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SUPERCONVERGENCE PROPERTIES OF DISCONTINUOUS GALERKIN METHODS FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. Three discontinuous Galerkin methods (SIPG, NIPG, DG) are considered for solving a one-dimensional elliptic problem. Superconvergence for the error at the interior node points and the derivative of the error at Gauss points are considered. All theorectical results obtained in the paper are supported by the results of numerical experiments.

Key Words. Discontinuous Galerkin methods, superconvergence, 1D problem.

1. Introduction

Discontinuous Galerkin (DG) methods are effective numerical methods for solving differential equations. DG methods dates back to the early 1970s when Nitche [21] introduced the concept of replacing the Lagrange multiplier used in hybrid formulations with averaged normal fluxes at the boundaries and added stabilization terms to produce optimal convergence rates. Early work on DG methods can be found in Reed and Hill [25], Percell and Wheeler [23], Arnold [1], Delves and Hall [14], etc. DG methods have been developed and analyzed for both hyperbolic and elliptic problems in parallel. There are two types of DG methods: one is in primal formulations and another is in mixed formulations. Both formulation may or may not include interior penalty terms. We refer to Chen [11] for a review of relationships on different DG methods for solving second order elliptic differential equations. Several papers have been published for rigorous a priori error estimates of DG methods. See, for examples, the paper by Arnold, Brezzi, Cockburn and Martin [2] for DG methods in primal formulations and Castillo, Cockburn, Perugia and Schötzau [7] for DG methods in mixed formulations.

Superconvergence in finite element methods have been studied for several decades. We refer to Krizek and Neittaanmaki [18], Wahlbin [28], Lin and Zhu [32] and the literature cited there. However, there are only a few papers dealing with superconvergence for discontinuous Galerkin methods. In Cockburn, Kanschat, Perugia and Schötau [12] and Castillo, Cockburn, Schötzau and Schwab [8], some superconvergence results have been obtained for the local discontinuous Galerkin method. There are no superconvergence results reported in the literature about the discontinuous Galerkin method in its primal formulation (the local DG method is in mixed formulation). It is the aim of this paper to study the superconvergence property of discontinuous Galerkin methods in non-mixed formulation. We will study a simple one-dimensional problem and analyze the superconvergence property of

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two types of DG methods. In the first type of DG methods we will study interior penalty Galerkin (IPG) methods; This includes the symmetric interior penalty (SIPG) method and the non-symmetric interior penalty (NIPG) method. In the second type of DG methods we consider a DG method without penalty terms. This method was developed by Baumann etc. in [3]. The main results obtained in this paper are the following: 1) superconvergence for the derivative of the error at Gauss points for SIPG and NIPG methods when super-penalty is used in the DG formulations; 2) superconvergence for the derivative of the error at Gauss points for all three methods when the mesh is uniform and the degree of the polynomials of finite element space is an odd number; 3) superconvergence for the averaged errors at node points for the SIPG and NIPG methods using at least piecewise quadratical polynomials.

The superconvergence property for DG methods solving partial differential equations is currently under investigation.

The paper is organized in the following way: In section 2, we derive the weak formulation used for discontinuous Galerkin methods, introduce some notation and define the interpolation operator; In section 3, superconvergence results are derived for SIPG and NIPG methods; The corresponding results for a DG method without penalty is in Section 4. Finally the numerical experiments that support our theoretical results are presented in Section 5.

2. Preliminaries

For simplicity, we consider the following two-point boundary value problem with mixed boundary conditions:

$$\begin{cases} -(p(x)u'(x))' = f(x), \quad x \in (0,1), \\ u(x) = u_d(x) \text{ at } x \in \Gamma_D, \\ u'(x) = u_n(x) \text{ at } x \in \Gamma_N, \end{cases} \tag{2.1}$$

where coefficients $p(x)$ and $f(x)$ are assumed to be sufficiently smooth and satisfy

$$p(x) \geq p_0 > 0, \qquad \text{for all } x \in (0,1). \tag{2.2}$$

And $\Gamma_D \subset \{0,1\}$, $\Gamma_N \subset \{0,1\}$ are the sets of points where boundary conditions are defined and satisfy $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D \cup \Gamma_N = \{0,1\}$, $\Gamma_D \neq \emptyset$.

To formulate the discontinuous Galerkin method for solving (2.1), we divide Ω into N subintervals:

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, \quad \text{with } h_i = x_i - x_{i-1}, \quad h = \max_{1\leq i\leq N} h_i,$$

and assume that the partition is quasi uniform in the sense that

$$h \leq C \min_{1\leq i\leq N} h_i.$$

Here, and throughout the paper, letter C denotes a generic constant independent of the mesh size h and the functions u, v, etc. Let $\Gamma_{int} = \{x_1, x_2, \cdots, x_{N-1}\}$ denote the set of interior nodes. Then

$$\{x_0, x_1, \cdots, x_N\} = \Gamma_{int} \cup \Gamma_D \cup \Gamma_N.$$

For any real numbers $a < b$, let $H^k(a,b)$ denote the standard Sobolev space with the standard norm $\|\cdot\|_{H^k(a,b)}$ and semi-norm $|\cdot|_{H^k(a,b)}$:

$$\|v\|^2_{H^k(a,b)} = \sum_{j=0}^{k} \|v^{(j)}\|^2_{L^2(a,b)}, \qquad \text{with } \|v\|^2_{L^2(a,b)} = \int_a^b v^2(x)dx.$$

In addition, for integer $k \geq 0$, we define the following broken Sobolev space

$$H^k_h(0,1) = \left\{ v \in L^2(0,1) : \ v|_{(x_{i-1},x_i)} \in H^k(x_{i-1},x_i), \ i = 1,\cdots,N \right\}$$

and the corresponding norm

$$\|v\|^2_{H^k_h(0,1)} = \sum_{i=1}^{N} \|v\|^2_{H^k(x_{i-1},x_i)}.$$

Multiplying the differential equation in (2.1) by a function $v \in H^2_h(0,1)$ and integrating over $(0,1)$, we have

$$\sum_{i=1}^{N} \left[\int_{x_{i-1}}^{x_i} p(x)u'(x)v'(x)dx - p(x_i)u'(x_i)v^-(x_i) + p(x_{i-1})u'(x_{i-1})v^+(x_{i-1}) \right] = \int_0^1 fvdx, \tag{2.3}$$

where $v^+(x_i) = \lim_{x\to x_i+} v(x)$ and $v^-(x_i) = \lim_{x\to x_i-} v(x)$. Introduce notation for jump and average operators:

$$[v(x_i)] = v^+(x_i) - v^-(x_i), \quad i = 1,\cdots,N-1,$$
$$\{x_i\} = \frac{1}{2}\left(v^+(x_i) + v^-(x_i)\right), \quad i = 1,\cdots,N-1$$

and

$$[v(x_0)] = v(x_0), \quad \{v(x_0)\} = v(x_0).$$
$$[v(x_N)] = -v(x_N), \quad \{v(x_N)\} = v(x_N).$$

Then, by a simple manipulation and the continuity of $u'(x)$, equation (2.3) can be re-written as

$$\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} p(x)u'(x)v'(x)dx + \sum_{x_i\in\Gamma_{int}\cup\Gamma_D} \{p(x_i)u'(x_i)\}[v(x_i)] = -\sum_{x_i\in\Gamma_N} p(x_i)u_n(x_i)[v(x_i)] + \int_0^1 fvdx. \tag{2.4}$$

Define the following bilinear forms and linear forms

$$D(u,v) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} p(x)u'(x)v'(x)dx, \tag{2.5}$$

$$J_1(u,v) = \sum_{x_i\in\Gamma_{int}\cup\Gamma_D} \{p(x_i)u'(x_i)\}[v(x_i)], \tag{2.6}$$

$$J_{2,\alpha}(u,v) = \sum_{x_i\in\Gamma_{int}\cup\Gamma_D} \frac{1}{\bar{h}_i^{1+2\alpha_i}}[u(x_i)][v(x_i)], \tag{2.7}$$

$$\mathcal{F}_N(v) = -\sum_{x_i\in\Gamma_N} p(x_i)u_n(x_i)[v(x_i)], \quad \mathcal{F}_0(v) = \int_0^1 fvdx. \tag{2.8}$$

Here, in (2.7), $\bar{h}_0 = h_0$, $\bar{h}_N = h_N$ and for $0 < i < N$, $\bar{h}_i$ is the average of h_i and h_{i-1}. In $J_{2,\alpha}$, all exponents $\alpha_i \geq 0$ and $\alpha = \min\{\alpha_i : \ 0 \leq i \leq N\}$. With these bilinear forms, the equation (2.4) becomes

$$D(u,v) + J_1(u,v) = \mathcal{F}_N(v) + \mathcal{F}_0(v). \tag{2.9}$$

According to (2.9) and the continuity of the solution u, the solution of (2.1) satisfies the following weak formulation: For any $v \in H_h^2(0,1)$

$$D(u,v) + J_1(u,v) + \tau J_1(v,u) + \eta J_{2,\alpha}(u,v) = \mathcal{F}(v), \tag{2.10}$$

where $\tau = 1, -1$, $\eta \geq 0$ and

$$\mathcal{F}(v) = \tau \mathcal{F}_D(v) + \mathcal{F}_N(v) + \eta \mathcal{F}_J(v) + \mathcal{F}_0(v),$$

$$\mathcal{F}_J(v) = \sum_{x_i \in \Gamma_D} \frac{1}{h_i^{1+2\alpha_i}} u_d(x_i) v(x_i), \quad \mathcal{F}_D(v) = \sum_{x_i \in \Gamma_D} p(x_i)[u_d(x_i)] v'(x_i).$$

To define the finite element approximation to (2.10), let $k > 0$ be a fixed integer and let S^h denote the finite element space which consists of discontinuous piecewise polynomials of degree less than or equal to k:

$$S^h = \left\{ v \in L^2(0,1) : v|_{(x_{i-1},x_i)} \in \mathcal{P}_k, i = 1, \cdots, N \right\}.$$

Here, $\mathcal{P}_k$ denotes the set of all polynomials of degree $\leq k$.

Different approaches have been used to define the discontinuous Galerkin approximation u_h for problem (2.1). Many of them can be included in the following formula: Find $u_h \in S^h$ such that for any $v \in S^h$

$$D(u,v) + J_1(u,v) + \tau J_1(v,u) + \eta J_{2,\alpha}(u,v) = \mathcal{F}(v). \tag{2.11}$$

Based on the weak formulation (2.10) for the continuous problem, we see that (2.11) is consistent with (2.1).

