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THE STABILIZER-FREE WEAK GALERKIN FINITE ELEMENT METHOD FOR THE BIHARMONIC EQUATION USING POLYNOMIALS OF REDUCED ORDER

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Abstract. In this article, we use the stabilizer-free weak Galerkin (SFWG) finite element method on general polytopal meshes to solve the biharmonic equation. By decreasing the degree of the polynomials on the boundary of weak functions and modifying the definition of the weak Laplacian, this paper not only removes the stabilizer in the weak Galerkin (WG) numerical scheme, but also minimizes the number of unknowns in the scheme. The optimal orders of error estimates in the H^2 and L^2 norms are obtained and the convergence results are verified by some numerical examples.

Key words. Stabilizer-free weak Galerkin finite element method, biharmonic equation, weak operator.

1. Introduction

In this paper, we consider the biharmonic equation

$$\Delta^2 u = f, \qquad \text{in } \Omega, \tag{1}$$

$$u = 0, \qquad \text{on } \partial\Omega, \tag{2}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \qquad \text{on } \partial\Omega, \tag{3}$$

where Ω is a bounded polygonal or polyhedral domain in $\mathbb{R}^d (d = 2, 3)$ and $\mathbf{n}$ is the outward unit normal vector along $\partial\Omega$.

The derivation of the variational form is straightforward: Find $u \in H_0^2(\Omega)$ such that

$$(\Delta u, \Delta v) = (f, v), \qquad \forall\ v \in H_0^2(\Omega), \tag{4}$$

where

$$H_0^2(\Omega) = \left\{ v \in H^2(\Omega) :\ v|_{\partial\Omega} = 0,\ \frac{\partial v}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0 \right\}.$$

Based on the above form, the conforming finite element methods have been utilized to solve the biharmonic equation [2, 12, 20]. However, these methods necessitate the construction of C^1-continuous finite elements to form subspaces of $H^2(\Omega)$ and the complexity involved in constructing such elements has led researchers to focus on alternative approaches.

The discontinuous Galerkin finite element method (DGFEM) constructs a numerical scheme for the variational form (4) using completely discontinuous functions, thereby avoiding the construction of H^2 conforming finite elements. As a type of DGFEM, the interior penalty discontinuous Galerkin (IPDG) method [6] employs the discontinuous function space to approximate the solution in $H^2(\Omega)$. Although the use of discontinuous functions offers greater flexibility, the numerical scheme introduces complex penalty terms that complicate the implementation.

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Another approach to circumvent the construction of C^1-continuous elements is to employ mixed variational formulations by introducing auxiliary variables. These auxiliary variables typically carry physical significance, making them adaptable to different types of physics problems, such as $\varphi = -\Delta u$ for hydrodynamics [1, 5] and $\sigma = -\nabla^2 u$ for plate bending problems [3, 4].

In the last decade, the weak Galerkin (WG) finite element method, as a novel method, has been well developed. The method uses weak functions and weak differential operators in numerical formulation to solve various equations, such as the Poisson equation [10, 13], the Stokes equation [15], the Brinkman equation [7], and the biharmonic equation [8, 11]. In the weak Galerkin framework, weak functions are defined by separate polynomial approximations: v_0 (on the cell interior) and v_b (on the cell boundary). Weak differential operators are constructed through integration by parts of classical differential operators. In addition, the stabilizers appear in WG numerical schemes to ensure weak continuity but increase the complexity of the algorithm to a certain extent. The subsequent improvement of this method is also carried out by modifying the definitions of weak functions or weak operators.

Scholars reduce the order of v_b to decrease the degrees of freedom to solve the Poisson equation in [9], the biharmonic equation in [19], among others. The stabilizer-free weak Galerkin (SFWG) finite element method eliminates the stabilizer by increasing the order of polynomials in the range of the weak operator, thereby simplifying the numerical scheme. The SFWG methods with weak functions $(P_k(T), P_k(e))$ $(k \geq 1)$ in [16] or $(P_k(T), P_{k-1}(e))$ $(k \geq 1)$ in [18] are used to solve the second-order elliptic equation. The SFWG method [17] focuses on eliminating the stabilizer from the WG numerical scheme for the biharmonic equation. To simplify the analysis, the authors constructed the SFWG method for the biharmonic equation using weak functions of the form $(P_k(T), P_k(e), P_{k-1}(e))$ $(k \geq 2)$. In this paper, we aim to enhance computational efficiency by developing an SFWG method with weak functions $(P_k(T), P_{k-1}(e), P_{k-1}(e))$ $(k \geq 2)$ for solving the biharmonic equation. By constructing such an SFWG method, we obtain a numerical scheme with fewer degrees of freedom. This results in a smaller-scale linear system to solve during numerical computations, thereby significantly reducing computational costs. Based on numerical experiments, if the degree of v_b is reduced by one and the definition of the weak Laplacian operator in [17] is retained, we observe that the convergence rates of the error in the H^2 and L^2 norms are degraded by one or two orders compared to the optimal rates. This phenomenon is primarily caused by the degradation in the convergence rate of $|||Q_h u - u|||$, where Q_h denotes the projection operator onto the weak finite element space. In order to achieve the highest possible convergence rate for the algorithm, it is imperative that we refine the definition of the weak operator accordingly. By doing so, we not only remove the stabilizer from the WG numerical scheme, but also utilize fewer degrees of freedom to attain the same convergence order as stated in [17].

The outline of this article is as follows. In Section 2, we make preparations and propose the numerical scheme. Section 3 is devoted to deriving the error equations and establishing the error estimates in the H^2 and L^2 norms. In Section 4, two numerical examples are conducted to validate the theoretical results. Finally, Section 5 summarizes the key findings and concludes the paper.

2. SFWG numerical scheme for the biharmonic equation

In this section, we begin by introducing some necessary notations and defining the weak function space and weak operator. Then we proceed to establish the numerical scheme and rigorously prove the existence and uniqueness of the numerical solution.

2.1. Notations for partitions. Suppose K is an open bounded domain in $\mathbb{R}^d$ and s is a positive integer. We utilize $\|\cdot\|_{s,K}$, $|\cdot|_{s,K}$, $(\cdot,\cdot)_{s,K}$ to represent the norm, seminorm and inner product of Sobolev space $H^s(K)$, respectively. If $K=\Omega$, we drop the subscript K and drop s if $s=2$.

Let the partition $\mathcal{T}_h$ of Ω satisfy assumptions in [14] and denote $\mathcal{E}_h$ as the set of all edges in $\mathbb{R}^2$ or faces in $\mathbb{R}^3$. $\mathcal{E}_h^0=\mathcal{E}_h\backslash\partial\Omega$ is defined as the set of all interior edges/faces. The mesh size of $\mathcal{T}_h$ is denoted by h. The following simplified notations are adopted:

$$\begin{aligned}(v,w)_{\mathcal{T}_h}&=\sum_{T\in\mathcal{T}_h}(v,w)_T=\sum_{T\in\mathcal{T}_h}\int_T vwdT,\\ \langle v,w\rangle_{\partial\mathcal{T}_h}&=\sum_{T\in\mathcal{T}_h}\langle v,w\rangle_{\partial T}=\sum_{T\in\mathcal{T}_h}\int_{\partial T} vwds.\end{aligned}$$

In addition, we define the unit normal vector for each edge (in 2D) or face (in 3D) in $\mathcal{E}_h$ as follows:

$$\mathcal{D}_h=\{\mathbf{n}_e:\mathbf{n}_e \text{ is unit and normal to } e,\ e\in\mathcal{E}_h\}.$$

This allows us to define the discrete function space as follows:

$$\begin{aligned}V_h=\{v=\{v_0,v_b,v_n\mathbf{n}_e\}:v_0\big|_T\in P_k(T),v_b\big|_e\in P_{k-1}(e),&\\ v_n\big|_e\in P_{k-1}(e),T\in\mathcal{T}_h,e\in\mathcal{E}_h\},&\end{aligned}\tag{5}$$

$$V_h^0=\left\{v=\{v_0,v_b,v_n\mathbf{n}_e\}\in V_h:\ v_b\big|_e=0,\ v_n\big|_e=0,\ e\subset\partial\Omega\right\},\tag{6}$$

where $k\geq 2$. $P_k(T)$ denotes the set of polynomials of degree at most k on element T and $P_k(e)$ represents the set of polynomials with degree no more than k on edge e.

