{m-1} \le x \le x_{m-1}^e, 0 \le y \le 1\}$, and assume that for $m = 1, M$, the corresponding domains $x_0 \le x \le x_0^e$ and $x_M^b \le x \le x_M$ are empty.

Introduce the notation

$$l = \max_{m\in\mathcal{M}_1} \left\{ \max_{1\le i\le i_m-1} \left[l_{(m)i} \right] \right\},\ l_{(m)i} = \|L_{m,i}\|, \quad \|L_{m,i}\| \equiv \max_{1\le j\le N_y-1} |l_{ij}|,$$

$$r = \max_{m\in\mathcal{M}_1} \left\{ \max_{1\le i\le i_m-1} \left[r_{(m)i} \right] \right\},\ r_{(m)i} = \|R_{m,i}\|, \quad \|R_{m,i}\| \equiv \max_{1\le j\le N_y-1} |r_{ij}|,$$

$$l_{(m)e} = \left\| L_{m,i_{m-1}^e} \right\|, \quad r_{(m)b} = \left\| R_{m,i_m^b} \right\|,$$

$$\kappa = \left(\frac{1}{c^* + \tau^{-1}} \right) \max_{1\le m\le M-1} \left[l_{(m)e}; r_{(m)b} \right],$$

where the indices i_{m-1}^e, i_m^b correspond to x_{m-1}^e and x_m^b, respectively, and $i = 0,\ i = i_m$ correspond to x_m and x_{m+1}, respectively.

Theorem 4. *For the block monotone domain decomposition algorithm (8)-(11), (14) the following estimate holds true*

$$\left\|Z^{(n)}(t_k)\right\|_{\overline{\Omega}^h} \leq (\rho+\lambda)\left\|Z^{(n-1)}(t_k)\right\|_{\overline{\Omega}^h}, \tag{28}$$

$$\lambda = \frac{l+r}{c^*+\tau^{-1}} + \kappa \max\left[1; \frac{l+r}{c^*+\tau^{-1}}\right],$$

where $Z^{(n)} = V^{(n)} - V^{(n-1)},\ \rho = c^*/\left(c^*+\tau^{-1}\right)$.

Proof. Let $\overline{V}^{(0)}$ be an upper solution. Then similar to (17), (18) and (20), by induction, we get for $n \geq 1$

$$A_{m,i}Z^{(n)}_{m,i}(t_k) + c^* Z^{(n)}_{m,i}(t_k) = -\left[\Lambda\overline{V}^{(n-1)}(P,t_k)\right]_{m,i},\ 1 \leq i \leq i_m - 1,\ m \in \mathcal{M}_1,$$

$$\mathcal{L}Z^{(n)}_m(P,t_k) + c^* Z^{(n)}_m(P,t_k) = -\Lambda\overline{V}^{(n-1)}(P,t_k),\ P \in \Omega^h_m,\ m \in \mathcal{M}_2,$$

$$\mathcal{L}\tilde{Z}^{(n)}_m(P,t_k) + c^* \tilde{Z}^{(n)}_m(P,t_k) = -\Lambda\overline{V}^{(n-1)}(P,t_k),\ P \in \omega^h_m,\ m = 1,\ldots,M-1.$$

Using (16) with $c = c^* + \tau^{-1}$, we get the following estimates on $Z^{(n)}_m$ and $\tilde{Z}^{(n)}_m$

$$|Z^{(n)}_m(P,t_k)| \leq \frac{1}{c^*+\tau^{-1}}\left\|\Lambda\overline{V}^{(n-1)}(t_k)\right\|_{\Omega^h_m},\ P \in \overline{\Omega}^h_m, \tag{29}$$

$$|\tilde{Z}^{(n)}_m(P,t_k)| \leq \max\left[\frac{1}{c^*+\tau^{-1}}\left\|\Lambda\overline{V}^{(n-1)}(t_k)\right\|_{\omega^h_m};\right.$$
$$\left.\left\|Z^{(n)}_m(t_k)\right\|_{\gamma^{hb}_m}; \left\|Z^{(n)}_{m+1}(t_k)\right\|_{\gamma^{he}_m}\right],\ P \in \overline{\omega}^h_m.$$

