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ANALYSIS OF TRANSMISSION LINE MODEL WITH UNCERTAIN PARAMETER USING THE PC–FDTD METHOD

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Abstract. Voltage and current wave propagation on transmission lines is commonly described by the telegrapher's equations. However, in practical applications, the parameters of the telegrapher's equations – such as inductance and capacitance – often exhibit uncertainty due to manufacturing tolerances, material inhomogeneities, and environmental variations. This study presents a numerical framework for solving the telegrapher's equations with uncertain parameters using a combination of Polynomial Chaos Expansion and the Finite Difference Time Domain (FDTD) method. The proposed approach enables efficient uncertainty quantification while maintaining computational tractability. In addition to the numerical formulation, the conditional stability of our method is analyzed, and the discrete dispersion relation is derived and compared with the continuous dispersion relation. The results demonstrate that the PC–FDTD method is a robust and accurate tool for modeling wave behavior in transmission lines under parameter uncertainty.

Key words. Maxwell's equations, telegrapher's equations, FDTD, polynomial chaos, uncertainty.

1. Introduction

Maxwell's equations are a set of fundamental equations in electromagnetism that describe how electric and magnetic fields propagate and interact with matter. These equations not only form the theoretical foundation of classical electrodynamics but also underpin a vast array of modern technologies, from wireless communication to power transmission. One notable application of Maxwell's equations is their reduction to simpler forms under specific assumptions, such as the derivation of the telegrapher's equations, which model the voltage and current in an electrical transmission line.

The telegrapher's equations are a partial differential equations that account for the distributed resistance, inductance, capacitance, and conductance of transmission lines. They plays a critical role in electrical engineering and mathematical physics, particularly in analyzing signal propagation, dispersion, and attenuation in conductive media. Mathematically, they are expressed in both hyperbolic and parabolic forms, depending on the modeling context, and have been extensively studied using both analytic and numerical methods.

Recent research efforts have focused on advancing solution techniques for the telegrapher's equations. Analytical approaches, such as the Laplace and Fourier transform methods, have been utilized to derive exact solutions under specific initial and boundary conditions [7, 10]. However, due to the complexity and variability of practical systems, numerical methods are increasingly being used to approximate solutions for more general and complex scenarios. Frequency domain methods have fewer numerical issues and are easier to implement. However, time domain approaches have a significant advantage in being able to model a broad range of frequencies in a single simulation. Time domain numerical methods – such as finite difference methods, finite element methods – [12, 13], and in particular

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the finite-difference time-domain (FDTD) method, are the most common way of approximating the time-domain response of transmission lines [4].

Extensions to the telegrapher's equations include incorporating interactions with other components which are modeled by coupling to additional equations such as Maxwell's equations, circuit theory, and thin-wire models [14]. Memory effects are modeled by coupling to additional constitutive laws, or modifying the telegrapher's equations by adding fractional derivatives [3]. In dispersive electromagnetic models, fractional derivative mechanisms can be accurately represented by endowing dielectric parameters with probability distributions accounting for microscale variation [6]. This is one motivation for the current work of treating random model parameters.

In practical applications, uncertainty in the parameters of the telegrapher's equations, such as resistance or inductance per unit length, can arise due to manufacturing tolerances, environmental effects, or material inconsistencies. This uncertainty poses significant challenges in modeling and simulation, as it can affect the reliability and accuracy of the solution. Many recent studies have begun to incorporate stochastic methods and uncertainty quantification frameworks in computational electromagnetics [5, 11, 15]. To our knowledge, random parameters in telegrapher's equations has not yet been considered.

The objective of this study is to develop a numerical method using polynomial chaos expansion for the telegrapher's equations with uncertain parameters. By treating these parameters as random variables, we aim to propose a robust computational framework that can accommodate uncertainty and improve predictive accuracy. The methodology will be validated through comparative studies and numerical experiments.

