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UNIFORM EQUIVALENCE OF L^2- AND DISCRETE ℓ^2-NORMS ON Q_1-FINITE ELEMENT SPACES WITH MASS LUMPING IN ANY FINITE DIMENSION

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Abstract. The Q_1-finite element spaces, in any finite d-dimension, are equipped with the discrete ℓ_h^2-inner product generated by the simple row-sum mass lumping. The equivalence of the discrete ℓ_h^2-norm and the L^2-norm on the Q_1-finite element spaces is uniform in mesh size h, in both cases of uniform and nonuniform partitions. Several representation formulae for these norms are derived. Using these, accurate bounds between these two norms are obtained, which is our major contribution. Examples show that these bounds are sharp. As an important application, the equivalence is established between discrete h_h^1-norm and H^1-norm. Numerical results are presented.

Key words. Finite element method, mass lumping, norm equivalence.

1. Introduction

In this section we will begin by the 1D simplest case for the motivation of our study. Then some notations and preliminaries will be given.

1.1. 1D motivation.

Let us consider the simplest Sobolev space

$$H_0^1(0,1) = \{f \in L^2(0,1) \mid f' \in L^2(0,1);\ \mathrm{tr}_0(f) = 0, \mathrm{tr}_1(f) = 0\},$$

where tr_ξ denotes the standard trace operator at ξ in the Sobolev space theory. A family of C^0 piecewise linear finite element spaces $(V_h)_{0<h<1}$ is defined in a standard fashion. For a positive integer l, let $\mathcal{T}_h$ denote the standard mesh

$$0 = x^{(0)} < x^{(1)} < \cdots < x^{(l-1)} < x^{(l)} = 1;$$

$$h^{(j)} = x^{(j)} - x^{(j-1)}, \quad j = 1, \cdots, l,$$

with $h = \max_{j=1}^{l} h^{(j)}$. It is quite convenient, in analysis and in actual programming, to extend the meshes outside the domain. Thus, we assume that $x^{(-1)} = -h^{(1)}$ and $x^{(l+1)} = 1 + h^{(l)}$. Let V_h be the finite element subspace of $H_0^1(0,1)$ associated with $\mathcal{T}_h$. For $j = 0, \cdots, l$,

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denoting by ϕ_j the basis function

$$\phi_j(x) = \left[\frac{x - x^{(j-1)}}{h^{(j)}} \chi_{[x^{(j-1)}, x^{(j)})}(x) + \frac{x^{(j+1)} - x}{h^{(j+1)}} \chi_{[x^{(j)}, x^{(j+1)}]}(x) \right] \chi_{[0,1]}(x), \tag{1}$$

where χ_S denotes the characteristic function of set S, one has $V_h = \text{span}\{\phi_1, \cdots, \phi_{l-1}\}$. Let us look at the $L^2(0,1)$-norm on V_h in detail. For any v_h in V_h, *i.e.* $v_h = \sum_{j=1}^{l-1} v^j \phi_j, v^j \in \mathbb{R}$, (throughout we assume $v^0 = v^l = 0$)

$$\begin{aligned}
\|v_h\|^2_{L^2(0,1)} &= \int_0^1 |v_h|^2 \mathrm{d}x = \sum_{j=1}^{l} \int_{x^{(j-1)}}^{x^{(j)}} \left| v^{j-1}\phi_{j-1}(x) + v^j \phi_j(x) \right|^2 \mathrm{d}x \\
&= \sum_{j=1}^{l} \int_{x^{(j-1)}}^{x^{(j)}} \Big[(v^{j-1})^2 \phi^2_{j-1}(x) \\
&\quad + 2v^{j-1} v^j \phi_{j-1}(x)\phi_j(x) + (v^j)^2 \phi_j^2(x) \Big] \mathrm{d}x \\
&= \sum_{j=1}^{l} \left[(v^{j-1})^2 \frac{h^{(j)}}{3} + 2v^{j-1} v^j \frac{h^{(j)}}{6} + (v^j)^2 \frac{h^{(j)}}{3} \right] \\
&= \frac{1}{3} \sum_{j=1}^{l} h^{(j)} \left[(v^{j-1})^2 + v^{j-1} v^j + (v^j)^2 \right] \\
&= \frac{1}{6} \sum_{j=1}^{l} h^{(j)} \left(|v^{j-1}|^2 + |v^j|^2 + (v^{j-1} + v^j)^2 \right). \qquad (2)
\end{aligned}$$

From this $L^2(0,1)$-norm representation, the $L^2(0,1)$-inner product on V_h is evidently deduced. Instead of the $L^2(0,1)$-inner product, the following (row-sum) mass-lumped ℓ^2_h-inner product on V_h is frequently used:

$$\left(u_h, v_h\right)_{\ell^2_h} = \sum_{j=1}^{l-1} u^j v^j \bar{h}^{(j)} \quad \forall\ u_h = \sum_{j=1}^{l-1} u^j \phi_j(x),\ v_h = \sum_{j=1}^{l-1} v^j \phi_j(x) \in V_h, \tag{3}$$

where $\bar{h}^{(j)} = \dfrac{h^{(j)} + h^{(j+1)}}{2}$.

It is immediate to see that, for the V_h-interpolant $v_h \in V_h$ of any $v \in H^1_0(0,1)$,

$$\|v_h\|^2_{\ell^2_h} = \sum_{j=1}^{l-1} (v^j)^2 \bar{h}^{(j)} = \frac{1}{2} \sum_{j=1}^{l} \left[(v^{j-1})^2 + (v^j)^2 \right] h^{(j)}, \tag{4}$$

and therefore, the comparison of (2) and (4) leads to

$$\begin{aligned} \|v_h\|^2_{\ell^2_h} - \|v_h\|^2_{L^2(0,1)} &= \frac{1}{2}\sum_{j=1}^{l}\Big((v^{j-1})^2+(v^j)^2\Big)h^{(j)} \\ &\quad -\frac{1}{3}\sum_{j=1}^{l}\Big((v^{j-1})^2+v^{j-1}v^j+(v^j)^2\Big)h^{(j)} \\ &= \frac{1}{6}\sum_{j=1}^{l}\left(v^{j-1}-v^j\right)^2 h^{(j)} \geqslant 0. \qquad (5) \end{aligned}$$

Moreover, by taking the limit of the difference as $h \to 0$,

$$\begin{aligned} \lim_{h\to 0}\Big[\|v_h\|^2_{\ell^2_h} - \|v_h\|^2_{L^2(0,1)}\Big] &\leqslant \lim_{h\to 0}\frac{h^2}{6}\sum_{j=1}^{l}\left|\frac{v^j-v^{j-1}}{h^{(j)}}\right|^2 h^{(j)} \\ &\leqslant \lim_{h\to 0}\frac{h^2}{6}\left\|\frac{\mathrm{d}v_h}{\mathrm{d}x}\right\|^2_{L^2(0,1)} = 0, \qquad (6) \end{aligned}$$

since $\left\|\frac{\mathrm{d}v_h}{\mathrm{d}x}\right\|_{L^2(0,1)} \leqslant C\|v\|_1$. From (2) and (4) it follows that

$$\|v_h\|^2_{\ell^2_h} \leqslant \frac{1}{2}\sum_{j=1}^{l} h^{(j)}\left(|v^{j-1}|^2+|v^j|^2+(v^{j-1}+v^j)^2\right) = 3\|v_h\|^2_{L^2(0,1)}. \qquad (7)$$

A combination of (5) and (7) states as follows:

$$\|v_h\|^2_{L^2(0,1)} \leqslant \|v_h\|^2_{\ell^2_h} \leqslant 3\|v_h\|^2_{L^2(0,1)} \quad \forall v_h \in V_h. \qquad (8)$$

We thus far observe the following facts.