From (10), (22), (23) and (24), on $\Omega^h_m, m = 1,\ldots,M$, we have

$$\Lambda\overline{V}^{(n-1)}(P,t_k) = \begin{cases} -\left(c^*-f_u\right)\tilde{Z}^{(n-1)}_{m-1}(P,t_k), & x_{m-1} \leq x < x^e_{m-1}; \\ -\left(c^*-f_u\right)Z^{(n-1)}_m(P,t_k), & x^e_{m-1} < x < x^b_m,\ m \in \mathcal{M}_2; \\ -\left(c^*-f_u\right)\tilde{Z}^{(n-1)}_m(P,t_k), & x^b_m < x \leq x_m, \end{cases}$$

$$\left[\Lambda\overline{V}^{(n-1)}(P,t_k)\right]_{m,i} = -\left(c^*-f_u\right)Z^{(n-1)}_{m,i}(t_k) - \Big(L_{m,i}Z^{(n-1)}_{m,i-1}(t_k)$$
$$+R_{m,i}Z^{(n-1)}_{m,i+1}(t_k)\Big),\ i^e_{m-1} < i < i^b_m,\ m \in \mathcal{M}_1,$$

where the indices i^e_{m-1}, i^b_m correspond to x^e_{m-1} and x^b_m, respectively. From here and (5) and taking into account $L_{m,i} \geq 0$, $R_{m,i} \geq 0$ and (19), we get

$$\frac{1}{c^*+\tau^{-1}}\left|\Lambda\overline{V}^{(n-1)}(P,t_k)\right| \leq \rho_1\left\|Z^{(n-1)}(t_k)\right\|_{\overline{\Omega}^h},\ P \in \Omega^h_m \setminus (\gamma^{he}_{m-1} \cup \gamma^{hb}_m), \tag{30}$$

where $\rho_1 = \rho + (l+r)/\left(c_*+\tau^{-1}\right)$.

Now, we prove the following estimates

$$\frac{1}{c^*+\tau^{-1}}\left\|\Lambda\overline{V}^{(n-1)}(t_k)\right\|_{\gamma^{hb}_m} \leq \left[\rho_1 + \frac{\rho_2 r_{(m)b}}{c^*+\tau^{-1}}\right]\left\|Z^{(n-1)}(t_k)\right\|_{\overline{\Omega}^h}, \tag{31}$$

$$\frac{1}{c^*+\tau^{-1}}\left\|\Lambda\overline{V}^{(n-1)}(t_k)\right\|_{\gamma^{he}_{m-1}} \leq \left[\rho_1 + \frac{\rho_2 l_{(m)e}}{c^*+\tau^{-1}}\right]\left\|Z^{(n-1)}(t_k)\right\|_{\overline{\Omega}^h},$$

where $\rho_2 = \max\left[1; (l+r)/\left(c^* + \tau^{-1}\right)\right]$. Using (10), on the boundary γ_m^{hb}, we can write the relation

$$\begin{aligned}\Lambda\overline{V}^{(n-1)}(P,t_k) &= \Lambda V_m^{(n-1)}(P,t_k) \\ &\quad -\frac{\mu^2}{\hbar_m^b h_m^{b+}}\left(\tilde{V}_m^{(n-1)}\left(P_m^{b+},t_k\right) - V_m^{(n-1)}\left(P_m^{b+},t_k\right)\right),\end{aligned}$$

$$P = \left(x_m^b, y\right) \in \gamma_m^{hb},\ P_m^{b+} = \left(x_m^b + h_m^{b+}, y\right) \in \gamma_m^{hb+}.$$

From (22) and (23), we can represent $\Lambda V_m^{(n-1)}(P,t_k)$ on γ_m^{hb} in the form

$$\begin{aligned}\left[\Lambda V_m^{(n-1)}(t_k)\right]_i &= -\left(c^* - f_u\right) Z_{m,i}^{(n-1)}(t_k) - \Big(L_{m,i} Z_{m,i-1}^{(n-1)}(t_k) \\ &\quad + R_{m,i} Z_{m,i+1}^{(n-1)}(t_k)\Big), \quad i = i_m^b, \quad m \in \mathcal{M}_1,\end{aligned}$$

where the index i_m^b corresponds to γ_m^{hb},

$$\Lambda V_m^{(n-1)}(P,t_k) = -\left(c^* - f_u\right) Z^{(n-1)}(P,t_k),\ P \in \gamma_m^{hb}, \quad m \in \mathcal{M}_2.$$

Thus, we conclude the estimate

$$\begin{aligned}\frac{1}{c^* + \tau^{-1}}\left|\Lambda\overline{V}^{(n-1)}(P,t_k)\right| &\le \rho_1 \left\|Z^{(n-1)}(t_k)\right\|_{\overline{\Omega}^h} + \frac{\mu^2}{\left(c^* + \tau^{-1}\right)\hbar_m^b h_m^{b+}} \times \\ &\quad \left|\overline{V}^{(n-1)}\left(P_m^{b+},t_k\right) - V_m^{(n-1)}\left(P_m^{b+},t_k\right)\right|,\end{aligned}$$

$$P = \left(x_m^b, y\right) \in \gamma_m^{hb},\ P_m^{b+} = \left(x_m^b + h_m^{b+}, y\right) \in \gamma_m^{hb+}.$$