We now define the standard Lagrange interpolation operator $\pi_h : \ H_h^2(0,1) \to S^h$. For each $i \in \{1, 2, \cdots, N\}$, let

$$F_i(t) = \frac{x_{i-1} + x_i}{2} + \frac{x_i - x_{i-1}}{2} t.$$

Then, F_i maps the interval $[-1,1]$ to $[x_{i-1}, x_i]$. Let

$$-1 = t_0^{(k)} < t_1^{(k)} < \cdots < t_{k-1}^{(k)} < t_k^{(k)} = 1$$

be the Lobatto points on $[-1,1]$, i.e., the $k+1$ zeros of the polynomial

$$\phi_{k+1}(t) = \frac{d^{k-1}}{dt^{k-1}} (t^2 - 1)^k.$$

Let $\Pi w \in \mathcal{P}_k$ denote the standard Lagrange interpolation of a continuous function w defined on $[-1,1]$ by using the Lobatto points $\{t_j^{(k)}\}_{j=0}^k$ as node points. Then we have

$$(\Pi w - w)(t) = \frac{w^{(k+1)}(t)}{(2k)!} \phi_{k+1}(t) + O(1) \int_{-1}^{1} |w^{(k+2)}(t)| dt \tag{2.12}$$

and

$$(\Pi w - w)'(t) = \frac{w^{(k+1)}(t)}{(2k)!} \psi_k(t) + O(1) \int_{-1}^{1} |w^{(k+2)}(t)| dt \tag{2.13}$$

Here, $\psi_k(t) = \dfrac{d^k}{dt^k}(t^2 - 1)^k$ is a multiple of the kth Legendre polynomial on $[-1,1]$. Thus, the zeros of $\psi_k(t)$ are the Gauss points.

For $i \in \{1, 2, \cdots, N\}$, let $x_j^{(k)} = F_i(t_j^{(k)})$, and $\pi_h u$ denote the piecewise Lagrange interpolation using $\{x_j^{(k)}\}_{j=0}^k$ as node points on each interval $[x_{i-1}, x_i]$. Then, from (2.12) and (2.13), we have for $x \in [x_{i-1}, x_i]$

$$(2.14) \qquad (\pi_h u - u)(x) = \frac{u^{(k+1)}(x)}{(2k)!}\Phi_{i,k+1}(x) + O(h_i^{k+1}) \int_{x_{i-1}}^{x_i} |u^{(k+2)}(x)| dx$$

and

$$(2.15) \qquad (\pi_h u - u)'(x) = \frac{u^{(k+1)}(x)}{(2k)!}\Psi_{i,k}(x) + O(h_i^{k}) \int_{x_{i-1}}^{x_i} |u^{(k+2)}(x)| dx,$$

where

$$\Phi_{i,k+1}(x) = \frac{d^{k-1}}{dx^{k-1}}\left((x - x_{i-\frac{1}{2}})^2 - \frac{h_i^2}{4}\right)^k, \quad \Psi_{i,k}(x) = \Phi'_{i,k+1}(x).$$

It is well known that we have the following error estimates for the interpolation error $u - \pi_h u$: for any $u \in H_h^{k+1}(0,1)$

$$(2.16) \qquad \|u - \pi_h u\|_{H_h^j(0,1)} \le Ch^{k+1-j}|u|_{H_h^{k+1}(0,1)}, \quad 0 \le j \le k+1.$$

3. DG methods with Interior Penalty

3.1. Main results. In this section, we consider two discontinuous Galerkin methods with interior penalty: the symmetric interior penalty Galerkin (SIPG) method and the non-symmetric interior penalty Galerkin (NIPG) method, which are obtained from (2.11) by choosing $\eta > 0$. The bilinear form for these methods is

$$B_{IPG}(u,v) = D(u,v) + J_1(u,v) + \tau J_1(v,u) + \eta J_{2,\alpha}(u,v)$$

and the discrete solution $u_h \in S^h$ is sought to satisfy

$$(3.1) \qquad B_{IPG}(u_h, v) = \tau \mathcal{F}_D(v) + \mathcal{F}_N(v) + \eta \mathcal{F}_J(v) + \mathcal{F}_0(v), \quad \text{for any } v \in S^h.$$

The method is consistent and we have the error equation

$$(3.2) \qquad B_{IPG}(u - u_h, v) = 0, \quad \text{for all } v \in S^h.$$

The SIPG method ($\tau = 1$) was introduced and studied by Arnold [1], Douglas and Dupont [15] and Wheeler [29]. The NIPG method ($\tau = -1$) was considered and analyzed by Rivière, Wheeler and Girault [26], Süli, Schwab and Houston [27] and [17]. The resulting linear algebraic system from SIPG is symmetric. The NIPG method is not symmetric, but it is more stable than the SIPG method.

To state the error estimates for these two methods, we define the following mesh-dependent norm: for any $v \in H_h^1(0,1)$

$$(3.3) \qquad \|v\|_{IPG}^2 = |v|_{H_h^1(0,1)}^2 + \sum_{x_i \in \Gamma_{int} \cup \Gamma_D} \bar{h}_i^{-1-2\alpha_i}[v(x_i)]^2.$$

To be convenient, we also introduce the space $H_D^1(0,1)$ which is defined by

$$H_D^1(0,1) = \{u \in H^1(0,1) : \ u(x) = 0 \text{ if } x \in \Gamma_D\}.$$

Then, it is clear that

$$(3.4) \qquad J_1(v,u) = 0, \quad \forall\ u \in H_D^1(0,1) \text{ and } \ v \in H_h^1(0,1) \cap H_h^2(0,1).$$

The optimal error estimate in H^1 and L^2 norms are obtained for SIPG method (see Arnold [1], Prudhomme, Pascal, Oden and Romkes [24] or Arnold, Brezzi, Cockburn and Martin [2]):

$$(3.5) \qquad \|u - u_h\|_{L^2(0,1)} + h\|u - u_h\|_{H_h^1(0,1)} \le Ch^{k+1}\|u\|_{H^{k+1}(0,1)}.$$

For NIPG method, optimal error estimate in H^1 semi norm has been obtained (see Rivière, Wheeler and Girault [26]):

$$|u - u_h|_{H_h^1(0,1)} \leq Ch^k \|u\|_{H^{k+1}(0,1)}. \tag{3.6}$$

The following sub-optimal error estimate in L^2 norm for NIPG method can also be found in [26]:

$$\|u - u_h\|_{L^2(0,1)} \leq Ch^{\min(k+1,k+\alpha)} \|u\|_{H^{k+1}(0,1)}. \tag{3.7}$$

We are concerned with the superconvergence property for both the error $u - u_h$ and the derivative of the error.

To state our superconvergence results, we define the following discrete semi-norm over Gauss points:

$$\|v\|_{G(k)}^2 = \sum_{i=1}^{N} \sum_{j=1}^{k} h_i |v(x_{i,j})|^2,$$

where $\{x_{i,j},\ j = 1, \cdots, k\}$ denotes the set of all Gauss points on the interval $[x_{i-1}, x_i]$. Then, it is easy to see that $\|v\|_{G(k)}$ is a semi-norm in space S^h. Moreover, there is a constant $C > 0$ such that

$$\|v'\|_{G(k)} \leq C \|v'\|_{L^2(0,1)}, \quad \forall\ v \in S^h. \tag{3.8}$$

Our main results are stated in the following theorems: Theorem 3.1 and 3.2.

Theorem 3.1. *(SIPG and NIPG) Let* $u \in H^{k+2}(0,1)$ *is the solution of (2.1) and* $u_h \in S^h$ *is the solution of (3.1). Then we have*

$$\|(u - u_h)'\|_{G(k)} \leq Ch^{\min(k+1,k+\alpha)} \|u\|_{H^{k+2}(0,1)}. \tag{3.9}$$

Furthermore, if the mesh is uniform and k *is odd, then we have*

$$\|(u - u_h)'\|_{G(k)} \leq Ch^{\min(k+1,k+\frac{1}{2}+\alpha_D)} \|u\|_{H^{k+2}(0,1)}, \tag{3.10}$$

where $\alpha_D = \min\{\alpha_i :\ x_i \in \Gamma_D\}$.

The results in Theorem 3.1 indicate a convergence rate higher than the optimal rate h^k for $(u-u_h)'$ at gauss points of the elements for any $k \geq 1$ and quasi-uniform meshes if $\alpha > 0$. The best superconvergence is achieved when $\alpha = 1$. With the minimum penalty ($\alpha = 0$) we still have superconvergence for both SIPG and NIPG methods when the partition is uniform and k is odd. Our numerical tests show that no superconvergence is observed when k is even and $\alpha = 0$.

Our next theorem establishes superconvergence for the error $u - u_h$ at interior nodes for the SIPG and NIPG methods.

Theorem 3.2. *(SIPG and NIPG) Let* $u \in H^{k+1}(0,1)$ *is the solution of (2.1) and* $u_h \in S^h$ *is the solution of (3.1). If* $\tau = 1$, *we have*

$$\max_{x_i \in \Gamma_{int}} |\{(u - u_h)(x_i)\}| \leq Ch^{2k} \|u\|_{H^{k+1}(0,1)}. \tag{3.11}$$

If $\tau = -1$, *we have*

$$\max_{x_i \in \Gamma_{int}} |\{(u - u_h)(x_i)\}| \leq Ch^{\min(2k,k+\alpha)} \|u\|_{H^{k+1}(0,1)}. \tag{3.12}$$

We can see from Theorem 3.2 that the averages of $u - u_h$ at the interior node points are superconvergent when $k > 1$ for any quasi-uniform meshes. However, (3.11) does not give a superconvergent estimate when $k = 1$.

As we can see from (3.7), the error estimate in L^2 norm for the NIPG method is sub-optimal when $\alpha = 0$. As a result of the superconvergence property for the derivative of the error with uniform partitions and odd k, we have the following improved error estimate in L^2 norm for the NIPG method:

Corollary 3.1. *(NIPG) Let $u \in H^{k+2}(0,1)$ is the solution of (2.1) and $u_h \in S^h$ is the solution of (3.1) with $\tau = -1$. If the partition is uniform and k is odd, then we have*

$$\|u - u_h\|_{L^2(0,1)} \le Ch^{\min(k+1,k+\frac{1}{2}+\alpha_D)}\|u\|_{H^{k+2}(0,1)}. \tag{3.13}$$

The rest of this subsection is devoted to the proofs of the above two theorems and the corollary. In these proofs, the results from the next subsection will be used.