1. This (8) implies that the ℓ^2_h-inner product approximates $L^2(\Omega)$-inner product reasonably well and the two norms, $\|\cdot\|_{\ell^2_h}$ and $\|\cdot\|_{L^2(\Omega)}$, are equivalent uniformly in h for V_h.
2. Furthermore, (6) indicates the first bounding coefficient "1" in (8) is sharp. We remark that for 1D, if the coefficients $v^j, j = (0), 1, \cdots, l$, in the representation of v_h are alternating, that is, $v^{j-1}+v^j = 0$ for all $j = 1, \cdots, l$, (2) and (7) imply that the second inequality in (8) turns out to be an equality.
3. However, the sharpness of the second bounding coefficient "3" in (8) seems to be investigated further, in particular, for general meshes in higher dimensions.

The aim of this paper is to establish sharp bounds for the equivalence of L^2-norm and ℓ^2_h-norm on multi-dimensional finite element subspaces of $H^1_0(\Omega_{\boldsymbol{x}})$ with values in an inner product space Y where $\Omega_{\boldsymbol{x}}$ is a rectangular domain in $\mathbb{R}^d$. We remark that Y can be either finite– or infinite–dimensional inner product spaces, which includes Euclidean spaces $\mathbb{R}^k$ and Sobolev–Hilbert spaces $H^k(\Omega_{\boldsymbol{y}})$.

In particular, we will also investigate in the special case of $Y = L^2(\Omega_{\boldsymbol{y}})$, and further that of $Y = H_0^1(\Omega_{\boldsymbol{y}})$.

The techniques for using mass lumping in the calculation of mass matrices have been widely used (see [**?**, **?**, **?**], and the references therein). Popular mass lumping techniques include the three main procedures: the row sum method, diagonal scaling; evaluation of the integration of the mass matrix at the nodal points only [**?**, pp474 & Appendix I]. Due to the diagonal form of lumped mass matrices, they are often used in time integration to obtain explicit methods [**?**, **?**]. For one-dimensional examples, see [**?**]. Mass-lumping techniques have been extended to mixed finite elements [**?**, **?**], to higher-order elements [**?**, **?**, **?**], and to spectral elements [**?**], and to eigenvalue problems (see [**?**] and references therein). The use of mass lumping has recently drawn growing attention in the field of isogeometric analysis (see [**?**, **?**, **?**, **?**, **?**] and the references therein).

Recently, estimates such as (**??**) and (**??**) for semi-discrete subspaces $\mathcal{L}_h^2 \subset L^2(\Omega)$ and $\mathcal{H}_h^1 \subset H^1(\Omega)$ (for the definitions, see (**??**) and (**??**) below) are frequently employed in the theory of difference finite element methods (DFEM) via dimensional splitting. Leveraging mass lumping techniques, works [**?**, **?**, **?**, **?**, **?**] introduce semi-discrete ℓ_h^2- and h_h^1-inner products in $\mathcal{L}_h^2$ and $\mathcal{H}_h^1$, respectively.

1.2. Notations and preliminaries.

For measurable set ω in $\mathbb{R}^k, k \geqslant 1$, by $(\cdot,\cdot)_\omega$ we denote the $L^2(\omega)$-inner product. For $m \in \mathbb{Z}$, standard notations for Sobolev spaces $H^m(\omega)$ and $H_0^m(\omega)$ will be employed with the norm $\|v\|_{m,\omega}$. Let Y be an inner product space over $\mathbb{R}$ with inner product, $(\cdot,\cdot)_Y$, and norm, $\|\cdot\|_Y$. We consider the space $L^2(\Omega_{\boldsymbol{x}};Y)$ where $\Omega_{\boldsymbol{x}} = [0,L_1]\times\cdots\times[0,L_d]$ with $L_j > 0, j = 1,\cdots,d$, endowed with inner-product, $(u,v)_{L^2(\Omega_{\boldsymbol{x}};Y)} = \int_{\Omega_{\boldsymbol{x}}} \big(u(\boldsymbol{x}),v(\boldsymbol{x})\big)_Y \,\mathrm{d}\boldsymbol{x}$, and L^2-norm, $\|u\|_{L^2(\Omega_{\boldsymbol{x}};Y)} = \sqrt{(u,u)_{L^2(\Omega_{\boldsymbol{x}};Y)}} = \sqrt{\int_{\Omega_{\boldsymbol{x}}} \|u(\boldsymbol{x})\|_Y^2 \mathrm{d}\boldsymbol{x}}$ for all $u,v \in L^2(\Omega_{\boldsymbol{x}};Y)$, where $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_d$. We stress that Y can be any inner-product space, such as $Y = H^s(\Omega_y), s \in \mathbb{R}$.

For positive integers l_j $(j = 1,\cdots,d)$, let $[0,L_j]$ be partitioned into $0 = x_j^{(0)} < x_j^{(1)} < \cdots < x_j^{(l_j)} = L_j$, with $h_j^{(i_j)} = x_j^{(i_j)} - x_j^{(i_j-1)}$ for $i_j = 1,\cdots,l_j$. Then consider the triangulation of $\Omega_{\boldsymbol{x}}$ into d-dimensional rectangular polytopes:

$$\overline{\Omega_{\boldsymbol{x}}} = \bigcup_{j=1}^{d}\bigcup_{i_j=1}^{l_j} K_{i_1\cdots i_d}, \quad K_{i_1\cdots i_d} = [x_1^{(i_1-1)}, x_1^{(i_1)}]\times\cdots\times[x_d^{(i_d-1)}, x_d^{(i_d)}]$$

and

$$|K_{i_1\cdots i_d}| = \prod_{j=1}^{d} h_j^{(i_j)}.$$

Throughout the paper, the multi-index notation $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$ is used. For notational convenience's sake, the extended meshes are defined for each $j = 1, \cdots, d$, such that $x_j^{(-1)} = -x_j^{(1)}$ and $x_j^{(l_j+1)} = L_j + (x_j^{(l_j)} - x_j^{(l_j-1)})$ for each j, and $h_j^{(i_j)} = x_j^{(i_j)} - x_j^{(i_j-1)}$ for $i_j = 0, \cdots, l_j + 1$, the linear basis functions $\phi_{i_j}(x_j)$ in x_j-direction are defined by

$$\begin{aligned}\phi_{i_j}(x_j) = \Bigg[&\frac{x_j - x_j^{(i_j-1)}}{h_j^{(i_j)}} \chi_{\left[x_j^{(i_j-1)}, x_j^{(i_j)}\right)}(x_j) \\ &+ \frac{x_j^{(i_j+1)} - x_j}{h^{(i_j+1)}} \chi_{\left[x_j^{(i_j)}, x_j^{(i_j+1)}\right]}(x_j) \Bigg] \chi_{[0,L_j]}(x_j),\end{aligned}$$

for $i_j = 1, \cdots, l_j$.