2. Model Formulation

2.1. Random Telegrapher's Equations. For voltage and current waves on transmission lines, application of Kirchoff's voltage and current laws, which are contained in Maxwell's equations, leads to the telegrapher's equations in the time domain:

$$\frac{\partial V}{\partial x} + L\frac{\partial I}{\partial t} + RI = 0, \tag{1a}$$

$$\frac{\partial I}{\partial x} + C\frac{\partial V}{\partial t} + GV = 0, \tag{1b}$$

where $V(x,t)$ and $I(x,t)$ are respectively the line voltage and current, while R, L, G, and C are respectively the distributed resistance, inductance, conductance, and capacitance of the line, in units of Ω/m, H/m, S/m, and F/m, respectively, and $(x,t) \in [0,X] \times [0,T]$. The system is completed by initial and boundary conditions, for example homogeneous initial conditions and homogeneous Dirichlet boundary conditions. The system of equations can be combined to derive a wave equation with only one dependent variable, either in terms of the voltage V or the current I:

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC + LG)\frac{\partial V}{\partial t} + RGV,$$

or

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC + LG)\frac{\partial I}{\partial t} + RGI.$$

If parameters R and G are zero, the equations model the lossless case.

Uncertainties in the primary parameters of transmission lines can arise due to material imperfections, environmental conditions, manufacturing tolerances, or operational conditions.

For simplicity, let us assume that the inductance is uncertain in this paper. Then, we can define the following random telegrapher's equations:

$$\frac{\partial V}{\partial x} + (L + r\xi)\frac{\partial I}{\partial t} + RI = 0, \tag{2a}$$

$$\frac{\partial I}{\partial x} + C\frac{\partial V}{\partial t} + GV = 0, \tag{2b}$$

where ξ is a random variable and r is a scaling constant.

2.2. Polynomial Chaos for Random Telegrapher's Equations. Polynomial Chaos (PC) is a mathematical method used to quantify uncertainty [6, 16], especially in systems influenced by random variables. It represents the input uncertainties as a series of orthogonal polynomials, enabling efficient computation of the resulting output uncertainties.

In Polynomial Chaos Expansion (PCE), stochastic processes or random variables are represented as a series of orthogonal polynomials (e.g., Hermite, Legendre, Laguerre), where each polynomial is associated with a specific probability distribution of the input random variable. These orthogonal polynomials satisfy a three-term recurrence relation, which is key to constructing the basis functions efficiently and computing integrals or stochastic Galerkin projections [17].

We utilize the Legendre Chaos expansion of degree p:

$$V(x,t,\xi) = \sum_{\ell=0}^{p} V_\ell(x,t)\phi_\ell(\xi), \tag{3a}$$

$$I(x,t,\xi) = \sum_{\ell=0}^{p} I_\ell(x,t)\phi_\ell(\xi), \tag{3b}$$

where $\phi_\ell(\xi)$ represents the ℓth degree Legendre polynomial. In this study, we assume that the random variable ξ follows a uniform distribution $\mathcal{U}[-1,1]$ (and $r < L$ to ensure positivity of $L + r\xi$). Then the random telegrapher's equations (2) become

$$\begin{aligned}\frac{\partial}{\partial x}\left(\sum_{\ell=0}^{p} V_\ell\phi_\ell\right) + (L+r\xi)\frac{\partial}{\partial t}\left(\sum_{\ell=0}^{p} I_\ell\phi_\ell\right) + R\left(\sum_{\ell=0}^{p} I_\ell\phi_\ell\right) &= 0,\\ \frac{\partial}{\partial x}\left(\sum_{\ell=0}^{p} I_\ell\phi_\ell\right) + C\frac{\partial}{\partial t}\left(\sum_{\ell=0}^{p} V_\ell\phi_\ell\right) + G\left(\sum_{\ell=0}^{p} V_\ell\phi_\ell\right) &= 0.\end{aligned}$$

Using the three-term recurrence relation for the orthogonal polynomials and a Galerkin projection onto the space of polynomials of degree at most p, we get

$$\begin{aligned}\frac{\partial V_\ell}{\partial x} + L\frac{\partial I_\ell}{\partial t} + r\left[\frac{\partial I_{\ell+1}}{\partial t}a_{\ell+1} + \frac{\partial I_\ell}{\partial t}b_\ell + \frac{\partial I_{\ell-1}}{\partial t}c_{\ell-1}\right] + RI_\ell &= 0,\\ \frac{\partial I_\ell}{\partial x} + C\frac{\partial V_\ell}{\partial t} + GV_\ell &= 0,\end{aligned}$$

for $\ell = 0, 1, \cdots, p$.