We denote by Q_0 the L^2 projection operator onto $P_k(T)$ for each element $T\in\mathcal{T}_h$ and by Q_b the locally defined L^2 projection operator onto $P_{k-1}(e)$ for each edge $e\in\mathcal{E}_h$. The operator $Q_h:H^2(\Omega)\to V_h$ is defined such that for any $u\in H^2(\Omega)$,

$$Q_hu=\{Q_0u,Q_bu,Q_b(\nabla u\cdot\mathbf{n}_e)\mathbf{n}_e\}.$$

Then we define the discrete weak Laplacian operator:

Definition 1. *For each* $v\in V_h+H^2(\Omega)$, $\Delta_wv\big|_T\in P_j(T)$ *satisfies*

$$\begin{aligned}(\Delta_wv,\varphi)_T=&(\Delta v_0,\varphi)_T+\langle Q_b(v_0-v_b),\nabla\varphi\cdot\mathbf{n}\rangle_{\partial T}\\ &-\langle(\nabla v_0-v_n\mathbf{n}_e)\cdot\mathbf{n},\varphi\rangle_{\partial T},\qquad\forall\,\varphi\in P_j(T),\end{aligned}\tag{7}$$

where $j=k+2n-1$, n *is the number of edges or faces of the element* T *and* $\mathbf{n}$ *denotes the unit outward normal vector.*

2.2. Numerical scheme. With the above preparations, we define the numerical scheme as follows: The numerical solution $u_h \in V_h^0$ of (1)-(3) satisfies

$$(\Delta_w u_h, \Delta_w v)_{\mathcal{T}_h} = (f, v_0)_{\mathcal{T}_h}, \qquad \forall\, v \in V_h^0. \tag{8}$$

When $v \in H^2(\Omega)$, there are $v_0\big|_T = v\big|_T$, $v_b\big|_{\partial T} = v\big|_{\partial T}$ and $v_n\big|_{\partial T} = (\nabla v \cdot \mathbf{n}_e)\big|_{\partial T}$ for any $T \in \mathcal{T}_h$. Therefore, we obtain the following lemma:

Lemma 1. *For any $v \in H^2(\Omega)$, we have*

$$\Delta_w v = \mathbb{Q}_h \Delta v, \qquad \forall\, T \in \mathcal{T}_h, \tag{9}$$

where we define $\mathbb{Q}_h$ as the L^2 projection onto $[P_j(T)]^d$ on each element $T \in \mathcal{T}_h$.

Proof. For any $T \in \mathcal{T}_h$, using the definitions of Δ_w and $\mathbb{Q}_h$, we have

$$\begin{aligned}(\Delta_w v, \varphi)_T =&(\Delta v, \varphi)_T + \langle Q_b(v - v), \nabla\varphi \cdot \mathbf{n}\rangle_{\partial T} - \langle(\nabla v - (\nabla v \cdot \mathbf{n}_e)\mathbf{n}_e) \cdot \mathbf{n}, \varphi\rangle_{\partial T}\\ =&(\Delta v, \varphi)_T\\ =&(\mathbb{Q}_h \Delta v, \varphi)_T\end{aligned}$$

for any $\varphi \in P_j(T)$, which implies $\Delta_w v = \mathbb{Q}_h \Delta v$. □

2.3. Existence and uniqueness. First, we introduce the two semi-norms.

Definition 2. *For $v \in V_h + H^2(\Omega)$,*

$$\begin{aligned}|||v|||^2 =& \sum_{T\in\mathcal{T}_h} (\Delta_w v, \Delta_w v)_T,\\ \|v\|_{2,h}^2 =& \sum_{T\in\mathcal{T}_h} \left(\|\Delta v_0\|_T^2 + h_T^{-3}\,\|Q_b(v_0 - v_b)\|_{\partial T}^2 + h_T^{-1}\,\|(\nabla v_0 - v_n \mathbf{n}_e) \cdot \mathbf{n}\|_{\partial T}^2\right).\end{aligned}$$

Lemma 2. *There exist two positive constants C_1 and C_2 such that*

$$C_1\,\|v\|_{2,h} \le |||v||| \le C_2\,\|v\|_{2,h}, \quad \forall\, v \in V_h. \tag{10}$$

Proof. For any $v \in V_h$, by choosing $\varphi = \Delta_w v$ in (7) and using the Cauchy-Schwarz inequality, the trace inequality [14, Lemma A.3] and the inverse inequality [14, Lemma A.6], we obtain

$$\|\Delta_w v\|_T^2 \le C\left(\|\Delta v_0\|_T + h_T^{-\frac{3}{2}}\,\|Q_b(v_0 - v_b)\|_{\partial T} + h_T^{-\frac{1}{2}}\,\|\nabla v_0 \cdot \mathbf{n}_e - v_n\|_{\partial T}\right)\|\Delta_w v\|_T.$$

Summing over all elements, we arrive at the following conclusion:

$$|||v|||^2 \le C\,\|v\|_{2,h}\,|||v|||,$$

which indicates $|||v||| \le C\,\|v\|_{2,h}$.

For any $v \in V_h$ and any $e \subset \partial T$, by Lemma 3.1 in [17], there exists a unique polynomial $q_1 \in P_{k+2n-1}(T)$ such that

$$\begin{aligned}&(\Delta v_0, q_1)_T = 0, \quad \langle Q_b(v_0 - v_b), \nabla q_1 \cdot \mathbf{n}\rangle_{\partial T} = \|Q_b(v_0 - v_b)\|_e^2,\\ &\langle(\nabla v_0 - v_n \mathbf{n}_e) \cdot \mathbf{n}, q_1\rangle_{\partial T} = 0,\end{aligned}$$

and $\|q_1\|_T \le C h_T^{\frac{3}{2}}\,\|Q_b(v_0 - v_b)\|_e$. From the definition of Δ_w, we know

$$\begin{aligned}\|Q_b(v_0 - v_b)\|_e^2 &= (\Delta_w v, q_1)_T\\ &\le \|\Delta_w v\|_T\,\|q_1\|_T\\ &\le C h_T^{\frac{3}{2}}\,\|\Delta_w v\|_T\,\|Q_b(v_0 - v_b)\|_e,\end{aligned}$$

which implies $\|Q_b(v_0-v_b)\|_e \le Ch_T^{\frac{3}{2}}\|\Delta_w v\|_T$. Further, we have

$$h_T^{-\frac{3}{2}}\|Q_b(v_0-v_b)\|_{\partial T} \le C\|\Delta_w v\|_T. \tag{11}$$

Then, we get

$$\sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_b(v_0-v_b)\|_{\partial T}^2 \le C|||v|||^2. \tag{12}$$

For any $v\in V_h$ and any $e\subset\partial T$, using Lemma 3.2 in [17], we have $q_2\in P_{k+2n-1}(T)$ such that

$$\begin{aligned}&(\Delta v_0, q_2)_T = 0,\ \langle Q_b(v_0-v_b), \nabla q_2\cdot\mathbf{n}\rangle_{\partial T} = 0,\\ &\langle(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}, q_2\rangle_{\partial T} = \|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_e^2,\end{aligned}$$

and $\|q_2\|_T \le Ch_T^{\frac{1}{2}}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_e$. By the definition of Δ_w, we obtain

$$\begin{aligned}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_e^2 &= (\Delta_w v, q_2)_T\\ &\le \|\Delta_w v\|_T\|q_2\|_T\\ &\le Ch_T^{\frac{1}{2}}\|\Delta_w v\|_T\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_e.\end{aligned}$$

Then we have

$$h_T^{-\frac{1}{2}}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T} \le C\|\Delta_w v\|_T, \tag{13}$$

which illustrates

$$\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}^2 \le C|||v|||^2, \tag{14}$$

For any $v\in V_h$, letting $\varphi = \Delta v_0$ in (7) and using the Cauchy-Schwarz inequality, the trace inequality, the inverse inequality, (11) and (13), we get

$$\begin{aligned}\|\Delta v_0\|_T^2 &= (\Delta_w v, \Delta v_0)_T - \langle Q_b(v_0-v_b), \nabla(\Delta v_0)\cdot\mathbf{n}\rangle_{\partial T} + \langle(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}, \Delta v_0\rangle_{\partial T}\\ &\le \Big(\|\Delta_w v\|_T + Ch_T^{-\frac{3}{2}}\|Q_b(v_0-v_b)\|_{\partial T}\\ &\quad + Ch_T^{-\frac{1}{2}}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}\Big)\|\Delta v_0\|_T\\ &\le C\|\Delta_w v\|_T\|\Delta v_0\|_T,\end{aligned}$$

which implies $\sum_{T\in\mathcal{T}_h}\|\Delta v_0\|_T^2 \le C|||v|||^2$. Combining with (12) and (14), we have

$$\|v\|_{2,h} \le C|||v|||, \quad \forall v\in V_h.$$

The proof is completed. ☐

Lemma 3. *$\|\cdot\|_{2,h}$ is the norm of V_h^0.*

Proof. According to the definition of $\|\cdot\|_{2,h}$, we shall only prove the positivity property. Assume that $v\in V_h^0$ such that $\|v\|_{2,h} = 0$. Then we have

$$\Delta v_0\big|_T = 0,\ Q_b(v_0-v_b)\big|_{\partial T} = 0,\ (\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\big|_{\partial T} = 0, \qquad \forall T\in\mathcal{T}_h.$$

