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NUMERICAL METHODS FOR THE EXTENDED FISHER-KOLMOGOROV (EFK) EQUATION

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Abstract. In the study of pattern formation in bi–stable systems, the extended Fisher–Kolmogorov (EFK) equation plays an important role. In this paper, some *a priori* bounds are proved using Lyapunov functional. Further, existence, uniqueness and regularity results for the weak solutions are derived. Using C^1-conforming finite element method, optimal error estimates are established for the semidiscrete case. Finally, fully discrete schemes like backward Euler, two step backward difference and Crank-Nicolson methods are proposed, related optimal error estimates are derived and some computational experiments are discussed.

Key Words. Extended Fisher-Kolmogorov (EFK) equation, Lyapunov functional, weak solution, existence, uniqueness and regularity results, finite element method, semidiscrete method, backward Euler, two step backward difference and Crank-Nicolson schemes, optimal estimates.

1. Introduction

In this paper, the C^1-conforming finite element method is analyzed for the following extended Fisher-Kolmogorov (EFK) equation :

$$u_t + \gamma\Delta^2 u - \Delta u + f(u) = 0, \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

either of the boundary conditions

$$u = 0, \frac{\partial u}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{1.3}$$

or

$$u = 0, \ \Delta u = 0, \ (x,t) \in \partial \times (0,T], \tag{1.4}$$

where $f(u) = u^3 - u$, $T > 0$ and Ω is a bounded domain in $\Re^d$, $d \leq 2$ with boundary $\partial\Omega$.

When $\gamma = 0$ in (1.1), we obtain the standard Fisher-Kolmogorov equation. However, by adding a stabilizing fourth order derivative term to the Fisher-Kolmogorov equation, Coullet *et al.* [4], Dee and van Saarloos [7, 19, 20] proposed (1.1) and called the model described in (1.1) as the extended Fisher-Kolmogorov equation.

The equation (1.1) occurs in a variety of applications such as pattern formation in bi-stable systems [7], propagation of domain walls in liquid crystals [22], travelling waves in reaction diffusion systems [2] and mezoscopic model of a phase transition in a binary system near the Lipschitz point [8]. In particular, in the

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phase transitions near critical points (Lipschitz points), the higher order gradient terms in the free energy functional can no longer be neglected and the fourth order derivative becomes important.

Recently, attention has been focused on the steady state equation of (1.1). The aim of considering the steady state equation of (1.1) is to study the heteroclinic solutions (so called kinks) connecting to the equilibria $u = -1$ and $u = 1$. Typically, the stationary problem displays a multitude of periodic, homoclinic and heteroclinic solutions [13, 15] depending on the parameter γ. The steady state equation of (1.1) has been analysed by Peletier and Troy [13, 14] using shooting methods and by Kalies, Kwapisz and Vander Vorst [9] with the help of variational methods.

As far as computational studies are concerned, there is hardly any literature for the numerical approximations to (1.1)–(1.3) or (1.1)–(1.2) and (1.4). Therefore, an attempt has been made here to discuss finite element Galerkin method for the EFK equation. In this article, we mainly concentrate on the equation (1.1) with the initial condition (1.2) and boundary conditions (1.3). Related to fourth order evolution equations, the C^1-conforming finite element method is analyzed by Pani and Chung [11] for the Rosenau equation, for "Good" Boussinesq equation by Pani and Haritha [12], for one dimensional Cahn-Hilliard equation by Elliott *et., al* [5, 6], for multidimensional Cahn-Hilliard equation by Qiang and Nicolaides [16] and for Kuramoto-Sivashinsky equation by Akrivis [1].

The outline of the paper is as follows. Section 2 deals with existence, uniqueness and regularity results. In section 3, we derive *a priori* error estimates for the semidiscrete Galerkin method using C^1-conforming finite elements. In section 4, we discretize the semidiscrete equation in the temporal direction and obtain optimal error estimates for the backward Euler, two step backward difference and Crank-Nicolson schemes. Finally in section 5, we discuss some computational experiments.

2. Existence, Uniqueness and Regularity results

In this section, we derive existence uniqueness and regularity results for the extended Fisher-Kolmogorov (EFK) equation. In literature, we observe that there is hardly any study on the existence, uniqueness and regularity results of weak solutions to the problem (1.1)–(1.3) or (1.1)–(1.2) and (1.4). Therefore, an attempt has been made in this section to derive existence, uniqueness and regularity results for the EFK equation (1.1)–(1.3).

Taking L^2-innerproduct of (1.1) with $\chi \in H_0^2$ and applying Green's formula, we obtain the following weak formulation. Find $u(\cdot,t) \in H_0^2$ for $t \in (0,T]$ such that

$$\begin{aligned} (u_t,\chi) + \gamma(\Delta u, \Delta\chi) + (\nabla u, \nabla\chi) \quad &+ \quad (f(u),\chi) = 0, \quad \chi \in H_0^2(\Omega), \\ u(0) \quad &= \quad u_0. \end{aligned} \tag{2.1}$$

For the proof of existence and uniqueness results, the following *a priori* bound will be useful.

Theorem 2.1. *Assume that $u_0 \in H_0^2$. Then there exists a positive constant C such that*

$$\|u(t)\|_2 \le C(\gamma, \|u_0\|_2), \quad t > 0.$$

Further,

$$\|u(t)\|_\infty \le C(\gamma, \|u_0\|_2), \quad t > 0.$$

Proof. We consider the Lyapunov functional $\mathcal{E}(\chi)$ as

$$\mathcal{E}(\chi) = \int_\Omega \{\frac{\gamma}{2}|\Delta\chi|^2 + \frac{1}{2}|\nabla\chi|^2 + F(\chi)\}\,dx, \tag{2.2}$$

where

$$F(\chi) = \frac{1}{4}(1-\chi^2)^2.$$

Note that $F' = f$. Differentiating (2.2) with respect to t, we obtain

$$\frac{d\mathcal{E}(u)}{dt} = \gamma(\Delta u, \Delta u_t) + (\nabla u, \nabla u_t) + (F'(u), u_t). \tag{2.3}$$

Choose $\chi = u_t$ in (2.1) and write the resulting equation as

$$\gamma(\Delta u, \Delta u_t) + (\nabla u, \nabla u_t) + (f(u), u_t) = -\|u_t\|^2. \tag{2.4}$$

Using (2.4) in (2.3), we find that

$$\frac{d\mathcal{E}(u)}{dt} = -\|u_t\|^2 \leq 0,$$

and hence,

$$\mathcal{E}(u) \leq \mathcal{E}(u_0).$$

Using the definition of $\mathcal{E}(\cdot)$, it follows that

$$\int_\Omega \{\frac{\gamma}{2}|\Delta u|^2 + \frac{1}{2}|\nabla u|^2 + F(u)\}\, dx \leq C\left(\|u_0\|_2\right).$$

Since $F(u) \geq 0$, using Poincaré inequality, we obtain

$$\|u(t)\|_2 \leq C\left(\gamma,\, \|u_0\|_2\right).$$

An application of Sobolev imbedding theorem yields

$$\|u(t)\|_{L^\infty} \leq C\|u(t)\|_{H^2} \leq C\left(\gamma,\, \|u_0\|_2\right),$$

and this completes the rest of the proof. □

Remark 2.1. *Note that*

$$\frac{d\mathcal{E}(u)}{dt} + \|u_t\|^2 = 0,$$

and hence, for $t > 0$

$$\int_0^t \|u_t(\tau)\|^2\, d\tau \leq C\left(\gamma, \|u_0\|_2\right). \tag{2.5}$$

Below, we discuss the global existence, uniqueness and regularity results using Faedo-Galerkin method.

Theorem 2.2. *Let* $u_0 \in H_0^2(\Omega)$. *For any* $T > 0$, *there exists a unique* $u = u(x,t)$ *in* $\Omega \times [0,T)$ *with*

$$u \in L^\infty(0,T; H_0^2(\Omega))$$

and

$$u_t \in L^\infty(0,T; L^2(\Omega)),$$

such that u *satisfies the initial condition* $u(0) = u_0$ *and the equation (2.1) in the sense that*

$$(u_t, \chi) + \gamma(\Delta u, \Delta\chi) + (\nabla u, \nabla\chi) + (f(u), \chi) = 0, \;\; \chi \in H_0^2(\Omega),\; t \in (0,T].$$

Proof. Let $\{w_j\}$ be a basis of H_0^2, and let $V^m =$ span $\{w_1, w_2, \cdots, w_m\}$. Define for each $t > 0$

$$u^m(t) = \sum_{i=1}^m g_{im}(t) w_i \;\in V^m$$

as a solution of

$$
\begin{aligned}
(u_t^m, \chi) + \gamma(\Delta u^m, \Delta\chi) + (\nabla u^m, \nabla\chi) \quad &+ \quad (f(u^m), \chi) = 0, \ \chi \in V^m, \\
u^m(0) \quad &= \quad u_{0,m},
\end{aligned} \tag{2.6}
$$

where $u_{0,m} = u^m(0) = \sum_{i=1}^m g_{im}(0)w_i$ is the orthogonal projection of u_0 onto V^m and $u_{0,m} \to u_0$ in $H_0^2(\Omega)$. Note that $\|u_{0,m}\|_2 \leq C\|u_0\|_2$.

Clearly (2.6) represents a system of nonlinear ordinary differential equations. Therefore, Picard's theorem ensures that there exists a unique solution locally, i.e., there exists a unique solution u^m in $(0, t_m)$ for some $t_m > 0$. For proving global existence, we use continuation arguement and hence, we need the following *a priori* bounds.

As in the proof of Theorem 2.1, we can easily obtain the following bounds using Lyapunov functional $\mathcal{E}(u^m)$:

$$
\|u^m(t)\|_2, \ \|\nabla u^m(t)\|, \ \|\Delta u^m(t)\| \leq C.
$$

Since $\|u^m(t)\|_{L^p} \leq C\|u^m(t)\|_{H^2}, \ 1 \leq p \leq \infty$, we note that

$$
\|f(u^m)\|^2 = \int_\Omega \left((u^m)^3 - u^m\right)^2 \, dx \leq 2\int_\Omega (u^m)^6 \, dx + 2\int_\Omega (u^m)^2 \, dx \leq C,
$$

and hence, $f(u^m)$ is bounded in $L^\infty(0, T; L^2(\Omega))$.

Now, let j be fixed and $m > j$,

$$
\begin{aligned}
(\Delta u^m(t), \Delta w_j) &\to (\Delta u, \Delta w_j) \ \text{ in } L^\infty(0,T) \text{ weak*} \\
(\nabla u^m, \nabla w_j) &\to (\nabla u, \nabla w_j) \ \text{ in } L^\infty(0,T) \text{ weak*} \\
(f(u^m), w_j) &\to (f(u), w_j) \ \text{ in } L^\infty(0,T) \text{ weak*} \\
(u^m, w_j) &\to (u, w_j) \ \text{ in } L^\infty(0,T) \text{ weak*}.
\end{aligned}
$$

Also, we find that

$$
(u_t^m, w_j) \to (u_t, w_j) \ \text{ in } L^\infty(0,T) \text{ weak*}.
$$

Finally, we obtain

$$
(u_t, w_j) + \gamma(\Delta u, \Delta w_j) + (\nabla u, \nabla w_j) + (f(u), w_j) = 0.
$$

The existence of the equation (2.1) for $t > 0$ follows from the denseness of the basis $\{w_j\}$ in $H_0^2(\Omega)$.