Let

$$\vec{V_p} = \begin{bmatrix} V_0(x,t) \\ V_1(x,t) \\ \vdots \\ V_p(x,t) \end{bmatrix} \text{ and } \vec{I_p} = \begin{bmatrix} I_0(x,t) \\ I_1(x,t) \\ \vdots \\ I_p(x,t) \end{bmatrix}.$$

Then the Jacobi matrix [17] M_p can be written as the following $(p+1)\times(p+1)$ matrix:

$$M_p = \begin{bmatrix} 0 & \frac{1}{3} & 0 & \cdots & 0 \\ 1 & 0 & \frac{2}{5} & \ddots & \vdots \\ 0 & \frac{2}{3} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{p}{2p+1} \\ 0 & \cdots & 0 & \frac{p}{2p-1} & 0 \end{bmatrix}.$$

Next, we define

$$A_p = L\,E_p + rM_p,$$

where E_p is the identity matrix of size $p+1$. Consequently, the polynomial chaos modal system can be expressed as

$$\frac{\partial \vec{V_p}}{\partial x} + A_p \frac{\partial \vec{I_p}}{\partial t} + R\vec{I_p} = \vec{0_p}, \tag{4a}$$

$$\frac{\partial \vec{I_p}}{\partial x} + C\frac{\partial \vec{V_p}}{\partial t} + G\vec{V_p} = \vec{0_p}, \tag{4b}$$

where $\vec{0_p}$ represents the zero vector of dimension $p+1$.

2.3. PC–FDTD method. The Finite Difference Time Domain (FDTD) method [9, 18], also known as the Yee scheme, is a numerical technique used to solve Maxwell's equations for electromagnetic wave propagation. It discretizes both time and space using finite differences, allowing for the direct computation of electric and magnetic fields over time. The method employs a staggered grid-based approach to update field values iteratively. As it applies centered differences, it is known to be second order accurate in space and time. FDTD is widely used in various applications, including antenna design, photonics, and electromagnetic compatibility analysis, due to its ability to handle complex geometries and broadband simulations.

Consider the spatial domain $[0, X]$ and the time interval $[0, T]$, where X and T are both greater than zero, with a spatial step size $\Delta x > 0$ and a time step $\Delta t > 0$. Define $M = X/\Delta x$ and $N = T/\Delta t$. For $j, n \in \mathbb{N}$, the primal grid is given by

$$0 = x_0 \le x_1 \le \cdots \le x_j \le \cdots \le x_M = X,$$

$$0 = t_0 \le t_1 \le \cdots \le t_n \le \cdots \le t_N = T,$$

where $x_j = j\Delta x$ and $t_n = n\Delta t$ for $j = 0, \cdots, M$ and $n = 0, \cdots, N$. The dual grid is given by $\{x_{j+\frac{1}{2}}\}$ for $j = 0, \cdots, M-1$ and $\{t_{n+\frac{1}{2}}\}$ for $n = 0, \cdots, N-1$. We take V to be defined on the primal grid, and I to be defined on the dual grid in the following.