Applying (16) to (25) and (26), and taking into account that $\tilde{V}_m^{(n-1)}(P,t_k) - V_m^{(n-1)}(P,t_k) = \overline{V}^{(n-1)}(P,t_k) - \overline{V}^{(n-2)}(P,t_k),\ P \in \Gamma_m^{he}$, we get the estimates

$$\left\|\tilde{V}_m^{(n-1)}(t_k) - V_m^{(n-1)}(t_k)\right\|_{\overline{\vartheta}_m^{hb}} \le \rho_2 \left\|Z^{(n-1)}(t_k)\right\|_{\Gamma_m^{he}}, \quad m \in \mathcal{M}_1,$$

$$\left\|\tilde{V}_m^{(n-1)}(t_k) - V_m^{(n-1)}(t_k)\right\|_{\overline{\vartheta}_m^{hb}} \le \left\|Z^{(n-1)}(t_k)\right\|_{\Gamma_m^{he}}, \quad m \in \mathcal{M}_2.$$

Thus, we prove (31) on γ_m^{hb}. Similarly, we can prove (31) on the boundary γ_{m-1}^{he}. From (29), (30) and (31) and using the definition of $Z^{(n)}(P,t_k)$, we prove the theorem. □

Theorem 5. *Let $V^{(0)}(P,t)$ be an upper or lower solution in the domain decomposition algorithm (8)-(11), (14) and let $f(P,t,u)$ satisfy (5). Suppose that on each time level, the number of iterates n_* satisfies $n_* \ge 2$. Then the following estimate on convergence rate holds*

$$\max_{1 \le k \le N_\tau} \|V(t_k) - U(t_k)\| \le C\left(c^* + \nu\right)(\rho + \lambda)^{n_* - 1}, \tag{32}$$

$$\nu = \left(c^* + \tau^{-1}\right)\lambda,$$

where $U(P,t)$ is the solution to (2), ρ and λ are defined in Theorem 4, and constant C is independent of τ. Furthermore, on each time level the sequence $\left\{V^{(n)}(P,t)\right\}$ converges monotonically.

Proof. Denote $W(P,t) = U(P,t) - V(P,t)$. Using the notation from (22) and (2), we can write the following difference problem for $W(P,t)$

$$\mathcal{L}W(P,t) + f_u(P,t)W(P,t) = -\Lambda V^{(n_*)}(P,t) + \tau^{-1}W(P,t-\tau),\ P \in \Omega^h, \tag{33}$$

$$W(P,t) = 0, \quad P \in \partial\Omega^h.$$

From here, (30), (31) and using (16), we obtain the estimate

$$\|W(t_k)\|_{\overline{\Omega}^h} \leq \tau (c^* + \nu) \left\| Z^{(n_*)}(t_k) \right\|_{\overline{\Omega}^h} + \|W(t_k - \tau)\|_{\overline{\Omega}^h} .$$

Using (28), we prove by induction the estimates

$$\|W(t_k)\| \leq \left(\sum_{l=1}^{k} C_l \right) \tau (c^* + \nu) (\rho + \lambda)^{n_* - 1}, \ k = 1, \dots, N_\tau,$$

$$\left\| Z^{(1)}(t_l) \right\|_{\overline{\Omega}^h} \leq \tau \left\| \mathcal{L} V^{(0)}(t_l) + f\left(V^{(0)}\right) - \tau^{-1} V(t_l - \tau) \right\|_{\overline{\Omega}^h} \leq C_l, \tag{34}$$

where all constants C_l are independent of τ. Since $N_\tau \tau = T$, we prove the estimate in the theorem with $C = TC_0$, where $C_0 = \max_{1 \leq l \leq N_\tau} C_l$. □

Remark 14. *We mentioned in Remark 12 that the block monotone iterative scheme (14) can be applied on the whole computational domain* $\overline{\Omega}^h$*, i.e. in the case of* $\mathcal{M}_1 = \{1\}$ *and* $\mathcal{M}_2 = \emptyset$*. As follows from the proofs of Theorems 4 and 5, these theorems hold true with* $\lambda = (l + r)/\left(c^* + \tau^{-1}\right)$.

Remark 15. *In the case of* $\mathcal{M}_1 = \emptyset$*, as follows from the proofs of Theorems 4 and 5, these theorems hold true with*

$$\lambda = \kappa, \quad \kappa = \max_{1 \leq m \leq M-1} \left\{ \kappa_m^b; \kappa_m^e \right\},$$

where we use the notation from (13) and (28). Since the solutions to (12) are bounded by $0 \leq \Phi^s(P) \leq 1$*, then in (13),* $0 \leq q_m^{b,e} \leq 1$*. Thus, the last estimate on* λ *follows from (13).*

4.2. Estimates on the rate of convergence of algorithm (8)-(11), (14). Here we analyze a convergence rate of algorithm (8)-(11), (14) applied to the difference scheme (2) defined on a piecewise equidistant mesh of Shishkin-type and on its modification. On the Shishkin mesh, the difference scheme (2) converges μ-uniformly to the solution of (1) (see [7] for details).