Uniqueness. Suppose u and v are two solutions of (2.1). Taking $w = u - v$, we obtain

$$
(w_t, \chi) + \gamma(\Delta w, \Delta\chi) + (\nabla w, \nabla\chi) = -(f(u) - f(v), \chi), \ \chi \in H_0^2.
$$

Setting $\chi = w$ and using the boundedness of $\|u\|_\infty$ and $\|v\|_\infty$, we obtain

$$
\|w(t)\|^2 \leq C\int_0^t \|w(\tau)\|^2 d\tau. \tag{2.7}
$$

Setting

$$
\int_0^t \|w(\tau)\|^2 \, d\tau = \Phi(t),
$$

we rewrite (2.7) as $\Phi'(t) - C\Phi(t) \leq 0$, and hence,

$$
(e^{-Ct}\Phi)' \leq 0.
$$

Finally, integrating with respect to t, we obtain $\Phi(t) \leq 0$.

Since $\Phi(t) \geq 0$, we, therefore, obtain $\Phi(t) = 0$. This implies, $w(t) = 0$ and hence, the uniqueness follows for $t > 0$. This completes the rest of the proof. □

Finally, we discuss the regularity results needed for the proof of *a priori* error bounds in the subsequent sections.

Theorem 2.3. *(Regularity) Suppose $u_0 \in H_0^2 \cap H^6$, then there exists a unique function u with*

$$u \in L^\infty(0,T; H^4 \cap H_0^2), \quad u_t \in L^2(0,T; H^4 \cap H_0^2) \quad \text{and} \quad u_{tt} \in L^2(0,T; L^2)$$

such that u satisfies (2.1).

Proof. Let $\{w_i\}$ be a basis of $H_0^2 \cap H^4$. Then we define

$$u^m(t) = \sum_{i=1}^{m} g_{im}(t) w_i \in V^m,$$

where $V^m = \text{span}\{w_1, w_2, \cdots, w_m\}$. Differentiating (2.6) with respect to t and taking L^2-innerproduct with $\chi \in V^m$ and applying Green's formula, we obtain the following equation

(2.8) $(u_{tt}^m, \chi) + \gamma(\Delta u_t^m, \Delta\chi) + (\nabla u_t^m, \nabla\chi) + (f'(u^m)u_t^m, \chi) = 0, \; \chi \in V^m.$

Setting $\chi = u_t^m$ in (2.8) and using the boundedness of $\|u^m\|_{L^\infty}$, we find that

$$\frac{1}{2}\frac{d}{dt}\|u_t^m\|^2 + \gamma\|\Delta u_t^m\|^2 + \|\nabla u_t^m\|^2 \leq C\|u_t^m\|^2.$$

We now integrate on both sides with respect to t to obtain

$$\|u_t^m(t)\|^2 + 2\int_0^t \left(\gamma\|\Delta u_t^m\|^2 + \|\nabla u_t^m\|^2\right) d\tau \leq \|u_t^m(0)\|^2 + C\int_0^t \|u_t^m(\tau)\|^2 \, d\tau.$$

Note that

$$\int_0^t \|u_t^m\|^2 \, d\tau \leq C\left(\|u_0\|_2^2\right),$$

and hence,

$$\|u_t^m(t)\|^2 + 2\int_0^t \{\gamma\|\Delta u_t^m\|^2 + \|\nabla u_t^m\|^2\} \, d\tau \leq C\left(\|u_0\|_2^2\right) + \|u_t^m(0)\|^2.$$

We now evaluate $\|u_t^m(0)\|$. A use of (1.1) yields

$$\begin{aligned} \|u_t^m(0)\| &\leq \gamma\|\Delta^2 u_0^m\| + \|\Delta u_0^m\| + \|f(u_0^m)\| \\ &\leq C\left(\|u_0^m\|_4\right) \leq C\left(\|u_0\|_4\right). \end{aligned}$$

Thus, we obtain

$$\|u_t^m\|_{L^\infty(L^2)} \leq C(\|u_0\|_4) \quad \text{and} \quad u_t^m \in L^2(H_0^2).$$

Using elliptic regularity, we find that

$$\begin{aligned} \|u^m(t)\|_4 &\leq C\|\Delta^2 u^m - \Delta u^m\| \\ &\leq \|u_t^m\| + \|f(u^m)\| \\ &\leq C\left(\|u_0\|_4\right). \end{aligned}$$

This implies that

$$u^m \in L^\infty(0,T; H^4 \cap H_0^2).$$

Setting $\chi = u_{tt}^m$ in (2.8), we obtain

$$\begin{aligned} \|u_{tt}^m\|^2 + \frac{1}{2}\frac{d}{dt}\left(\gamma\|\Delta u_t^m\|^2 + \|\nabla u_t^m\|^2\right) &= -(f'(u^m)u_t^m, u_{tt}^m) \\ &\leq C\|u_t^m\|^2 + \frac{1}{2}\|u_{tt}^m\|^2, \end{aligned}$$

and hence,

$$\|u_{tt}^m\|^2 + \frac{d}{dt}\left(\gamma\|\Delta u_t^m\|^2 + \|\nabla u_t^m\|^2\right) \leq C\|u_t^m\|^2.$$

Integrating both sides with respect to t, it now follows that

$$\begin{aligned} \int_0^t \|u_{tt}^m\|^2\,ds + \gamma\|\Delta u_t^m\|^2 + \|\nabla u_t^m\|^2 &\leq \gamma\|\Delta u_t^m(0)\|^2 + \|\nabla u_t^m(0)\|^2 \\ &+ C\int_0^t \|u_t^m\|^2\,ds. \end{aligned}$$

A use of (2.5) yields

$$\begin{aligned} \int_0^t \|u_{tt}^m\|^2\,ds + \gamma\|\Delta u_t^m\|^2 + \|\nabla u_t^m\|^2 &\leq C\left(\|u_0^m\|_2\right) + \gamma\|\Delta u_0^m\|_4^2 + \|\nabla u_0^m\|^2 \\ &\leq C\left(\|u_0\|_6\right). \end{aligned}$$

Thus, we derive the following bounds

$$\int_0^t \|u_{tt}^m\|^2\,d\tau \leq C \quad \text{and} \quad \gamma\|\Delta u_t^m\|^2 + \|\nabla u_t^m\|^2 \leq C,$$

and hence,

$$u_{tt}^m \in L^2(0,T;L^2).$$

Again, using elliptic regularity, we obtain

$$\begin{aligned} C\int_0^t \|u_t^m\|_4^2 &\leq \int_0^t \|\Delta^2 u_t^m - \Delta u_t^m\|^2\,d\tau \leq \int_0^t \|u_{tt}^m\|^2\,d\tau + \int_0^t \|f'(u^m)u_t^m\|^2\,ds \\ &\leq C\left(\|u_0\|_6\right), \end{aligned}$$

and hence,

$$u_t^m \in L^2(0,T;H_0^4 \cap H_0^2).$$

Finally, using the compactness arguments as in the proof of the previous Theorem 2.2, we prove the existence of u with

$$u \in L^\infty(0,T;H^4 \cap H_0^2), \;\; u_t \in L^2(0,T;H^4 \cap H_0^2) \;\text{ and }\; u_{tt} \in L^2(0,T;L^2).$$

This completes the proof of the theorem. □

Remark 2.2. *For deriving higher regularity, again we differentiate (2.8) with respect to t and choose* $\chi = u_{ttt}^m$. *As in the proof of Theorem 2.3 we use higher regularity condition on the initial data to obtain the following bounds:*

$$u_{ttt} \in L^2(L^2), \quad u_{tt} \in L^\infty(H_0^2).$$

3. Semidiscrete Galerkin Approximations

In this section, we apply Galerkin procedure in the spatial direction for the EFK equation and obtain the semidiscrete scheme. Further, we derive *a priori* error estimates for the semidiscrete method.

Let S_h^0, $0 < h < 1$ be a family of finite dimensional subspace of H_0^2 with the following approximation property: For $v \in H^4(\Omega) \cap H_0^2(\Omega)$, there exists a constant C independent of h such that

$$\inf_{\chi \in S_h^0} \|v - \chi\|_j \leq Ch^{4-j} \|v\|_4, \ j = 0, 1, 2. \tag{3.1}$$

As an example of the finite element space, let $\mathcal{T}_h$ be a regular triangulation of $\overline{\Omega}$ which consists of nonoverlapping simplexes. Define

$$S_h^0 = \{v_h \in C^1(\bar{\Omega}) : v_h|_K \in P_3(K), \ v_h = 0, \frac{\partial v_h}{\partial \nu} = 0 \text{ on } \partial\Omega, K \in \mathcal{T}_h\}.$$

This finite element space satisfies the property (3.1). For details, see Ciarlet [3]. The semidiscrete Galerkin approximation of (1.1)–(1.3) is defined to be a function $u_h : [0, T] \to S_h^0$ such that

$$\begin{aligned} (u_{ht}, \chi) + \gamma(\Delta u_h, \Delta \chi) + (\nabla u_h, \nabla \chi) \quad &+ \quad (f(u_h), \chi) = 0, \ \chi \in S_h^0, \\ u_h(0) \quad &= \quad u_{0,h}, \end{aligned} \tag{3.2}$$

where $u_{0,h} \in S_h^0$ is an appropriate approximation to u_0 to be defined later.
Since S_h^0 is a finite dimensional space, the equation (3.2) yields a system of nonlinear ordinary differential equations. Picard's theorem ensures that there exists a unique local solution in $(0, t^*)$ for some $t^* > 0$. For proving the global existence, we need an *a priori* bound like $\|u_h(t)\|_{L^\infty(H^2)} \leq C$. Then using continuation argument, it is easy to show the existence of a unique solution u_h to (3.2) for all $t > 0$.
As in the case of continuous problem, we again use Lyapunov functional $\mathcal{E}(u_h)$ to derive the following *a priori* bound:

$$\|u_h(t)\|_{L^\infty} \leq C\|u_h(t)\|_{H^2} \leq C\left(\|u_{0,h}\|_{H^2}\right).$$

We now introduce the bilinear form

$$A(v, w) = \gamma(\Delta v, \Delta w) + (\nabla v, \nabla w), \ v, w \in H_0^2,$$

for our subsequent use note that $A(\cdot, \cdot)$ satisfies the following properties:
(i) *Boundedness:* There is a positive constant M such that

$$|A(v, w)| \leq M\|v\|_2 \, \|w\|_2, \ v, w \in H_0^2.$$

(ii) *Coercivity:* There is a constant $\alpha_0 > 0$ such that

$$A(v, v) \geq \alpha_0 \|v\|_2^2, \ v \in H_0^2.$$

3.1. Error estimates. Very often a direct comparision between u and u_h does not yield optimal rate of convergence. Therefore, there is a need to introduce an appropriate auxiliary or intermediate function $\tilde{u}$ so that the optimal estimate of $u - \tilde{u}$ is easy to obtain and the comparision between u_h and $\tilde{u}$ yields a sharper estimate which leads to optimal rate of convergence for $u - u_h$. In literature, wheeler [21] for the first time introduced this technique in the context of parabolic problem.
Following wheeler, we introduce $\tilde{u}$ be as an auxiliary projection of u defined by

$$A(u - \tilde{u}, \chi) = 0, \ \chi \in S_h^0. \tag{3.3}$$

We now split the error $e = u - u_h$ as

$$\begin{aligned} e &:= u - u_h = (u - \tilde{u}) - (u_h - \tilde{u}) \\ &:= \eta - \theta, \end{aligned}$$

where $\eta = u - \tilde{u}$ and $\theta = u_h - \tilde{u}$. Below, we derive the error estimates of η and its temporal derivatives.