And $(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\big|_{\partial T} = 0$ implies $(\nabla v_0\cdot\mathbf{n}_e - v_n)\big|_{\partial T} = 0$ for any $T\in\mathcal{T}_h$. Next we verify that $\nabla v_0\big|_T = 0$, $\forall T\in\mathcal{T}_h$. From integration by parts, we get

$$\|\nabla v_0\|_T^2 = (\nabla v_0, \nabla v_0)_T = -(\Delta v_0, v_0)_T + \langle\nabla v_0\cdot\mathbf{n}, v_0\rangle_{\partial T} = \langle\nabla v_0\cdot\mathbf{n}, v_0\rangle_{\partial T}.$$

By summing over all elements $T \in \mathcal{T}_h$, the following holds

$$\sum_{T\in\mathcal{T}_h} \|\nabla v_0\|_T^2 = \sum_{T\in\mathcal{T}_h} \langle \nabla v_0 \cdot \mathbf{n}, v_0\rangle_{\partial T}$$

When $e \in \mathcal{E}_h^0$, let T_1 and T_2 be two elements sharing e, v_0^1, v_0^2 be the values of v on T_1 and T_2, $\mathbf{n}_1, \mathbf{n}_2$ be the unit outward normal vectors of T_1, T_2 on e, and we obtain

$$\begin{aligned}
\langle \nabla v_0^1 \cdot \mathbf{n}_1, v_0^1\rangle_e + \langle \nabla v_0^2 \cdot \mathbf{n}_2, v_0^2\rangle_e &= \pm\left(\langle \nabla v_0^1 \cdot \mathbf{n}_e, v_0^1\rangle_e - \langle \nabla v_0^2 \cdot \mathbf{n}_e, v_0^2\rangle_e\right)\\
&= \pm\left(\langle \nabla v_0^1 \cdot \mathbf{n}_e, Q_b v_0^1\rangle_e - \langle \nabla v_0^2 \cdot \mathbf{n}_e, Q_b v_0^2\rangle_e\right)\\
&= \pm\left(\langle v_n, v_b\rangle_e - \langle v_n, v_b\rangle_e\right)\\
&= 0,
\end{aligned}$$

where we used $Q_b v_0\big|_{\partial T} = v_b\big|_{\partial T}$ and $\nabla v_0 \cdot \mathbf{n}_e\big|_{\partial T} = v_n\big|_{\partial T}$.

With $\nabla v_0 \cdot \mathbf{n}_e = v_n = 0$ on any boundary edge, we get

$$\sum_{T\in\mathcal{T}_h} \|\nabla v_0\|_T^2 = 0, \tag{15}$$

which deduces $\nabla v_0 = 0$ on any element T. Using $Q_b v_0\big|_e = v_b\big|_e$, $\forall\, e \in \mathcal{E}_h$ and $v \in V_h^0$, we obtain $v = \{0, 0, \mathbf{0}\}$. ☐

Furthermore, by the norm equivalence of Lemma 2, we have $|||\cdot|||$ is the norm in V_h^0.

Theorem 1. *The numerical scheme (8) has a unique solution.*

Proof. Taking $f = 0$ and $v = u_h$ in (8), we have $|||u_h|||^2 = 0$. Since $|||\cdot|||$ is the norm of V_h^0, we get $u_h = 0$. ☐

3. Error analysis

In this section, we introduce the errors $e_h = u - u_h$ and $\varepsilon_h = Q_h u - u_h$ to derive the relevant error equations. By analyzing and estimating the terms in error equations, we arrive at the optimal convergence order of errors in the H^2 and L^2 norms.

3.1. Error equations.

We first give the error equations and estimate each term in the error equations.

Theorem 2. *For any $v \in V_h^0$, the following error equations hold true*

$$(\Delta_w e_h, \Delta_w v)_{\mathcal{T}_h} = -\, l_1(u,v) + l_2(u,v) - l_3(u,v), \tag{16}$$

$$(\Delta_w \varepsilon_h, \Delta_w v)_{\mathcal{T}_h} = -\, l_1(u,v) + l_2(u,v) - l_3(u,v) + (\Delta_w(Q_h u - u), \Delta_w v)_{\mathcal{T}_h}, \tag{17}$$

where

$$\begin{aligned}
l_1(u,v) &= \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u) \cdot \mathbf{n}, Q_b(v_0 - v_b)\rangle_{\partial\mathcal{T}_h},\\
l_2(u,v) &= \langle \Delta u - \mathbb{Q}_h \Delta u, (\nabla v_0 - v_n \mathbf{n}_e) \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h},\\
l_3(u,v) &= \langle \nabla(\Delta u) \cdot \mathbf{n}, v_0 - Q_b v_0\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

Proof. For any $v \in V_h^0$, from (1) and using integration by parts, we have

$$\begin{aligned}(f, v_0)_{\mathcal{T}_h} =&(\Delta^2 u, v_0)_{\mathcal{T}_h}\\ =&-(\nabla(\Delta u), \nabla v_0)_{\mathcal{T}_h} + \langle \nabla(\Delta u)\cdot \mathbf{n}, v_0\rangle_{\partial\mathcal{T}_h}\\ =&(\Delta u, \Delta v_0)_{\mathcal{T}_h} - \langle \Delta u, \nabla v_0 \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h} + \langle \nabla(\Delta u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial\mathcal{T}_h}\\ =&(\Delta u, \Delta v_0)_{\mathcal{T}_h} - \langle \Delta u, (\nabla v_0 - v_n\mathbf{n}_e)\cdot \mathbf{n}\rangle_{\partial\mathcal{T}_h} + \langle \nabla(\Delta u)\cdot\mathbf{n}, v_0 - Q_b v_0\rangle_{\partial\mathcal{T}_h}\\ &+ \langle \nabla(\Delta u)\cdot\mathbf{n}, Q_b v_0 - v_b\rangle_{\partial\mathcal{T}_h},\end{aligned}$$

where we used $v_b\big|_{\partial\Omega} = 0$ and $v_n\big|_{\partial\Omega} = 0$. This implies

$$\begin{aligned}(\Delta u, \Delta v_0)_{\mathcal{T}_h} =&(f, v_0)_{\mathcal{T}_h} + \langle \Delta u, (\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} - \langle \nabla(\Delta u)\cdot\mathbf{n}, v_0 - Q_b v_0\rangle_{\partial\mathcal{T}_h}\\ &- \langle \nabla(\Delta u)\cdot\mathbf{n}, Q_b v_0 - v_b\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Utilizing the above equation, Lemma 1 and the definition of Δ_w, for any $v \in V_h^0$, we obtain

$$\begin{aligned}&(\Delta_w u, \Delta_w v)_{\mathcal{T}_h}\\ =&(\mathbb{Q}_h \Delta u, \Delta_w v)_{\mathcal{T}_h}\\ =&(\Delta v_0, \Delta u)_{\mathcal{T}_h} + \langle Q_b(v_0 - v_b), \nabla(\mathbb{Q}_h\Delta u)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\ &- \langle(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}, \mathbb{Q}_h\Delta u\rangle_{\partial\mathcal{T}_h}\\ =&(f, v_0)_{\mathcal{T}_h} - l_1(u,v) + l_2(u,v) - l_3(u,v).\end{aligned} \tag{18}$$

Combining with the numerical scheme (8), we have

$$(\Delta_w(u - u_h), \Delta_w v)_{\mathcal{T}_h} = -l_1(u,v) + l_2(u,v) - l_3(u,v), \quad \forall\, v \in V_h^0.$$

By adding $(\Delta_w(Q_h u - u), \Delta_w v)_{\mathcal{T}_h}$ to both sides of the above equation, we get (17). $\square$

To estimate $l_1(\cdot,\cdot)$, $l_2(\cdot,\cdot)$, $l_3(\cdot,\cdot)$, we first give the following projection inequalities:

Lemma 4. *For $w \in H^{k+2}(\Omega)$, we have*

$$\left(\sum_{T\in\mathcal{T}_h} h_T^{2s}\left\|w - Q_0 w\right\|_{s,T}^2\right)^{\frac{1}{2}} \le Ch^{k+1}\left\|w\right\|_{k+1}, \tag{19}$$

$$\left(\sum_{T\in\mathcal{T}_h} h_T^{2s}\left\|\Delta w - \mathbb{Q}_h\Delta w\right\|_{s,T}^2\right)^{\frac{1}{2}} \le Ch^{k}\left\|w\right\|_{k+2}, \tag{20}$$

$$\left(\sum_{T\in\mathcal{T}_h} h_T\left\|\Delta w - \mathbb{Q}_h\Delta w\right\|_{\partial T}^2\right)^{\frac{1}{2}} \le Ch^{k}\left\|w\right\|_{k+2}, \tag{21}$$

$$\left(\sum_{T\in\mathcal{T}_h} h_T^3\left\|\nabla(\Delta w - \mathbb{Q}_h\Delta w)\right\|_{\partial T}^2\right)^{\frac{1}{2}} \le Ch^{k}\left\|w\right\|_{k+2}. \tag{22}$$