We will use the following identities:

$$\int_{x_j^{(i_j-1)}}^{x_j^{(i_j)}} \phi_{i_j-1}^2(x_j)\,\mathrm{d}x_j = \int_{x_j^{(i_j-1)}}^{x_j^{(i_j)}} \phi_{i_j}^2(x_j)\,\mathrm{d}x_j = \int_0^{h_j^{(i_j)}} \frac{x_j^2}{(h_j^{(i_j)})^2}\,\mathrm{d}x_j = \frac{h_j^{(i_j)}}{3}, \tag{9a}$$

$$\int_{x_j^{(i_j-1)}}^{x_j^{(i_j)}} \phi_{i_j-1}(x_j)\phi_{i_j}(x_j)\,\mathrm{d}x_j = \int_0^{h_j^{(i_j)}} \frac{h_j^{(i_j)} - x_j}{h_j^{(i_j)}} \frac{x_j}{h_j^{(i_j)}}\,\mathrm{d}x_j = \frac{h_j^{(i_j)}}{6}. \tag{9b}$$

We denote by $(\mathcal{T}_h)_{h\in(0,1)}$ the family of the above d-dimensional rectangular triangularization of Ω and by $\mathcal{V}(\mathcal{T}_h)$ the set of all vertices in $\mathcal{T}_h$; in particular, by $\mathcal{V}^i(\mathcal{T}_h)$ and $\mathcal{V}^b(\mathcal{T}_h)$ the set of all interior and boundary vertices $\mathbf{V}_{\boldsymbol{\alpha}}$ in $\mathcal{T}_h$, respectively.

The tensor products $\psi_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \psi_{i_1\cdots i_d}(\boldsymbol{x}) = \phi_{i_1}(x_1)\cdots\phi_{i_d}(x_d)$ ($i_j = 1, \cdots, l_j - 1,\ j = 1, \cdots, d$) form a basis for the standard conforming Q_1-finite element subspace, say $\mathcal{X}_h$, of $\subset H_0^1(\Omega_{\boldsymbol{x}}; \mathbb{R}) \subset L^2(\Omega_{\boldsymbol{x}})$. That is,

$$\begin{aligned}\mathcal{X}_h &= \operatorname{span}\left\{\psi_{i_1\cdots i_d} \in L^2(\Omega_x),\ i_j = 1, \cdots, l_j - 1,\ j = 1, \cdots, d\right\} \\ &= \operatorname{span}\left\{\psi_{\boldsymbol{\alpha}} \in L^2(\Omega_x) \ \middle|\ \mathbf{V}_{\boldsymbol{\alpha}} \in \mathcal{V}^i(\mathcal{T}_h)\right\}. \end{aligned}\tag{10}$$

We consider the semi-finite dimensional subspace $V_h = L^2(\mathcal{X}_h; Y)$ of $L^2(\Omega_x; Y)$ ("semi" means finite dimensional subspace only in $L^2(\Omega_x)$), that is,

$$\begin{aligned} V_h &= \Big\{ v_h = \sum_{\mathbf{V}_{\boldsymbol{\alpha}} \in \mathcal{V}^i(\mathcal{T}_h)} v^{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}} \;\mid\; v^{\boldsymbol{\alpha}} \in Y \Big\} \\ &= \Big\{ \sum_{i_1=1}^{l_1-1} \cdots \sum_{i_d=1}^{l_d-1} v^{i_1\cdots i_d} \psi_{i_1\cdots i_d} \;\mid\; v^{i_1\cdots i_d} \in Y, \\ &\qquad i_j = 1, \cdots, l_j - 1,\; j = 1, \cdots, d \Big\}. \end{aligned} \tag{11}$$

We emphasize that for a typical element $v_h \in V_h$, $v_h(\boldsymbol{x}) \in Y$ $\forall \boldsymbol{x} \in \Omega_{\boldsymbol{x}}$ with

$$v_h(\boldsymbol{x}) \;=\; \sum_{i_1=1}^{l_1-1} \cdots \sum_{i_d=1}^{l_d-1} v^{i_1\cdots i_d} \psi_{i_1\cdots i_d}(\boldsymbol{x}), \text{ where } v^{i_1\cdots i_d} \in Y, \psi_{i_1\cdots i_d}(\boldsymbol{x}) \in \mathbb{R},$$
$$i_j = 1, \cdots, l_j - 1,\; j = 1, \cdots, d,$$

and

$$\begin{aligned} &(u_h, v_h)_{L^2(\Omega_{\boldsymbol{x}};Y)} \\ &= \sum_{i_1=1}^{l_1-1} \cdots \sum_{i_d=1}^{l_d-1} \sum_{i'_1=1}^{l_1-1} \cdots \sum_{i'_d=1}^{l_d-1} \left(u^{i_1\cdots i_d}, v^{i'_1\cdots i'_d}\right)_Y \int\limits_{\Omega_{\boldsymbol{x}}} \psi_{i_1\cdots i_d}(\boldsymbol{x}) \psi_{i'_1\cdots i'_d}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \end{aligned} \tag{12}$$

for $\forall u_h, v_h \in V_h$.

Instead of the usual $L^2(\Omega_{\boldsymbol{x}}; Y)$–inner product, the mass lumping will be employed for the inner product on V_h, and the discrete ℓ_h^2-inner product $(\cdot,\cdot)_{\ell_h^2}$ and the corresponding ℓ_h^2-norm $\|\cdot\|_{\ell_h^2}$ will be used for V_h:

$$\begin{aligned} (u_h, v_h)_{\ell_h^2} &= \sum_{i_1=1}^{l_1-1} \cdots \sum_{i_d=1}^{l_d-1} (u^{i_1\cdots i_d}, v^{i_1\cdots i_d})_Y \big|\widetilde{K}_{i_1\cdots i_d}\big| \\ \|v_h\|_{\ell_h^2} &= \sqrt{(v_h, v_h)_{\ell_h^2}} \;\forall u_h, v_h \in V_h, \end{aligned} \tag{13}$$

where

$$\big|\widetilde{K}_{i_1\cdots i_d}\big| = \prod_{j=1}^{d} \bar{h}_j^{(i_j)}, \quad \bar{h}_j^{(i_j)} = \frac{h_j^{(i_j)} + h_j^{(i_j+1)}}{2}, \quad \big|K_{i_1\cdots i_d}\big| = \prod_{j=1}^{d} h_j^{(i_j)}.$$

Owing to the finite dimensionality of V_h, the ℓ_h^2-norm and $L^2(\Omega_{\boldsymbol{x}}; Y)$-norm are equivalent on V_h. Our aim is to establish accurate bounds in the equivalence of these two norms on V_h, which are uniform in mesh size h.

We recall and define some definitions and notations to be used.

Definition 1.1. Let K be a d-polytope. For $k = 0, \cdots, d$, we use $\mathcal{F}_k(K)$ the set of all k-faces of K. Evidently, a k-face f of K is a k-face of some $(k+1)$-face g of K when $k+1 \leqslant d$. If $d \geqslant 2$, $\mathcal{F}_0(K)$, $\mathcal{F}_1(K)$, and $\mathcal{F}_2(K)$ are the vertices, edges, and 2-faces of K, respectively. $(d-1)$-faces are called facets of K. For all $f \in \mathcal{F}_k(K)$ we will denote by $\mathcal{V}(f)$ the set of all 0-face (vertices) of f, that is $\mathcal{V}(f) = \mathcal{F}_0(f) = \operatorname{ext} f$, the set of extreme points of f.