Lemma 3.1. *For $t \in [0, T]$, there exists a constant C such that for any nonnegative integer l*

$$\|\frac{\partial^l \eta}{\partial t^l}\|_j \leq Ch^{4-j} \sum_{k=0}^{l} \|\frac{\partial^k u}{\partial t^k}\|_4, \; 0 \leq j \leq 2. \tag{3.4}$$

Proof. Using coercivity property and (3.3), we note that

$$\begin{aligned} \alpha_0 \|u - \tilde{u}\|_2^2 &\leq A(u - \tilde{u}, u - \tilde{u}) \\ &= A(u - \tilde{u}, u - \chi), \; \chi \in S_h^0. \end{aligned}$$

Since the bilinear form $A(\cdot, \cdot)$ is bounded, we find that

$$\|u - \tilde{u}\|_2 \leq C \inf_{\chi \in S_h^0} \|u - \chi\|_2.$$

Using approximation property, we obtain the required result for $j = 2$.
For $j = 0$, we use Aubin-Nitsche duality argument. Let Φ be a solution of

$$\gamma \Delta^2 \Phi - \Delta \Phi = \eta, \; x \in \Omega, \tag{3.5}$$

$$\Phi = 0, \frac{\partial \Phi}{\partial \nu} = 0, \; x \in \partial \Omega.$$

The solution Φ satisfies the regularity condition

$$\|\Phi\|_4 \leq C(\gamma^{-1}) \|\eta\|.$$

Taking L^2-innerproduct of the equation (3.5) with η, using Green's formula and (3.3), we find that

$$\begin{aligned} \|\eta\|^2 &= A(\eta, \Phi - \chi) \\ &\leq C \|\eta\|_2 \|\Phi - \chi\|_2, \;\; \chi \in S_h^0, \end{aligned}$$

and hence,

$$\|\eta\|^2 \leq C \|\eta\|_2 \inf_{\chi \in S_h^0} \|\Phi - \chi\|_2.$$

Using approximation property and the regularity condition we obtain the required result for $j = 0$. Finally, for $j = 1$, we use the interpolation inequality to complete the proof for $l = 0$. For $l \geq 1$, we differentiate (3.3) l times to obtain

$$A\left(\frac{\partial^l \eta}{\partial t^l}, \chi\right) = 0.$$

Now repeat the above arguments to complete the rest of the proof. □
Assuming quasi-uniformity condition on the triangulation, it is easy to check that

$$\|\eta(t)\|_{W^{j,\infty}} \leq Ch^{4-j} \|u\|_{4,\infty}, \;\; j = 0, 1. \tag{3.6}$$

For a proof see [17]. Below, we choose the initial approximation $u_{0,h}$ as H_0^2 projection of u_0 that is $u_{0,h} = \tilde{u}(0)$. Then $\|u_{0,h}\|_2 \leq C \|u_0\|_2$ and $\theta(0) = 0$.

Theorem 3.1. *Let u_h be a solution of (3.2) and let $u_{0,h}$ be the H_0^2 projection of u_0 onto S_h^0. Then, there exists a positive constant C independent of h such that*

$$\|u-u_h\|_{L^\infty(0,T;H^j(\Omega))} \leq C(T,\gamma^{-1})h^{4-j}\left(\|u\|_{L^\infty(H^4)}+\|u_t\|_{L^2(H^4)}\right), \quad 0\leq j\leq 2.$$

Moreover, assuming quasi-uniformity condition on the triangulation $\mathcal{T}_h$, there exists a positive constant C independent of h such that

$$\|u-u_h\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C(T,\gamma^{-1})h^{4}\left(\|u\|_{L^\infty(W^{4,\infty})}+\|u_t\|_{L^2(H^4)}\right).$$

Proof. Note that $u-u_h=\eta-\theta$. From (3.4), the estimates of η are known and for completing the proof, it is enough to derive the estimates for θ. Then, a use of triangle inequality completes the proof. Substracting (2.1) from (3.2) and using auxiliary projection, we obtain the following equation in θ

$$(\theta_t,\chi)+\gamma(\Delta\theta,\Delta\chi)+(\nabla\theta,\nabla\chi)=(\eta_t,\chi)+(f(u)-f(u_h),\chi). \tag{3.7}$$

Choose $\chi=\theta$ in (3.7) and using Cauchy Schwarz inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|^2+\gamma\|\Delta\theta\|^2+\|\nabla\theta\|^2 \leq (\|\eta_t\|+\|f(u)-f(u_h)\|)\,\|\theta(t)\|.$$

For the nonlinear term $\|f(u)-f(u_h)\|$, we use the boundedness of $\|u\|_{L^\infty}$ and $\|u_h\|_{L^\infty}$ to find that

$$\begin{aligned}\|f(u)-f(u_h)\|^2 &= \int_\Omega (u-u_h)^2(u^2+uu_h+u_h^2-1)^2\,dx\\ &\leq C\|u-u_h\|^2 = C\left(\|\eta\|^2+\|\theta\|^2\right),\end{aligned}$$

and hence,

$$\|f(u)-f(u_h)\| \leq C\left(\|\eta\|+\|\theta\|\right). \tag{3.8}$$

This implies that

$$\frac{d}{dt}\|\theta(t)\|^2+2(\gamma\|\Delta\theta\|^2+\|\nabla\theta\|^2) \leq C\left(\|\eta_t\|^2+\|\eta\|^2+\|\theta\|^2\right).$$

Integrating from 0 to t, it follows that

$$\begin{aligned}\|\theta(t)\|^2+2\int_0^t\left(\gamma\|\Delta\theta\|^2+\|\nabla\theta\|^2\right)ds &\leq \|\theta(0)\|^2\\ &+ C\int_0^t\left(\|\eta_t\|^2+\|\eta\|^2+\|\theta\|^2\right)ds.\end{aligned} \tag{3.9}$$

Note that $\theta(0)=0$ and an application of Gronwall's Lemma yields the following estimate

$$\|\theta\|_{L^\infty(0,T;L^2(\Omega))} \leq C(T)h^4\left(\|u\|_{L^2(H^4)}+\|u_t\|_{L^2(H^4)}\right).$$

For obtaining $\|\theta\|_{L^\infty(0,T;H^2(\Omega))}$ estimate, setting $\chi=\theta_t$ in (3.7), we obtain

$$\|\theta_t(t)\|^2+\frac{1}{2}\frac{d}{dt}\left(\gamma\|\Delta\theta\|^2+\|\nabla\theta\|^2\right)=(\eta_t,\theta_t)+(f(u)-f(u_h),\theta_t). \tag{3.10}$$

Using Cauchy Schwarz inequality in (3.10) gives the following inequality

$$\begin{aligned}\|\theta_t(t)\|^2+\frac{1}{2}\frac{d}{dt}\left(\gamma\|\Delta\theta\|^2+\|\nabla\theta\|^2\right) &\leq C(\|\eta_t\|\\ &+ \|f(u)-f(u_h)\|)\|\theta_t\|.\end{aligned} \tag{3.11}$$

Substituting (3.8) in (3.11), we arrive at

$$\|\theta_t\|^2+\frac{1}{2}\frac{d}{dt}\{\gamma\|\Delta\theta\|^2+\|\nabla\theta\|^2\} \leq C\left(\|\eta_t\|^2+\|\eta\|^2+\|\theta\|^2\right)+\frac{1}{2}\|\theta_t\|^2.$$

Integrating from 0 to t, it follows that

$$\begin{aligned}\int_0^t \|\theta_t(s)\|^2\,ds + \gamma\|\Delta\theta\|^2 + \|\nabla\theta\|^2 &\leq C\left(\gamma\|\Delta\theta(0)\|^2 + \|\nabla\theta(0)\|^2\right)\\ &+ C\int_0^t \left(\|\eta_t\|^2 + \|\eta\|^2 + \|\theta\|^2\right)\,ds. \end{aligned}\tag{3.12}$$

Substituting the estimates of $\|\eta_t\|, \|\eta\|$, $\|\theta(t)\|$ and $\theta(0) = 0$ in (3.12), we obtain using Poincaré inequality the following super-convergence result for $\|\theta(t)\|_2$

$$\|\theta\|_{L^\infty(0,T;H^2(\Omega))} \leq C(T,\gamma^{-1})h^4\left(\|u\|_{L^\infty(H^4)} + \|u_t\|_{L^2(H^4)}\right).$$

Using Sobolev Imbedding theorem, we find that

$$\|\theta(t)\|_{L^\infty} \leq C\|\theta(t)\|_2,$$

and hence,

$$\|\theta\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C(T,\gamma^{-1})h^4\left(\|u\|_{L^\infty(H^4)} + \|u_t\|_{L^2(H^4)}\right).$$

Using (3.4) and (3.6) along with triangle inequality, we complete the rest of the proof. □

Remark 3.1. *For optimal estimate of the error $u - u_h$ in L^2-norm, it is possible to choose $u_{0,h}$ as L^2-projection, i.e., $u_{0,h} = P_h u_0$, or $u_{0,h} = \mathcal{I}_h u_0$, where $\mathcal{I}_h u_0$ is the interpolant of u_0 onto S_h^0. In both the cases, $\|u_{0,h}\|_2$ is bounded by $\|u_0\|_2$ as*

$$\begin{aligned}\|u_{0,h}\|_2 &\leq \|u_{0,h} - u_0\|_2 + \|u_0\|_2\\ &\leq C\|u_0\|_2.\end{aligned}$$

Moreover, for $j = 0, 1, 2$

$$\|\theta(0)\|_j \leq \|u_{0,h} - u_0\|_j + \|u_0 - \tilde{u}(0)\|_j \leq Ch^{4-j}\|u_0\|_4.$$

4. Completely Discrete Scheme.

In this section, we discretize the semidiscrete equation (3.2) in the temporal direction using backward Euler method, Crank-Nicolson scheme and two step backward difference method. We derive existence and uniqueness results by using a varient of Brouwer fixed point theorem. Finally, we establish optimal error estimates for all the three schemes.

4.1. Backward Euler Method. We consider a discretization in time based on backward Euler's method. For any given positive integer N, let $k = T/N$ denote the size of time discretization and $t_n = nk, n = 0, 1, 2, \cdots, N$. For a continuous function φ, let $\varphi^n = \varphi(t_n)$ and

$$\bar{\partial}_t\varphi^n = \frac{\varphi^n - \varphi^{n-1}}{k}.$$

The discrete time Galerkin approximation $U^n \in S_h^0$ of $u(t_n)$ is defined as a solution of

$$\begin{aligned}(\bar{\partial}_t U^n, \chi) + \gamma(\Delta U^n, \Delta\chi) + (\nabla U^n, \nabla\chi) &+ (f(U^n), \chi) = 0,\ \chi \in S_h^0,\\ U^0 &= u_{0,h},\end{aligned}\tag{4.1}$$

where $u_{0,h} \in S_h^0$ is an appropriate approximation to u_0 to be defined later.
For proving existence of a unique solution U^n to (4.1) at each time level t_n, the following *a priori* bound is useful.