Proof. The proofs of (19) and (20) are the same as [14]. By the trace inequality and (20), we get

$$\begin{aligned}&\left(\sum_{T\in\mathcal{T}_h} h_T \left\|\Delta w-\mathbb{Q}_h\Delta w\right\|_{\partial T}^2\right)^{\frac{1}{2}}\\ \le&\left(\sum_{T\in\mathcal{T}_h}\left(\left\|\Delta w-\mathbb{Q}_h\Delta w\right\|_T^2+h_T^2\left\|\nabla(\Delta w-\mathbb{Q}_h\Delta w)\right\|_T^2\right)\right)^{\frac{1}{2}}\\ \le& Ch^k\left\|w\right\|_{k+2},\end{aligned}$$

and

$$\begin{aligned}&\left(\sum_{T\in\mathcal{T}_h} h_T^3 \left\|\nabla(\Delta w-\mathbb{Q}_h\Delta w)\right\|_{\partial T}^2\right)^{\frac{1}{2}}\\ \le&\left(\sum_{T\in\mathcal{T}_h}\left(h_T^2\left\|\nabla(\Delta w-\mathbb{Q}_h\Delta w)\right\|_T^2+h_T^4\left\|\nabla^2(\Delta w-\mathbb{Q}_h\Delta w)\right\|_T^2\right)\right)^{\frac{1}{2}}\\ \le& Ch^k\left\|w\right\|_{k+2}.\end{aligned}$$

□

With these preparations, we give estimates of certain terms in the error equations as follows:

Lemma 5. *If $w\in H^{k+2}(\Omega)$, then for any $v\in V_h^0$, we have*

$$\left|l_1(w,v)\right| \le Ch^k\left\|w\right\|_{k+2}|||v|||, \tag{23}$$

$$\left|l_2(w,v)\right| \le Ch^k\left\|w\right\|_{k+2}|||v|||, \tag{24}$$

$$\left|l_3(w,v)\right| \le Ch^{k-1}\left\|w\right\|_{k+2}|||v|||, \tag{25}$$

$$\left|l_3(w,v)\right| \le \lambda^{\frac{1}{2}}Ch^{k-2}\left\|w\right\|_{k+2}\left\|v\right\|+\lambda^{-\frac{1}{2}}Ch^k\left\|w\right\|_{k+2}|||v|||. \tag{26}$$

In particular, when $w\in H^4(\Omega)$, for any $v\in V_h^0$, we get

$$\left|l_1(w,v)\right| \le Ch^2\left\|w\right\|_4|||v|||, \tag{27}$$

$$\left|l_2(w,v)\right| \le Ch^2\left\|w\right\|_4|||v|||, \tag{28}$$

$$\left|l_3(w,v)\right| \le Ch\left\|w\right\|_4|||v|||, \tag{29}$$

$$\left|l_3(w,v)\right| \le \lambda^{\frac{1}{2}}C\left\|w\right\|_4\left\|v\right\|+\lambda^{-\frac{1}{2}}Ch^2\left\|w\right\|_4|||v|||. \tag{30}$$

Proof. From the Cauchy-Schwarz inequality, (21)-(22) and Lemma 2, we derive

$$\begin{aligned}\left|l_1(w,v)\right| =&\left|\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta w-\mathbb{Q}_h\Delta w)\cdot\mathbf{n},Q_b(v_0-v_b)\rangle_{\partial T}\right|\\ \le&\left(\sum_{T\in\mathcal{T}_h}h_T^3\left\|\nabla(\Delta w-\mathbb{Q}_h\Delta w)\right\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}h_T^{-3}\left\|Q_bv_0-v_b\right\|_{\partial T}^2\right)^{\frac{1}{2}}\\ \le& Ch^k\left\|w\right\|_{k+2}\left\|v\right\|_{2,h}\\ \le& Ch^k\left\|w\right\|_{k+2}|||v|||,\end{aligned}$$

and

$$\begin{aligned}
\left|l_2(w,v)\right| =&\left|\sum_{T\in\mathcal{T}_h}\langle \Delta w-\mathbb{Q}_h\Delta w, (\nabla v_0-v_n\mathbf{n}_e)\cdot\mathbf{n}\rangle_{\partial T}\right|\\
\le&\left(\sum_{T\in\mathcal{T}_h} h_T\left\|\Delta w-\mathbb{Q}_h\Delta w\right\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\left\|(\nabla v_0-v_n\mathbf{n}_e)\cdot\mathbf{n}\right\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le&Ch^k\left\|w\right\|_{k+2}\left\|v\right\|_{2,h}\\
\le&Ch^k\left\|w\right\|_{k+2}|||v|||.
\end{aligned}$$

Since $\langle\nabla(Q_0\Delta w)\cdot\mathbf{n}, v_0-Q_bv_0\rangle_{\partial\mathcal{T}_h}=0$, the Cauchy-Schwarz inequality, the trace inequality, the projection inequality and (A.4), we obtain

$$\begin{aligned}
&\left|l_3(w,v)\right|\\
=&\left|\langle\nabla(\Delta w)\cdot\mathbf{n}, v_0-Q_bv_0\rangle_{\partial\mathcal{T}_h}\right|\\
=&\left|\langle\nabla(\Delta w-Q_0\Delta w)\cdot\mathbf{n}, v_0-Q_bv_0\rangle_{\partial\mathcal{T}_h}\right|\\
\le&Ch^{-\frac{3}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^3\left\|\nabla(\Delta w-Q_0\Delta w)\right\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}\left\|v_0-Q_bv_0\right\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le&Ch^{-\frac{3}{2}}\left(\sum_{T\in\mathcal{T}_h}(h_T^2\left\|\nabla(\Delta w-Q_0\Delta w)\right\|_T^2+h_T^4\left\|\nabla^2(\Delta w-Q_0\Delta w)\right\|_T^2)\right)^{\frac{1}{2}}Ch^{\frac{1}{2}}|||v|||\\
\le&Ch^{-\frac{3}{2}}Ch^k\left\|w\right\|_{k+2}Ch^{\frac{1}{2}}|||v|||\\
\le&Ch^{k-1}\left\|w\right\|_{k+2}|||v|||.
\end{aligned}$$

If we use (A.5) instead of (A.4), we have

$$\begin{aligned}
\left|l_3(w,v)\right|\le&Ch^{-\frac{3}{2}}Ch^k\left\|w\right\|_{k+2}(\lambda^{\frac{1}{2}}Ch^{-\frac{1}{2}}\left\|v\right\|+\lambda^{-\frac{1}{2}}Ch^{\frac{3}{2}}|||v|||)\\
\le&\lambda^{\frac{1}{2}}Ch^{k-2}\left\|w\right\|_{k+2}\left\|v\right\|+\lambda^{-\frac{1}{2}}Ch^k\left\|w\right\|_{k+2}|||v|||.
\end{aligned}$$

For $k=2$, the estimates (27)-(30) hold. □

Remark 1. *From (25) and (29), we have the estimates of $l_3(\cdot,\cdot)$, which are used to bound errors in the H^2 norm. However, using them to analyze errors in the L^2 norm will result in a lower order of error convergence. For higher convergence order in the L^2 norm, we derive (26) and (30).*