The piecewise equidistant mesh of Shishkin-type is formed by the following manner. We divide each of the intervals $\overline{\Omega}^x = [0, 1]$ and $\overline{\Omega}^y = [0, 1]$ into three parts each $[0, \sigma_x]$, $[\sigma_x, 1 - \sigma_x]$, $[1 - \sigma_x, 1]$, and $[0, \sigma_y]$, $[\sigma_y, 1 - \sigma_y]$, $[1 - \sigma_y, 1]$, respectively. Assuming that N_x, N_y are divisible by 4, in the parts $[0, \sigma_x]$, $[1 - \sigma_x, 1]$ and $[0, \sigma_y]$, $[1 - \sigma_y, 1]$ we use a uniform mesh with $N_x/4 + 1$ and $N_y/4 + 1$ mesh points, respectively, and in the parts $[\sigma_x, 1 - \sigma_x]$, $[\sigma_y, 1 - \sigma_y]$ with $N_x/2 + 1$ and $N_y/2 + 1$ mesh points, respectively. This defines the piecewise equidistant meshes in the x- and y-directions condensed in the boundary layers at $x = 0, 1$ and $y = 0, 1$:

$$x_i = \begin{cases} i h_{x\mu}, & i = 0, 1, \dots, N_x/4; \\ \sigma_x + (i - N_x/4) h_x, & i = N_x/4 + 1, \dots, 3N_x/4; \\ 1 - \sigma_x + (i - 3N_x/4) h_{x\mu}, & i = 3N_x/4 + 1, \dots, N_x, \end{cases}$$

$$y_j = \begin{cases} j h_{y\mu}, & j = 0, 1, \dots, N_y/4; \\ \sigma_y + (j - N_y/4) h_y, & j = N_y/4 + 1, \dots, 3N_y/4; \\ 1 - \sigma_y + (j - 3N_y/4) h_{y\mu}, & j = 3N_y/4 + 1, \dots, N_y, \end{cases}$$

$$h_x = 2(1 - 2\sigma_x) N_x^{-1}, \ h_{x\mu} = 4\sigma_x N_x^{-1}, \ h_y = 2(1 - 2\sigma_y) N_y^{-1}, \ h_{y\mu} = 4\sigma_y N_y^{-1},$$

where $h_{x\mu}, h_{y\mu}$ and h_x, h_y are the step sizes inside and outside the boundary layers, respectively. We choose the transition points σ_x, $(1 - \sigma_x)$ and σ_y, $(1 - \sigma_y)$ in Shishkin's sense (see [7] for details), i.e.

$$\sigma_x = \min\left\{4^{-1}, \upsilon_1 \mu \ln N_x\right\}, \quad \sigma_y = \min\left\{4^{-1}, \upsilon_2 \mu \ln N_y\right\},$$

where υ_1 and υ_2 are positive constants. If $\sigma_{x,y} = 1/4$, then $N_{x,y}^{-1}$ are very small relative to μ, and in this case the difference scheme (2) can be analyzed using standard techniques. We therefore assume that

$$\sigma_x = \upsilon_1 \mu \ln N_x, \quad \sigma_y = \upsilon_2 \mu \ln N_y.$$

In this case the meshes $\overline{\Omega}^{hx}$ and $\overline{\Omega}^{hy}$ are piecewise equidistant with the step sizes

$$\begin{aligned} &N_x^{-1} < h_x < 2N_x^{-1}, \quad h_{x\mu} = 4\upsilon_1 \mu N_x^{-1} \ln N_x, \\ &N_y^{-1} < h_y < 2N_y^{-1}, \quad h_{y\mu} = 4\upsilon_2 \mu N_y^{-1} \ln N_y. \end{aligned} \tag{35}$$

The difference scheme (2) on the piecewise uniform mesh (35) converges μ-uniformly to the solution of (1):

$$\max_{(P,t)\in \overline{\Omega}^h \times \overline{\Omega}^\tau} |U(P,t) - u(P,t)| \le K\left(\left(N^{-1} \ln N\right)^2 + \tau\right), \ N = \min\left\{N_x; N_y\right\}, \tag{36}$$

where constant K is independent of μ, N and τ. The proof of this result can be found in [7].