Proof of Theorem 3.1. Consider the decomposition

$$u - u_h = (u - \pi_h u) + (\pi_h u - u_h). \tag{3.14}$$

For the first term in the decomposition (3.14), let us recall the identity (2.15). The leading term of (2.15) vanishes at the Gauss points $x = x_{i,j}$. Hence, we have

$$|(u - \pi_h u)'(x_{i,j})| \le Ch_i^k \int_{x_{i-1}}^{x_i} |u^{(k+2)}(x)|dx.$$

From the definition of norm $\|\cdot\|_{G(k)}$ and the Hölder's inequality, it follows

$$\begin{aligned} \|(u - \pi_h u)'\|^2_{G(k)} &\le C\sum_{i=1}^{N} h_i^{2k+1}\left(\int_{x_{i-1}}^{x_i} |u^{(k+2)}(x)|dx\right)^2 \\ &\le Ch^{2k+2}|u|^2_{H^{k+2}(0,1)}. \end{aligned} \tag{3.15}$$

By virtue of the coercivity (3.36) and the error equation (3.2), we have

$$\begin{aligned} C_0\|\pi_h u - u_h\|^2_{IPG} &\le B_{IPG}(\pi_h u - u_h, \pi_h u - u_h) \\ &= B_{IPG}(\pi_h u - u, \pi_h u - u_h). \end{aligned} \tag{3.16}$$

Now, we use a key result from the next subsection. An application of the estimate (3.41) yields

$$\|\pi_h u - u_h\|^2_{IPG} \le Ch^{\min(k+1,k+\alpha)}\|u\|_{H^{k+2}(0,1)}\|\pi_h u - u_h\|_{IPG},$$

which implies

$$\|\pi_h u - u_h\|_{IPG} \le Ch^{\min(k+1,k+\alpha)}\|u\|_{H^{k+2}(0,1)}. \tag{3.17}$$

Thus, it follows

$$\begin{aligned} \|(\pi_h u - u_h)'\|_{G(k)} &\le C\|(\pi_h u - u_h)'\|_{L^2(0,1)} \le C\|\pi_h u - u_h\|_{IPG} \\ &\le Ch^{\min(k+1,k+\alpha)}\|u\|_{H^{k+2}(0,1)}. \end{aligned} \tag{3.18}$$

The desired estimate (3.9) follows from (3.18), (3.15) and (3.14). To show (3.10), we assume $h_i = h$ for all $0 \le i \le N-1$ and k is odd. we apply (3.47) in (3.16) to obtain

$$\|\pi_h u - u_h\|_{IPG} \le Ch^{\min(k+1,k+\frac{1}{2}+\alpha_D)}\|u\|_{H^{k+2}(0,1)}. \tag{3.19}$$

This yields

$$\|(\pi_h u - u_h)'\|_{G(k)} \le Ch^{\min(k+1,k+\frac{1}{2}+\alpha_D)}\|u\|_{H^{k+2}(0,1)}. \tag{3.20}$$

(3.20) and (3.15) prove (3.10). The proof of Theorem 3.1 is therefore complete. □

Proof of Theorem 3.2. We shall employ the Green's function for problem (2.1). For simplicity, we assume that $\Gamma_D = \{0\}$, $\Gamma_N = \{1\}$. For each $x_i \in \Gamma_{int}$, let $g_i(x)$ denote the Green's function of (2.1) at x_i satisfying the homogeneous boundary conditions: $g_i(0) = g_i'(1) = 0$. More precisely, $g_i(x)$ is given by

$$g_i(x) = \begin{cases} \displaystyle\int_0^x \frac{dt}{p(t)} & \text{if } 0 \le x \le x_i, \\ \displaystyle\int_0^{x_i} \frac{dt}{p(t)} & \text{if } x_i \le x \le 1. \end{cases}$$

We note that g_i is continuous in $(0,1)$ and is smooth in $(0, x_i)$ and $(x_i, 1)$ so that $\|g_i\|_{H_h^{k+1}(0,1)}$ is finite. A straightforward calculation shows

$$D(g_i, v) + J_1(g_i, v) = \{v(x_i)\}, \qquad \text{for any } v \in H_h^2(0,1). \tag{3.21}$$

Hence, by (3.21), the continuity of function $g_i(x)$, the error equation (3.2) and the symmetry of B_{IPG} when $\tau = 1$, we have

$$\begin{aligned} \{(u - u_h)(x_i)\} &= D(g_i, u - u_h) + J_1(g_i, u - u_h) \\ &= B_{IPG}(u - u_h, g_i - \pi_h g_i) + (1 - \tau) J_1(g_i, u - u_h). \end{aligned} \tag{3.22}$$

Using the inequality (3.34) and error estimate (2.16) for the interpolation, we obtain

$$\begin{aligned} & B_{IPG}(u - u_h, g_i - \pi_h g_i) \\ = \;& D(u - \pi_h u, g_i - \pi_h g_i) + B_{IPG}(g_i - \pi_h g_i, \pi_h u - u_h) \\ \le \;& C|u - \pi_h u|_{H^1(0,1)} |g_i - \pi_h g_i|_{H^1(0,1)} \\ & + C(|g_i - \pi_h g_i|_{H^1(0,1)} + |g_i - \pi_h g_i|_{H_h^2(0,1)}) \|\pi_h u - u_h\|_{IPG} \\ \le \;& C h^{2k} \|u\|_{H^{k+1}(0,1)} \|g_i\|_{H_h^{k+1}(0,1)} \\ \le \;& C h^{2k} \|u\|_{H^{k+1}(0,1)}. \end{aligned} \tag{3.23}$$

We note that, in (3.23) the following error estimate has been used:

$$\|\pi_h u - u_h\|_{IPG} \le C h^k \|u\|_{H^{k+1}(0,1)}, \tag{3.24}$$

which can be obtained from (3.16). In fact, using (3.34), (3.16) and then applying (2.16), we have

$$\begin{aligned} \|\pi_h u - u_h\|_{IPG}^2 &\le C(|u - \pi_h u|_{H^1(0,1)} + h|u - \pi_h u|_{H_h^2(0,1)}) \|\pi_h u - u_h\|_{IPG} \\ &\le C h^k \|u\|_{H^{k+1}(0,1)} \|\pi_h u - u_h\|_{IPG}, \end{aligned}$$

which yields (3.24). Now, we deduce (3.11) from (3.22) (with $\tau = 1$) and (3.23). To show (3.12), we must estimate the last term $2J_1(g_i, u - u_h)$ in (3.22) when $\tau = -1$. For this, we use (3.33) and (3.24) to get

$$\begin{aligned} 2J_1(g_i, u - u_h) &= 2J_1(g_i, \pi_h u - u_h) \\ &\le C h^{\alpha} (|g_i|_{H^1(0,1)} + h|g_i|_{H_h^2(0,1)}) \|\pi_h u - u_h\|_{IPG} \\ &\le C h^{k+\alpha} \|u\|_{H^{k+1}(0,1)}. \end{aligned} \tag{3.25}$$

Inserting (3.25) and (3.23) in (3.22), we obtain (3.12). The proof is complete. □

Now, we give a proof for Corollary 3.1.

Proof of Corollary 3.1. We shall employ a standard duality argument. To this end, let $w \in H_D^1(0,1) \cap H^2(0,1)$ be the solution of the problem

$$\begin{cases} -(p(x)w')' = u - u_h, & x \in (0,1) \\ w(x) = 0 \text{ if } x \in \Gamma_D, & w'(x) = 0 \text{ if } x \in \Gamma_N. \end{cases} \tag{3.26}$$

Then, the equation (2.9) is satisfied by w replacing u for $\mathcal{F}_N = 0$. Recall $\tau = -1$. Hence, for any $v \in H_h^2(0,1)$, it follows

$$\begin{aligned} B_{IPG}(v,w) &= D(v,w) + J_1(v,w) - J_1(w,v) + \eta J_{2,\alpha}(v,w) \\ &= D(w,v) - J_1(w,v) = \int_0^1 (u-u_h)v dx - 2J_1(w,v). \end{aligned} \tag{3.27}$$

Therefore, letting $v = u - u_h$ in (3.27), we obtain

$$\|u-u_h\|^2_{L^2(0,1)} = B_{IPG}(u-u_h, w-\pi_h w) + 2J_1(w, u-u_h). \tag{3.28}$$

The first term on the right-hand side of (3.28) can be bounded in a similar way as (3.22). Thus,

$$B_{IPG}(u-u_h, w-\pi_h w) \le Ch^{k+1}\|u\|_{H^{k+1}(0,1)}\|w\|_{H^2(0,1)}. \tag{3.29}$$

To estimate the last term on (3.28), we decompose it into two terms:

$$J_1(w, u-u_h) = J_1(w-\pi_h w, u-u_h) + D(u-u_h, \pi_H w). \tag{3.30}$$

In (3.30), we have used the equation $D(u-u_h, \pi_h w) = J_1(\pi_h w, u-u_h)$ which is a direct result of the error equation (3.2). Since it is a part of $B_{IPG}(u-u_h, w-\pi_h w)$, the first term on the right hand side of (3.30) is bounded by the same upper bound in (3.29):

$$J_1(w-\pi_h w, u-u_h) \le Ch^{k+1}\|u\|_{H^{k+1}(0,1)}\|w\|_{H^2(0,1)}. \tag{3.31}$$

It remains to deal with the last term in (3.30). Applying the inequalities (3.40) and (3.19), we have

$$\begin{aligned} D(u-u_h, \pi_h w) &= D(u-\pi_h u, \pi_h w) + D(\pi_h u - u_h, \pi_h w) \\ &\le Ch^{k+1}\|u\|_{H^{k+2}(0,1)}|\pi_h w|_{H^1(0,1)} \\ &\quad + Ch^{\min(k+1, k+\frac{1}{2}+\alpha_D)}\|u\|_{H^{k+2}(0,1)}|\pi_h w|_{H^1(0,1)} \\ &\le Ch^{\min(k+1, k+\frac{1}{2}+\alpha_D)}\|u\|_{H^{k+2}(0,1)}\|w\|_{H^1(0,1)}. \end{aligned} \tag{3.32}$$

Finally, taking into account of all the above estimates (3.29), (3.30), (3.31) and (3.32), we deduce the desired estimate (3.13). □

3.2. Auxiliary lemmas. In this subsection, we show the results that have been used in the proofs of the main results in the preceding subsection.