Lemma 6. *If $u\in H^{k+2}(\Omega)$, we have*

$$|||Q_hu-u|||\le Ch^{k-1}\left\|u\right\|_{k+2}. \tag{31}$$

Proof. Using the definition of Δ_w, integration by parts, the Cauchy-Schwarz inequality, the projection inequalities, the trace inequality and the inverse inequality, we get

$$
\begin{aligned}
&(\Delta_w(Q_h u - u), q)_T\\
=&(\Delta(Q_0 u - u), q)_T + \langle Q_b(Q_0 u - u - (Q_b u - u)), \nabla q \cdot \mathbf{n}\rangle_{\partial T}\\
&-\langle \nabla(Q_0 u - u)\cdot \mathbf{n} - (Q_b(\nabla u \cdot \mathbf{n}_e)\mathbf{n}_e \cdot \mathbf{n} - \nabla u \cdot \mathbf{n}), q\rangle_{\partial T}\\
=&(Q_0 u - u, \Delta q)_T - \langle Q_0 u - u, \nabla q \cdot \mathbf{n}\rangle_{\partial T} + \langle \nabla(Q_0 u - u)\cdot \mathbf{n}, q\rangle_{\partial T}\\
&+\langle Q_b(Q_0 u - Q_b u), \nabla q \cdot \mathbf{n}\rangle_{\partial T} - \langle \nabla(Q_0 u - u)\cdot \mathbf{n} - (Q_b(\nabla u \cdot \mathbf{n}) - \nabla u \cdot \mathbf{n}), q\rangle_{\partial T}\\
=&(Q_0 u - u, \Delta q)_T + \langle Q_b(Q_0 u - Q_b u) - (Q_0 u - u), \nabla q \cdot \mathbf{n}\rangle_{\partial T}\\
&+\langle Q_b(\nabla u \cdot \mathbf{n}) - \nabla u \cdot \mathbf{n}, q\rangle_{\partial T}\\
\le&\left\|Q_0 u - u\right\|_T \left\|\Delta q\right\|_T + (\left\|Q_b(Q_0 u - u)\right\|_{\partial T} + \left\|Q_0 u - u\right\|_{\partial T}) \left\|\nabla q\right\|_{\partial T}\\
&+\left\|Q_b(\nabla u \cdot \mathbf{n}) - \nabla u \cdot \mathbf{n}\right\|_{\partial T} \left\|q\right\|_{\partial T}\\
\le&Ch^{k+1}|u|_{k+1,T}h^{-2}\left\|q\right\|_T + 2\left\|Q_0 u - u\right\|_{\partial T} Ch^{-\frac{3}{2}}\left\|q\right\|_T + Ch^{-\frac{1}{2}}h^k|\nabla u|_{k,T}h^{-\frac{1}{2}}\left\|q\right\|_T\\
\le&Ch^{k-1}|u|_{k+1,T}\left\|q\right\|_T + Ch^{-\frac{1}{2}}h^{k+1}|u|_{k+1,T}h^{-\frac{3}{2}}\left\|q\right\|_T + Ch^{k-1}|u|_{k+1,T}\left\|q\right\|_T\\
\le&Ch^{k-1}|u|_{k+1,T}\left\|q\right\|_T
\end{aligned}
$$

for any $q \in P_j(T)$. By choosing $q = \Delta_w(Q_h u - u)$, we prove (31). □

Remark 2. *If $\phi \in H^4(\Omega)$, the same proof procedure can be employed to obtain*

$$
|||\phi - Q_h\phi||| \le Ch^{k_0-1}\left\|\phi\right\|_4, \tag{32}
$$

where $k_0 = \min\{k, 3\}$.

3.2. Error estimates. Next, we derive the estimates of the errors in the discrete H^2 and L^2 norms. The following theorem shows the convergence rates of e_h and ε_h in the H^2 norm.

Theorem 3. *Suppose $u \in H^{k+2}(\Omega)$. Then we have*

$$
|||Q_h u - u_h||| \le Ch^{k-1}\left\|u\right\|_{k+2}, \tag{33}
$$

$$
|||u - u_h||| \le Ch^{k-1}\left\|u\right\|_{k+2}. \tag{34}
$$

Proof. Setting $v = Q_h u - u_h$ in (17) and using (23)-(25) and (31), we find

$$
\begin{aligned}
|||\varepsilon_h|||^2 \le&\left|l_1(u,\varepsilon_h)\right| + \left|l_2(u,\varepsilon_h)\right| + \left|l_3(u,\varepsilon_h)\right| + |||Q_h u - u|||\,|||\varepsilon_h|||\\
\le&Ch^k\left\|u\right\|_{k+2}|||\varepsilon_h||| + Ch^{k-1}\left\|u\right\|_{k+2}|||\varepsilon_h|||\\
\le&Ch^{k-1}\left\|u\right\|_{k+2}|||\varepsilon_h|||,
\end{aligned}
$$

which implies (33). This yields

$$
\begin{aligned}
|||u - u_h||| \le& |||u - Q_h u||| + |||Q_h u - u_h|||\\
\le&Ch^{k-1}\left\|u\right\|_{k+2}.
\end{aligned}
$$

□

For estimating the error convergence order in the L^2 norm, we need to consider the following dual problem:

$$
\Delta^2\phi = \varepsilon_0, \qquad \text{in } \Omega, \tag{35}
$$

$$
\phi = 0, \qquad \text{on } \partial\Omega, \tag{36}
$$

$$
\frac{\partial\phi}{\partial\mathbf{n}} = 0, \qquad \text{on } \partial\Omega, \tag{37}
$$

and assume $\left\|\phi\right\|_4 \le C\left\|\varepsilon_0\right\|$. Then we derive the estimates as follows.

Theorem 4. *When* $u \in H^{k+2}(\Omega)$*, we arrive at*

$$\|Q_0 u - u_0\| \le Ch^{k+k_0-2} \|u\|_{k+2}, \tag{38}$$

$$\|u - u_0\| \le Ch^{k+k_0-2} \|u\|_{k+2}, \tag{39}$$

where $k_0 = \min\{k, 3\}$*.*

Proof. From (18) and (17), we obtain

$$\begin{aligned}
&\|\varepsilon_0\|^2 \\
=&(\Delta^2\phi, \varepsilon_0)_{\mathcal{T}_h} \\
=&(\Delta_w\phi, \Delta_w\varepsilon_h)_{\mathcal{T}_h} + l_1(\phi, \varepsilon_h) - l_2(\phi, \varepsilon_h) + l_3(\phi, \varepsilon_h) \\
=&(\Delta_w Q_h\phi, \Delta_w\varepsilon_h)_{\mathcal{T}_h} + (\Delta_w(\phi - Q_h\phi), \Delta_w\varepsilon_h)_{\mathcal{T}_h} + l_1(\phi, \varepsilon_h) - l_2(\phi, \varepsilon_h) + l_3(\phi, \varepsilon_h) \\
=&-l_1(u, Q_h\phi) + l_2(u, Q_h\phi) - l_3(u, Q_h\phi) + (\Delta_w(Q_h u - u), \Delta_w Q_h\phi)_{\mathcal{T}_h} \\
&+ (\Delta_w(\phi - Q_h\phi), \Delta_w\varepsilon_h)_{\mathcal{T}_h} + l_1(\phi, \varepsilon_h) - l_2(\phi, \varepsilon_h) + l_3(\phi, \varepsilon_h) \\
=&I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\end{aligned}$$

For I_1, I_2 and I_3, by employing the Cauchy-Schwarz inequality, (21)-(22), the definition of the projection operator Q_b, the trace inequality, and the projection inequality, we get

$$\begin{aligned}
\left|I_1\right| =&\left|\langle\nabla(\Delta u - \mathbb{Q}_h\Delta u)\cdot\mathbf{n}, Q_b(Q_0\phi - Q_b\phi)\rangle_{\partial\mathcal{T}_h}\right| \\
\le&\left(\sum_{T\in\mathcal{T}_h} h_T^3 \|\nabla(\Delta u - \mathbb{Q}_h\Delta u)\|_{\partial T}^2\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^{-3} \|Q_b(Q_0\phi - Q_b\phi)\|_{\partial T}^2\right)^{\frac{1}{2}} \\
\le&Ch^k \|u\|_{k+2} \left(\sum_{T\in\mathcal{T}_h} h_T^{-3} \|Q_0\phi - \phi\|_{\partial T}^2\right)^{\frac{1}{2}} \\
\le&Ch^k \|u\|_{k+2} \left(\sum_{T\in\mathcal{T}_h} (h_T^{-4} \|Q_0\phi - \phi\|_T^2 + h_T^{-2} \|\nabla(Q_0\phi - \phi)\|_T^2)\right)^{\frac{1}{2}} \\
\le&Ch^k \|u\|_{k+2} Ch^{k_0-1} \|\phi\|_4 \\
\le&Ch^{k+k_0-1} \|u\|_{k+2} \|\phi\|_4,
\end{aligned}$$

$$\begin{aligned}
\left|I_2\right| =&\left|\langle\Delta u - \mathbb{Q}_h\Delta u, (\nabla Q_0\phi - Q_b(\nabla\phi\cdot\mathbf{n}_e)\mathbf{n}_e)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\right| \\
\le&\left(\sum_{T\in\mathcal{T}_h} h_T \|\Delta u - \mathbb{Q}_h\Delta u\|_{\partial T}^2\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|\nabla(Q_0\phi)\cdot\mathbf{n} - Q_b(\nabla\phi\cdot\mathbf{n})\|_{\partial T}^2\right)^{\frac{1}{2}} \\
\le&Ch^k \|u\|_{k+2} \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|\nabla(Q_0\phi - \phi)\cdot\mathbf{n}\|_{\partial T}\right)^{\frac{1}{2}} \\
\le&Ch^k \|u\|_{k+2} h^{-2} Ch^{k_0+1} \|\phi\|_4 \\
\le&Ch^{k+k_0-1} \|u\|_{k+2} \|\phi\|_4,
\end{aligned}$$