Consider algorithm (8)-(11), (14) on the piecewise uniform mesh (35) with the subdomains $\overline{\Omega}_m^h$, $m \in \mathcal{M}_1$ and the interfacial subdomains $\overline{\omega}_m^h$, $m = 1, \dots, M-1$ located in the x-direction outside the boundary layer, where the step size h_x from (35) is in use. In this case, l, r and κ in (28) are bounded by $\tau\mu^2/h_x^2$, and we estimate λ in (28) by

$$\lambda \le \frac{2\tau\mu^2}{h_x^2} + \frac{\tau\mu^2}{h_x^2} \max\left[1; \frac{2\tau\mu^2}{h_x^2}\right]. \tag{37}$$

Thus, if $\mu \le h_x$ and $2\tau \le 1$, then $\lambda \le 3\tau$, hence, the right hand side in (32) is estimated by

$$C\left(c^* + \nu\right)\left(\rho + \lambda\right)^{n_*-1} \le \tilde{C}\left(\rho + 3\tau\right)^{n_*-1}, \tag{38}$$

where constant $\tilde{C}$ is independent of τ.

Remark 16. *We mention that the implicit difference scheme (2) is of the first order with respect to* τ *and* $\rho = c^*/\left(c^* + \tau^{-1}\right) \le c^*\tau$. *Thus, to guarantee the consistency of the global errors in the difference scheme (2) and in the block monotone domain decomposition algorithm (8)-(11), (14) it is enough to choose* $n_* = 2$.

Remark 17. *Without loss of generality, we assume that the boundary condition* $g(P,t) = 0$. *This assumption can always be obtained via a change of variables. Let on each time level the initial function* $V^{(0)}(P,t)$ *be chosen in the form of (7), i.e.* $V^{(0)}(P,t)$ *is the solution of the following difference problem*

$$\begin{aligned} &\mathcal{L}V^{(0)}(P,t) = q\left|f(P,t,0) - \tau^{-1}V(P,t-\tau)\right|, \ P \in \Omega^h, \\ &V^{(0)}(P,t) = 0, \ P \in \partial\Omega^h, \quad q = 1, -1, \end{aligned} \tag{39}$$

where $R(P,t) = 0$. *Then the functions* $\overline{V}^{(0)}(P,t)$, $\underline{V}^{(0)}(P,t)$ *corresponding to* $q = 1$ *and* $q = -1$ *are upper and lower solutions, respectively. From here and (34), it follows that*

$$\begin{aligned} \left\|Z^{(1)}\left(t_l\right)\right\|_{\overline{\Omega}^h} &\le \tau\left[\left\|\mathcal{L}V^{(0)}\left(t_l\right)\right\|_{\overline{\Omega}^h} + \left\|f\left(P, t_l, 0\right) - \tau^{-1}V(t_l - \tau)\right\|_{\overline{\Omega}^h}\right] \\ &\le 2\tau\left\|f\left(P, t_l, 0\right) - \tau^{-1}V(t_l - \tau)\right\|_{\overline{\Omega}^h} \\ &\le 2\tau\left\|f\left(P, t_l, 0\right)\right\|_{\overline{\Omega}^h} + 2\left\|V(t_l - \tau)\right\|_{\overline{\Omega}^h} \le C_l. \end{aligned}$$

To prove that all constants C_l *are independent of the small parameter* μ, *we have to prove that* $\|V(t_l - \tau)\|_{\overline{\Omega}^h}$ *are* μ*-uniformly bounded. For* $l = 1$, $V(P,0) = u^0(P)$, *where u^0 is the initial condition in the differential problem (1), and, hence, C_1 is independent of μ and τ. For $l = 2$, we have*

$$\left\| Z^{(1)}(t_2) \right\|_{\overline{\Omega}^h} \leq 2\tau \left\| f(P, t_1, 0) \right\|_{\overline{\Omega}^h} + 2 \left\| V(t_1) \right\|_{\overline{\Omega}^h} \leq C_2,$$

where $V(P, t_1) = V^{(n_*)}(P, t_1)$. As follows from Theorem 3, the monotone sequences $\left\{ \overline{V}^{(n)}(P, t_1) \right\}$ and $\left\{ \underline{V}^{(n)}(P, t_1) \right\}$ are μ-uniformly bounded from above by $\overline{V}^{(0)}(P, t_1)$ and from below by $\underline{V}^{(0)}(P, t_1)$. Applying (16) to the problem (39) at $t = t_1$, we have

$$\left\| V^{(0)}(t_1) \right\|_{\overline{\Omega}^h} \leq \tau \left\| f(P, t_1, 0) - \tau^{-1} u^0(P) \right\|_{\overline{\Omega}^h} \leq K_1,$$

where constant K_1 is independent of μ and τ. Thus, we prove that C_2 is independent of μ and τ. Now by induction on l, we prove that all constants C_l in (34) are independent of μ, and, hence, constant $C = T \max_{1 \leq l \leq N_\tau} C_l$ in (32) is independent of μ and τ. Thus, if $\mu \leq h_x$ and $2\tau \leq 1$, then from (32), (38) and (36), we conclude that the monotone domain decomposition algorithm (8)-(11), (14) converges μ-uniformly to the solution of the differential problem (1).