Let U_j^n represent an approximate solution of $U(x_j, t_n)$. The temporal and spatial centered difference operators and a discrete time averaging operation are defined, respectively, as

$$\delta_t\,(U)_j^n = \frac{U_j^{n+\frac{1}{2}} - U_j^{n-\frac{1}{2}}}{\Delta t},$$

$$\delta_x (U)_j^n = \frac{U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n}{\Delta x},$$

and

$$\left(\overline{U}\right)_j^n = \frac{U_j^{n+\frac{1}{2}} + U_j^{n-\frac{1}{2}}}{2}.$$

By applying the FDTD method to the random telegrapher's equations (2), we obtain the following system referred to as the random FDTD method:

$$\delta_x (V)_{j+\frac{1}{2}}^n + (L + r\xi)\delta_t (I)_{j+\frac{1}{2}}^n + R\left(\overline{I}\right)_{j+\frac{1}{2}}^n = 0, \tag{5a}$$

$$\delta_x (I)_j^{n+\frac{1}{2}} + C\,\delta_t (V)_j^{n+\frac{1}{2}} + G\left(\overline{V}\right)_j^{n+\frac{1}{2}} = 0. \tag{5b}$$

Similarly, applying the FDTD method on the polynomial chaos (PC) modal system (4) yields the following system, which is referred to as the PC–FDTD method:

$$\delta_x \left(\vec{V}_p\right)_{j+\frac{1}{2}}^n + A_p\,\delta_t \left(\vec{I}_p\right)_{j+\frac{1}{2}}^n + R\left(\overline{\vec{I}_p}\right)_{j+\frac{1}{2}}^n = \vec{0}_p \tag{6a}$$

$$\delta_x \left(\vec{I}_p\right)_j^{n+\frac{1}{2}} + C\,\delta_t \left(\vec{V}_p\right)_j^{n+\frac{1}{2}} + G\left(\overline{\vec{V}_p}\right)_j^{n+\frac{1}{2}} = \vec{0}_p. \tag{6b}$$

Given appropriate initial conditions, the PC-FDTD method is implemented by first solving the linear system in (6a) for $\left(\vec{I}_p\right)_{j+\frac{1}{2}}^{\frac{1}{2}}$, and then (6b) for $\left(\vec{V}_p\right)_j^1$ for all j, repeating the alternating time stepping. As expected, our method achieves second order accuracy in both space and time.

3. Stability Analysis

In this section, we examine the numerical stability of the PC–FDTD method for transmission lines with uncertain inductance. The most commonly used procedure for assessing the stability of a finite difference scheme is the so-called von Neumann method. We shall assume plane wave solutions with constant amplitudes $\widetilde{V}$ and $\widetilde{I}$ [6].

We begin by analyzing the stability criterion of the random FDTD method. Substituting $V_j^n = \widetilde{V}\zeta^n e^{ikj\Delta x}$ and $I_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \widetilde{I}\zeta^{n+\frac{1}{2}} e^{ik(j+\frac{1}{2})\Delta x}$ into (5), we obtain the random stability polynomial

$$a_2\zeta^2 + a_1\zeta + a_0 = 0,$$

where the coefficients of the stability polynomial are given by

$$a_2 = \left(\frac{\Delta x}{\Delta t}\right)^2 (L + r\xi)C + \frac{\Delta x}{\Delta t} C \frac{R\Delta x}{2} + \frac{\Delta x}{\Delta t}(L + r\xi)\frac{G\Delta x}{2} + \frac{RG}{4}(\Delta x)^2,$$

$$a_1 = -2\left(\frac{\Delta x}{\Delta t}\right)^2 (L + r\xi)C + \frac{RG}{2}(\Delta x)^2 + 4\sin^2\left(\frac{k\Delta x}{2}\right),$$

and

$$a_0 = \left(\frac{\Delta x}{\Delta t}\right)^2 (L + r\xi)C - \frac{\Delta x}{\Delta t} C \frac{R\Delta x}{2} - \frac{\Delta x}{\Delta t}(L + r\xi)\frac{G\Delta x}{2} + \frac{RG}{4}(\Delta x)^2.$$

By applying Theorem 2 from the stability analysis in [2], we obtain $|\zeta| \leq 1$ if

$$\frac{1}{\sqrt{(L + r\xi)C}} \cdot \frac{\Delta t}{\Delta x} \leq 1. \tag{7}$$

In the case when $r = 0$, we recover the stability condition for the deterministic, lossy telegrapher's equations.