Lemma 3.1. *For any* $u \in H_h^2(0,1)$ *and* $v \in H_h^1(0,1)$*, we have*

$$|J_1(u,v)| \le Ch^\alpha(|u|_{H_h^1(0,1)} + h|u|_{H_h^2(0,1)})\|v\|_{IPG}. \tag{3.33}$$

If $u \in H_D^1(0,1) \cap H_h^2(0,1)$*, we have*

$$|B_{IPG}(u,v)| + |B_{IPG}(v,u)| \le C(\|u\|_{H^1(0,1)} + h\|u\|_{H_h^2(0,1)})\|v\|_{IPG}. \tag{3.34}$$

Proof. By a trace theorem, we have

$$\sum_{x_i \in \Gamma_{int} \cup \Gamma_D} \bar{h}_i^{1+2\alpha}|\{u'(x_i)\}|^2 \le Ch^{2\alpha}|u|^2_{H^1(0,1)} + Ch^{2+2\alpha}|u|^2_{H_h^2(0,1)}.$$

This and the Cauchy-Schwarz inequality leads to

$$\begin{aligned}
&|J_1(u,v)| \\
\le\ & C\left(\sum_{x_i\in\Gamma_{int}\cup\Gamma_D} \bar{h}_i^{1+2\alpha_i}\{u'(x_i)\}^2\right)^{\frac12}\left(\sum_{x_i\in\Gamma_{int}\cup\Gamma_D} \bar{h}_i^{-1-2\alpha_i}[v(x_i)]^2\right)^{\frac12} \\
\le\ & Ch^{\alpha}(|u|_{H^1(0,1)}+h|u|_{H_h^2(0,1)})\|v\|_{IPG},
\end{aligned}$$

which proves (3.33). By (3.4), we have

$$\begin{aligned}
B_{IPG}(u,v) &= D(u,v)+J_1(u,v) \\
&\le C|u|_{H_h^1(0,1)}|v|_{H_h^1(0,1)}+|J_1(u,v)|.
\end{aligned} \tag{3.35}$$

By (3.35) and (3.33), we have

$$|B_{IPG}(u,v)|\le C(\|u\|_{H^1(0,1)}+h\|u\|_{H_h^2(0,1)})\|v\|_{IPG}.$$

The same estimate for $B_{IPG}(v,u)$ can be proved similarly. Therefore, we complete the proof. □

Next lemma provides the necessary coercivity of the bilinear form $B_{IPG}(\cdot,\cdot)$.

Lemma 3.2. *If $\tau=-1$, then there is a constant $C_0>0$ such that for any $v\in S^h$, there holds*

$$B_{IPG}(v,v)\ge C_0\|v\|_{IPG}^2. \tag{3.36}$$

If $\eta>0$ is chosen sufficiently large, then (3.36) is also true for $\tau=1$.

Proof. If $\tau=-1$, then for any $v\in S^h$,

$$B_{IPG}(v,v)=D(v,v)+\eta J_{2,0}(v,v)\ge\min(p_0,\eta)\|v\|_{IPG}^2.$$

This proves (3.36) when $\tau=-1$. Now for $\tau=1$, we observe that

$$\begin{aligned}
B_{IPG}(v,v) &= D(v,v)+2J_1(v,v)+\eta J_{2,\alpha}(v,v) \\
&\ge p_0|v|_{H_h^1(0,1)}^2+2J_1(v,v)+\eta J_{2,\alpha}(v,v).
\end{aligned} \tag{3.37}$$

By Cauchy-Schwarz inequality and the inverse inequality

$$|\{v'(x_i)\}|^2\le\|v'\|_{L^\infty(x_{i-1},x_{i+1})}^2\le C\bar{h}_i^{-1}|v|_{H_h^1(x_{i-1},x_{i+1})}^2, \tag{3.38}$$

we have for any $\epsilon>0$

$$\begin{aligned}
J_1(v,v) &\le \frac{C}{\epsilon}\sum_{x_i\in\Gamma_{int}\cup\Gamma_D}\bar{h}_i^{-1-2\alpha_i}[v(x_i)]^2+\epsilon\sum_{x_i\in\Gamma_{int}\cup\Gamma_D}\bar{h}_i^{1+2\alpha_i}\{v'(x_i)\}^2 \\
&\le \frac{C}{\epsilon}J_{2,0}(v,v)+Ch^{2\alpha}\epsilon|v|_{H_h^1(0,1)}^2.
\end{aligned}$$

From this and (3.37), we deduce that

$$B_{IPG}(v,v)\ \ge\ (p_0-2C\epsilon h^{2\alpha})|v|_{H_h^1(0,1)}^2+(\eta-\frac{2C}{\epsilon})J_{2,\alpha}(v,v). \tag{3.39}$$

Let ϵ be sufficiently small and η sufficiently large so that

$$p_0-2C\epsilon h^{2\alpha}>\frac{p_0}{2},\quad \eta-\frac{C}{\epsilon}>\frac{p_0}{2},$$

then we arrive at the desired estimate (3.36) from (3.39) with $C_0=\dfrac{p_0}{2}$. □

In the next two lemmas, we provide some key estimates used in the proof of Theorem 3.1.

Lemma 3.3. *For any $u \in H^{k+2}(0,1)$ and any $v \in S^h$, we have*

$$|D(\pi_h u - u, v)| \le Ch^{k+1}\|u\|_{H^{k+2}(0,1)}|v|_{H^1(0,1)} \tag{3.40}$$

and

$$B_{IPG}(\pi_h u - u, v) \le Ch^{\min(k+1,k+\alpha)}\|u\|_{H^{k+2}(0,1)}\|v\|_{IPG}. \tag{3.41}$$

Proof. Let $x_{i-1/2}$ denote the middle point of the interval $[x_{i-1}, x_i]$. Applying (2.15), we have

$$\begin{aligned}
D(\pi_h u - u, v) &= \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} p(x)(\pi_h u - u)'v'dx \\
&= \frac{1}{(2k)!}\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} p(x_{i-\frac{1}{2}})u^{(k+2)}(x_{i-\frac{1}{2}})\Psi_{i,k}(x)v'dx \\
&\quad + \sum_{i=0}^{N-1}\int_{x_{i-1}}^{x_i}\left(p(x)u^{(k+1)}(x) - p(x_{i-\frac{1}{2}})u^{(k+1)}(x_{i-\frac{1}{2}})\right)\Psi_{i,k}(x)v'dx \\
&\quad + O(h^{k+1})\|u\|_{H^{k+2}(0,1)}|v|_{H_h^1(0,1)}.
\end{aligned} \tag{3.42}$$

In view of $\int_{x_{i-1}}^{x_i}\Psi_{i,k}(x)q(x)dx = 0$ for any $q \in \mathcal{P}_{k-1}$, the first term on the right hand side of (3.42) vanishes:

$$\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} p(x_{i-\frac{1}{2}})u^{(k+2)}(x_{i-\frac{1}{2}})\Psi_{i,k}(x)v'dx = 0. \tag{3.43}$$

By using the inequality

$$|p(x)u^{(k+1)}(x) - p(x_{i-\frac{1}{2}})u^{(k+1)}(x_{i-\frac{1}{2}})| \le Ch^{\frac{1}{2}}\|u\|_{H^{k+2}(x_{i-1},x_i)},$$

the second term on the right hand side of (3.42) can be estimated as follows:

$$\begin{aligned}
&\left|\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(p(x)u^{(k+1)}(x) - p(x_{i-\frac{1}{2}})u^{(k+1)}(x_{i-\frac{1}{2}}))\Psi_{i,k}(x)v'dx\right| \\
&\quad \le Ch^{k+1}\|u\|_{H^{k+2}(0,1)}|v|_{H^1(0,1)}.
\end{aligned} \tag{3.44}$$

Consequently, (3.42), (3.43) and (3.44) yield (3.40). For (3.41), we note that $u - \pi_h u \in H_D^1(0,1)$ and obtain

$$B_{IPG}(\pi_h - u, v) = D(\pi_h u - u, v) + J_1(\pi_h u - u, v). \tag{3.45}$$

By (3.40), it suffices to estimate the second term on the right-hand side of (3.45). We use the inequality (3.33) and the error estimate (2.16) for the interpolation: to obtain

$$\begin{aligned}
|J_1(\pi_h u - u, v)| &\le Ch^{\alpha}\left(|\pi_h u - u|_{H^1(0,1)} + h|\pi_h u - u|_{H_h^2(0,1)}\right)\|v\|_{IPG} \\
&\le Ch^{k+\alpha}\|u\|_{H^{k+1}(0,1)}\|v\|_{IPG}.
\end{aligned} \tag{3.46}$$

Inserting (3.40) and (3.46) in (3.45), we complete the proof.

When the partition is uniform, we shall establish the following result.

Lemma 3.4. *If the finite element partition is uniform and k is odd, then for any $u \in H^{k+2}(0,1)$ and $v \in S^h$, we have*

$$B_{IPG}(\pi_h u - u, v) \le Ch^{\min(k+1,k+\frac{1}{2}+\alpha_D)}\|u\|_{H^{k+2}(0,1)}\|v\|_{IPG}, \tag{3.47}$$

where $\alpha_D = \min\{\alpha_i : x_i \in \Gamma_D\}$.