Now we modify the piecewise equidistant mesh of Shishkin-type in the x-direction. Let the number of mesh points $N_{x\mu}$ and the step size $h_{x\mu}$ in the boundary layers be chosen in the form

$$N_{x\mu} = \alpha \ln(1/\mu), \quad h_{x\mu} = \upsilon \mu, \tag{40}$$

where α and υ are positive constants. In this case, the transition points σ_x and $(1 - \sigma_x)$ are defined by

$$\sigma_x = h_{x\mu} N_{x\mu} = (\alpha \upsilon) \mu \ln\left(\mu^{-1}\right).$$

We note that, in general, the difference scheme (2) on the modified piecewise equidistant mesh (35), (40) does not converge μ-uniformly to the solution of (1).

Consider algorithm (8)-(11), (14) on the modified mesh (35), (40) and assume $\mu \leq h_x$. Using (37) with the step size $h_{x\mu}$, we estimate λ in the boundary layers in the form

$$\lambda \leq 2\upsilon^{-2}\tau + \upsilon^{-2}\tau \max\left[1; 2\upsilon^{-2}\tau\right].$$

If $2\upsilon^{-2}\tau \leq 1$, then $\lambda \leq 3\upsilon^{-2}\tau$, and from here and (37), the right hand side in (32) is estimated by

$$C\left(c^* + \nu\right)\left(\rho + \lambda\right)^{n_*-1} \leq \tilde{C}\left(\rho + r\tau\right)^{n_*-1}, \quad r = 3\max\left[1; \upsilon^{-2}\right],$$

where constant $\tilde{C}$ is independent of τ.

We mention that Remark 16 holds true for the block monotone domain decomposition algorithm (8)-(11), (14) on the modified mesh (35), (40).

5. Numerical experiments

Consider problem (1) with $f(P, t, u) = (u-4)/(5-u)$, $g(P, t) = 1$ and $u^0(P) = 1$, which models the biological Michaelis-Menton process without inhibition [5]. This problem gives

$$\overline{V}(P, t_1) = \begin{cases} 4, & x \in \Omega^h; \\ 1, & x \in \partial\Omega^h, \end{cases} \qquad \underline{V}(P, t_1) = \begin{cases} 0, & x \in \Omega^h; \\ 1, & x \in \partial\Omega^h, \end{cases}$$

where $\overline{V}(P, t_1)$ and $\underline{V}(P, t_1)$ are the upper and lower solutions on the time level $t_1 = \tau$ corresponding to $V(P, 0) = 1$, $P \in \overline{\Omega}^h$. Suppose that we initiate our algorithms with $\underline{V}^{(0)}(P, t_1) = \underline{V}(P, t_1)$ or with $\overline{V}^{(0)}(P, t_1) = \overline{V}(P, t_1)$ and thus generate sequences of lower and upper solutions $\{\underline{V}^{(n)}(P,t_1)\}$ and $\{\overline{V}^{(n)}(P,t_1)\}$, respectively, with respect to $V(P,0)$. It follows from Theorem 1 in [3] that

$$0 \leq \underline{V}(P,t_1) \leq \underline{V}^{(n)}(P,t_1) \leq \overline{V}^{(n)}(P,t_1) \leq \overline{V}(P,t_1) \leq 4, \quad P \in \overline{\Omega}^h, \quad n \geq 0.$$

Now for $k \geq 2$, let $V_l(P,t_{k-1}) = \underline{V}^{(n_*)}(P,t_{k-1})$, $V_r(P,t_{k-1}) = \underline{V}^{(n_*)}(P,t_{k-1})$. Since the boundary condition g and the function f in the test problem are independent of time, the mesh functions $\underline{V}^{(0)}(P,t_k)$, $\overline{V}^{(0)}(P,t_k)$ defined by $\underline{V}^{(0)}(P,t_k) = V_l(P,t_{k-1})$, $\overline{V}^{(0)}(P,t_k) = \overline{V}_r(P,t_{k-1})$ are lower and upper solutions with respect to $V_l(P,t_{k-1})$ and $V_r(P,t_{k-1})$, respectively. Applying Theorem 1 from [3], one has by induction on k that

$$0 \leq \underline{V}^{(n)}(P,t_k) \leq \overline{V}^{(n)}(P,t_k) \leq 4,\ P \in \overline{\Omega}^h, \quad 0 \leq n \leq n_*, \quad 1 \leq k \leq N_\tau.$$

Since each of our computed mesh functions satisfies the above inequalities, we may suppose that f_u is bounded above by $c^* = 1$.

On each time level t_k, the stopping criterion is chosen in the form

$$\left\| V^{(n)}(t_k) - V^{(n-1)}(t_k) \right\|_{\overline{\Omega}^h} \leq \delta,$$

where $\delta = 10^{-5}$. All the discrete linear systems corresponding to set $\mathcal{M}_2$ are solved by $GMRES$-solver.