Definition 1.2. Denote the index set

$$\mathcal{A}_{i_1\cdots i_d} = \big\{ \boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) \mid \alpha_j \in \{i_j - 1, i_j\},\, j = 1, \cdots, d \,\big\},$$

$$|\mathcal{A}_{i_1\cdots i_d}| = 2^d \tag{14}$$

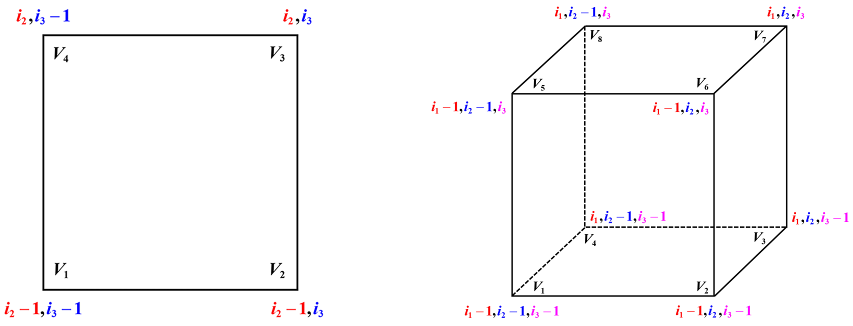
with $\big|\mathcal{A}_{i_1\cdots i_d}\big|$ designating the cardinality of $\mathcal{A}_{i_1\cdots i_d}$. For a rectangular d-polytope $K := K_{i_1\cdots i_d}$, recall that $\mathcal{F}_0(K)$ denotes the set of 2^d number of vertices $\mathbf{V}_{\boldsymbol{\alpha}}$ of K. Denoting by $\|\cdot\|_{\ell^1}$ the ℓ^1-norm on $\mathbb{Z}^d$, we introduce the notation for the set of ordered pairs of indices whose ℓ^1-distance is k as follows: for $k = 0, 1, \cdots, d$,

$$\begin{aligned} &\mathcal{A}^k_{i_1\cdots i_d} \\ &=\big\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{A}_{i_1\cdots i_d} \times \mathcal{A}_{i_1\cdots i_d} \mid \mathbf{V}_{\boldsymbol{\alpha}}, \mathbf{V}_{\boldsymbol{\beta}} \in \mathcal{F}_0(K),\ \|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_{\ell^1} = k \big\}. \end{aligned} \tag{15}$$

Since $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{A}^k_{i_1\cdots i_d}$ is an ordered pair, $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \neq (\boldsymbol{\beta}, \boldsymbol{\alpha})$ if $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ in general. It is obvious that $\big|\mathcal{A}^k_{i_1\cdots i_d}\big| = 2^d\binom{d}{k}$, $k = 1, \cdots, d$. For $\mathbf{V}_{\boldsymbol{\alpha}} \in \mathcal{F}_0(K)$, define, for $k = 0, 1, \cdots, d$,

$$\mathcal{A}^k_{i_1\cdots i_d}(\boldsymbol{\alpha}) = \big\{ \boldsymbol{\beta} \in \mathcal{A}_{i_1\cdots i_d} \mid (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{A}^k_{i_1\cdots i_d} \big\}, \quad \big|\mathcal{A}^k_{i_1\cdots i_d}(\boldsymbol{\alpha})\big| = \binom{d}{k}. \tag{16}$$

Figures **??** and **??** illustrate d dimension polytopes $K_{i_1\cdots i_d}$ with vertices $\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N, N = 2^d$, for $d = 2, 3, 4$, respectively. Here, only the superscripts of components of vertex coordinates are shown. Corresponding to each vertex $\mathbf{V}_{\boldsymbol{\alpha}}$, the coefficient and the basis functions are denoted by $v^{\boldsymbol{\alpha}} \in Y$ and $\psi_{\boldsymbol{\alpha}}(\boldsymbol{x})$, respectively.



(A) K is a 2D polytope (B) K is a 3D polytope

FIGURE 1. The nodes relationship in 2D and 3D polytopes.

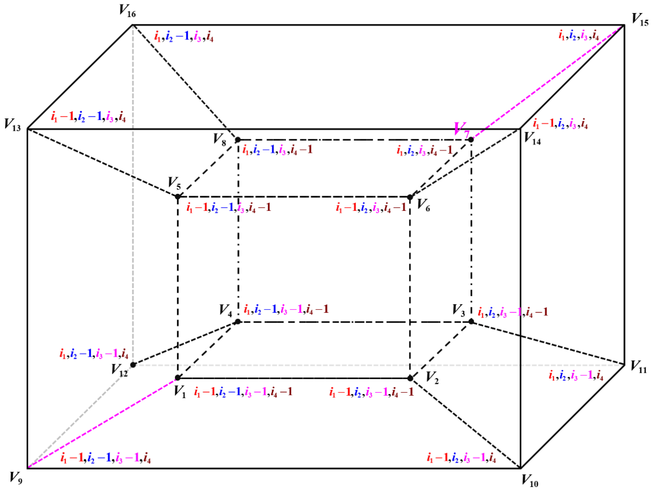


FIGURE 2. The nodes relationship in 4D polytope K.

2. The case where Y is an inner product space

This section is devoted to the case where Y is an inner product space. We prove the case of symmetric inner product space, but the extension to the non-symmetric case is trivial.

Before state and prove our main theorems, some useful technical lemmas will be given as follows.

Lemma 2.1. *The following equality holds, for* $k = 0, 1, \cdots, d$,

$$\sum_{f \in \mathcal{F}_k(K_{i_1 \cdots i_d})} \Big\| \sum_{\mathbf{V}_{\boldsymbol{\alpha}} \in \mathcal{V}(f)} v^{\boldsymbol{\alpha}} \Big\|_Y^2 = \sum_{j=0}^{k} \binom{d-j}{k-j} \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{A}^j_{i_1 \cdots i_d}} \left(v^{\boldsymbol{\alpha}}, v^{\boldsymbol{\beta}}\right)_Y . \tag{17}$$

Also, the following inequality holds: for $k = 0, 1, \cdots, d$,

$$\Big| \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{A}^k_{i_1\cdots i_d}} \big(v^{\boldsymbol{\alpha}}, v^{\boldsymbol{\beta}}\big)_Y \Big| \leqslant \binom{d}{k} \sum_{\boldsymbol{\alpha}\in\mathcal{A}_{i_1\cdots i_d}} \|v^{\boldsymbol{\alpha}}\|^2_Y. \tag{18}$$

Proof. For the sake of simplicity, in this proof we will use the simplification of notations as follows: $K = K_{i_1\cdots i_d}$ and $\mathcal{A} = \mathcal{A}_{i_1\cdots i_d}$ omitting the subindices $i_1 \cdots i_d$. Notice that (**??**) is rotationally symmetric for the 2^d number of indices $\boldsymbol{\alpha} \in \mathcal{A}$. Therefore, it suffices to prove (**??**) partially such that both sides contain equal amount of quantities in $v^{\boldsymbol{\alpha}_0}$ where $\boldsymbol{\alpha}_0 = (i_1, i_2, \cdots, i_d) - (1, 1, \cdots, 1)$ plays a role of a pivot index.

For this purpose, we restrict the faces $f \in \mathcal{F}_k(K)$ in the LHS of (**??**) to those which contain the pivot vertex $\mathbf{V}_{\boldsymbol{\alpha}_0}$. Conveniently, by $T_f(v^{\boldsymbol{\alpha}_0})$ we denote all terms containing $v^{\boldsymbol{\alpha}_0}$ in the the LHS of (**??**) in the k-face f.

Accordingly, in the RHS of (**??**), it suffices to restrict to and investigate in the cases such that at least one of the indices from $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{A}^j$ contains the pivot index $\boldsymbol{\alpha}_0$.