Next, we turn our attention to the stability criterion of the PC–FDTD method. Suppose we utilize the Legendre Polynomial Chaos expansion of degree p as in (3). By using a similar approach to the random FDTD stability analysis and applying the substitution as $\left(\vec{V}_p\right)_j^n = \widetilde{\vec{V}_p}\zeta^n e^{ikj\Delta x}$ and $\left(\vec{I}_p\right)_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \widetilde{\vec{I}_p}\zeta^{n+\frac{1}{2}}e^{ik(j+\frac{1}{2})\Delta x}$ into (6), we obtain the following PCFDTD stability criterion:

$$\frac{1}{\sqrt{(L + r\eta_{p,\ell})C}} \cdot \frac{\Delta t}{\Delta x} \leq 1, \tag{8}$$

for $\ell = 0, 1, \cdots, p$. In this inequality, $\eta_{p,\ell}$ is an eigenvalue of the Jacobi matrix M_p. Note that $\eta_{p,\ell}$ is also a zero of the Legendre polynomial of degree $p + 1$, and its value lies within the interval $(-1, 1)$.

The above inequality must hold for all $\eta_{p,\ell}$, and in particular, it must also hold at the smallest zero $\eta_{p,\min}$ such that

$$\frac{1}{\sqrt{(L + r\eta_{p,\min})C}} \cdot \frac{\Delta t}{\Delta x} \leq 1, \tag{9}$$

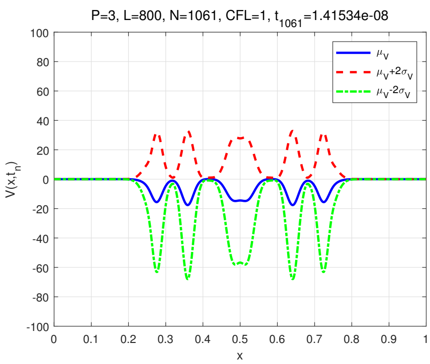
which represents the stability criterion of the PC–FDTD scheme of degree p.

Therefore, for the given p and Δx, we choose Δt that satisfies

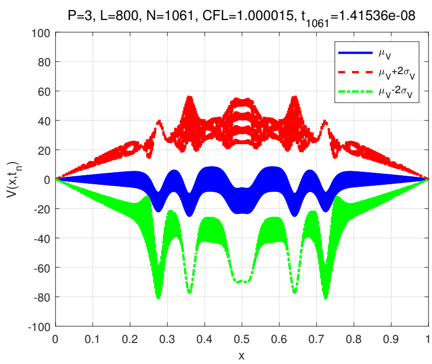
$$\Delta t = \mathrm{CFL}\sqrt{(L + r\eta_{p,\min})C}\,\Delta x,$$

where the CFL number is less than or equal to 1. We note that if we replace $\eta_{p,\min}$ with its smallest theoretical value of -1, we find a stability criteria valid for all degree p.

Figure 1 shows the mean and confidence interval of the voltage function $V(x, t)$ in the random telegrapher's equations (2) using the PC–FDTD method of degree 3. The PC–FDTD method is stable for all CFL ≤ 1, but unstable for CFL > 1. In this simulation, the parameters are $L = 2\mu\mathrm{H/m}, C = 100\mathrm{pF/m}, R = 100\Omega/\mathrm{m}, G = 0.001\mathrm{S/m}$, and the source function used is a Gaussian pulse. The scaling parameter r was set to $L/2$ and simulations using higher than degree 3 yielded similar results.



(a) CFL=1.0



(b) CFL=1.000015

Figure 1. Mean and confidence intervals of $V(x, t)$.

4. Dispersion Analysis

In this section, we analyze the numerical dispersion characteristics [1, 8] of the PC–FDTD method applied to transmission lines with uncertain inductance.