Theorem 4.1. *The solution (4.1) satisfies*

$$\mathcal{E}(U^n) \le \mathcal{E}(U^0),\ n \ge 1,$$

where $\mathcal{E}(U^n)$ is a Lyapunov functional defined by

$$\mathcal{E}(U^n) = \int_\Omega \{\frac{\gamma}{2}|\Delta U^n|^2 + \frac{1}{2}|\nabla U^n|^2 + F(U^n)\}\,dx$$

with $F' = f$. Further, there exists a positive constant C such that

$$\|U^n\|_\infty \le C\|U^n\|_2 \le C\left(\gamma^{-1}, \|U^0\|_2\right),\ \ n \ge 1.$$

Proof. Setting $\chi = U^n - U^{n-1}$ in (4.1), it follows that

$$\frac{1}{k}\|U^n - U^{n-1}\|^2 + \gamma\left(\Delta U^n, \Delta(U^n - U^{n-1})\right) + \left(\nabla U^n, \nabla(U^n - U^{n-1})\right) + (f(U^n), U^n - U^{n-1}) = 0.$$

Now, we use $a(a-b) = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$, we arrive at

$$\begin{aligned}\frac{1}{k}\|U^n - U^{n-1}\|^2 &+ \frac{\gamma}{2}\left(\|\Delta U^n\|^2 - \|\Delta U^{n-1}\|^2\right) + \frac{\gamma}{2}\|\Delta U^n - \Delta U^{n-1}\|^2\\ &+ \frac{1}{2}\left(\|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2\right) + \frac{1}{2}\|\nabla U^n - \nabla U^{n-1}\|^2\\ &+ (f(U^n), U^n - U^{n-1}) = 0.\end{aligned} \tag{4.2}$$

Since $F'(U^n) = f(U^n)$, we obtain using the Taylor series expansion of $F(\cdot)$

$$\begin{aligned}\left(F(U^n) - F(U^{n-1}), 1\right) &= (f(U^n), U^n - U^{n-1})\\ &- \left(\frac{F''(\xi_n)}{2}(U^n - U^{n-1})^2, 1\right),\end{aligned} \tag{4.3}$$

where ξ_n is a point on the line joining U^n and U^{n-1}. We note that

$$\left(-\frac{F''(\xi_n)}{2}(U^n - U^{n-1})^2, 1\right) \le \frac{1}{2}\|U^n - U^{n-1}\|^2. \tag{4.4}$$

Now, taking the difference between $\mathcal{E}(U^n)$ and $\mathcal{E}(U^{n-1})$, we obtain

$$\begin{aligned}\mathcal{E}(U^n) - \mathcal{E}(U^{n-1}) &= \frac{\gamma}{2}(\|\Delta U^n\|^2 - \|\Delta U^{n-1}\|^2) + \frac{1}{2}(\|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2)\\ &+ (F(U^n) - F(U^{n-1}), 1).\end{aligned} \tag{4.5}$$

Substituting (4.2)–(4.3) in (4.5), we arrive at the following expression

$$\begin{aligned}\mathcal{E}(U^n) - \mathcal{E}(U^{n-1}) &+ \frac{1}{k}\|U^n - U^{n-1}\|^2 + \frac{\gamma}{2}\|\Delta U^n - \Delta U^{n-1}\|^2\\ &+ \frac{1}{2}\|\nabla U^n - \nabla U^{n-1}\|^2 = -\left(\frac{F''(\xi_n)}{2}(U^n - U^{n-1})^2, 1\right).\end{aligned} \tag{4.6}$$

From (4.4), it follows that

$$\mathcal{E}(U^n) - \mathcal{E}(U^{n-1}) + \frac{1}{k}\|U^n - U^{n-1}\|^2 - \frac{1}{2}\|U^n - U^{n-1}\|^2 \le 0,$$

and

$$\mathcal{E}(U^n) - \mathcal{E}(U^{n-1}) + \frac{(2-k)k}{2}\|\bar{\partial}_t U^n\|^2 \le 0.$$

Hence,

$$\mathcal{E}(U^n) \le \mathcal{E}(U^{n-1}).$$

Finally, we obtain

$$\mathcal{E}(U^n) \le \cdots \le \mathcal{E}(U^0).$$

Using the definition of $\mathcal{E}(U^n)$, we find that

$$\int_\Omega \{\frac{\gamma}{2}|\Delta U^n|^2 + \frac{1}{2}|\nabla U^n|^2 + F(U^n)\}\, dx \le \int_\Omega \{\frac{\gamma}{2}|\Delta U^0|^2 + \frac{1}{2}|\nabla U^0|^2 + F(U^0)\}\, dx.$$

Since $F(U^n) \ge 0$, using Poincaré inequality, we obtain

$$\|U^n\|_2 \le C\left(\gamma^{-1}, \|U^0\|_2\right).$$

An application of the Sobolev Imbedding theorem yields

$$\|U^n\|_{L^\infty} \le C\left(\gamma^{-1}, \|U^0\|_2\right).$$

This completes the rest of the proof. □

Below, we discuss the existence of a solution U^n to (4.1) using the following Lemma 4.1 which is a consequence of the Brouwer fixed point theorem. For a proof, see Kesavan [10].

Lemma 4.1. *Let H be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)_H$ and induced norm $\|\cdot\|_H$. Further, let J be a continuous mapping from H into itself and be such that $(J(\xi),\xi)_H > 0$, for all $\xi \in H$ with $\|\xi\|_H = \alpha > 0$. Then there exists $\xi^* \in H$ with $\|\xi^*\|_H \le \alpha$ such that $J(\xi^*) = 0$.*

Theorem 4.2. *Assume that $U^0, U^1, \cdots, U^{n-1}$ are given, then there exists a unique solution U^n, satisfying (4.1) for small k.*

Proof. With H as S_h^0, define $J(U^n)$ as

$$(J(U^n),\chi) = (U^n,\chi) + kA(U^n,\chi) + k\left(f(U^n),\chi\right) - (U^{n-1},\chi).$$

Taking $\chi = U^n$, we obtain

$$(J(U^n),U^n) \ge \|U^n\|^2 + k\left(\gamma\|\Delta U^n\|^2 + \|\nabla U^n\|^2\right) + k(f(U^n),U^n) - \|U^n\|\,\|U^{n-1}\|,$$

and hence, using $k|(f(U^n),U^n)| \le Ck\|U^n\|^2$, we find that

$$(J(U^n),U^n) \ge (1-Ck)\|U^n\|^2 - \|U^{n-1}\|\,\|U^n\|.$$

Choose k sufficiently small so that $1 - Ck = \frac{1}{2}$. Thus,

$$(J(U^n),U^n) \ge \left(\frac{1}{2}\|U^n\| - \|U^{n-1}\|\right)\|U^n\|.$$

Setting $\|U^n\| > 3\|U^{n-1}\| = \alpha$, it follows that

$$(J(U^n),U^n) > 0.$$

An application of the Lemma 4.1 yields the existence of U^{n*} such that $J(U^{n*}) = 0$. Infact, $U^{n*} = U^n$ satisfies the Lemma 4.1 and this completes the proof of existence. For uniqueness, let U^n and V^n be two distinct solutions of (4.1). Taking $W^n = U^n - V^n$, it follows that

$$(\bar\partial_t W^n,\chi) + \gamma(\Delta W^n,\Delta\chi) + (\nabla W^n,\nabla\chi) + (f(U^n) - f(V^n),\chi) = 0.$$

Choose $\chi = W^n$, and using

$$(\bar\partial_t W^n, W^n) \ge \frac{1}{2}\bar\partial_t\|W^n\|^2,$$

we obtain

$$\frac{1}{2}\bar\partial_t\|W^n\|^2 + \gamma\|\Delta W^n\|^2 + \|\nabla W^n\|^2 + (f(U^n) - f(V^n), W^n) \le 0. \tag{4.7}$$

For the last term on the right hand side of (4.7), we use the boundedness of $\|U^n\|_\infty$ and $\|V^n\|_\infty$. Thus,

$$(f(U^n) - f(V^n), W^n) \le C\|W^n\|^2.$$

On substituting in (4.7), we obtain

$$\bar{\partial}_t \|W^n\|^2 \leq C\|W^n\|^2,$$

and hence,

$$\|W^n\|^2 \leq \frac{1}{(1-Ck)}\|W^{n-1}\|^2.$$

Assuming $W^{n-1} = 0$, the above inequality implies $W^n = 0$ for sufficiently small k with $(1 - Ck) > 0$ i.e., $U^n = V^n$. This leads to a contradiction and hence, the solution is unique. This complete the rest of the proof. □

Below, we derive optimal error estimates for the backward Euler method. Now, using the elliptic projection $\tilde{u}$ at $t = t_n$, we split the error e^n as

$$\begin{aligned} e^n &:= u(t_n) - U^n = (u(t_n) - \tilde{u}(t_n)) - (U^n - \tilde{u}(t_n)) \\ &:= \eta^n - \theta^n. \end{aligned}$$

Theorem 4.3. *Let $U^0 = \tilde{u}(0)$ so that $\theta^0 = 0$ and $\|U^0\|_2 \leq C\|u_0\|_2$. Then, there exists a positive constant C independent of the discretization parameters h and k such that for small k and $J = 1, 2, \cdots, N$,*

$$\begin{aligned} \|u(t_J) - U^J\|_j &\leq C(T, \gamma^{-1})(h^{4-j}\left(\|u\|_{L^\infty(0,T;H^4)} + \|u_t\|_{L^2(0,T;H^4)}\right) \\ &+ k\|u_{tt}\|_{L^2(0,T;L^2)}),\ j = 0, 1, 2. \end{aligned}$$

Moreover, assuming the quasi-uniformity condition on the triangulation $\mathcal{T}_h$, the following estimate holds :

$$\begin{aligned} \|u(t_J) - U^J\|_{L^\infty} &\leq C(T, \gamma^{-1})(h^4\left(\|u\|_{L^\infty(0,T;W^{4,\infty})} + \|u_t\|_{L^2(0,T;H^4)}\right) \\ &+ k\|u_{tt}\|_{L^2(0,T;L^2)}). \end{aligned}$$

Proof. Since the estimates of η^n are known, so for completing the proof, we need to estimate θ^n. We substract the equation (4.1) from (2.1) and using the auxiliary projection, we obtain the equation in θ^n as

$$\begin{aligned} (\bar{\partial}_t\theta^n, \chi) + \gamma(\Delta\theta^n, \Delta\chi) + (\nabla\theta^n, \nabla\chi) &= (f(u^n) - f(U^n), \chi) - (\bar{\partial}_t\tilde{u}(t_n) - u_t(t_n), \chi) \\ &= (f(u^n) - f(U^n), \chi) - (w^n, \chi), \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} w^n &= \bar{\partial}_t\tilde{u}(t_n) - u_t(t_n) = \left(\bar{\partial}_t\tilde{u}(t_n) - \bar{\partial}_t u(t_n)\right) + \left(\bar{\partial}_t u(t_n) - u_t(t_n)\right), \\ &= -\bar{\partial}_t\eta^n + \left(\bar{\partial}_t u(t_n) - u_t(t_n)\right) = w_1^n + w_2^n. \end{aligned}$$