It is easy to see, by expansion and rearrangement, that for given k-face f the terms which contain $v^{\boldsymbol{\alpha}_0}$ in $\Big\| \sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}(f)} v^{\boldsymbol{\alpha}} \Big\|^2_Y$ are given by

$$\begin{aligned} T_f(v^{\boldsymbol{\alpha}_0}) :=&(v^{\boldsymbol{\alpha}_0}, v^{\boldsymbol{\alpha}_0})_Y + 2\sum_{j=1}^{k} \sum_{\substack{\mathbf{V}_{\boldsymbol{\beta}}\in\mathcal{V}(f)\\ (\boldsymbol{\alpha}_0,\boldsymbol{\beta})\in\mathcal{A}^j}} (v^{\boldsymbol{\alpha}_0}, v^{\boldsymbol{\beta}})_Y \\ =&(v^{\boldsymbol{\alpha}_0}, v^{\boldsymbol{\alpha}_0})_Y + \sum_{j=1}^{k} \sum_{\substack{\mathbf{V}_{\boldsymbol{\beta}}\in\mathcal{V}(f)\\ (\boldsymbol{\alpha}_0,\boldsymbol{\beta})\in\mathcal{A}^j}} \Big[(v^{\boldsymbol{\alpha}_0}, v^{\boldsymbol{\beta}})_Y + (v^{\boldsymbol{\beta}}, v^{\boldsymbol{\alpha}_0})_Y\Big]. \end{aligned} \tag{19}$$

We observe that there are several classes of k-faces that share the pivot vertex $\boldsymbol{\alpha}_0$, such as (0) the vertex $\boldsymbol{\alpha}_0$; (1) one edge containing $\boldsymbol{\alpha}_0$; and (j) one j-face containing $\boldsymbol{\alpha}_0$, $j = 2, 3, \cdots$. Let us call by $\mathcal{G}_j, j = 0, 1, \cdots, k$, these classes of k-faces that share $\boldsymbol{\alpha}_0$ and j-face in common. Notice that $\mathcal{G}_j \subset \mathcal{G}_{j-1}$ for $j = 1, \cdots, k$. We write $\boldsymbol{\alpha}_j = \boldsymbol{\alpha}_0 + \mathbf{e}_j \in \mathcal{A}, j = 1, \cdots, k$, where $\mathbf{e}_j, j = 1, \cdots, d$, are the j-th unit vector in $\mathbb{R}^d$. Thus, a k-face f is represented by $f = \Big\{ \sum_{l=1}^k t_l \boldsymbol{\alpha}_{j_l} \;\big|\; t_l \in [0, 1], l = 1, \cdots, k\Big\}$ for some k number of indices $\{j_l, l = 1, \cdots, k\}$ from the index set $\{1, \cdots, d\}$. To each $(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_j)$ there corresponds an edge $\overline{\boldsymbol{\alpha}_0\boldsymbol{\alpha}_j}$.

For $j = 0, 1, \cdots, k$, we investigate each class $\mathcal{G}_j$ of k-faces that share a j-face each other

Case ($\mathcal{G}_0$) In the d-polytope K there are $\binom{d}{k}$ k-faces that share the pivot vertex $\boldsymbol{\alpha}_0$. These k-faces from $\mathcal{G}_0$ contribute to $T_f(v^{\boldsymbol{\alpha}_0})$ in quadratic terms in the amount of $\binom{d}{k}(v^{\boldsymbol{\alpha}_0}, v^{\boldsymbol{\alpha}_0})$. We have $|\mathcal{G}_0| = \binom{d}{k}$.

Case ($\mathcal{G}_1$) Next, we consider the linear terms in $T_f(v^{\boldsymbol{\alpha}_0})$ of form $\sum_{\substack{\mathbf{v}_{\boldsymbol{\beta}}\in\mathcal{V}(f)\\(\boldsymbol{\alpha}_0,\boldsymbol{\beta})\in\mathcal{A}^1}}$ $\Big[(v^{\boldsymbol{\alpha}_0},v^{\boldsymbol{\beta}})_Y+(v^{\boldsymbol{\beta}},v^{\boldsymbol{\alpha}_0})_Y\Big]$. It is clear that $\boldsymbol{\beta}=\boldsymbol{\alpha}_j$ for some $j=1,\cdots,d$, in order to have $(\boldsymbol{\alpha}_0,\boldsymbol{\beta})\in\mathcal{A}^1$. For fixed j, those k-faces which can contain the vertex pair $(\boldsymbol{\alpha}_0,\boldsymbol{\alpha}_j)\in\mathcal{A}^1$ should contain the edge $\overline{\boldsymbol{\alpha}_0\boldsymbol{\alpha}_j}$ and other $k-1$ edges $\overline{\boldsymbol{\alpha}_0\boldsymbol{\alpha}_{j_l}}$ where $j_l, l=1,\cdots,k-1$, are $k-1$ indices from $\{1,\cdots,d\}$. This means that there are $\binom{d-1}{k-1}$ such k-faces which contain $(v^{\boldsymbol{\alpha}_0},v^{\boldsymbol{\alpha}_j})_Y+(v^{\boldsymbol{\alpha}_j},v^{\boldsymbol{\alpha}_0})_Y$. We have $|\mathcal{G}_1|=\binom{d-1}{k-1}$.

Case ($\mathcal{G}_2$) Next, we consider the linear terms in $T_f(v^{\boldsymbol{\alpha}_0})$ of form $\sum_{\substack{\mathbf{v}_{\boldsymbol{\beta}}\in\mathcal{V}(f)\\(\boldsymbol{\alpha}_0,\boldsymbol{\beta})\in\mathcal{A}^2}}$ $\Big[(v^{\boldsymbol{\alpha}_0},v^{\boldsymbol{\beta}})_Y+(v^{\boldsymbol{\beta}},v^{\boldsymbol{\alpha}_0})_Y\Big]$. In order for a face f to have $(\boldsymbol{\alpha}_0,\boldsymbol{\beta})\in\mathcal{A}^2$, one must have $\boldsymbol{\beta}=\boldsymbol{\alpha}_0+\mathbf{e}_{j_1}+\mathbf{e}_{j_2}$ for some j_1,j_2 from the index set $\{1,\cdots,d\}$. Similarly to the the above **Case($\mathcal{G}_1$)**, for fixed j_1,j_2, those k-faces which can contain $(\boldsymbol{\alpha}_0,\boldsymbol{\alpha})\in\mathcal{A}^2$ should contain the edges $\overline{\boldsymbol{\alpha}_0\boldsymbol{\alpha}_{j_l}}, l=1,2$, and other $k-2$ edges $\overline{\boldsymbol{\alpha}_0\boldsymbol{\alpha}_{j'}}$ with $j'=1,\cdots,d, j'\neq j_l, l=1,2$. This means that there are $\binom{d-2}{k-2}$ such k-faces which contain $(v^{\boldsymbol{\alpha}_0},v^{\boldsymbol{\alpha}_{j_l}})_Y+(v^{\boldsymbol{\alpha}_{j_l}},v^{\boldsymbol{\alpha}_0})_Y, l=1,2$. We have $|\mathcal{G}_1|=\binom{d-2}{k-2}$.

Case ($\mathcal{G}_j$) In general, for $j=1,\cdots,d$, we consider the linear terms in $T_f(v^{\boldsymbol{\alpha}_0})$ of form $\sum_{\substack{\mathbf{v}_{\boldsymbol{\beta}}\in\mathcal{V}(f)\\(\boldsymbol{\alpha}_0,\boldsymbol{\beta})\in\mathcal{A}^k}}$ $\Big[(v^{\boldsymbol{\alpha}_0},v^{\boldsymbol{\beta}})_Y+(v^{\boldsymbol{\beta}},v^{\boldsymbol{\alpha}_0})_Y\Big]$. In order for a face f to have $(\boldsymbol{\alpha}_0,\boldsymbol{\beta})\in\mathcal{A}^j$, one see that $\boldsymbol{\beta}=\boldsymbol{\alpha}_0+\sum_{l=1}^{j}\mathbf{e}_{j_l}$ for some $j_l, l=1,\cdots,j$, from the index set $\{1,\cdots,d\}$. For fixed $j_l, l=1,\cdots,j$, those k-faces which can contain $(\boldsymbol{\alpha}_0,\boldsymbol{\alpha})\in\mathcal{A}^j$ must contain the edges $\overline{\boldsymbol{\alpha}_0\boldsymbol{\alpha}_{j_l}}, l=1,\cdots,j$, and other $k-j$ edges $\overline{\boldsymbol{\alpha}_0\boldsymbol{\alpha}_{j'}}$ with $j'=1,\cdots,d, j'\neq j_l, l=1,\cdots,j$. This means that there are $\binom{d-j}{k-j}$ such k-faces which contain $(v^{\boldsymbol{\alpha}_0},v^{\boldsymbol{\alpha}_{j_l}})_Y+(v^{\boldsymbol{\alpha}_{j_l}},v^{\boldsymbol{\alpha}_0})_Y, l=1,\cdots,j$. We have $|\mathcal{G}_j|=\binom{d-j}{k-j}$, for $j=1,\cdots,k$.