and

$$\begin{aligned}
\left|I_3\right| =&\left|\langle\nabla(\Delta u)\cdot\mathbf{n}, Q_0\phi - Q_bQ_0\phi\rangle_{\partial\mathcal{T}_h}\right|\\
=&\left|\langle\nabla(\Delta u - Q_0\Delta u)\cdot\mathbf{n}, Q_0\phi - Q_bQ_0\phi\rangle_{\partial\mathcal{T}_h}\right|\\
=&\left|\langle\nabla(\Delta u - Q_0\Delta u)\cdot\mathbf{n}, Q_0\phi - \phi + \phi - Q_b\phi + Q_b\phi - Q_bQ_0\phi\rangle_{\partial\mathcal{T}_h}\right|\\
=&\left|\langle\nabla(\Delta u - Q_0\Delta u)\cdot\mathbf{n}, Q_0\phi - \phi\rangle_{\partial\mathcal{T}_h} + \langle\nabla(\Delta u - Q_0\Delta u)\cdot\mathbf{n}, Q_b\phi - Q_bQ_0\phi\rangle_{\partial\mathcal{T}_h}\right|\\
\le& Ch^{-\frac{3}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^3\left\|\nabla(\Delta u - Q_0\Delta u)\right\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}\left\|Q_0\phi-\phi\right\|_{\partial T}^2\right)^{\frac{1}{2}}\\
\le& Ch^{-\frac{3}{2}}Ch^{k}\left\|\Delta u\right\|_k Ch^{-\frac{1}{2}}Ch^{k_0+1}\left\|\phi\right\|_4\\
\le& Ch^{k+k_0-1}\left\|u\right\|_{k+2}\left\|\phi\right\|_4,
\end{aligned}$$

where we utilized

$$\begin{aligned}
&\langle\nabla(\Delta u - Q_0\Delta u)\cdot\mathbf{n}, \phi - Q_b\phi\rangle_{\partial\mathcal{T}_h}\\
=&\langle\nabla(\Delta u)\cdot\mathbf{n}, \phi - Q_b\phi\rangle_{\partial\mathcal{T}_h} - \langle\nabla(Q_0\Delta u)\cdot\mathbf{n}, \phi - Q_b\phi\rangle_{\partial\mathcal{T}_h} = 0.
\end{aligned}$$

If we assume P_1 is the L^2 projection onto the polynomial space $P_1(T)$, the following result holds:

$$\begin{aligned}
&(\Delta_w(Q_hu-u), P_1\Delta\phi)_{\mathcal{T}_h}\\
=&(\Delta(Q_0u-u), P_1\Delta\phi)_{\mathcal{T}_h} + \langle Q_b(Q_0u - u - (Q_bu-u)), \nabla(P_1\Delta\phi)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&-\langle\nabla(Q_0u-u)\cdot\mathbf{n} - Q_b(\nabla u\cdot\mathbf{n}_e)\mathbf{n}_e\cdot\mathbf{n} + \nabla u\cdot\mathbf{n}, P_1\Delta\phi\rangle_{\partial\mathcal{T}_h}\\
=&(Q_0u-u, \Delta P_1\Delta\phi)_{\mathcal{T}_h} - \langle Q_0u-u, \nabla P_1\Delta\phi\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}\\
&+\langle Q_b(Q_0u-Q_bu), \nabla(P_1\Delta\phi)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} + \langle Q_b(\nabla u\cdot\mathbf{n}) - \nabla u\cdot\mathbf{n}, P_1\Delta\phi\rangle_{\partial\mathcal{T}_h}\\
=&0 - \langle Q_0u-u, \nabla P_1\Delta\phi\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} + \langle Q_0u-u, \nabla(P_1\Delta\phi)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} + 0\\
=&0,
\end{aligned}$$

where we used the definition of Δ_w, integration by parts, and the definitions of projection operators.

By the above equation, Lemma 1, the Cauchy-Schwarz inequality, the definition of $\mathbb{Q}_h$, (31)-(32), and the projection inequality, we know

$$\begin{aligned}
\left|I_4\right| =&\left|(\Delta_w(Q_hu-u), \Delta_wQ_h\phi)_{\mathcal{T}_h}\right|\\
\le&\left|(\Delta_w(Q_hu-u), \Delta_w(Q_h\phi-\phi))_{\mathcal{T}_h}\right| + \left|(\Delta_w(Q_hu-u), \Delta_w\phi)_{\mathcal{T}_h}\right|\\
\le& |||Q_hu-u|||\,|||Q_h\phi-\phi||| + \left|(\Delta_w(Q_hu-u), \mathbb{Q}_h\Delta\phi)_{\mathcal{T}_h}\right|\\
\le& Ch^{k-1}\left\|u\right\|_{k+2}Ch^{k_0-1}\left\|\phi\right\|_4 + \left|(\Delta_w(Q_hu-u), \Delta\phi)_{\mathcal{T}_h}\right|\\
\le& Ch^{k+k_0-2}\left\|u\right\|_{k+2}\left\|\phi\right\|_4 + \left|(\Delta_w(Q_hu-u), \Delta\phi - P_1\Delta\phi)_{\mathcal{T}_h}\right|\\
\le& Ch^{k+k_0-2}\left\|u\right\|_{k+2}\left\|\phi\right\|_4 + |||Q_hu-u|||\left\|\Delta\phi - P_1\Delta\phi\right\|\\
\le& Ch^{k+k_0-2}\left\|u\right\|_{k+2}\left\|\phi\right\|_4 + Ch^{k-1}\left\|u\right\|_{k+2}Ch^2\left\|\phi\right\|_4\\
\le& Ch^{k+k_0-2}\left\|u\right\|_{k+2}\left\|\phi\right\|_4.
\end{aligned}$$

From (32) and (33), there exists

$$\begin{aligned}\left|I_5\right| =&\left|(\Delta_w(\phi-Q_h\phi),\Delta_w\varepsilon_h)_{\mathcal{T}_h}\right|\\ \leq& \,|||\phi-Q_h\phi|||\,|||\varepsilon_h|||\\ \leq& Ch^{k_0-1}\left\|\phi\right\|_4 Ch^{k-1}\left\|u\right\|_{k+2}\\ \leq& Ch^{k+k_0-2}\left\|u\right\|_{k+2}\left\|\phi\right\|_4.\end{aligned}$$

Using (27)-(30) and (33), we find

$$\begin{aligned}\left|I_6\right| =&\left|l_1(\phi,\varepsilon_h)\right|\\ \leq& Ch^2\left\|\phi\right\|_4|||\varepsilon_h|||\\ \leq& Ch^2\left\|\phi\right\|_4 Ch^{k-1}\left\|u\right\|_{k+2}\\ \leq& Ch^{k+1}\left\|u\right\|_{k+2}\left\|\phi\right\|_4,\end{aligned}$$

$$\begin{aligned}\left|I_7\right| =&\left|l_2(\phi,\varepsilon_h)\right|\\ \leq& Ch^2\left\|\phi\right\|_4|||\varepsilon_h|||\\ \leq& Ch^2\left\|\phi\right\|_4 Ch^{k-1}\left\|u\right\|_{k+2}\\ \leq& Ch^{k+1}\left\|u\right\|_{k+2}\left\|\phi\right\|_4,\end{aligned}$$

and

$$\begin{aligned}\left|I_8\right| =&\left|l_3(\phi,\varepsilon_h)\right|\\ \leq&\lambda^{\frac{1}{2}}C\left\|\phi\right\|_4\left\|\varepsilon_h\right\|+\lambda^{-\frac{1}{2}}Ch^2\left\|\phi\right\|_4|||\varepsilon_h|||\\ \leq&\lambda^{\frac{1}{2}}C\left\|\phi\right\|_4\left\|\varepsilon_0\right\|+\lambda^{-\frac{1}{2}}Ch^{k+1}\left\|\phi\right\|_4\left\|u\right\|_{k+2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left\|\varepsilon_0\right\|^2 \leq&\lambda^{\frac{1}{2}}C\left\|\phi\right\|_4\left\|\varepsilon_0\right\|+\lambda^{-\frac{1}{2}}Ch^{k+1}\left\|\phi\right\|_4\left\|u\right\|_{k+2}+Ch^{k+1}\left\|u\right\|_{k+2}\left\|\phi\right\|_4\\ &+Ch^{k+k_0-2}\left\|u\right\|_{k+2}\left\|\phi\right\|_4+Ch^{k+k_0-1}\left\|u\right\|_{k+2}\left\|\phi\right\|_4.\end{aligned}$$

If λ is chosen such that $\lambda^{\frac{1}{2}}C\left\|\phi\right\|_4\leq\frac{1}{2}\left\|\varepsilon_0\right\|$, we get

$$\begin{aligned}\left\|\varepsilon_0\right\|^2 \leq&\frac{1}{2}\left\|\varepsilon_0\right\|^2+Ch^{k+1}\left\|u\right\|_{k+2}\left\|\phi\right\|_4+Ch^{k+k_0-2}\left\|u\right\|_{k+2}\left\|\phi\right\|_4\\ \leq&\frac{1}{2}\left\|\varepsilon_0\right\|^2+Ch^{k+1}\left\|u\right\|_{k+2}\left\|\varepsilon_0\right\|+Ch^{k+k_0-2}\left\|u\right\|_{k+2}\left\|\varepsilon_0\right\|,\end{aligned}$$

which implies

$$\left\|\varepsilon_0\right\|\leq Ch^{k+k_0-2}\left\|u\right\|_{k+2}.$$

□

4. Numerical experiments

In this section, we shall employ numerical examples to validate the theoretical results using triangular meshes in Figure 1 and polygonal meshes in Figure 2.