Setting $\chi = \theta^n$ in (4.8), we arrive at the following expression

$$(\bar{\partial}_t\theta^n, \theta^n) + \gamma\|\Delta\theta^n\|^2 + \|\nabla\theta^n\|^2 = (f(u^n) - f(U^n), \theta^n) - (w^n, \theta^n).$$

Note that

$$(\bar{\partial}_t\theta^n, \theta^n) \geq \frac{1}{2}\bar{\partial}_t\|\theta^n\|^2,$$

and hence,

$$\frac{1}{2}\bar{\partial}_t\|\theta^n\|^2 + \gamma\|\Delta\theta^n\|^2 + \|\nabla\theta^n\|^2 \leq \left(\|f(u^n) - f(U^n)\| + \|w^n\|\right)\|\theta^n\|. \tag{4.9}$$

To bound for $\|f(u^n) - f(U^n)\|$, we use the boundedness of $\|u^n\|_{L^\infty}$ and $\|U^n\|_{L^\infty}$ to obtain

$$\|f(u^n) - f(U^n)\| \leq C(\|\eta^n\| + \|\theta^n\|). \tag{4.10}$$

Substituting (4.10) in (4.9), we arrive at

$$\bar{\partial}_t\|\theta^n\| \leq C\left(\|\eta^n\| + \|\theta^n\| + \|w^n\|\right).$$

On summing from $n = 1$ to J, we find that

$$(1 - Ck)\|\theta^J\| \leq C\left(k\sum_{n=1}^{J}\|\eta^n\| + k\sum_{n=1}^{J}\|w^n\| + k\sum_{n=1}^{J-1}\|\theta^n\|\right).$$

Choose k sufficient small so that $(1 - Ck) > 0$ and an application of discrete Gronwall's Lemma yields

$$\|\theta^J\| \leq C(T)\left(k\sum_{n=1}^{J}\|\eta^n\| + k\sum_{n=1}^{J}\|w^n\|\right). \tag{4.11}$$

For completing the proof, it remains to estimate $\|w^n\|$. For w_1^n, we note that

$$w_1^n = -k^{-1}\int_{t_{n-1}}^{t_n} \eta_t(s)\,ds,$$

and hence,

$$k\sum_{n=1}^{J}\|w_1^n\| \leq Ch^4\sum_{n=1}^{J}\int_{t_{n-1}}^{t_n}\|u_t\|_4\,ds = Ch^4\int_0^{t_J}\|u_t\|_4\,ds.$$

For w_2^n, we observe that

$$\begin{aligned} w_2^n &= \frac{u(t_n) - u(t_{n-1})}{k} - u_t(t_n) \\ &= -k^{-1}\int_{t_{n-1}}^{t_n}(s - t_{n-1})u_{tt}(s)\,ds, \end{aligned}$$

and therefore,

$$\begin{aligned} k\sum_{n=1}^{J}\|w_2^n\| &\leq \sum_{n=1}^{J}\left(\int_{t_{n-1}}^{t_n}(s - t_{n-1})\|u_{tt}(s)\|\,ds\right) \\ &\leq k\int_0^{t_J}\|u_{tt}\|\,ds. \end{aligned}$$

Substituting the estimates of $\|\eta^n\|, \|w_1^n\|$ and $\|w_2^n\|$ in (4.11), we obtain the estimate for $\|\theta^J\|$. For $\|\theta^J\|_2$, we choose $\chi = \bar{\partial}_t\theta^n$ in (4.8) to obtain

$$\begin{aligned} \|\bar{\partial}_t\theta^n\|^2 + \frac{1}{2}\bar{\partial}_t\left(\gamma\|\Delta\theta^n\|^2 + \|\nabla\theta^n\|^2\right) &\leq \left(f(u^n) - f(U^n), \bar{\partial}_t\theta^n\right) - (w^n, \bar{\partial}_t\theta^n), \\ &\leq C\left(\|\theta^n\|^2 + \|\eta^n\|^2 + \|w^n\|^2\right) + \frac{1}{2}\|\bar{\partial}_t\theta^n\|^2. \end{aligned}$$

Now, we summing from $n = 1$ to J, we arrive at

$$\begin{aligned} k\sum_{n=1}^{J}\|\bar{\partial}_t\theta^n\|^2 + \gamma\|\Delta\theta^J\|^2 + \|\nabla\theta^J\|^2 &\leq Ck\sum_{n=1}^{J}(\|\theta^n\|^2 \\ &+ \|\eta^n\|^2 + \|w^n\|^2). \end{aligned} \tag{4.12}$$

Using Poincaré inequality and substituting the estimates of $\|\theta^n\|, \|\eta^n\|$ and $\|w^n\|$ in (4.12), we obtain a super-convergence result for $\|\theta^J\|_2$. Finally, a use of Sobolev Imbedding theorem yields

$$\|\theta^J\|_{L^\infty} \leq C\|\theta^J\|_2.$$

Using triangle inequality with estimates of η, we complete the rest of the proof. □

We note that the backward Euler method is of first order convergence in time. For obtaining second order convergence in time, we consider, below, the Crank-Nicolson scheme and two step backward difference method.

4.2. Crank-Nicolson Scheme. For obtaining second order accuracy in time, we now consider Crank-Nicolson scheme. For a continuous function $\varphi \in C[0,T]$, let

$$\bar{\partial}_t \varphi^n = \frac{\varphi^n - \varphi^{n-1}}{k}, \quad \varphi^{n-1/2} = \frac{\varphi^n + \varphi^{n-1}}{2}.$$

Following Qiang and Nicolaides [16], we define

$$\tilde{f}(U^{n-1}, U^n) = \begin{cases} \frac{F(U^{n-1}) - F(U^n)}{U^{n-1} - U^n}, & U^{n-1} \neq U^n, \\ F'(U^{n-1}), & U^{n-1} = U^n, \end{cases}$$

where, $\tilde{f}(\cdot,\cdot)$ in our case has the following explicit form

$$\tilde{f}(w,z) = \frac{1}{4}\left(w^3 + w^2 z + w z^2 + z^3\right) - \frac{1}{2}(w+z).$$

It is easy to verify that $\tilde{f}(w,z) \to f(z)$ as $w \to z$. Now, the discrete time finite element Galerkin approximation U^n of $u(t_n)$ is defined as a solution of

$$\begin{aligned} (\bar{\partial}_t U^n, \chi) + \gamma(\Delta U^{n-1/2}, \Delta\chi) + (\nabla U^{n-1/2}, \nabla\chi) \\ +(\tilde{f}(U^{n-1}, U^n), \chi) = 0, \chi \in S_h^0,\ n \geq 1, \\ U^0 = u_{0,h}, \end{aligned} \tag{4.13}$$

where $u_{0,h} \in S_h^0$ is an appropriate approximation to u_0 to be defined later. For proving optimal error estimates, the following *a priori* bound is useful.

Theorem 4.4. *Let U^n be a solution of (4.13). Then, there exists a positive constant C such that*

$$\|U^n\|_\infty \leq C\left(\gamma^{-1}, \|U^0\|_2\right),\ n \geq 1.$$

Proof. Setting $\chi = U^n - U^{n-1}$ in (4.13), we obtain

$$\begin{aligned} \frac{1}{k}\|U^n - U^{n-1}\|^2 \quad &+ \quad \gamma(\Delta U^{n-1/2}, \Delta(U^n - U^{n-1})) + (\nabla U^{n-1/2}, \nabla(U^n - U^{n-1})) \\ &+ \quad (\tilde{f}(U^{n-1}, U^n), U^n - U^{n-1}) = 0. \end{aligned} \tag{4.14}$$

Using the definition of $\tilde{f}(\cdot,\cdot)$ in (4.14), we arrive at

$$\begin{aligned} \frac{1}{k}\|U^n - U^{n-1}\|^2 + \frac{\gamma}{2}\left(\|\Delta U^n\|^2 - \|\Delta U^{n-1}\|^2\right) + \frac{1}{2}\left(\|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2\right) \\ + \left(F(U^n) - F(U^{n-1}), 1\right) = 0. \end{aligned} \tag{4.15}$$

Note that

$$\begin{aligned} \mathcal{E}(U^n) - \mathcal{E}(U^{n-1}) = \frac{\gamma}{2}\left(\|\Delta U^n\|^2 - \|\Delta U^{n-1}\|^2\right) + \frac{1}{2}\left(\|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2\right) \\ + \left(F(U^n) - F(U^{n-1}), 1\right). \end{aligned} \tag{4.16}$$

Using (4.15) in (4.16), it gives the following expression

$$\mathcal{E}(U^n) - \mathcal{E}(U^{n-1}) = -\frac{1}{k}\|U^n - U^{n-1}\|^2 \leq 0,$$

and hence,

$$\mathcal{E}(U^n) \leq \mathcal{E}(U^0).$$

Using the definition of $\mathcal{E}(U^n)$ and $F(U^n) = \frac{1}{4}\left(1 - (U^n)^2\right)^2 \geq 0$, we finally obtain

$$\|U^n\|_2 \leq C\left(\gamma^{-1}, \|U^0\|_2\right).$$

An application of Sobolev Imbedding theorem yields

$$\|U^n\|_\infty \leq C\|U^n\|_2 \leq C\left(\gamma^{-1}, \|U^0\|_2\right),$$

and this completes the proof. □

Note that the existence of a unique solution U^n to (4.13) follows easily following the analysis of the Theorem 4.2. Therefore, we omit the proof. Below, we discuss the error analysis. For error analysis, the following Lemma 4.2 will be useful. In the context of Cahn-Hilliard equation [16], similar result is proved by Qiang and Nicolaides [16]. For completeness, we briefly sketch the proof.