Summing over (**??**) over all $f\in\mathcal{F}_k$, considering all the cases of $\mathcal{G}_j, j=0,\cdots,k$, we see that the terms containing $v^{\boldsymbol{\alpha}_0}$ are

$$\begin{aligned}\sum_{f\in\mathcal{F}_k}T_f(v^{\boldsymbol{\alpha}_0}) &= \sum_{j=0}^{k}\binom{d-j}{k-j}\sum_{l=1}^{j}\left[(v^{\boldsymbol{\alpha}_0},v^{\boldsymbol{\alpha}_{j_l}})_Y+(v^{\boldsymbol{\alpha}_{j_l}},v^{\boldsymbol{\alpha}_0})_Y\right]\\ &= \sum_{j=0}^{k}\binom{d-j}{k-j}\sum_{(\boldsymbol{\alpha}_0,\boldsymbol{\beta})\in\mathcal{A}^j_{i_1\cdots i_d}}\left[\left(v^{\boldsymbol{\alpha}_0},v^{\boldsymbol{\beta}}\right)_Y+\left(v^{\boldsymbol{\beta}},v^{\boldsymbol{\alpha}_0}\right)_Y\right].\end{aligned}\tag{20}$$

Invoking the rotational symmetry of indices $\boldsymbol{\alpha}\in\mathcal{A}$, we conclude (**??**) from (**??**).

Now, let us turn to prove (**??**). Since

$$\begin{aligned}\Big|\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{A}^k}\big(v^{\boldsymbol{\alpha}},v^{\boldsymbol{\beta}}\big)_Y\Big| &\leqslant \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{A}^k}\|v^{\boldsymbol{\alpha}}\|_Y\|v^{\boldsymbol{\beta}}\|_Y\\ &\leqslant \frac{1}{2}\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{A}^k}(\|v^{\boldsymbol{\alpha}}\|_Y^2+\|v^{\boldsymbol{\beta}}\|_Y^2),\end{aligned}$$

it suffices to prove that

$$\frac{1}{2}\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{A}^k}(\|v^{\boldsymbol{\alpha}}\|_Y^2+\|v^{\boldsymbol{\beta}}\|_Y^2)=\binom{d}{k}\sum_{\boldsymbol{\alpha}\in\mathcal{A}}\|v^{\boldsymbol{\alpha}}\|_Y^2. \tag{21}$$

Again, we notice that (**??**), is rotationally symmetric for the 2^d number of indices $\boldsymbol{\alpha}\in\mathcal{A}$. Therefore, it suffices prove (**??**) comparing both sides partially the quantities in $v^{\boldsymbol{\alpha}_0}$ where $\boldsymbol{\alpha}_0=(i_1,i_2,\cdots,i_d)-(1,1,\cdots,1)$ is a pivot index as before. But it is immediate to see that the term $\|v^{\boldsymbol{\alpha}_0}\|_Y^2$ appears exactly $\binom{d}{k}$ times in the LHS of (**??**), which is the coefficient in the RHS of (**??**). The rotational symmetry implies that any quantity $\|v^{\boldsymbol{\alpha}}\|_Y^2$ for $\boldsymbol{\alpha}\in\mathcal{A}$ appear $\binom{d}{k}$ times in both sides of (**??**). Thus (**??**) holds, which implies (**??**). This completes the proof. □

Lemma 2.2. *The following identity holds:*

$$\sum_{k=0}^{d}2^{d-k}\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{A}^k_{i_1\cdots i_d}}\big(v^{\boldsymbol{\alpha}},v^{\boldsymbol{\beta}}\big)_Y=\sum_{k=0}^{d}\sum_{f\in\mathcal{F}_k(K_{i_1\cdots i_d})}\Big\|\sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big\|_Y^2. \tag{22}$$

Proof. First, we notice that if $g_j\in\mathbb{R}$, for $j=0,\cdots,d$, the following holds:

$$\begin{aligned}\sum_{k=0}^{d}\sum_{j=0}^{k}\binom{d-j}{k-j}g_j &= \binom{d}{0}g_0+\left[\binom{d}{1}g_0+\binom{d-1}{0}g_1\right]+\cdots\\ &\quad+\left[\binom{d}{d}g_0+\binom{d-1}{d-1}g_1+\cdots+\binom{0}{0}g_d\right]\\ &= \sum_{k=0}^{d}\binom{d}{k}g_0+\sum_{k=0}^{d-1}\binom{d-1}{k}g_1\\ &= 2^dg_0+2^{d-1}g_1+\cdots+2^0g_d=\sum_{k=0}^{d}2^{d-k}g_k.\end{aligned}$$

Using this identity, thanks to the above Lemma **??**, we have

$$\begin{aligned}
&\sum_{k=0}^{d}\sum_{f\in\mathcal{F}_k(K_{i_1\cdots i_d})}\Big\|\sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big\|_Y^2\\
=\;&\sum_{k=0}^{d}\sum_{j=0}^{k}\binom{d-j}{k-j}\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{A}^j_{i_1\cdots i_d}}\big(v^{\boldsymbol{\alpha}},v^{\boldsymbol{\beta}}\big)_Y\\
=\;&\sum_{k=0}^{d}2^{d-k}\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{A}^k_{i_1\cdots i_d}}\big(v^{\boldsymbol{\alpha}},v^{\boldsymbol{\beta}}\big)_Y.
\end{aligned}$$

This completes the proof. □

We are now ready to state and prove the following representations for the norms $\|\cdot\|_{L^2(\Omega_{\boldsymbol{x}};Y)}$ and $\|\cdot\|_{\ell^2_h}$.

Theorem 2.3. *Let Y be an inner product space with inner product $(\cdot,\cdot)_Y$. Let the conforming Q_1-(semi-)finite element subspace V_h and ℓ^2_h-norm be defined as in (**??**) and (**??**). For $v_h=\sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}^i(\mathcal{T}_h)}v^{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}\in V_h$, we have the following representations:*

$$\begin{aligned}
&\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}};Y)}\\
=&\frac{1}{6^d}\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\big|K_{i_1\cdots i_d}\big|\sum_{k=0}^{d}\sum_{f\in\mathcal{F}_k(K_{i_1\cdots i_d})}\Big\|\sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big\|_Y^2\\
=&\frac{1}{6^d}\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\big|K_{i_1\cdots i_d}\big|\sum_{\boldsymbol{\alpha}\in\mathcal{F}_0(K_{i_1\cdots i_d})}\sum_{k=0}^{d}2^{d-k}\sum_{\boldsymbol{\beta}\in\mathcal{A}^k_{i_1\cdots i_d}(\boldsymbol{\alpha})}(v^{\boldsymbol{\alpha}},v^{\boldsymbol{\beta}})_Y
\end{aligned}$$

$$=\frac{1}{6^d}\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\big|K_{i_1\cdots i_d}\big|\sum_{k=0}^{d}2^{d-k}\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathcal{A}^k_{i_1\cdots i_d}}(v^{\boldsymbol{\alpha}},v^{\boldsymbol{\beta}})_Y,\tag{23}$$

$$\|v_h\|^2_{\ell^2_h}=\frac{1}{2^d}\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\big|K_{i_1\cdots i_d}\big|\sum_{\boldsymbol{\alpha}\in\mathcal{A}_{i_1\cdots i_d}}\big\|v^{\boldsymbol{\alpha}}\big\|_Y^2.\tag{24}$$