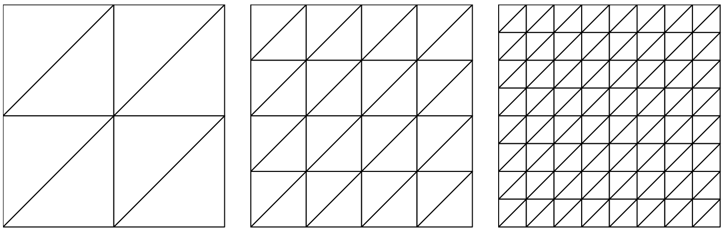

FIGURE 1. The uniform triangular meshes with $n = 2,\ 4,\ 8$.

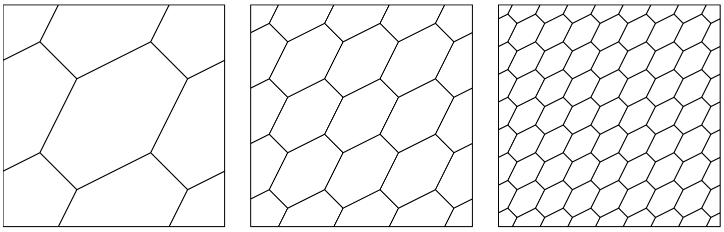

FIGURE 2. The polygonal meshes with $n = 2,\ 4,\ 8$.

4.1. Example 1. We consider the square domain $\Omega = (0,1)^2$ and the exact solution

$$u = x^2(1-x)^2y^2(1-y)^2. \tag{40}$$

On triangular meshes, we set $j = k + 2$. When $k = 2$, Table 1 shows that the convergence orders of e_h are $O(h)$ in the H^2 norm and $O(h^2)$ in the L^2 norm. For $k = 3$, the error convergence rates in the H^2 and L^2 norms are of orders $O(h^2)$ and $O(h^4)$ in Table 2. Furthermore, the results shown in Tables 1-2 coincide with the theorems in the previous section.

TABLE 1. Results for (40) on triangular meshes with $k = 2$.

n	$\|\|\|u-u_h\|\|\|$	Rate	$\|u-u_h\|_{2,h}$	Rate	$\|u-u_0\|$	Rate
8	9.5700E-03	—	1.5165E-02	—	4.6178E-05	—
16	4.7707E-03	1.00	7.5996E-03	1.00	1.1483E-05	2.01
32	2.3665E-03	1.01	3.7959E-03	1.00	2.8221E-06	2.02
64	1.1760E-03	1.01	1.8959E-03	1.00	6.9606E-07	2.02
128	5.8582E-04	1.01	9.4727E-04	1.00	1.7229E-07	2.01

TABLE 2. Results for (40) on triangular meshes with $k = 3$.

n	$\|\|\|u-u_h\|\|\|$	Rate	$\|u-u_h\|_{2,h}$	Rate	$\|u-u_0\|$	Rate
2	1.6283E-02	—	2.5831E-02	—	1.4332E-04	—
4	4.0486E-03	2.01	7.0678E-03	1.87	9.8818E-06	3.86
8	1.0417E-03	1.96	1.8704E-03	1.92	6.7580E-07	3.87
16	2.6142E-04	1.99	4.7555E-04	1.98	4.2635E-08	3.99
32	6.5097E-05	2.01	1.1948E-04	1.99	2.6431E-09	4.01

On polygonal meshes, let $j = k + 4$, and Tables 3-4 show the convergence rates consistent with our theoretical analysis.

TABLE 3. Results for (40) on polygonal meshes with $k = 2$.

n	$\|\|\|u - u_h\|\|\|$	Rate	$\|u - u_h\|_{2,h}$	Rate	$\|u - u_0\|$	Rate
16	1.2242E-02	—	1.0176E-02	—	7.4200E-05	—
32	6.5755E-03	0.90	5.2017E-03	0.97	2.1480E-05	1.79
64	3.3740E-03	0.96	2.6315E-03	0.98	5.6675E-06	1.92
128	1.7046E-03	0.99	1.3245E-03	0.99	1.4475E-06	1.97
256	8.5618E-04	0.99	6.6462E-04	0.99	3.6900E-07	1.97

TABLE 4. Results for (40) on polygonal meshes with $k = 3$.

n	$\|\|\|u - u_h\|\|\|$	Rate	$\|u - u_h\|_{2,h}$	Rate	$\|u - u_0\|$	Rate
4	6.9507E-03	—	8.4116E-03	—	2.5430E-05	—
8	2.2107E-03	1.65	2.5972E-03	1.70	2.5638E-06	3.31
16	6.1553E-04	1.84	7.1587E-04	1.86	2.0686E-07	3.63
32	1.5974E-04	1.95	1.8464E-04	1.96	1.4303E-08	3.85
64	4.0278E-05	1.99	4.6334E-05	1.99	8.2573E-10	4.11

4.2. Example 2.

We choose the same square domain as in the previous example, and the exact solution is

$$u = \sin(\pi x)\sin(\pi y). \tag{41}$$

Set $j = k+2$ on triangular meshes and $j = k+4$ on polygonal meshes; the results are shown in Tables 5-6 and 7-8, respectively. These convergence orders match the theoretical results.

TABLE 5. Results for (41) on triangular meshes with $k = 2$.

n	$\|\|\|u - u_h\|\|\|$	Rate	$\|u - u_h\|_{2,h}$	Rate	$\|u - u_0\|$	Rate
8	9.4121E-01	—	1.3644E+00	—	3.4904E-03	—
16	4.7017E-01	1.00	6.8351E-01	1.00	8.9894E-04	1.96
32	2.3481E-01	1.00	3.4193E-01	1.00	2.2798E-04	1.98
64	1.1732E-01	1.00	1.7099E-01	1.00	5.7391E-05	1.99
128	5.8633E-02	1.00	8.5497E-02	1.00	1.4433E-05	1.99

TABLE 6. Results for (41) on triangular meshes with $k = 3$.

n	$\|\|\|u - u_h\|\|\|$	Rate	$\|u - u_h\|_{2,h}$	Rate	$\|u - u_0\|$	Rate
2	1.0474E+00	—	1.6703E+00	—	8.5304E-03	—
4	2.7187E-01	1.95	4.5396E-01	1.88	5.6923E-04	3.91
8	6.8241E-02	1.99	1.1647E-01	1.96	3.6278E-05	3.97
16	1.7027E-02	2.00	2.9360E-02	1.99	2.2723E-06	4.00
32	4.2486E-03	2.00	7.3603E-03	2.00	1.4099E-07	4.01

5. Conclusions

In the paper, we use weak functions $(P_k(T), P_{k-1}(e), P_{k-1}(e))$ to build the SFWG numerical scheme for the biharmonic equation. To achieve the optimal convergence orders of the errors, we modify the definition of the weak Laplacian in [17]. Finally,

Table 7. Results for (41) on polygonal meshes with $k = 2$.

n	$\|\|\|u - u_h\|\|\|$	Rate	$\|u - u_h\|_{2,h}$	Rate	$\|u - u_0\|$	Rate
8	2.9734E+00	—	1.8766E+00	—	1.2437E-02	—
16	1.5486E+00	0.94	9.9298E-01	0.92	3.7323E-03	1.74
32	7.8404E-01	0.98	5.0677E-01	0.97	9.8116E-04	1.93
64	3.9390E-01	0.99	2.5551E-01	0.99	2.4850E-04	1.98
128	1.9736E-01	1.00	1.2822E-01	0.99	6.2082E-05	2.00

Table 8. Results for (41) on polygonal meshes with $k = 3$.

n	$\|\|\|u - u_h\|\|\|$	Rate	$\|u - u_h\|_{2,h}$	Rate	$\|u - u_0\|$	Rate
4	5.2002E-01	—	5.9866E-01	—	1.1222E-03	—
8	1.3220E-01	1.98	1.6706E-01	1.84	8.8280E-05	3.67
16	3.3449E-02	1.98	4.3703E-02	1.93	6.1111E-06	3.85
32	8.4305E-03	1.99	1.1141E-02	1.97	3.9362E-07	3.96
64	2.1173E-03	1.99	2.8113E-03	1.99	1.7194E-08	4.52

the convergence rates in the H^2 and L^2 norms are of order $O(h^{k-1})$ and $O(h^{k+k_0-2})$ ($k_0 = \min\{k, 3\}$), which are supported by numerical examples.