Lemma 4.2. *Let $u(t_n)$ and U^n be a solution of (2.1) and (4.13), respectively. Then, there exists a positive constant C independent of the discretization parameters h and k such that*

$$\begin{aligned} \|f(u(t_{n-1/2})) - \tilde{f}(U^{n-1}, U^n)\| &\leq Ck^2(\|u_t\|_{L^\infty(0,T;L^2)} \\ &+ \|u_{tt}\|_{L^\infty(0,T;L^2)}). \end{aligned} \tag{4.17}$$

Proof. We rewrite as

$$\begin{aligned} \|f(u(t_{n-1/2}) - \tilde{f}(U^{n-1}, U^n)\| &\leq \|f(u(t_{n-1/2})) - f(u_{n-1/2})\| \\ &+ \|f(u_{n-1/2}) - \tilde{f}(u(t_{n-1}), u(t_n))\| \\ &+ \|\tilde{f}(u(t_{n-1}), u(t_n)) - \tilde{f}(u(t_{n-1}, U^n)\| \\ &+ \|\tilde{f}(u(t_{n-1}, U^n)) - \tilde{f}(U^{n-1}, U^n)\| \\ &= T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{4.18}$$

Using the smoothness of f, boundedness of $\|u\|_{L^\infty}$ and Taylor series expansion, we estimate T_1 as

$$T_1 \leq C\|u(t_{n-1/2}) - \left(\frac{u^n + u^{n-1}}{2}\right)\| \leq Ck^2\|u_{tt}\|_{L^\infty(0,T;L^2)}.$$

Now, using the definition of $f(\cdot)$ and $\tilde{f}(\cdot,\cdot)$, we derive the bound for T_2 :

$$\begin{aligned} T_2 &= \|\frac{1}{8}\left(u_n + u_{n-1}\right)^3 - \frac{1}{4}\left(u_{n-1}^3 + u_{n-1}^2 u_n + u_{n-1}u_n^2 + u_n^3\right)\| \\ &= \|\frac{1}{8}\left(u_n - u_{n-1}\right)\left(u_n^2 - u_{n-1}^2\right)\| \\ &\leq C\|u_n - u_{n-1}\|^2 \leq Ck^2\|u_t\|_{L^\infty(0,T;L^2)}. \end{aligned}$$

Similarly, using the boundedness of $\|u\|_{L^\infty}$ and $\|U^n\|_{L^\infty}$, we easily derive the following estimates

$$\begin{aligned} T_3 &\leq C\|u(t_n) - U^n\|, \\ T_4 &\leq C\|u(t_{n-1}) - U^{n-1}\|. \end{aligned}$$

Substituting the estimates for T_1, T_2, T_3 and T_4 in (4.18), we obtain the required resuly for (4.17). □

Theorem 4.5. *Let $U^0 = \tilde{u}(0)$ so that $\theta^0 = 0$. Then, there exists a positive constant C independent of the discretization parameters h and k such that for $j = 0, 1, 2$*

$$\begin{aligned} \|u(t_J) - U^J\|_j &\leq C(T)h^{4-j}\left(\|u\|_{L^\infty(0,T;H^4)} + \|u_t\|_{L^2(0,T;H^4)}\right) \\ &+ C(T)k^2\left(\|u\|_{W^{2,\infty}(0,T;L^2)} + \|u_{tt}\|_{L^2(0,T;H^2)}\right), \ J \geq 1. \end{aligned}$$

In addition, assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. Then

$$\begin{aligned} \|u(t_J) - U^J\|_{L^\infty} &\leq C(T)h^4\left(\|u\|_{L^\infty(0,T;W^{4,\infty})} + \|u_t\|_{L^2(0,T;H^4)}\right) \\ &+ C(T)k^2\left(\|u\|_{W^{2,\infty}(0,T;L^2)} + \|u_{tt}\|_{L^2(0,T;H^2)}\right), \ J \geq 1. \end{aligned}$$

Proof. Substracting (4.13) from (2.1) and using the auxiliary projection, we obtain the error equation in θ^n as

$$\begin{aligned}(\bar{\partial}_t\theta^n,\chi)+\gamma(\Delta\theta^{n-1/2},\Delta\chi) &+ (\nabla\theta^{n-1/2},\nabla\chi)=(\bar{\partial}_t\eta^n,\chi)+(\sigma^{n-1/2},\chi)\\ &+ \gamma(w_1^{n-1/2},\Delta\chi)+(w_2^{n-1/2},\nabla\chi)\\ &+ \Big(f(u(t_{n-1/2}))-\tilde{f}(U^{n-1},U^n),\chi\Big), \end{aligned} \tag{4.19}$$

where

$$\begin{aligned}\sigma^{n-1/2} &= u_t(t_{n-1/2})-\bar{\partial}_t u^n,\\ w_1^{n-1/2} &= \Delta\left(u(t_{n-1/2})-(\frac{u^n+u^{n-1}}{2})\right),\\ w_2^{n-1/2} &= \nabla\left(u(t_{n-1/2})-(\frac{u^n+u^{n-1}}{2})\right).\end{aligned}$$

Setting $\chi=\theta^{n-1/2}$ in (4.19), we arrive at

$$\begin{aligned}(\bar{\partial}_t\theta^n,\theta^{n-1/2})+\gamma\|\Delta\theta^{n-1/2}\|^2 &+ \|\nabla\theta^{n-1/2}\|^2=(\bar{\partial}_t\eta^n,\theta^{n-1/2})+(\sigma^{n-1/2},\theta^{n-1/2})\\ &+ \gamma(w_1^{n-1/2},\Delta\theta^{n-1/2})+(w_2^{n-1/2},\nabla\theta^{n-1/2})\\ &+ \Big(f(u(t_{n-1/2}))-\tilde{f}(U^{n-1},U^n),\theta^{n-1/2}\Big).\end{aligned}$$

Note that

$$\Big(\bar{\partial}_t\theta^n,\theta^{n-1/2}\Big)=\frac{1}{2}\bar{\partial}_t\|\theta^n\|^2,$$

and hence,

$$\begin{aligned}\frac{1}{2}\bar{\partial}_t\|\theta^n\|^2+\frac{\gamma}{2}\|\Delta\theta^{n-1/2}\|^2 &+ \frac{1}{2}\|\nabla\theta^{n-1/2}\|^2\le C(\gamma,\epsilon^{-1})(\|\bar{\partial}_t\eta^n\|^2\\ &+ \|\sigma^{n-1/2}\|^2+\|w_1^{n-1/2}\|^2+\|w_2^{n-1/2}\|^2+\|\theta^{n-1/2}\|^2)\\ &+ \|\Big(f(u(t_{n-1/2}))-\tilde{f}(U^{n-1},U^n)\Big)\|^2.\end{aligned} \tag{4.20}$$

Using Lemma 4.2 in (4.20), we arrive at

$$\begin{aligned}\frac{1}{2}\bar{\partial}_t\|\theta^n\|^2+\frac{\gamma}{2}\|\Delta\theta^{n-1/2}\|^2 &+ \frac{1}{2}\|\nabla\theta^{n-1/2}\|^2\le C(k^4+\|\eta^{n-1/2}\|^2+\|\bar{\partial}_t\eta^n\|^2\\ &+ \|\sigma^{n-1/2}\|^2+\|w_1^{n-1/2}\|^2+\|w_2^{n-1/2}\|^2+\|\theta^{n-1/2}\|^2).\end{aligned} \tag{4.21}$$

On summing from $n=1$ to J and using Poincaré inequality, it follows that

$$\begin{aligned}(1-Ck)\|\theta^J\|^2+k\sum_{n=1}^{J}\|\theta^{n-1/2}\|_2^2 &\le Ck\sum_{n=1}^{J}(k^4+\|\eta^{n-1/2}\|^2+\|\bar{\partial}_t\eta^n\|^2+\|\sigma^{n-1/2}\|^2\\ &+ \|w_1^{n-1/2}\|^2+\|w_2^{n-1/2}\|^2)+Ck\sum_{n=1}^{J-1}\|\theta^n\|^2.\end{aligned} \tag{4.22}$$

We note that

$$\begin{aligned}\|w_1^{n-1/2}\|^2 &\le Ck^3\int_0^{t_n}\|\Delta u_{tt}(s)\|^2\,ds,\\ \|w_2^{n-1/2}\|^2 &\le Ck^3\int_0^{t_n}\|\nabla u_{tt}(s)\|^2\,ds,\\ \|\sigma^{n-1/2}\|^2 &\le Ck^3\int_0^{t_n}\|u_{ttt}(s)\|^2\,ds.\end{aligned}$$

For small k with $(1-Ck) > 0$ and substituting $\|\sigma^{n-1/2}\|$, $\|w_1^{n-1/2}\|$, $\|w_2^{n-1/2}\|$, $\|\bar{\partial}_t\eta^n\|$, $\|\eta^n\|$ in (4.22) and an application of discrete Gronwall's Lemma yields the $\|\theta^J\|$ estimate. Finally, the result $j = 0$ follows from the triangle inequality.
For $j = 2$, we choose $\chi = \bar{\partial}_t\theta^n$ in (4.19) gives the following expression

$$\begin{aligned} \|\bar{\partial}_t\theta^n\|^2 + \gamma(\Delta\theta^{n-1/2}, \Delta\bar{\partial}_t\theta^n) &+ (\nabla\theta^{n-1/2}, \nabla\bar{\partial}_t\theta^n) = (\bar{\partial}_t\eta^n, \bar{\partial}_t\theta^n) + (\sigma^{n-1/2}, \bar{\partial}_t\theta^n) \\ &+ \gamma(w_1^{n-1/2}, \Delta\bar{\partial}_t\theta^n) + (w_2^{n-1/2}, \nabla\bar{\partial}_t\theta^n) \\ &+ \left(f(u(t_{n-1/2})) - \tilde{f}(U^{n-1}, U^n), \bar{\partial}_t\theta^n\right). \end{aligned} \tag{4.23}$$

Note that

$$\begin{aligned} \gamma(\Delta\theta^{n-1/2}, \Delta\bar{\partial}_t\theta^n) + (\nabla\theta^{n-1/2}, \nabla\bar{\partial}_t\theta^n) \\ = \frac{1}{2}\bar{\partial}_t\left(\gamma\|\Delta\theta^n\|^2 + \|\nabla\theta^n\|^2\right). \end{aligned} \tag{4.24}$$

Substituting (4.24) in (4.23), we arrive at

$$\begin{aligned} \frac{1}{2}\|\bar{\partial}_t\theta^n\|^2 &+ \frac{1}{2}\bar{\partial}_t\left(\gamma\|\Delta\theta^n\|^2 + \|\nabla\theta^n\|^2\right) \le C(\gamma)(\|\bar{\partial}_t\eta^n\|^2 + \|\sigma^{n-1/2}\|^2 \\ &+ \|w_1^{n-1/2}\|^2 + \|w_2^{n-1/2}\|^2 + \|\theta^{n-1/2}\|_2^2 \\ &+ \|\eta^{n-1/2}\|^2 + \|f(u(t_{n-1/2})) - \tilde{f}(U^{n-1}, U^n)\|^2). \end{aligned} \tag{4.25}$$

Again using Lemma 4.2 in (4.25) and summing from $n = 1$ to J, we obtain

$$\begin{aligned} k\sum_{n=1}^{J}\|\bar{\partial}_t\theta^n\| + \gamma\|\Delta\theta^J\| + \|\nabla\theta^J\| &\le Ck\sum_{n=1}^{J}(k^2 + \|\bar{\partial}_t\eta^n\| + \|\sigma^{n-1/2}\| + \|w_1^{n-1/2}\| \\ &+ \|w_2^{n-1/2}\| + \|\theta^{n-1/2}\| + \|\eta^{n-1/2}\|). \end{aligned} \tag{4.26}$$

Substituting the estimates of $\|\theta^n\|$, $\|\eta^n\|$, $\|\sigma^{n-1/2}\|$, $\|w_1^{n-1/2}\|$, $\|w_2^{n-1/2}\|$ and $\|\bar{\partial}_t\eta^n\|$ in (4.26), we obtain the super-convergent result for $\|\theta^J\|_2$. An application of Sobolev Imbedding theorem yields

$$\|\theta^J\|_\infty \le C\|\theta^J\|_2.$$

Finally, the result follows from triangle inequality. □

4.3. Second Order Backward Difference Method.

For a second order accuracy in time, we consider a two-step backward method. Let

$$D_t^{(2)}U^n = \bar{\partial}_t U^n + \frac{1}{2}k\bar{\partial}_t^2 U^n,$$

and let $U^n, n = 0, 1, 2, \cdots, N$ be the discrete time finite element Galerkin solution defined by

$$\begin{aligned} (D_t^{(2)}U^n, \chi) + \gamma(\Delta U^n, \Delta\chi) + (\nabla U^n, \nabla\chi) + (f(U^n), \chi) \\ = 0,\ \chi \in S_h^0, n \ge 2 \end{aligned} \tag{4.27}$$

and for $n = 1$

$$(\bar{\partial}_t U^1, \chi) + \gamma(\Delta U^1, \Delta\chi) + (\nabla U^1, \nabla\chi) + (f(U^1), \chi) = 0,\ \chi \in S_h^0 \tag{4.28}$$

with

$$U^0 = u_{0,h}.$$

For proving optimal error estimates the following *a priori* bound is useful.