First, let us examine the continuous dispersion relation of the random telegrapher's equations. Let k be the wavenumber and ω be the angular frequency of the wave. Substituting $V(x,t) = \widetilde{V}e^{i\omega t}e^{ikx}$ and $I(x,t) = \widetilde{I}e^{i\omega t}e^{ikx}$ into the random telegrapher's equations (2), we derive the corresponding random continuous dispersion relation:

$$k^2 = (L + r\xi)C\omega^2 - RG - i\left[(L + r\xi)G + RC\right]\omega, \tag{10}$$

where the random variable $\xi \sim \mathcal{U}[-1,1]$, or equivalently,

$$k^2 = \widetilde{L}(C\omega^2 - i\omega G) - R(G + i\omega C), \tag{11}$$

with $\widetilde{L} = (L + r\xi)$.

Next, we will consider the discrete dispersion relation of the random FDTD method. Inserting $V_j^n = \widetilde{V}e^{i\omega n\Delta t}e^{ikj\Delta x}$ and $I_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \widetilde{I}e^{i\omega(n+\frac{1}{2})\Delta t}e^{ik(j+\frac{1}{2})\Delta x}$ into (5), we obtain the random FDTD dispersion relation:

$$\begin{aligned}\left(\frac{2}{\Delta x}\right)^2 \sin^2\left(\frac{k\Delta x}{2}\right) &= (L + r\xi)C\left(\frac{2}{\Delta t}\right)^2 \sin^2\left(\frac{\omega\Delta t}{2}\right) \\ &-RG\cos^2\left(\frac{\omega\Delta t}{2}\right) - i\left[(L + r\xi)G + RC\right]\left(\frac{2}{\Delta t}\right)\sin\left(\frac{\omega\Delta t}{2}\right)\cos\left(\frac{\omega\Delta t}{2}\right).\end{aligned} \tag{12}$$

We note that with appropriate definitions of misrepresented physical parameters as follows (similarly to [1])

$$\begin{aligned}
k_\Delta &= \left(\frac{2}{\Delta x}\right)\sin\left(\frac{k\Delta x}{2}\right) = \sin\left(\frac{k\Delta x}{2}\right) / \left(\frac{\Delta x}{2}\right), \\
\omega_\Delta &= \left(\frac{2}{\Delta t}\right)\sin\left(\frac{\omega\Delta t}{2}\right) = \sin\left(\frac{\omega\Delta t}{2}\right) / \left(\frac{\Delta t}{2}\right), \\
R_\Delta &= R\cos\left(\frac{\omega\Delta t}{2}\right), \\
G_\Delta &= G\cos\left(\frac{\omega\Delta t}{2}\right),
\end{aligned}$$

the above relation becomes

$$k_\Delta^2 = (L + r\xi)C\omega_\Delta^2 - R_\Delta G_\Delta - i\left[(L + r\xi)G_\Delta + R_\Delta C\right]\omega_\Delta. \tag{13}$$

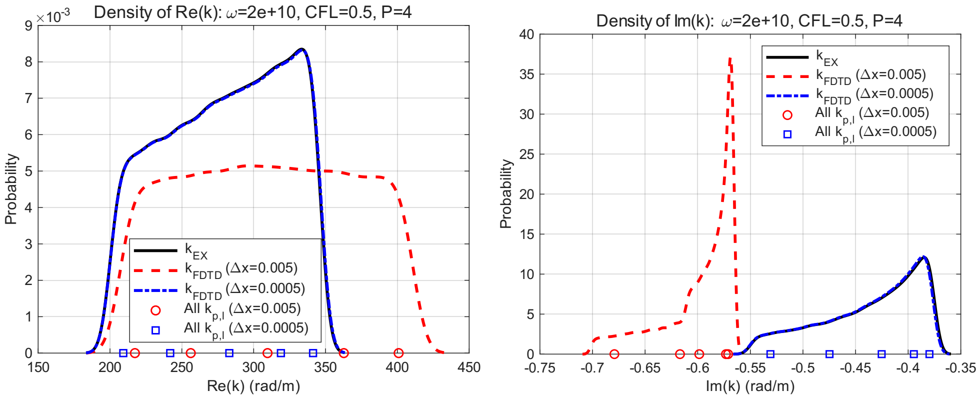
Alternatively, it can be written in the equivalent form:

$$k_\Delta^2 = \widetilde{L}(C\omega_\Delta^2 - i\omega_\Delta G_\Delta) - R_\Delta(G_\Delta + i\omega_\Delta C). \tag{14}$$

Let k_{EX} be the k that satisfies the random continuous dispersion relation (10), and let k_{FDTD} be the k that satisfies the random FDTD dispersion relation (12).