Proof. Let $v_h=\sum_{i_1=1}^{l_1-1}\cdots\sum_{i_d=1}^{l_d-1}v^{i_1\cdots i_d}\psi_{i_1\cdots i_d}\in V_h$ be arbitrary. Assume the notations $v^{i_1\cdots i_d}=0$ for $i_j=0$ or $i_j=l_j$, $j=1,\cdots,d$. More conveniently we also use the notation $v_h=\sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}^i(\mathcal{T}_h)}v^{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}\in V_h$ and exploit the property that $v^{\boldsymbol{\alpha}}=0\ \forall\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}^b(\mathcal{T}_h)$ so that

$$v_h=\sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}^i(\mathcal{T}_h)}v^{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}=\sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}(\mathcal{T}_h)}v^{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}\in V_h.$$

First, we will prove (**??**). It follows from (**??**) and (**??**)-(**??**) that

$$
\begin{aligned}
&\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}};Y)}\\
=&\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\int_{K_{i_1\cdots i_d}}\Big\|\sum_{i_1=0}^{l_1}\cdots\sum_{i_d=0}^{l_d}v^{i_1\cdots i_d}\psi_{i_1\cdots i_d}(\boldsymbol{x})\Big\|^2_Y\mathrm{d}\boldsymbol{x}\\
=&\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\int_{K_{i_1\cdots i_d}}\Big\|\sum_{\boldsymbol{\alpha}\in\mathcal{A}_{i_1\cdots i_d}}v^{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}(\boldsymbol{x})\Big\|^2_Y\mathrm{d}\boldsymbol{x}\\
=&\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\int_{K_{i_1\cdots i_d}}\left(\sum_{\boldsymbol{\alpha}\in\mathcal{F}_0(K_{i_1\cdots i_d})}v^{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}(\boldsymbol{x}),\quad\sum_{\boldsymbol{\beta}\in\mathcal{F}_0(K_{i_1\cdots i_d})}v^{\boldsymbol{\beta}}\psi_{\boldsymbol{\beta}}(\boldsymbol{x})\right)_Y\mathrm{d}\boldsymbol{x}\\
=&\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{F}_0(K_{i_1\cdots i_d})}(v^{\boldsymbol{\alpha}},v^{\boldsymbol{\beta}})_Y\big(\psi_{\boldsymbol{\alpha}},\psi_{\boldsymbol{\beta}}\big)_{K_{i_1\cdots i_d}}\\
=&\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\sum_{\boldsymbol{\alpha}\in\mathcal{F}_0(K_{i_1\cdots i_d})}\sum_{k=0}^{d}\sum_{\boldsymbol{\beta}\in\mathcal{A}^k_{i_1\cdots i_d}(\boldsymbol{\alpha})}(v^{\boldsymbol{\alpha}},v^{\boldsymbol{\beta}})_Y\big(\psi_{\boldsymbol{\alpha}},\psi_{\boldsymbol{\beta}}\big)_{K_{i_1\cdots i_d}}\\
=&\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\sum_{\boldsymbol{\alpha}\in\mathcal{F}_0(K_{i_1\cdots i_d})}\sum_{k=0}^{d}\sum_{\boldsymbol{\beta}\in\mathcal{A}^k_{i_1\cdots i_d}(\boldsymbol{\alpha})}(v^{\boldsymbol{\alpha}},v^{\boldsymbol{\beta}})_Y\frac{|K_{i_1\cdots i_d}|}{3^{d-k}6^k}\\
(25)&\\
=&\frac{1}{6^d}\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}|K_{i_1\cdots i_d}|\sum_{k=0}^{d}\sum_{f\in\mathcal{F}_k(K_{i_1\cdots i_d})}\Big\|\sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big\|^2_Y.
\end{aligned}
$$

where Lemma **??** is applied for the last equality (**??**). This proves (**??**). One notices that (**??**) and are equivalent to (**??**) owing to (**??**) and Lemma **??**, respectively.

Next, it is easy to prove (**??**) by shifting summation indices. Indeed, the definition of $\|\cdot\|_{\ell^2_h}$-norm given by (**??**) implies that

$$
\begin{aligned}
\|v_h\|^2_{\ell^2_h} &= \sum_{i_1=1}^{l_1-1}\cdots\sum_{i_d=1}^{l_d-1}\big|\widetilde{K}_{i_1\cdots i_d}\big|\|v^{i_1\cdots i_d}\|^2_Y\\
&= \frac{1}{2^d}\sum_{i_1=1}^{l_1-1}\cdots\sum_{i_d=1}^{l_d-1}\prod_{j=1}^{d}\left(h_j^{(i_j)}+h_j^{(i_j+1)}\right)\|v^{i_1\cdots i_d}\|^2_Y\\
&= \frac{1}{2^d}\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}|K_{i_1\cdots i_d}|\sum_{f\in\mathcal{F}_0(K_{i_1\cdots i_d})}\Big\|\sum_{\boldsymbol{\alpha}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big\|^2_Y
\end{aligned}
$$

$$= \frac{1}{2^d}\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\left|K_{i_1\cdots i_d}\right|\sum_{\boldsymbol{\alpha}\in\mathcal{A}_{i_1\cdots i_d}}\|v^{\boldsymbol{\alpha}}\|_Y^2.$$

This completes the proof. □

From the above norm representations in Theorem **??** it follows that the two norms are equivalent as stated in the following theorem.

Theorem 2.4. *Let Y be an inner product space with inner product $(\cdot,\cdot)_Y$. Let the conforming Q_1-(semi-)finite element subspace V_h and ℓ_h^2-norm be defined as in (**??**) and (**??**). Then, the following inequalities hold:*

$$\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}};Y)} \leqslant \|v_h\|^2_{\ell_h^2} \leqslant 3^d\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}};Y)} \quad \forall v_h \in V_h, \tag{26}$$

where the bounding constants, 1 *and* 3^d, *are independent of mesh size* h.

Proof. For the first inequality, the application of (**??**) for each k in (**??**) and the identity $\sum_{k=0}^{d} 2^{d-k}\binom{d}{k} = 3^d$ for $d \in \mathbb{Z}_+$ leads to

(27)

$$\begin{aligned}\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}};Y)} \leqslant& \frac{1}{6^d}\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\left|K_{i_1\cdots i_d}\right|\sum_{k=0}^{d}2^{d-k}\binom{d}{k}\sum_{\boldsymbol{\alpha}\in\mathcal{A}_{i_1\cdots i_d}}\|v^{\boldsymbol{\alpha}}\|_Y^2\\ =&\frac{1}{6^d}\sum_{i_1=1}^{l_1}\cdots\sum_{i_d=1}^{l_d}\left|K_{i_1\cdots i_d}\right|3^d\sum_{\boldsymbol{\alpha}\in\mathcal{A}_{i_1\cdots i_d}}\|v^{\boldsymbol{\alpha}}\|_Y^2 = \|v_h\|^2_{\ell_h^2},\end{aligned}$$

where (**??**) is applied in the last equality.

The second inequality follows from (**??**) and (**??**). □

Likewise in the 1D example described in Subsection **1.1**, the following 2D example illustrates the bounds in (**??**) are sharp in 2D.