Theorem 4.6. *Let U^n be a solution of (4.27)–(4.28). Then, there exists a positive constant C such that*

$$\|U^j\|_2^2 + \frac{k}{4}\|\bar{\partial}_t U^j\|^2 \leq C(\gamma^{-1}, \|U^0\|_2),\ j = 1, 2, \cdots, N.$$

Moreover,

$$\|U^j\|_\infty \leq C(\gamma^{-1}, \|U^0\|_2),\ j = 1, 2, \cdots, N.$$

Proof. Setting $\chi = U^1 - U^0$ in (4.28), we obtain

$$k\|\bar{\partial}_t U^1\|^2 + \gamma(\Delta U^1, \Delta(U^1 - U^0)) + (\nabla U^1, \nabla(U^1 - U^0)) + (f(U^1), U^1 - U^0) = 0.$$

Using $a(a-b) = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$, it gives the following expression

$$\begin{aligned} k\|\bar{\partial}_t U^1\|^2 &+ \frac{\gamma}{2}(\|\Delta U^1\|^2 - \|\Delta U^0\|^2) + \frac{\gamma}{2}\|\Delta U^1 - \Delta U^0\|^2 + \frac{1}{2}(\|\nabla U^1\|^2 - \|\nabla U^0\|^2) \\ &+ \frac{1}{2}\|\nabla U^1 - \nabla U^0\|^2 + (f(U^1), U^1 - U^0) = 0. \end{aligned} \tag{4.29}$$

Taking the difference between $\mathcal{E}(U^1)$ and $\mathcal{E}(U^0)$, we find that

$$\begin{aligned} \mathcal{E}(U^1) - \mathcal{E}(U^0) &= \frac{\gamma}{2}(\|\Delta U^1\|^2 - \|\Delta U^0\|^2) + \frac{1}{2}(\|\nabla U^1\|^2 - \|\nabla U^0\|^2) \\ &+ (F(U^1) - F(U^0), 1). \end{aligned} \tag{4.30}$$

Using the Taylor's series expansion, we obtain

$$\begin{aligned} \big(F(U^1) - F(U^0), 1\big) &= (f(U^1), U^1 - U^0) \\ &- \left(\frac{F''(\xi_1)}{2}(U^1 - U^0)^2, 1\right), \end{aligned} \tag{4.31}$$

where ξ_1 is a point on the line joining U^1 and U^0. Note that

$$\left(-\frac{F''(\xi_1)}{2}(U^1 - U^0)^2, 1\right) \leq \frac{1}{2}\|U^1 - U^0\|^2. \tag{4.32}$$

Substituting (4.29) and (4.31)–(4.32) in (4.30), we arrive at

$$\mathcal{E}(U^1) - \mathcal{E}(U^0) + (4 - 2k)\frac{k}{4}\|\bar{\partial}_t U^1\|^2 \leq 0.$$

Choose k with $0 < k < 1$ so that $(4 - 2k) > 1$, we obtain

$$\mathcal{E}(U^1) + \frac{k}{4}\|\bar{\partial}_t U^1\|^2 \leq \mathcal{E}(U^0). \tag{4.33}$$

Using the definition of $\mathcal{E}(\cdot)$ with $F(U^1) \geq 0$, we obtain

$$\|U^1\|_2 \leq C(\gamma^{-1}, \|U^0\|_2).$$

An application of Soblolev Imbedding theorem yields

$$\|U^1\|_{L^\infty} \leq C(\gamma^{-1}, \|U^0\|_2).$$

This completes the proof for $n = 1$. For $n \geq 2$, we choose $\chi = \bar{\partial}_t U^n$ in (4.27) and using the fact that

$$\left(D_t^{(2)} U^n, \bar{\partial}_t U^n\right) \geq \|\bar{\partial}_t U^n\|^2 + \frac{k}{4}\bar{\partial}_t(\|\bar{\partial}_t U^n\|^2),$$

we obtain

$$\begin{aligned} k\|\bar{\partial}_t U^n\|^2 + \frac{k^2}{4}\bar{\partial}_t(\|\bar{\partial}_t U^n\|^2) &+ \frac{\gamma}{2}(\|\Delta U^n\|^2 - \|\Delta U^{n-1}\|^2) + \frac{\gamma}{2}\|\Delta U^n - \Delta U^{n-1}\|^2 \\ &+ \frac{1}{2}(\|\nabla U^n\|^2 - \|\nabla U^{n-1}\|^2) + \frac{1}{2}\|\nabla U^n - \nabla U^{n-1}\|^2 \\ &+ (f(U^n), U^n - U^{n-1}) \leq 0. \end{aligned} \tag{4.34}$$

Taking the difference between $\mathcal{E}(U^n)$ and $\mathcal{E}(U^{n-1})$, using (4.3)–(4.4) and (4.34), we derive the following expression

$$\left(k - \frac{k^2}{2}\right)\|\bar{\partial}_t U^n\|^2 + \frac{k^2}{4}\bar{\partial}_t(\|\bar{\partial}_t U^n\|^2) + \mathcal{E}(U^n) - \mathcal{E}(U^{n-1}) \leq 0.$$

Choose k with $0 < k < 1$ so that $(1 - \frac{k}{2}) > 0$, Now, we arrive at

$$\mathcal{E}(U^n) + \frac{k}{4}\|\bar{\partial}_t U^n\|^2 \leq \mathcal{E}(U^{n-1}) + \frac{k}{4}\|\bar{\partial}_t U^{n-1}\|^2,$$

and hence, using (4.33)

$$\mathcal{E}(U^n) + \frac{k}{4}\|\bar{\partial}_t U^n\|^2 \leq \cdots \leq \mathcal{E}(U^1) + \frac{k}{4}\|\bar{\partial}_t U^1\|^2 \leq \mathcal{E}(U^0). \tag{4.35}$$

From the definition of $\mathcal{E}(\cdot)$ and Poincaré inequality, we find that

$$\|U^n\|_2 \leq C(\gamma^{-1}, \|U^0\|_2).$$

An application of Sobolev Imbedding theorem yields

$$\|U^n\|_\infty \leq C\|U^n\|_2 \leq C(\gamma^{-1}, \|U^0\|_2),$$

and this completes the rest of the proof. □

Using analysis similar to that of the Theorem 4.2, the existence of a unique solution U^n to (4.27)–(4.28) follows easily. Below, we derive optimal error estimates.

Theorem 4.7. *Let $U^0 = \tilde{u}(0)$ so that $\theta^0 = 0$. Then, there exists a positive constant C independent of the discretization parameters h and k such that for $J = 1, 2, \cdots, N$*

$$\begin{aligned} \|u(t_J) - U^J\|_j &\leq C(T)(h^{4-j}\left(\|u\|_{L^\infty(0,T;H^4)} + \|u_t\|_{L^2(0,T;H^4)}\right) \\ &+ k^2\|u_{tt}\|_{L^2(0,T;L^2)}),\ j = 0, 1, 2. \end{aligned}$$

In addition, assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. Then

$$\begin{aligned} \|u(t_J) - U^J\|_{L^\infty} &\leq C(T)\left(h^4\left(\|u\|_{L^\infty(0,T;W^{4,\infty})} + \|u_t\|_{L^2(0,T;H^4)}\right)\right. \\ &+ \left. k^2\|u_{tt}\|_{L^2(0,T;L^2)}\right),\ J = 1, 2, \cdots, N. \end{aligned}$$

Proof. Since the estimates of η^n are known, it is sufficient to estimate θ^n. Substracting the equations (4.27)–(4.28) from (2.1) and using auxiliary projection, we obtain the following equation in θ^n. For $n \geq 2$

$$\begin{aligned} (D_t^{(2)}\theta^n, \chi) + \gamma(\Delta\theta^n, \Delta\chi) + (\nabla\theta^n, \nabla\chi) &= (f(u^n) - f(U^n), \chi) \\ &+ (\sigma^n, \chi) + (D_t^{(2)}\eta^n, \chi), \end{aligned} \tag{4.36}$$

and for $n = 1$

$$\begin{aligned} (\bar{\partial}_t\theta^1, \chi) + \gamma(\Delta\theta^1, \Delta\chi) + (\nabla\theta^1, \nabla\chi) &= (f(u^1) - f(U^1), \chi) \\ &+ (\sigma^1, \chi) + (\bar{\partial}_t\eta^1, \chi), \end{aligned} \tag{4.37}$$

where

$$\begin{aligned} \sigma^n &= u_t(t_n) - D_t^{(2)}u(t_n),\ n \geq 2, \\ \sigma^1 &= u_t(t_1) - \bar{\partial}_t u(t_1). \end{aligned}$$

Setting $\chi = \theta^n$ in (4.36), we arrive at the following expression

$$(D_t^{(2)}\theta^n, \theta^n) + \gamma\|\Delta\theta^n\|^2 + \|\nabla\theta^n\|^2 \le \Big(\|f(u^n) - f(U^n)\| + \|D_t^{(2)}\eta^n\| + \|\sigma^n\|\Big)\|\theta^n\|.$$

From (4.10), we find that

$$\|f(u^n) - f(U^n)\| \le C(\|\theta^n\| + \|\eta^n\|),$$

and hence,

$$\begin{aligned}(D_t^{(2)}\theta^n, \theta^n) + \gamma\|\Delta\theta^n\|^2 + \|\nabla\theta^n\|^2 &\le C(\|\theta^n\| + \|\eta^n\| + \|D_t^{(2)}\eta^n\| \\ &+ \|\sigma^n\|)\|\theta^n\|. \end{aligned} \tag{4.38}$$

We note that

$$k(D_t^{(2)}\theta^n, \theta^n) = \Delta_1\|\theta^n\|^2 - \frac{1}{4}\Delta_2\|\theta^n\|^2 + \|\Delta_1\theta^n\|^2 - \frac{1}{4}\|\Delta_2\theta^n\|^2, \ \text{ for } \ n \ge 2,$$

where $\Delta_k = \theta^n - \theta^{n-k}$, for k = 1,2. As in McLean and Thomée [18], it is easy to find that

$$k\sum_{n=2}^{J}(D_t^{(2)}\theta^n, \theta^n) \ge \frac{3}{4}\|\theta^J\|^2 - \frac{1}{4}\|\theta^{J-1}\|^2 - \frac{1}{4}\|\theta^1\|^2. \tag{4.39}$$

Multiplying (4.38) by k and taking summation from $n = 2$ to J, we obtain

$$\begin{aligned} k\sum_{n=2}^{J}(D_t^{(2)}\theta^n, \theta^n) &\le Ck\sum_{n=2}^{J}(\|\theta^n\| + \|\eta^n\| \\ &+ \|D_t^{(2)}\eta^n\| + \|\sigma^n\|)\|\theta^n\|. \end{aligned} \tag{4.40}$$