Figure 2 shows the density of the random exact k and the random FDTD k for a fixed frequency (e.g., $\omega = 2e + 10$). In this figure, when Δx is very small, it can be seen that k_{FDTD} closely represents k_{EX}.

Now, we will analyze the discrete dispersion relation of the PC–FDTD method using a Legendre polynomial chaos expansion of degree p. Substituting $\left(\vec{V}_p\right)_j^n =$



(a) Density of $\mathrm{Re}(k)$ (b) Density of $\mathrm{Im}(k)$

Figure 2. Density of k_{EX} and k_{FDTD}.

$\widetilde{\vec{V}_p} e^{i\omega n\Delta t} e^{ikj\Delta x}$ and $\left(\vec{I_p}\right)^{n+\frac{1}{2}}_{j+\frac{1}{2}} = \widetilde{\vec{I}_p} e^{i\omega(n+\frac{1}{2})\Delta t} e^{ik(j+\frac{1}{2})\Delta x}$ into (6), we obtain the PC–FDTD dispersion relation:

$$k_\Delta^2 \widetilde{\vec{V}_p} = A_p (C\omega_\Delta^2 - i\omega_\Delta G_\Delta)\widetilde{\vec{V}_p} - R_\Delta (G_\Delta + i\omega_\Delta C)\widetilde{\vec{V}_p}. \tag{15}$$

This homogeneous linear system has $p+1$ possible solutions, each corresponding to an eigenvalue of the Jacobi matrix M_p. The relations become

$$k_\Delta^2 = (L + r\eta_{p,\ell})(C\omega_\Delta^2 - i\omega_\Delta G_\Delta) - R_\Delta (G_\Delta + i\omega_\Delta C), \quad \ell = 0, 1, \cdots, p. \tag{16}$$

Let $k_{p,\ell}$ be the k that satisfies this PC–FDTD dispersion relation, i.e.,

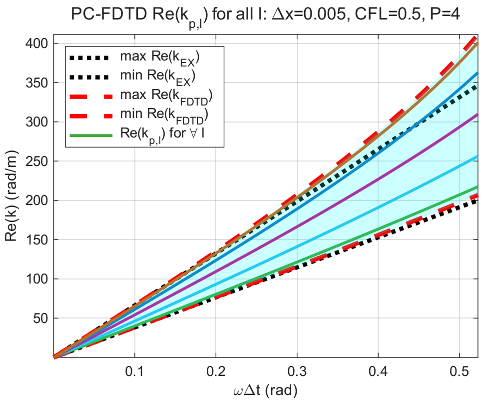
$$\left(\frac{2}{\Delta x}\right)^2 \sin^2\left(\frac{k\Delta x}{2}\right) = (L + r\eta_{p,\ell})C\left(\frac{2}{\Delta t}\right)^2 \sin^2\left(\frac{\omega\Delta t}{2}\right) \tag{17}$$

$$-RG\cos^2\left(\frac{\omega\Delta t}{2}\right) - i\left[(L + r\eta_{p,\ell})G + RC\right]\left(\frac{2}{\Delta t}\right)\sin\left(\frac{\omega\Delta t}{2}\right)\cos\left(\frac{\omega\Delta t}{2}\right),$$

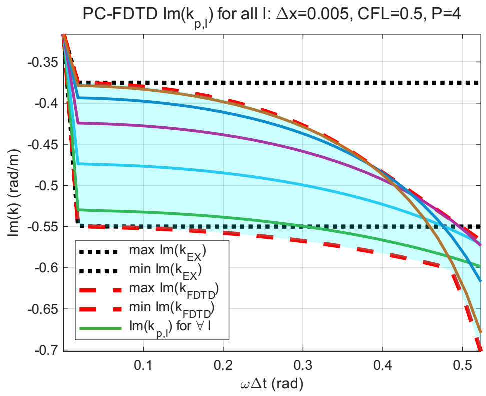
for $\ell = 0, 1, \cdots, p$.