Proof. Let us recall (3.45) in the proof of previous lemma. The first term is still estimated by (3.40). It remains only to re-estimate the second term $J_1(\pi_h u - u, v)$ in (3.45). To this end, we shall first show

$$|\{(\pi_h u - u)'(x_i)\}| \le Ch^{k+\frac{1}{2}}|u|_{H^{k+2}(x_{i-1},x_{i+1})}, \qquad \text{for any } x_i \in \Gamma_{int}. \tag{3.48}$$

Applying (2.15) and the fact

$$\Psi_{i+1,k}(x_i) = -\Psi_{i,k}(x_i), \text{ if } k \text{ is odd and } \; x_{i+1} - x_i = x_i - x_{i-1},$$

we have

$$(\pi_h u - u)^+(x_i) = -\frac{u^{(k+1)}(x_i)}{(2k)!}\Psi_{i,k}(x_i) + O(h_{i+1}^k)\int_{x_i}^{x_{i+1}} |u^{(k+2)}(x)|dx,$$

$$(\pi_h u - u)^-(x_i) = \frac{u^{(k+1)}(x_i)}{(2k)!}\Psi_{i,k}(x_i) + O(h_i^k)\int_{x_{i-1}}^{x_i} |u^{(k+2)}(x)|dx.$$

Hence, if $x_i \in \Gamma_{int}$

$$\{(\pi_h u - u)(x_i)\} = O(h^k)\int_{x_{i-1}}^{x_{i+1}} |u^{(k+2)}(x)|dx,$$

which results in (3.48) by Cauchy-Schwarz inequality. As a result of (3.48), we have

$$\begin{aligned}
|J_1(\pi_h u - u, v)| \le & \sum_{x_i \in \Gamma_D} |p(x_i)(\pi_h u - u)'(x_i)v(x_i)| \\
& + C\left(\sum_{x_i\in\Gamma_{int}} \bar{h}_i^{1+2\alpha_i}\{(\pi_h u - u)'(x_i)\}^2\right)^{\frac{1}{2}} \left(\sum_{x_i\in\Gamma_{int}} \bar{h}_i^{-1-2\alpha_i}[v(x_i)]^2\right)^{\frac{1}{2}} \\
\le \quad & Ch^{k+\frac{1}{2}+\alpha_D}|u^{(k+1)}|_{L^\infty(0,1)}\left(\sum_{x_i\in\Gamma_D} h_i^{-1-2\alpha_i}|v(x_i)|^2\right)^{\frac{1}{2}} \\
& + Ch^{k+1+\alpha}\|u\|_{H^{k+2}(0,1)}\left(\sum_{x_i\in\Gamma_{int}} \bar{h}_i^{-1-2\alpha_i}[v(x_i)]^2\right)^{1/2} \\
\le \quad & Ch^{\min(k+1,k+\frac{1}{2}+\alpha_D)}\|u\|_{H^{k+2}(0,1)}\left(\sum_{x_i\in\Gamma_{int}\cup\Gamma_D} \bar{h}_i^{-1}[v(x_i)]^2\right)^{1/2}.
\end{aligned} \tag{3.49}$$

Using inequalities (3.40) and (3.49) in (3.45), we deduce (3.47). This completes the proof. □

Remark. The following fact can be observed from the proof of Lemma 3.4. If $\Gamma_D = \emptyset$, then we have the following estimate

$$B_{IPG}(\pi_h u - u, v) \le Ch^{k+1}\|u\|_{H^{k+2}(0,1)}\|v\|_{IPG}.$$

This means that when no Dirichlet boundary conditions exist for the problem (i.e., pure Neumann boundary condition), a better superconvergence estimate can be proved. However, a reaction term should be included in the differential equation to ensure that the solution of (2.1) is unique and $\|\cdot\|_{IPG}$ is a norm.

4. A DG method without penalty

In this section, we discuss the convergence and superconvergence properties of a discontinuous Galerkin method without the penalty terms. This method is obtained from (2.11) by choosing $\tau = -1$ and $\eta = 0$. The bilinear form for this method is

$$B_{DG}(u,v) = D(u,v) + J_1(u,v) - J_1(v,u)$$

and the discrete solution $u_h \in S^h$ is sought to satisfy

$$B_{DG}(u_h, v) = -\mathcal{F}_D(v) + \mathcal{F}_N(v) + \mathcal{F}_0(v), \qquad \text{for any } v \in S^h. \tag{4.1}$$

The method is consistent and we have the error equation

$$B_{DG}(u - u_h, v) = 0, \qquad \text{for all } v \in S^h. \tag{4.2}$$

This method was introduced and analyzed by Oden, Babuška and Baumann in [22] and [3], and by Rivière, Wheeler, and Girault [26].

For this method, we do not have the coercivity in the norm $\|\cdot\|_{IPG}$ but in H^1 semi-norm:

$$p_0 |v|^2_{H^1_h(0,1)} \le B_{DG}(v,v). \tag{4.3}$$

Because of this weak coercivity, the analysis used for SIPG or NIPG is not applicable for this method. In Babuška , Baumann and Oden [3], optimal error estimate in the norm $\|\cdot\|_{DG}$ defined by:

$$\|v\|^2_{DG} = |v|^2_{H^1_h(0,1)} + \sum_{x_i \in \Gamma_{int} \cup \Gamma_D} \bar{h}_i^{-1} [v(x_i)]^2 + \sum_{x_i \in \Gamma_{int} \cup \Gamma_D} \bar{h}_i \{v'(x_i)\}^2 \tag{4.4}$$

have been obtained for $k \ge 3$. Riviere, Wheeler and Girault in [26] proved optimal error estimates in H^1 semi-norm (without the L^2 norm) for $k \ge 2$ in two dimensional case. In this section, we shall derive error estimates in the full H^1 norm for all $k \ge 1$. We have not seen error estimates for $k = 1$ for this DG method in the literature. In fact, the numerical experiments reported in the literature indicated that this method does not converge if $k = 1$. According to our results, this method converges if it is not solving a pure Dirichlet problem. Moreover, for uniform partitions and odd $k \ge 3$, we also obtain superconvergence error estimate for the derivative of the error for this DG method. As a result of this superconvergence for the derivative of the error, the error estimate in L^2 has been improved.

Let us start with the optimal error estimates in H^1 norm.

Theorem 4.1. *Let $u \in H^{k+1}(0,1)$ is the solution of (2.1) and $u_h \in S^h$ is the solution of (4.1). Then we have for $k \ge 2$*

$$\|u - u_h\|_{H^1_h(0,1)} \le Ch^k \|u\|_{H^{k+1}(0,1)}. \tag{4.5}$$

Furthermore, if $\Gamma_N \neq \emptyset$, then (4.5) holds true for $k = 1$ too.

Proof. We first prove the following error estimate in the semi H^1 norm:

$$|u - u_h|_{H^1_h(0,1)} \le Ch^k \|u\|_{H^{k+1}(0,1)}. \tag{4.6}$$

Suppose $x_N \notin \Gamma_D$. Construct a new approximation u^I of u as follows: $u^I_i = u^I|_{(x_i, x_{i+1})}$ and

$$(u - u^I_0)(x) \;=\; (u - \pi_h u)(x) - (u - \pi_h u)'(x_0)(x - x_0);$$

and for any $1 < i \le N - 1$,

$$(u - u^I_i)(x) \;=\; (u - \pi_h u)(x) - \big((u - \pi_h u)'(x_i) + (u - u^I_{i-1})'(x_i)\big)(x - x_i).$$

Then, we can easily see that

$$\{(u - u^I)'(x_i)\} = 0, \quad \text{for any } \; 0 \le i \le N-1. \tag{4.7}$$

Thus, for any $v \in H_h^1(0,1)$

$$J_1(u^I - u, v) = 0, \quad \text{if } \; x_N \notin \Gamma_D. \tag{4.8}$$

From (4.3), (4.2) and (4.8), we deduce that

$$\begin{aligned} p_0 |u^I - u_h|^2_{H_h^1(0,1)} &\le B_{DG}(u^I - u_h, u^I - u_h) = B_{DG}(u^I - u, u^I - u_h) \\ &= D(u^I - u, u^I - u_h) - J_1(u^I - u_h, u^I - u). \end{aligned} \tag{4.9}$$

By the construction of the interpolation u^I, it is easy to show that

$$\|u - u^I\|_{L^2(0,1)} + h\|u - u^I\|_{H_h^1(0,1)} \le Ch^{k+1}|u|_{H^{k+1}(0,1)}. \tag{4.10}$$

By a trace theorem and the inverse property of space S^h, we have

$$J_1(u^I - u_h, u^I - u) \le Ch^{-1}((\|u - u^I\|_{L^2(0,1)} + h\|u - u^I\|_{H_h^1(0,1)})|u^I - u_h|_{H_h^1(0,1)}.$$

Inserting this in (4.9) and using Cauchy-Schwarz inequality for $D(u^I - u, u^I - u_h)$ in (4.9) followed by the application of (4.10), we deduce that

$$|u^I - u_h|^2_{H_h^1(0,1)} \le Ch^k |u|_{H^{k+1}(0,1)} |u^I - u_h|_{H_h^1(0,1)}, \tag{4.11}$$

which proves (4.6) when $x_N \notin \Gamma_D$. Similarly, we can show (4.6) when $x_0 \notin \Gamma_D$ by constructing u^I starting at x_N instead x_0. Therefore we prove (4.6) if $\Gamma_N \neq \emptyset$ for any $k \ge 1$. Next, we prove (4.6) if $\Gamma_N = \emptyset$ and $k \ge 2$. To this end, we can modify the definition of u^I in the last interval (x_{N-1}, x_N) so that

$$\begin{aligned} (u - u^I_{N-1})(x) &= (u - \pi_h u)(x) - \frac{(u - \pi_h u)'(x_N)}{2(x_N - x_{N-1})}(x - x_{N-1})^2 \\ &\quad - \frac{\big((u - \pi_h u)'(x_{N-1}) + (u - u^I_{N-2})'(x_{N-1})\big)}{2(x_N - x_{N-1})}(x - x_N)^2. \end{aligned}$$

A direct calculation shows that $\{(u - u^I)(x_i)\} = 0$ for any $0 \le i \le N$. Namely, (4.8) holds true even if $x_N \in \Gamma_D$. Therefore, (4.6) holds true no matter whether $x_N \in \Gamma_D$ or not if $k \ge 2$. After we have proved the error estimate (4.6) in the semi-H^1 norm, we are now in a position to show error estimate in the L^2 norm. To this end, let $w \in H_D^1(0,1) \cap H^2(0,1)$ be the solution of the problem (3.26). Then, for any $v \in H_h^2(0,1)$, it follows

$$\begin{aligned} B_{DG}(v, w) &= D(v, w) + J_1(v, w) - J_1(w, v) \\ &= D(w, v) - J_1(w, v) = \int_0^1 (u - u_h) v dx - 2J_1(w, v). \end{aligned}$$

Therefore, letting $v = u - u_h$ in (4.12), we obtain

$$\|u - u_h\|^2_{L^2(0,1)} = B_{DG}(u - u_h, w - w^I) + 2J_1(w, u - u_h). \tag{4.12}$$