It is found that in all the numerical experiments the basic feature of monotone convergence of the upper and lower sequences is observed. In fact, the monotone property of the sequences holds at every mesh point in the domain. This is, of course, to be expected from the analytical consideration.

Consider the block monotone domain decomposition algorithm (8)-(11), (14) on the piecewise uniform mesh (35) with $N_x = N_y$. The domain decomposition of the computational domain $\overline{\Omega}^h$ consists of the three subdomains $\overline{\Omega}^h_m, m = 1,2,3$ and the two interfacial subdomains $\overline{\omega}^h_m, m = 1,2$, such that $\mathcal{M}_1 = \{2\}$, $\mathcal{M}_2 = \{1,3\}$. The interfacial subdomains $\overline{\omega}^h_m$, $m = 1,2$ contain only three mesh points in the x-direction and lie straightaway outside the boundary layers in the x-direction. In Table 1, for $\mu = 10^{-2}$, 10^{-3} and 10^{-4} and for various values of N_x, we give the average (over ten time levels) numbers of iterations n_{τ_1}, n_{τ_2}, ($\tau_1 = 5 \times 10^{-2}$, $\tau_2 = 10^{-2}$) required to satisfy the stopping criterion. The μ-dependence of the step sizes h_x and $h_{x\mu}$ of the piecewise uniform mesh (35) is tabulated in Table 2. Since for our data set we allow $\sigma_x > 0.25$, the step size $h_{x\mu}$ is calculated as

$$h_{x\mu} = \frac{4 \min\{0.25, \sigma_x\}}{N_x}. \tag{41}$$

From the data presented in Tables 1 and 2, it follows that if the condition $\mu \lessapprox h_x$ holds true then the numbers of iterations are equal to the numbers of iterations for the undecomposed monotone iterative algorithm (6), where $GMRES$-solver is in use on the whole domain $\overline{\Omega}^h$, i.e. $\mathcal{M}_1 = \emptyset$, $\mathcal{M}_2 = \{1\}$. If we violate this condition as in the case with $\mu = 10^{-2}$, $N_x = 512$, 1024 and $\tau_1 = 5 \times 10^{-2}$, then the number of iterations n_{τ_1} exceeds the number of iteration for the undecomposed monotone algorithm. Thus, the numerical experiments confirm our theoretical estimates that the monotone domain decomposition algorithm (8)-(11), (14) can be most efficiently used if the condition $\mu \lessapprox h_x$ holds true. For $\mu \leq 10^{-3}$, n_τ is independent of μ, which confirms the uniform convergence results shown in Remark 17.

In Table 3, we present the numerical results from [3] for the monotone domain decomposition algorithm (8)-(11) on the piecewise uniform mesh (35) corresponding

Table 1. Average numbers of iterations for $\tau_1 = 5 \times 10^{-2}$, $\tau_2 = 10^{-2}$ for the block monotone domain decomposition algorithm (8)-(11), (14) on the piecewise uniform mesh (35).

N_x	n_{τ_1}; n_{τ_2}	
64	3.8; 2.8	3.8; 2.8
128; 256	4.0; 3.0	4.0; 3.0
512	4.7; 3.0	4.0; 3.0
1024	6.6; 3.0	4.0; 3.0
μ	10^{-2}	10^{-3}; 10^{-4}

Table 2. The μ-dependence of h_x, $h_{x\mu}$.

N_x	h_x; $h_{x\mu}$		
64	1.82E-02; 1.30E-02	2.99E-02; 1.30E-03	3.11E-02; 1.30E-04
128	8.03E-03; 7.59E-03	1.49E-02; 7.59E-04	1.55E-02; 7.59E-05
256	3.91E-03; 3.91E-03	7.38E-03; 4.34E-04	7.77E-03; 4.34E-05
512	1.95E-03; 1.95E-03	3.66E-03; 2.44E-04	3.88E-03; 2.44E-05
1024	9.77E-04; 9.77E-04	1.82E-03; 1.35E-04	1.94E-03; 1.35E-05
μ	10^{-2}	10^{-3}	10^{-4}

to the numerical experiments reported in Table 1. As follows from Tables 1 and 3, the block monotone domain decomposition algorithm (8)-(11), (14) requires at most the same number of iterations as in the monotone domain decomposition (14) to reach the given accuracy. Since in algorithm (8)-(11), (14) the Thomas algorithm is in use on the subdomain located outside the boundary layers, it is clear that algorithm (8)-(11), (14) executes more quickly than algorithm (8)-(11).

Table 3. Average numbers of iterations for $\tau_1 = 5 \times 10^{-2}$, $\tau_2 = 10^{-2}$ for the monotone domain decomposition algorithm (8)-(11) on the piecewise uniform mesh (35).