Figure 2 above includes the value of each $k_{p,\ell}$, along the horizontal axis, for a fixed frequency (e.g., $\omega = 2e+10$).

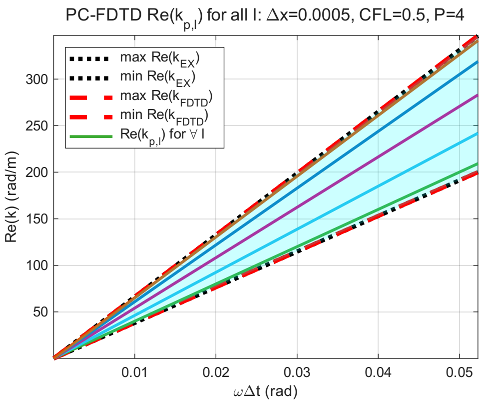
Figure 3 shows all the $p+1$ possible $k_{p,\ell}$ of the PC–FDTD, vs. $\omega\Delta t$, compared with the maximum and minimum values (computed via Monte Carlo) of k_{EX} and k_{FDTD}. We see that the fully discrete dispersion relation derived from the PC–FDTD method is a good numerical approximation of the discrete random k_{FDTD}. As the physical grid is refined, and the random FDTD more accurately models the random continuous dispersion relation, the PC–FDTD method also becomes more accurate. Note that the rightmost $\omega\Delta t$ value in these plots corresponds to the fixed ω value used in Figure 2. Thus, Figure 2 depicts a cross section from the plots in Figure 3, with a profile view to display the density variation. We can clearly see that one or two quadrature nodes would not be sufficient for capturing the shape of these probability density functions. As in [6], we suggest the degree p of the PC-FDTD method to be 3 or higher.



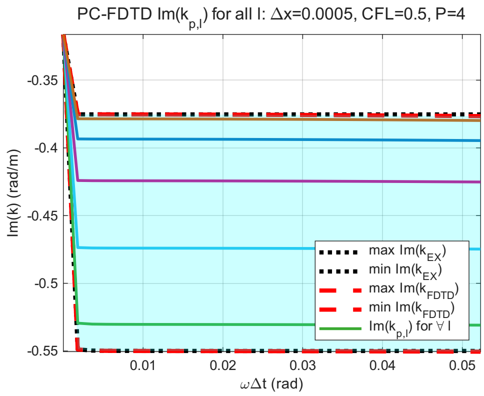
(a) Dispersion of Re(k): Δx=0.005



(b) Dispersion of Im(k): Δx=0.005



(c) Dispersion of Re(k): Δx=0.0005



(d) Dispersion of Im(k): Δx=0.0005

Figure 3. Discrete dispersion relation of the PC–FDTD.

5. Conclusions

In this study, we have presented a numerical method for solving the random telegrapher's equations with uncertain inductance. The key tool employed in our approach is the Finite Difference Time Domain (FDTD) method combined with the Polynomial Chaos (PC) expansion.

We have demonstrated that the proposed PC–FDTD scheme preserves the conditional stability of the FDTD scheme applied to the deterministic lossy model. Numerical demonstrations validate the theoretical analysis. Furthermore, we derived discrete dispersion relations for both the FDTD approach applied to the deterministic model and the PC–FDTD method for the random model. For each, we were able to write the relations in a form resembling the continuous dispersion relation thus emphasizing the numerical misrepresentations of the model parameters.

Our results indicate that the PC–FDTD method is highly effective for handling uncertainty in parameters, making it a robust and accurate approach for solving the random telegrapher's equations.

Acknowledgments

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