Note that $J_1(w - w^I, v) = 0$ for any $v \in H_h^2(0,1)$. For the second term on the right-hand side of (4.12), we use the error equation (4.2) to obtain

$$\begin{aligned} J_1(w, u - u_h) &= J_1(w^I, u - u_h) = D(u - u_h, w^I) + J_1(u - u_h, w^I) \\ &= D(u - u_h, w^I - w) + J_1(u - u_h, w^I - w) \\ &\quad + D(u - u_h, w) \\ &= B_{DG}(u - u_h, w^I - w) + D(u - u_h, w). \end{aligned}$$

Hence, it follows

$$\|u-u_h\|^2_{L^2(0,1)} = D(u-u_h, w^I - w) + J_1(u^I - u_h, w^I - w) + 2D(u-u_h, w). \tag{4.13}$$

The fist two terms on the right-hand side of (4.13) can be estimated as follows:

$$\begin{aligned} & D(u-u_h, w^I - w) + J_1(u^I - u_h, w^I - w) \\ \leq\ & C|u-u_h|_{H^1_h(0,1)}|w-w^I|_{H^1_h(0,1)} \\ & + Ch^{-1}|u^I - u_h|_{H^1_h(0,1)}(\|w-w^I\|_{L^2(0,1)} + h|w-w^I|_{H^1_h(0,1)}) \\ \leq\ & Ch^{k+1}\|u\|_{H^{k+1}(0,1)}\|w\|_{H^2(0,1)}. \end{aligned} \tag{4.14}$$

Here, we have used the estimate (4.6). The last term in (4.13) is bounded by $C|u-u_h|_{H^1_h(0,1)}\|w\|_{H^1(0,1)}$. Thus, by (4.6), we have

$$D(u-u_h, w) \leq Ch^k\|u\|_{H^{k+1}(0,1)}\|w\|_{H^1(0,1)},$$

which, together with (4.13), (4.14) and the a priori regularity of w:

$$\|w\|_{H^2(0,1)} \leq C\|u-u_h\|_{L^2(0,1)}, \tag{4.15}$$

implies

$$\|u-u_h\|_{L^2(0,1)} \leq Ch^k\|u\|_{H^{k+1}(0,1)}. \tag{4.16}$$

Finally, (4.6) and (4.16) prove (4.5). The proof is complete. □

Let us make a remark here. According to the results in Theorem 4.1, the DG method (4.1) is uniquely solvable for any $k \geq 2$. The existence and uniqueness of DG method for $k = 1$ is guaranteed when a Neumann boundary condition exists.

We now investigate the superconvergence property for the DG method. The error estimate in L^2 norm obtained in Theorem 4.1 is sub-optimal. Improved error estimate in L^2 can be obtained from the superconvergence results for the derivative of the error. We shall use an inf-sup condition proved by Babuška , Baumann and Oden in [3]. Before we state this result, we recall the definition of the discrete norm $\|\cdot\|_{DG}$ given in (4.4).

Lemma 4.1. *Suppose $k \geq 3$. Then, there is a constant $\beta > 0$ such that for any $u \in S^h$, we have*

$$\beta\|u\|_{DG} \leq \sup_{v\in S^h} \frac{B_{DG}(u,v)}{\|v\|_{DG}}. \tag{4.17}$$

Proof. See Theorem 3.1 in [3]. Although the proof in [3] is for $p(x) = const$. The proof there can be easily modified for smooth $p(x)$. □

The main result of this subsection is stated in the following theorem.

Theorem 4.2. *Let $u \in H^{k+2}(0,1)$ is the solution of (2.1) and $u_h \in S^h$ is the solution of (4.1). If the partition is uniform, $k \geq 3$ is odd, we have*

$$\|u-u_h\|_{L^2(0,1)} + \|(u-u_h)'\|_{G(k)} \leq Ch^{k+\frac{1}{2}}\|u\|_{H^{k+2}(0,1)}. \tag{4.18}$$

Proof. We shall first prove

$$|\pi_h u - u_h|_{H^1_h(0,1)} \leq Ch^{k+\frac{1}{2}}\|u\|_{H^{k+2}(0,1)}. \tag{4.19}$$

In view of $B_{DG}(\pi_h u - u, v) = B_{SIPG}(\pi_h u - u, v)$ and using the result of Lemma 3.4 for $\alpha_D = 0$, we have

$$B_{DG}(\pi_h u - u, v) \leq Ch^{k+\frac{1}{2}}\|u\|_{H^{k+2}(0,1)}\|v\|_{DG},$$

which implies

$$\sup_{v\in S^h} \frac{B_{DG}(\pi_h u - u_h, v)}{\|v\|_{DG}} \le Ch^{k+\frac{1}{2}}\|u\|_{H^{k+2}(0,1)}. \tag{4.20}$$

Applying (4.17) and using (4.20), we get

$$\|\pi_h u - u_h\|_{DG} \le \sup_{v\in S^h} \frac{B_{DG}(\pi_h u - u, v)}{\|v\|_{DG}} \le Ch^{k+\frac{1}{2}}\|u\|_{H^{k+2}(0,1)}. \tag{4.21}$$

Consequently, (4.19) follows (4.21). Now, using (3.15) and (4.21), we deduce that

$$\|(u-u_h)'\|_{G(k)} \le Ch^{k+\frac{1}{2}}\|u\|_{H^{k+2}(0,1)}. \tag{4.22}$$

The error estimate in the L^2 norm in (4.18) can be proved with the same procedure for the proof of Corollary 3.1. We omit the details. □

5. Numerical experiments

5.1. The test problem. In this section, we present the results of numerical experiments to support the theoretical results obtained in this paper. Let $u = e^x$. Then u is the solution of the following boundary value problem:

$$\begin{cases} -((1+x)u')' &= -(2+x)e^x \text{ in } (0,1) \\ u(x) &= e^x \text{ if } x \in \Gamma_D \\ u'(x) &= e^x \text{ if } x \in \Gamma_N \end{cases}. \tag{5.1}$$

Here $\Gamma_D = \{0\}$ and $\Gamma_N = \{1\}$. Numerical results are reported in all tables. In these tables, we show the errors and the convergence rates in L^2 norm, at interior nodes and at Gauss points, which are obtained by successive refinements of an initial grid. Let $e_h(x) = u(x) - u_h(x)$. The rates of convergence is computed by

$$\mathrm{R}_{L^2} = \frac{\ln(\|e_{2h}\|_{L^2(0,1)}/\|e_h\|_{L^2(0,1)})}{\ln 2},$$

$$\mathrm{R}_{G(k)} = \frac{\ln(\|(e_{2h})'\|_{G(k)}/\|(e_h)'\|_{G(k)})}{\ln 2}.$$

$$\mathrm{R}_{av} = \frac{\ln\left(\max\limits_{1\le i\le N-1}|\{(e_{2h})(x_i)\}|/\max\limits_{1\le i\le N-1}|\{(e_h)(x_i)\}|\right)}{\ln 2}.$$

5.2. The SIPG method. Tables S-1 to S-9 contain the results for the SIPG method. From Table S-1, we can see that with minimum penalty ($\alpha = \alpha_D = 0$) and $k = 1$ the convergence rate in L^2 norm is 2; the convergence rate for the averaged errors at interior nodes is also 2; but the derivatives of the errors at Gauss points is superconvergent with a rate = 1.5 (the optimal convergence rate in H^1 norm is 1). The superconvergent rate 1.5 can be further improved to 2 by additional penalty at the Dirichlet boundary condition ($\alpha_D = \frac{1}{2}$) as we can see from Table S-2.

Tables S-3 and S-4 contain the results for $k = 2$. While the optimal convergence rate in L^2 norm is 3, we have a superconvergent rate 4 at the interior nodes as it has been indicated in Theorem 3.2. In Table S-3 we can also see that there is no superconvergence at Gauss points for the derivative of the error if the penalty is the minimum. Table S-4 shows superconvergent rate 3 at Gauss points if super-penalty is applied at all node points.

The results for $k = 3$ and $k = 4$ are in Table S-5 to S-9. These results are similar to those for $k = 1$ and $k = 2$ and agree with Theorem 3.1 and 3.2 in this paper.

The missing errors or rates in some tables are because they exceed the accuracy of our computer program.

Table S-1. Errors and Rates for SIPG with $k = 1, \eta = 5, \alpha = 0, \alpha_D = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	4.282(−3)	2.47	8.959(−3)	2.20	2.907(−2)	1.81
8	9.080(−4)	2.24	2.100(−3)	2.09	8.556(−3)	1.76
16	2.120(−4)	2.10	5.095(−4)	2.04	2.628(−3)	1.70
32	5.161(−5)	2.04	1.256(−4)	2.02	8.433(−4)	1.64
64	1.277(−5)	2.02	3.117(−5)	2.01	2.609(−4)	1.59
128	3.178(−6)	2.01	7.764(−6)	2.01	9.603(−5)	1.55
256	7.928(−7)	2.00	1.938(−6)	2.00	3.335(−5)	1.53
512	1.981(−7)	2.00	4.840(−7)	2.00	1.168(−5)	1.51
1024	4.962(−8)	2.00	1.210(−7)	2.00	4.110(−6)	1.51
2048	1.250(−8)	2.00	3.023(−8)	2.00	1.450(−6)	1.50

Table S-2. SIPG with $k = 1, \eta = 5, \alpha = 0, \alpha_D = 0.5$

$\frac{1}{h}$	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	4.142(−3)	1.11	1.968(−2)	1.96
8	1.250(−3)	1.73	5.198(−3)	1.92
16	3.392(−4)	1.88	1.352(−3)	1.94
32	8.816(−5)	1.94	3.457(−4)	1.97
64	2.243(−5)	1.97	8.746(−5)	1.98
128	5.670(−6)	1.99	2.200(−5)	1.99
256	1.424(−6)	1.99	5.517(−6)	2.00
512	3.570(−7)	2.00	1.382(−6)	2.00
1024	8.951(−8)	2.00	3.458(−7)	2.00
2048	2.252(−8)	1.99	8.654(−8)	2.00

Table S-3. Errors and Rates for SIPG with $k = 2, \eta = 25, \alpha = 0, \alpha_D = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	6.167(−5)	2.97	4.232(−6)	3.27	7.610(−4)	2.25
8	8.059(−6)	2.94	3.123(−7)	3.76	1.772(−4)	2.10
16	1.037(−6)	2.96	2.094(−8)	3.90	4.302(−5)	2.04
32	1.316(−7)	2.98	1.365(−9)	3.94	1.062(−5)	2.02
64	1.659(−8)	2.99	1.213(−10)	3.49	2.637(−6)	2.01