N_x	n_{τ_1}; n_{τ_2}	
64	3.8; 2.8	3.8; 2.8
128; 256	4.0; 3.0	4.0; 3.0
512	6.6; 3.0	4.0; 3.0
μ	10^{-2}	10^{-3}; 10^{-4}

Now consider the case $\mathcal{M}_1 = \{1\}$, $\mathcal{M}_2 = \emptyset$, i.e. the block monotone iterative method (14) solves the difference scheme (2) on the whole computational domain $\overline{\Omega}^h$. In Table 4, for $\tau_1 = 5 \times 10^{-2}$, $\tau_2 = 10^{-2}$ and for various values of N_x and μ, we give the average (over ten time levels) numbers of iterations n_{τ_1}, n_{τ_2} required to satisfy the stopping criterion. In the case of the numerical experiments presented in Table 4, we violate the condition $\mu \lessapprox h_x$. For $\mu \leq 10^{-3}$, n_τ is independent of μ, which confirms the uniform convergence results shown in Remark 17. In the contrast to the block monotone domain decomposition algorithm (compare with Table 1), here n_τ is a monotone increasing function in N_x. For $\mu \geq 10^{-1}$, the block monotone iterative method (14) converges very slowly.

On the modified mesh (35), (40), consider the block monotone domain decomposition algorithm (8)-(11), (14) and the block monotone iterative method (14).

TABLE 4. Average numbers of iterations for $\tau_1 = 5 \times 10^{-2}$, $\tau_2 = 10^{-2}$ for the block monotone iterative method (14) on the piecewise uniform mesh (35).

N_x	$n_{\tau_1};\ n_{\tau_2}$			
64	33.2; 11.2	4.0; 3.0	4.0; 3.0	4.0; 3.0
128	101.0; 28.8	5.6; 4.0	5.6; 4.0	5.6; 4.0
256	311.4; 86.4	10.1; 5.0	9.0; 5.0	9.0; 5.0
512	942.6; 271.3	23.8; 9.1	17.5; 7.2	17.5; 7.2
1024	2592.7; 823.4	70.1; 26.8	41.3; 13.4	41.3; 13.4
μ	10^{-1}	10^{-2}	10^{-3}	10^{-4}

TABLE 5. Average numbers of iterations on the modified mesh (35), (40).

ALGORITHM	$n_{\tau_1};\ n_{\tau_2}$		
(8)-(11), (14)	4.0; 3.0	3.9; 2.9	3.9; 2.9
(14)	9.0; 5.0	5.0; 3.0	5.0; 3.0
$N_{x\mu}$	32	64	96
μ	e^{-4}	e^{-8}	e^{-12}

For algorithm (8)-(11), (14), we decompose each of the boundary layers in the x-direction $[0, \sigma_x]$ and $[1-\sigma_x, 1]$ into M_μ subdomains and solve the problems (8) on these subdomains by $GMRES$-solver, the problem on the subdomain $[\sigma_x, 1-\sigma_x]$ is solved by the block iterative method (14). Thus, the total number of subdomains is $M = 2M_\mu + 1$. The interfacial subdomains $\overline{\omega}^h_m,\ m = 1, \ldots, M-1$ contain only three mesh points in the x-direction, and the problems (9) on the interfacial subdomains are solved by the Thomas algorithm. In Table 5, for values of $\mu = \exp(-k),\ k = 4, 8, 12$ and $N_{x\mu} = 8\ln(1/\mu),\ N_x = 4N_{x\mu}$, we give the average (over ten time levels) numbers of iterations $n_{\tau_1},\ n_{\tau_2}$, ($\tau_1 = 5\times10^{-2}$, $\tau_2 = 10^{-2}$) required to satisfy the stopping criterion. Since for our data set we allow $\sigma_x > 0.25$, the step size $h_{x\mu}$ is calculated by (41). We mention that n_{τ_1} and n_{τ_2} for algorithm (8)-(11), (14) are independent of the number of subdomains M and equal to the number of iterations for the undecomposed monotone iterative algorithm (6), where $GMRES$-solver is in use on the whole domain $\overline{\Omega}^h$. For $\mu \le \exp(-5)$, n_τ in the block monotone domain decomposition algorithm (8)-(11), (14) and in the block monotone iterative method (14) is independent of μ. Thus, the main features of algorithm (8)-(11), (14) on the piecewise uniform mesh (35) highlighted in Table 1 hold true for the algorithm on the modified mesh (35), (40). An advantage of the modified mesh (35), (40) is that the boundary layers may be decomposed into a set of subdomains without deteriorating the convergence rate of iterations, and the problems (8) on these subdomains can be solved in parallel. In the contrast to the block monotone iterative method (14) on the piecewise uniform mesh (35) (compare Tables 4 and 5), the algorithm (14) on the modified mesh (35), (40) converges μ- and N-uniformly in (32).

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