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Appendix A. Some Technical Results

Lemma 7. [19] *There are positive constants* C *and small enough* λ *such that*

$$\sum_{T\in\mathcal{T}_h} \|\nabla v_0\|_T^2 \le C\,\|v\|_{2,h}^2\,, \quad \forall\, v \in V_h^0, \tag{A.1}$$

and

$$\sum_{T\in\mathcal{T}_h} \|\nabla v_0\|_T^2 \le \lambda h^{-2}\,\|v\|^2 + \frac{C}{\lambda} h^2\,\|v\|_{2,h}^2\,, \quad \forall\, v \in V_h^0. \tag{A.2}$$

Proof. For each $v \in V_h^0$, denote by T_L and T_R the two elements sharing $e \in \mathcal{E}_h^0$, v_{0L} and v_{0R} are the traces of v_0 on e from T_L and T_R, $\mathbf{n}_L$ and $\mathbf{n}_R$ are respectively the unit outward normal vectors on e with respect to T_L and T_R. If $e \subset \partial\Omega$, the traces of v_0 and $\nabla v_0 \cdot \mathbf{n}$ seen from outside of Ω are defined as 0. Thus, when $e \in \mathcal{E}_h^0$, the following inequality holds true:

$$\begin{aligned}\int_e \left(\frac{\partial v_{0L}}{\partial \mathbf{n}_L} + \frac{\partial v_{0R}}{\partial \mathbf{n}_R}\right)^2 ds &= \int_e \left(\frac{\partial v_{0L}}{\partial \mathbf{n}_L} - v_n\mathbf{n}_e\cdot\mathbf{n}_L - v_n\mathbf{n}_e\cdot\mathbf{n}_R + \frac{\partial v_{0R}}{\partial \mathbf{n}_R}\right)^2 ds \\ &\le C\int_e |(\nabla v_{0L} - v_n\mathbf{n}_e)\cdot\mathbf{n}_L|^2 + |(\nabla v_{0R} - v_n\mathbf{n}_e)\cdot\mathbf{n}_R|^2\,ds.\end{aligned}$$

When $e \subset \partial\Omega$, since $v_n\big|_e = 0$, we obtain

$$\int_e \left(\frac{\partial v_{0L}}{\partial \mathbf{n}_L} + \frac{\partial v_{0R}}{\partial \mathbf{n}_R}\right)^2 ds = \int_e \left(\frac{\partial v_{0L}}{\partial \mathbf{n}_L}\right)^2 ds = \int_e |(\nabla v_{0L} - v_n\mathbf{n}_e)\cdot\mathbf{n}_L|^2\,ds.$$

The above two inequalities imply

$$\sum_{e\in\mathcal{E}_h}\int_e\left(\frac{\partial v_{0L}}{\partial\mathbf{n}_L}+\frac{\partial v_{0R}}{\partial\mathbf{n}_R}\right)^2 ds\le C\sum_{T\in\mathcal{T}_h}\|\nabla v_0\cdot\mathbf{n}_e-v_n\|_{\partial T}^2 .$$

Similarly, we have

$$\sum_{e\in\mathcal{E}_h}\int_e\left(Q_bv_{0R}-Q_bv_{0L}\right)^2 ds\le C\sum_{T\in\mathcal{T}_h}\|Q_bv_0-v_b\|_{\partial T}^2 .$$

Then, we have

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2\\
\le&\varepsilon\sum_{T\in\mathcal{T}_h}\|v_0\|_T^2+C\varepsilon^{-1}\sum_{T\in\mathcal{T}_h}\|\Delta v_0\|_T^2+C\varepsilon^{-1}h^{-1}\sum_{e\in\mathcal{E}_h}\int_e\left(\frac{\partial v_{0L}}{\partial\mathbf{n}_L}+\frac{\partial v_{0R}}{\partial\mathbf{n}_R}\right)^2 ds\\
&+Ch^{-1}\left(\sum_{e\in\mathcal{E}_h}\int_e\left(Q_bv_{0R}-Q_bv_{0L}\right)^2 ds\right)\\
\le&\varepsilon\sum_{T\in\mathcal{T}_h}\|v_0\|_T^2+C\varepsilon^{-1}\sum_{T\in\mathcal{T}_h}\|\Delta v_0\|_T^2+C\varepsilon^{-1}h^{-1}\left(\sum_{T\in\mathcal{T}_h}\|\nabla v_0\cdot\mathbf{n}_e-v_n\|_{\partial T}^2\right)\\
&+Ch^{-1}\left(\sum_{T\in\mathcal{T}_h}\|Q_bv_0-v_b\|_{\partial T}^2\right)\\
\le&\varepsilon\sum_{T\in\mathcal{T}_h}\|v_0\|_T^2+C\varepsilon^{-1}\sum_{T\in\mathcal{T}_h}\|\Delta v_0\|_T^2+C\varepsilon^{-1}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\nabla v_0\cdot\mathbf{n}_e-v_n\|_{\partial T}^2\right)\\
&+C\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|Q_bv_0-v_b\|_{\partial T}^2\right),
\end{aligned}$$

When $\varepsilon<1$ and $h_T<1$, we have

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2\le&\varepsilon^{-1}C\sum_{T\in\mathcal{T}_h}\left(\|\Delta v_0\|_T^2+h_T^{-3}\|Q_bv_0-v_b\|_{\partial T}^2+h_T^{-1}\|\nabla v_0\cdot\mathbf{n}_e-v_n\|_{\partial T}^2\right)\\
&+\varepsilon\sum_{T\in\mathcal{T}_h}\|v_0\|_T^2 ,
\end{aligned}$$

which implies

$$\sum_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2\le\varepsilon\sum_{T\in\mathcal{T}_h}\|v_0\|_T^2+\varepsilon^{-1}C\|v\|_{2,h}^2 . \tag{A.3}$$

Further, by Lemma A.7 in [19], we obtain

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2\le&\varepsilon^{-1}C\sum_{T\in\mathcal{T}_h}\left(\|\Delta v_0\|_T^2+h_T^{-3}\|Q_bv_0-v_b\|_{\partial T}^2+h_T^{-1}\|\nabla v_0\cdot\mathbf{n}_e-v_n\|_{\partial T}^2\right)\\
&+\varepsilon C\sum_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2 .
\end{aligned}$$

That is $\sum\limits_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2\le\varepsilon C\sum\limits_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2+\varepsilon^{-1}C\|v\|_{2,h}^2$. When ε is enough small, we get (A.1).

If λ is enough small, setting $\varepsilon=\lambda h^{-2}$ in (A.3), (A.2) holds true. □

Further, using the projection inequality and Lemma 2, we have

$$\left(\sum_{T\in\mathcal{T}_h}\|v_0-Q_bv_0\|_{\partial T}^2\right)^{\frac{1}{2}}\le Ch^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2\right)^{\frac{1}{2}},$$

which implies

$$\left(\sum_{T\in\mathcal{T}_h}\|v_0-Q_bv_0\|_{\partial T}^2\right)^{\frac{1}{2}}\le Ch^{\frac{1}{2}}\,|||v|||,\quad \forall\, v\in V_h^0, \tag{A.4}$$

$$\left(\sum_{T\in\mathcal{T}_h}\|v_0-Q_bv_0\|_{\partial T}^2\right)^{\frac{1}{2}}\le \lambda^{\frac{1}{2}}Ch^{-\frac{1}{2}}\,\|v\|+\lambda^{-\frac{1}{2}}Ch^{\frac{3}{2}}\,|||v|||,\quad \forall\, v\in V_h^0. \tag{A.5}$$

The above inequalities are utilized to estimate $l_3(\cdot,\cdot)$ in error equations. The results, like [19], fail to achieve the desired inequality as follows:

$$\left(\sum_{T\in\mathcal{T}_h}\|v_0-Q_bv_0\|_{\partial T}^2\right)^{\frac{1}{2}}\le Ch^{\frac{3}{2}}\,|||v|||,\quad \forall\, v\in V_h^0,$$

which is left to someone interested to prove.

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