Table S-4. SIPG, $k = 2, \eta = 25, \alpha = \alpha_D = 1$

$\frac{1}{h}$	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	6.332(−6)	3.36	2.678(−4)	3.12
8	4.600(−7)	3.78	3.262(−5)	3.04
16	3.064(−8)	3.91	4.049(−6)	3.01
32	2.231(−9)	3.78	5.053(−7)	3.00

Table S-5. Errors and Rates for SIPG with $k = 3, \eta = 20, \alpha = 0, \alpha_D = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	1.396(−6)	4.44	4.940(−9)	4.99	2.832(−5)	3.78
8	7.406(−8)	4.24	1.031(−10)	5.58	2.185(−6)	3.70
16	4.262(−9)	4.12			1.759(−7)	3.64
32	2.233(−10)	4.25			1.465(−8)	3.59
					1.343(−9)	3.45

Table S-6. SIPG with $k = 3, \eta = 20, \alpha = 0, \alpha_D = 0.5$

$\frac{1}{h}$	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	6.531(−9)	5.46	1.478(−5)	3.93
8	1.183(−10)	5.79	1.029(−6)	3.84
16	1.7593(−11)	2.75	6.907(−8)	3.90
32			4.518(−9)	3.93
			4.922(−10)	3.20

Table S-7. SIPG with $k = 3, \eta = 20, \alpha = \alpha_D = 1$

$\frac{1}{h}$	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	6.493(−9)	5.48	5.348(−6)	4.44
8	1.158(−10)	5.81	3.077(−7)	4.12

Table S-8. Errors and Rates for SIPG with $k = 4, \eta = 20, \alpha = 0, \alpha_D = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	6.488(−8)	4.51	8.603(−12)	6.84	2.504(−5)	3.47
8	2.107(−9)	4.94			1.608(−7)	3.96
16	8.019(−11)	4.72			9.704(−9)	4.05
32					5.961(−10)	4.02

Table S-9. SIPG $k = 4, \eta = 50, \alpha = \alpha_D = 1$

$\frac{1}{h}$	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	8.369(−8)	5.46
8	2.292(−9)	5.19
16	9.093(−11)	4.66
32	5.961(−10)	4.02

5.3. The NIPG method. Tables N-1 to N-9 contain the results for the NIPG method. Again, all these computational results agree with the theoretical results. For examples, In Table N-1, we have the superconvergence at rate 1.5 for $k = 1$ with the minimum penalty $\alpha = \alpha_D = 0$. Table N-2 shows that the superconvergence at rate 2 is achieved if the additional penalty $\alpha_D = \frac{1}{2}$ is applied at Dirichlet boundary condition.

Table N-3 shows that when $k = 2$ the superconvergence property does not exists for both the averaged errors at interior nodes and the derivatives of the errors at Gauss points if $\alpha = \alpha_D = 0$. In fact, the convergence rate in L^2 norm is 2 which is not optimal even though the optimal approximation in L^2 norm for the finite element space is of degree 3. However, when there is a super-penalty at every node,

Table N-4 shows superconvergence for both the averaged errors at interior nodes and the derivatives of the errors at Gauss points. This phenomenon agrees again with Theorem 3.1 and 3.2.

When $k = 3$, Table N-5 reveals superconvergence at rate $3+\frac{1}{2}$ for the derivatives of the errors at Gauss points. This rate is raised to 4 when $\alpha_D = \frac{1}{2}$ in Table N-6. Unlike in the SIPG method, we do not have superconvergence for the averaged errors at the interior nodes if no super-penalty at these nodes according to Table N-5 and N-6. This agrees with Theorem 3.2. Table N-7 shows that the superconvergence at the interior nodes is achieved when super-penalty is used.

Table N-1. Errors and Rates for NIPG with $k = 1, \eta = 1, \alpha = 0, \alpha_D = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	3.452(−2)	2.01	3.991(−2)	1.95	2.742(−2)	1.63
8	8.542(−3)	2.01	1.077(−2)	1.89	9.880(−3)	1.47
16	2.125(−3)	2.01	2.813(−3)	1.94	3.542(−3)	1.48
32	5.300(−4)	2.00	7.191(−4)	1.97	1.261(−3)	1.49
64	1.324(−4)	2.00	1.818(−4)	1.98	4.475(−4)	1.49
128	3.307(−5)	2.00	4.572(−5)	1.99	1.585(−4)	1.50
256	8.265(−6)	2.00	1.146(−5)	2.00	5.609(−5)	1.50
512	2.066(−6)	2.00	2.870(−6)	2.00	1.984(−5)	1.50
1024	5.165(−7)	2.00	7.180(−7)	2.00	7.016(−6)	1.50
2048	1.291(−7)	2.00	1.800(−7)	2.00	2.481(−6)	1.50

Table N-2. NIPG with $k = 1, \eta = 1, \alpha = 0, \alpha_D = 0.5$

$\frac{1}{h}$	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	2.331(−2)	2.20	1.008(−2)	2.49
8	5.769(−3)	2.01	2.081(−3)	2.28
16	1.439(−3)	2.00	4.399(−4)	2.24
32	3.590(−4)	2.00	9.571(−5)	2.20
64	8.864(−5)	2.00	2.169(−5)	2.14
128	2.240(−5)	2.00	5.101(−6)	2.09
256	5.597(−6)	2.00	1.232(−6)	2.05
512	1.399(−6)	2.00	3.023(−7)	2.03
1024	3.499(−7)	2.00	7.490(−8)	2.01
2048	8.743(−8)	2.00	1.866(−8)	2.01

Table N-3. Errors and Rates for NIPG with $k = 2, \eta = 1, \alpha = 0, \alpha_D = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e'_h\|_{G(1)}$	$R_{G(1)}$
4	3.760(−3)	1.94	5.018(−3)	1.45	3.530(−3)	1.74
8	9.500(−4)	1.98	1.552(−3)	1.69	9.638(−4)	1.87
16	2.378(−4)	2.00	4.352(−4)	1.83	2.517(−4)	1.94
32	5.934(−5)	2.00	1.154(−4)	1.92	6.432(−5)	1.97
64	1.481(−5)	2.00	2.972(−5)	1.96	1.626(−5)	1.98
128	3.699(−6)	2.00	7.543(−6)	1.98	4.086(−6)	1.99
256	9.237(−7)	2.00	1.899(−6)	1.99	1.024(−6)	2.00
512	2.291(−7)	2.01	4.741(−7)	2.00	2.545(−7)	2.00
1024	5.032(−8)	2.19	1.087(−7)	2.12	5.722(−8)	2.15

Table N-4. NIPG, $k=2,\eta=1,\alpha=\alpha_D=1$

$\frac{1}{h}$	$\max_{1\le i\le N-1}\lvert e_h(x_i)\rvert$	R_{av}	$\lVert e_h'\rVert_{G(1)}$	$R_{G(1)}$
4	1.039(−3)	3.12	8.137(−3)	3.25
8	8.596(−5)	3.60	6.463(−5)	3.65
16	6.110(−6)	3.81	5.422(−6)	3.58
32	4.057(−7)	3.91	5.538(−7)	3.29
64	2.682(−8)	3.92	6.482(−8)	3.09

Table N-5. Errors and Rates for NIPG with $k=3,\eta=1,\alpha=0,\alpha_D=0$

$\frac{1}{h}$	$\lVert e_h\rVert_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1}\lvert e_h(x_i)\rvert$	R_{av}	$\lVert e_h'\rVert_{G(1)}$	$R_{G(1)}$
4	1.091(−5)	4.06	1.204(−5)	3.83	2.417(−5)	3.60
8	6.643(−7)	4.04	8.293(−7)	3.86	2.022(−6)	3.58
16	4.096(−8)	4.02	5.450(−8)	3.93	1.730(−7)	3.55
32	2.489(−9)	4.04	3.420(−9)	3.99	1.500(−8)	3.53
64	6.017(−11)	5.37	9.103(−11)	5.23	1.313(−9)	3.51

Table N-6. NIPG with $k=3,\eta=1,\alpha=0,\alpha_D=0.5$

$\frac{1}{h}$	$\max_{1\le i\le N-1}\lvert e_h(x_i)\rvert$	R_{av}	$\lVert e_h'\rVert_{G(1)}$	$R_{G(1)}$
4	1.054(−5)	3.91	1.871(−5)	3.83
8	6.794(−7)	3.96	1.215(−6)	3.94
16	4.164(−8)	4.03	7.243(−8)	4.07
32	2.439(−9)	4.09	3.980(−9)	4.19
64	1.593(−10)	3.94	3.151(−10)	3.66

Table N-7. NIPG with $k=3,\eta=1,\alpha=\alpha_D=1$

$\frac{1}{h}$	$\max_{1\le i\le N-1}\lvert e_h(x_i)\rvert$	R_{av}	$\lVert e_h'\rVert_{G(1)}$	$R_{G(1)}$
4	5.098(−6)	4.70	8.635(−6)	4.66
8	1.237(−7)	5.37	2.942(−7)	4.88
16	2.259(−9)	5.78	1.715(−8)	4.10

Table N-8. Errors and Rates for NIPG with $k=4,\eta=1,\alpha=0,\alpha_D=0$

$\frac{1}{h}$	$\lVert e_h\rVert_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1}\lvert e_h(x_i)\rvert$	R_{av}	$\lVert e_h'\rVert_{G(1)}$	$R_{G(1)}$
4	5.968(−7)	3.96	7.913(−7)	3.42	1.522(−6)	3.78
8	3.765(−8)	3.99	6.073(−8)	3.70	1.038(−7)	3.87
16	2.405(−9)	3.97	4.283(−9)	3.83	6.830(−9)	3.93
32	3.690(−10)	2.70	5.844(−10)	2.87	6.657(−10)	3.36

Table N-9. Errors and Rates for NIPG with $k=4,\eta=1,\alpha=\alpha_D=1$

$\frac{1}{h}$	$\lVert e_h\rVert_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1}\lvert e_h(x_i)\rvert$	R_{av}	$\lVert e_h'\rVert_{G(1)}$	$R_{G(1)}$
4	2.457(−7)	4.92	3.241(−7)	4.38	6.519(−7)	4.67
8	5.246(−9)	5.55	8.626(−9)	5.23	1.584(−8)	5.36
16	1.372(−10)	5.26	2.33(−10)	5.21	3.289(−10)	5.59

5.4. The DG method without penalty. In this subsection, we examine the computational results from using the DG method studied in Section 4. Table D-1 contains the result for the piecewise linear element. The results in Table D-1 agree with Theorem 4.1. Our experiments (not presented in this paper) also indicate that this DG method does not convergent if no Neumann boundary conditions are specified. Although the convergence rate in L^2 norm is not optimal, we can see that the averaged errors at nodes converge with optimal rate. This numerical result cannot be verified by our theoretical results. In Table D-2, we see that all rates are two. There is no superconvergence for the DG method if $k = 2$. We can only see superconvergence if k is odd. For examples, Table D-3 and Table D-5 show a rate of convergence at $k + \frac{1}{2}$ which agrees with Theorem 4.18.