Substituting (4.39) in (4.40), we arrive that

$$\frac{3}{4}\|\theta^J\|^2 \le \frac{1}{4}\|\theta^1\|^2 + \frac{1}{4}\|\theta^{J-1}\|^2 + Ck\sum_{n=2}^{J}\Big(\|\theta^n\| + \|\eta^n\| + \|D_t^{(2)}\eta^n\| + \|\sigma^n\|\Big)\|\theta^n\|.$$

Assume that $\|\theta^M\| = \max_{0\le n\le J}\|\theta^n\|$. Then

$$\frac{3}{4}\|\theta^M\|^2 \le \left(\frac{1}{4}\|\theta^1\| + Ck\sum_{n=2}^{J}\Big(\|\theta^n\| + \|\eta^n\| + \|D_t^{(2)}\eta^n\| + \|\sigma^n\|\Big)\right)\|\theta^M\| + \frac{1}{4}\|\theta^M\|^2$$

and hence,

$$\|\theta^J\| \le \|\theta^M\| \le C(\|\theta^1\| + k\sum_{n=2}^{J}(\|\theta^n\| + \|\eta^n\| + \|D_t^{(2)}\eta^n\| + \|\sigma^n\|)). \tag{4.41}$$

For completing the proof, it is enough to find $\|\theta^1\|$ estimate. We choose $\chi = \theta^1$ in (4.37), and use

$$\big(\bar{\partial}_t\theta^1, \theta^1\big) \ge \frac{1}{2}\bar{\partial}_t\|\theta^1\|^2,$$

to obtain

$$\frac{1}{2}\bar{\partial}_t\|\theta^1\|^2 + \gamma\|\Delta\theta^1\|^2 + \|\nabla\theta^1\|^2 \le \big(\|f(u^1) - f(U^1)\| + \|\bar{\partial}_t\eta^1\| + \|\sigma^1\|\big)\|\theta^1\|.$$

Note that

$$\|f(u^1) - f(U^1)\| \le C\left(\|\theta^1\| + \|\eta^1\|\right),$$

and hence,

$$\bar{\partial}_t\|\theta^1\| \le C\left(\|\theta^1\| + \|\eta^1\| + \|\bar{\partial}_t\eta^1\| + \|\sigma^1\|\right).$$

Finally, we find that

$$\|\theta^1\| \le \|\theta^0\| + Ck\left(\|\theta^1\| + \|\eta^1\| + \|\bar{\partial}_t\eta^1\| + \|\sigma^1\|\right).$$

Using $\theta^0 = 0$, we have the following inequality

$$(1 - Ck)\|\theta^1\| \le Ck\left(\|\eta^1\| + \|\bar{\partial}_t\eta^1\| + \|\sigma^1\|\right). \tag{4.42}$$

It is easy to find the following estimates

$$\|\sigma^1\| \le k\int_0^k \|u_{tt}\|\,ds,$$

and

$$k\|\bar{\partial}_t\eta^1\| \le Ch^4\int_0^k \|u_t\|_4\,ds.$$

For sufficiently small k with $(1 - Ck) > 0$ and substituting the above estimates in (4.42), we obtain the $\|\theta^1\|$ estimate. Finally, substituting $\|\theta^1\|$ estimate in (4.41), we obtain $\|\theta^J\|$ estimate. Using triangle inequality, we complete the rest of the proof for $j = 0$.

For $j = 2$, we choose $\chi = \bar{\partial}_t\theta^n$ in (4.36), we obtain

$$\begin{aligned}(D_t^{(2)}\theta^n, \bar{\partial}_t\theta^n) + \frac{1}{2}\bar{\partial}_t\left(\gamma\|\Delta\theta^n\|^2 + \|\nabla\theta^n\|^2\right) &\le (f(u^n) - f(U^n), \bar{\partial}_t\theta^n)\\ &\quad + (D_t^{(2)}\eta^n, \bar{\partial}_t\theta^n) + (\sigma^n, \bar{\partial}_t\theta^n),\ n \ge 2.\end{aligned} \tag{4.43}$$

Note that

$$\left(D_t^{(2)}\theta^n, \bar{\partial}_t\theta^n\right) \ge \|\bar{\partial}_t\theta^n\|^2 + \frac{k}{4}\bar{\partial}_t(\|\bar{\partial}_t\theta^n\|^2),\ n \ge 2. \tag{4.44}$$

Using (4.44) in (4.43), it yields the following expression

$$\begin{aligned}\|\bar{\partial}_t\theta^n\|^2 + \frac{k}{4}\bar{\partial}_t(\|\bar{\partial}_t\theta^n\|^2) &+ \frac{1}{2}\bar{\partial}_t(\gamma\|\Delta\theta^n\|^2 + \|\nabla\theta^n\|^2) \le C(\|\theta^n\|^2 + \|\eta^n\|^2\\ &+ \|D_t^{(2)}\eta^n\|^2 + \|\sigma^n\|^2) + \frac{1}{2}\|\bar{\partial}_t\theta^n\|^2.\end{aligned} \tag{4.45}$$

Multiplying (4.45) by k and summing from $n = 2$ to J, we arrive at

$$\begin{aligned}k\sum_{n=2}^{J}\|\bar{\partial}_t\theta^n\|^2 &+ \frac{k}{2}\|\bar{\partial}_t\theta^J\|^2 + \gamma\|\Delta\theta^J\|^2 + \|\nabla\theta^J\|^2 \le \gamma\|\Delta\theta^1\|^2 + \|\nabla\theta^1\|^2\\ &+ \frac{k}{2}\|\bar{\partial}_t\theta^1\|^2 + Ck\sum_{n=2}^{J}\left(\|\theta^n\|^2 + \|\eta^n\|^2 + \|D_t^{(2)}\eta^n\|^2 + \|\sigma^n\|^2\right).\end{aligned} \tag{4.46}$$

To complete the proof, it is enough to find estimates $\|\Delta\theta^1\|$, $\|\nabla\theta^1\|$ and $\|\bar{\partial}_t\theta^1\|$. Now, setting $\chi = \bar{\partial}_t\theta^1$ in (4.37) and use the boundedness of $\|u\|_\infty$ and $\|U\|_\infty$, we derive the following expression

$$\begin{aligned}\|\bar{\partial}_t\theta^1\|^2 + \frac{1}{2}\bar{\partial}_t\left(\gamma\|\Delta\theta^1\|^2 + \|\nabla\theta^1\|^2\right) &\le C\left(\|\theta^1\|^2 + \|\eta^1\|^2 + \|\bar{\partial}_t\eta^1\|^2 + \|\sigma^1\|^2\right)\\ &+ \frac{3}{4}\|\bar{\partial}_t\theta^1\|^2.\end{aligned}$$

Finally, we have the following inequality

$$\begin{aligned}\frac{k}{2}\|\bar{\partial}_t\theta^1\|^2 + \gamma\|\Delta\theta^1\|^2 + \|\nabla\theta^1\|^2 &\le Ck(\|\theta^1\|^2 + \|\eta^1\|^2\\ &+ \|\bar{\partial}_t\eta^1\|^2 + \|\sigma^1\|^2).\end{aligned} \tag{4.47}$$

Substituting (4.47) in (4.46) with known estimates $\|\theta^1\|, \|\eta^1\|, \|\bar{\partial}_t\eta^1\|$ and $\|\sigma^1\|$, we obtain the super-convergence result for $\|\theta^J\|_2$. An application of Sobolev Imbedding theorem yields

$$\|\theta^J\|_\infty \le C\|\theta^J\|_2.$$

Finally, we complete the proof using triangle inequality. □

5. Computational Experiments

We have seen in sections 3 and 4 that for obtaining the approximate solution for the EFK equation (1.1), we need ploynomials of the degree ≥ 3. It means that we have to construct minimum 10 node triangle for approximating the solution. Computationally, it is very expensive and difficult to impose inter-element C^1-continuity condition. If the boundary is curved, imposition of boundary conditions causes some more difficulties. Therefore, in this section, we discuss computational results for the following one dimensional EFK equation using C^1-piecewise cubic elements. Now, the one dimensional EFK equation is given by

$$u_t + \gamma u_{xxxx} - u_{xx} + f(u) = 0, \ (x,t) \in \Omega \times (0,T]$$

with initial condition

$$u(0) = u_0 = x^2(1-x)^2, \ x \in \Omega,$$

and the boundary conditions

$$\begin{aligned} u(0,t) &= u(1,t) = 0, \ (x,t) \in \partial\Omega \times (0,T], \\ u_x(0,t) &= u_x(1,t) = 0, \ (x,t) \in \partial\Omega \times (0,T], \end{aligned}$$

where $f(u) = u^3 - u$.

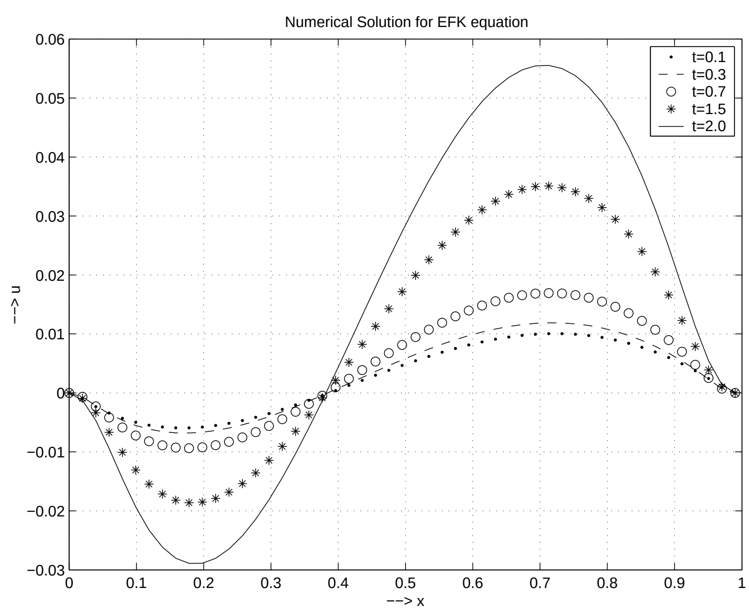

FIGURE 1. The profile of $u(x,t)$ vs x for $\gamma = 0.01$

Divide the domain into $N_i = 5, 10, 20$ with each of equal intervals h_i, where

$$h_i = \frac{1}{N_i} \quad i = 1, \cdots, 3.$$

With S_h^0 consisting of C^1-piecewise cubic polynomials, we consider the Galerkin approximation u_h. In Fig. 1, we obtain the graph of the approximate solution with $h = \frac{1}{50}$ at different time levels $t = 0.1, 0.3, 0.7, 1.5, 2.0$. Since the exact solution of the EFK equation is not known, it has been replaced by numerical solution u_h with $h = 160$. The order of convergence for the numerical method has been computed by the formula

$$\text{order} = \frac{\log\left[\dfrac{\|u_h - u_{h_i}\|_{L^j}}{\|u_h - u_{h_{i+1}}\|_{L^j}}\right]}{\log(2)}, \quad i = 1, 2, \quad j = 2, \infty,$$

where u_{h_i} is the numerical solution with step size h_i and $h_{i+1} = \frac{h_i}{2}$.
The order of the convergence in L^∞ norm:

N	$\|u_h - u_{h_i}\|_{L^\infty}$	order
10	0.7753074169158936E-03	
20	0.4878640174865723E-04	3.9902
40	0.3010034561157227E-05	4.0186

TABLE 1. The order of convergence for $u(x,t)$ at $t = 1.0$

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