Table D-1. Errors and Rates for DG with $k = 1, \eta = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e_h'\|_{G(1)}$	$R_{G(1)}$
4	1.227(−1)	1.47	1.113(−1)	1.78	1.235(−1)	0.95
8	5.167(−2)	1.25	3.009(−2)	1.90	6.231(−2)	0.97
16	2.438(−2)	1.08	7.752(−3)	1.96	3.123(−2)	1.00
32	1.120(−2)	1.02	1.966(−3)	1.98	1.562(−2)	1.00
64	5.975(−3)	1.01	4.950(−4)	1.99	7.812(−3)	1.00
128	2.985(−3)	1.00	1.242(−4)	2.00	3.906(−3)	1.00
256	1.492(−3)	1.00	3.110(−5)	2.00	1.953(−3)	1.00
512	7.459(−4)	1.00	7.781(−6)	2.00	9.766(−4)	1.00
1024	3.729(−4)	1.00	1.946(−6)	2.00	4.883(−4)	1.00
2048	1.865(−4)	1.00	4.866(−7)	2.00	2.441(−4)	1.00

Table D-2. Errors and Rates for DG with $k = 2, \eta = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e_h'\|_{G(1)}$	$R_{G(1)}$
4	8.462(−3)	1.78	1.123(−2)	1.30	7.386(−3)	1.70
8	2.220(−3)	1.93	3.527(−3)	1.67	2.063(−3)	1.84
16	5.635(−4)	1.98	9.917(−4)	1.83	5.472(−4)	1.91
32	1.415(−4)	1.99	2.632(−4)	1.91	1.410(−4)	1.96
64	3.541(−5)	2.00	6.783(−5)	1.96	3.580(−5)	1.98
128	8.855(−5)	2.00	1.722(−5)	1.98	9.020(−6)	1.99
256	2.214(−5)	2.00	4.336(−6)	1.99	2.263(−6)	1.99
512	5.516(−6)	2.00	1.086(−6)	2.00	5.649(−7)	2.00
1024	1.308(−6)	2.01	2.619(−7)	2.05	1.336(−7)	2.08
2048	9.254(−9)	3.82	2.603(−8)	3.33	3.539(−8)	1.91

Table D-3. Errors and Rates for DG with $k = 3, \eta = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e_h'\|_{G(1)}$	$R_{G(1)}$
4	1.190(−5)	4.07	1.321(−5)	3.84	2.729(−5)	3.59
8	7.218(−7)	4.04	9.072(−7)	3.86	2.283(−6)	3.58
16	4.441(−8)	4.02	5.955(−8)	3.93	1.953(−7)	3.55
32	2.700(−9)	4.04	3.741(−9)	3.99	1.694(−8)	3.53
64	5.013(−11)	5.75	9.005(−11)	5.38	1.479(−9)	3.52

Table D-4. Errors and Rates for DG with $k = 4, \eta = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max\limits_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e_h'\|_{G(1)}$	$R_{G(1)}$
4	6.612(−7)	3.95	8.768(−7)	3.41	1.675(−6)	3.78
8	4.192(−8)	3.98	6.738(−8)	3.70	1.145(−7)	3.87
16	2.678(−9)	3.97	4.743(−9)	3.83	7.528(−9)	3.93
32	3.872(−10)	2.79	6.160(−10)	2.94	7.080(−10)	3.41

Table D-5. Errors and Rates for DG with $k = 5, \eta = 0$

$\frac{1}{h}$	$\|e_h\|_{L^2(0,1)}$	R_{L^2}	$\max\limits_{1\le i\le N-1} \|e_h(x_i)\|$	R_{av}	$\|e_h'\|_{G(1)}$	$R_{G(1)}$
4	1.183(−9)	6.05	1.319(−9)	5.78	4.628(−9)	5.64
8	2.050(−11)	5.85	2.738(−11)	5.59	9.897(−11)	5.61

References

[1] D. N. Arnold, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal., 19 (4) (1982), 742–760.

[2] D. N. Arnold, F. Brezzi, B. Cockburn and L. D. Martin, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., **39** (2002), 1749–1779.

[3] I. Babuška, C.E.Baumann and J.T.Oden, *A discontinuous hp finite element method for diffusion problems: 1-D analysis*, Computers & Mathematics with Application, 37 (1999), 103-122.

[4] I. Babuška and M. Suri, *The h-p version of the finite element method with quasi uniform meshes*, Math. Modeling Numer. Anal., 21 (1987), 199-238.

[5] G. A. Baker, *Finite element methods for elliptic equations using nonconforming elements*, Math. Comp., **31** (1977), 45–59.

[6] F. Brezzi, G. Manzini, D. Marini, P. Pietra and A. Russo, *Discontinuous Galerkin approximations for elliptic problems*, Numer. Methods for Partial Differential Equations, **16** (2000), 365–378.

[7] P. Castillo, B. Cockburn, I. Perugia and D. Schötzau, *An a priori error analysis of the local discontinuous Galerkin method for elliptic problems*, SIAM J. Numer. Anal., **38** (2000), 1676–1706.

[8] P. Castillo, B. Cockburn, D. Schötzau and C. Schwab, *An optimal priori error estimate for the hp-version of the local discontinuous Galerkin method for convection-diffusion problems*, Math. Comp.

[9] H. Chen and Z. Chen, *Stability and convergence of mixed discontinuous finite element methods for second order differential problems*, J. Numer. Math., Vol 11 (4), 253-324, 2003.

[10] H. Chen, Z. Chen and B. Li, *Numerical study of hp version of mixed discontinuous finite element methods for reaction diffusion problems: 1D case*, Numerical Methods for Partial Differential Equations **19** (2003), 525–553.

[11] Z. Chen, *On the relationship of various discontinuous finite element methods for second-Order elliptic equations*, East-West Numer. Math., **9** (2001), 99–122.

[12] B. Cockburn, G. Kanschat, I. Perugia and D. Schötau, *Superconvergence of the local discontinuous Galerkin method for elliptic problems on Cartesian grids*, SIAM J. Numer. Anal., 39 (1) (2001), 264-285.

[13] B. Cockburn, G. E. Karniadakis and C. W. Shu, *Discontinuous Galerkin Methods, Theory, Computation and Applications*, Lecture Notes in Computational Science and Engineering, Vol. 11, Springer-Verlag, Berlin, 2000.

[14] L. M. Delves and C. A. Hall, *An implicit matching principle for global element calculations*, J. Inst. Math. Appl., 23 (1979), 223-234.

[15] J. Douglas and T. Dupont, *Interior penalty procedures for elliptic and parabolic Galerkin methods*, Lecture Notes in Physics, Vol. 58, Springer-Verlag, Berlin, 1976, 207–216.

[16] J. Douglas, R. E. Ewing and M. Wheeler, *the approximation of the pressure by a mixed method in the simulation of miscible displacement*, RAIRO Anal. Numér., 17 (1983), 17-33.

[17] P. Houston, C. Schwab and E. Süli, *Discontinuous hp-finite element methods for advection-diffusion-reaction problems*, SIAM J. Numer. Anal., 39(6) (2002), 2133-2163.

[18] M. Krizek and P. Neittaanmaki, *On Superconvergence techniques*, Acta Appl. Math., 9(1987), 175–198.

[19] N. D. Levine, *Superconvergence recovery of the gradient from piecewise linear finite element approximations*, IMA J. Numer. Anal., 5(1985), 407–427.

[20] Q. Lin, J. Xu, *Linear finite elements with high accuracy*, J. Comp. Math., 3(1985), 115–133.

[21] J. Nitche, *Über ein variationsprinzip zur lösung von Dirichlet problem bei verwendung von teilräumen, die keinen randbedingungungen unterworfen sind*, abh. Math. Sem. Univ. Hamburg, 36 (1971), 9-15.

[22] J. T. Oden, I. Babuška and C. E. Baumann, *A discontinuous hp finite element method for diffusion problems*, J. Comput. Phys., **146** (1998), 491–519.

[23] P. Percell and M. F. Wheeler, *A local residual finite element procedure for elliptic equations*, SIAM J. Numer. Anal., 15 (4) (1978), 705-714.

[24] S. Prudhomme, F. Pascal, J. T. Oden and A. Romkes, *Review of a priori error estimation for discontinuous Galerkin methods*, TICAM Report 00-27, October 17, 2000.

[25] W. H. Reed and T. R. Hill, *Triangular mesh methods for the neutron transport equation*, Technical Report, LA-UR-73-479, Los Alamos Scientific Laboratory, 1973.

[26] B. Rivière, M. F. Wheeler, and V. Girault, *Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems, Part I*, Computational Geoscience, **3** (1999), 337–360.

[27] E. Süli, C. Schwab and P. Houston, *hp-DGFEM for partial differential equations with nonnegative characteristic form*, in Cockburn, Karniadakis and Shu, editors, Discontinuous Galerkin Methods, vol 11 of Lectures Notes in Computational Science and Engineering, 221-230, Springer, Berklin, 2000.

[28] L. B. Wahlbin, *Superconvergence in Galerkin finite element methods*, Lecture Notes on Mathematics, Vol. 1065, Springer-Verlag, New York, 1995.

[29] M. F. Wheeler, *an elliptic collocation-finite element method with interior penalties*, SIAM J. Numer. Anal., **15** (1978), 152–161.

[30] M. F. Wheeler and J. R. Whiteman, *Superconvergence recovery of gradient on subdomains from piecewise linear finite element approximations*, Numer. Methods PDEs, 3(1987), 65–82.

[31] Z. Zhang, *Ultraconvergence of the patch recovery technique*, Math. Comp., 65(1996), 1431–1437.

[32] Q. Zhu and Q. Lin, *Superconvergence Theory of the finite element Methods*, Hunan Science Press., 1989 (in chinese).

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