Example 2.5. First, let $\Omega_x = (a,b)\times(c,d) \subset \mathbb{R}^2$ and $\mathcal{T}_h$ be a uniform triangulation of Ω_x into $l_1 \times l_2$ rectangles for some $l_j \in \mathbb{Z}_+, j = 1,2$. Assume that the coefficients $v^{i_1 i_2}$ in $v_h = \sum_{\mathbf{V}_{\boldsymbol{\alpha}}\in\mathcal{V}^i(\mathcal{T}_h)} v^{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}$ are of checkerboard pattern on $\mathcal{V}^i(\mathcal{T}_h)$ so that

$$\sum_{\boldsymbol{\alpha}\in\mathcal{V}(f)} v^{\boldsymbol{\alpha}} = 0 \text{ for } f \in \mathcal{F}_k(K_{i_1 i_2}), k = 1,2, \tag{28}$$

$$\forall i_j = 2,\cdots,l_j-1, j = 1,2,$$

$$v^{\boldsymbol{\alpha}} = 0 \ \forall \mathbf{V}_{\boldsymbol{\alpha}} \in \mathcal{V}^b(\mathcal{T}_h). \tag{29}$$

Evidently, such a checkerboard pattern of form (**??**) is a 2D tensor product of $1D$ checkerboard patterns:

$$v^{i_1 i_2} = c\begin{cases}(-1)^{i_1}(-1)^{i_2}, & i_j = 1,\cdots,l_j-1, j=1,2,\\ 0, & i_j = 0, \text{ or } i_j = l_j, j = 1,2,\end{cases} \tag{30}$$

for some nonzero constant c. Denote $|K| = |K_{i_1,i_2}| = \frac{1}{l_1 l_2}$ $\forall i_j = 1,\cdots,l_j$, $j = 1,2$.

Then, thanks to (**??**), (**??**) reduces to

$$\begin{aligned}
&\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}\\
=&\frac{1}{6^2}\sum_{i_1=1}^{l_1}\sum_{i_2=1}^{l_2}|K|\sum_{k=0}^{2}\sum_{f\in\mathcal{F}_k(K_{i_1,i_2})}\Big|\sum_{\boldsymbol{\alpha}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big|^2\\
=&\frac{|K|}{6^2}\sum_{i_1=2}^{l_1-1}\sum_{i_2=2}^{l_2-1}\sum_{f\in\mathcal{F}_0(K_{i_1,i_2})}\Big|\sum_{\boldsymbol{\alpha}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big|^2\\
&+\frac{|K|}{6^2}\sum_{i_1=2}^{l_1-1}\sum_{k=0}^{2}\Big(\sum_{f\in\mathcal{F}_k(K_{i_1,1})}+\sum_{f\in\mathcal{F}_k(K_{i_1,l_2})}\Big)\Big|\sum_{\boldsymbol{\alpha}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big|^2\\
&+\frac{|K|}{6^2}\sum_{i_2=2}^{l_2-1}\sum_{k=0}^{2}\Big(\sum_{f\in\mathcal{F}_k(K_{1,i_2})}+\sum_{f\in\mathcal{F}_k(K_{l_1,i_2})}\Big)\Big|\sum_{\boldsymbol{\alpha}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big|^2\\
&+\frac{|K|}{6^2}\sum_{k=0}^{2}\Big(\sum_{f\in\mathcal{F}_k(K_{1,1})}+\sum_{f\in\mathcal{F}_k(K_{1,l_2})}+\sum_{f\in\mathcal{F}_k(K_{l_1,1})}+\sum_{f\in\mathcal{F}_k(K_{l_1,l_2})}\Big)\Big|\sum_{\boldsymbol{\alpha}\in\mathcal{V}(f)}v^{\boldsymbol{\alpha}}\Big|^2\\
=&\frac{4c^2|K|}{6^2}(l_1-2)(l_2-2)+\frac{8c^2|K|}{6^2}(l_1-2)+\frac{8c^2|K|}{6^2}(l_2-2)+\frac{16c^2|K|}{6^2}\\
=&\frac{c^2|K|}{3^2}l_1 l_2.
\end{aligned}$$

In the meanwhile, by Definition (**??**),

$$\|v_h\|^2_{\ell^2_h} = \sum_{i_1=1}^{l_1-1}\sum_{i_2=1}^{l_2-1}\big|\widetilde{K}_{i_1 i_2}\big||v^{i_1 i_2}|^2 = c^2|K|(l_1-1)(l_2-1).$$

Hence, we have

$$\frac{\|v_h\|^2_{\ell^2_h}}{3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}} = \frac{(l_1-1)(l_2-1)}{l_1 l_2} \nearrow 1 \text{ as } l_1, l_2 \to \infty.$$

Remark 2.6. The exactness of bounding coefficients 3^d in (**??**) can be achieved in any d-dimensional rectangular domain by extending the 2D function of checkerboard pattern in Example **??** to d dimensions. Indeed, for $\Omega_x = (0,1)^d$, let $\mathcal{T}_h$ be a uniform triangulation of Ω_x into

$l_1 \times \cdots \times l_d$ d-rectangles. Then for $v_h = \sum_{\mathbf{V}_{\boldsymbol{\alpha}} \in \mathcal{V}^i(\mathcal{T}_h)} v^{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}$, choose the coefficients given by

$$\sum_{\boldsymbol{\alpha} \in \mathcal{V}(f)} v^{\boldsymbol{\alpha}} = 0 \text{ for } f \in \mathcal{F}_k(K_{i_1 \cdots i_d}), k = 1, \cdots, d, \tag{31}$$

$$\forall i_j = 2, \cdots, l_j - 1, j = 1, \cdots, d,$$

$$v^{\boldsymbol{\alpha}} = 0 \ \forall \mathbf{V}_{\boldsymbol{\alpha}} \in \mathcal{V}^b(\mathcal{T}_h). \tag{32}$$

The same argument as in Example **??** leads to

$$\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})} = \frac{c^2 |K| l_1 \cdots l_d}{3^d} \quad \text{and } \|v_h\|^2_{\ell^2_h} = c^2 |K| (l_1 - 1) \cdots (l_d - 1).$$

so that

$$\frac{\|v_h\|^2_{\ell^2_h}}{3^d \|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}} = \frac{(l_1 - 1) \cdots (l_d - 1)}{l_1 \cdots l_d} \nearrow 1 \text{ as } l_1, \cdots, l_d \to \infty.$$

3. Applications to the case of $Y = L^2(\Omega_y)$

In the special case of $Y = L^2(\Omega_{\boldsymbol{y}})$ for some open set $\Omega_{\boldsymbol{y}} \subset \mathbb{R}^n$, we can interpret $L^2(\Omega_{\boldsymbol{x}}; Y)$ as a subspace of $L^2(\Omega_{\boldsymbol{x}} \times \Omega_{\boldsymbol{y}})$. In this case, for points in $\Omega := \Omega_{\boldsymbol{x}} \times \Omega_{\boldsymbol{y}}$ we employ the coordinate system $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, \cdots, x_d, y_1, \cdots, y_n)$, and the gradient operator split as follows:

$$\boldsymbol{\nabla} = (\boldsymbol{\nabla}_{\boldsymbol{x}}, \boldsymbol{\nabla}_{\boldsymbol{y}})^\top, \quad \boldsymbol{\nabla}_{\boldsymbol{x}} = (\partial_{x_1}, \cdots, \partial_{x_d})^\top, \quad \boldsymbol{\nabla}_{\boldsymbol{y}} = (\partial_{y_1}, \cdots, \partial_{y_n})^\top.$$

Based on the above settings, we introduce the following semi-discrete subspace $\mathcal{L}^2_h \subset L^2(\Omega)$ in $\boldsymbol{x}$-directions defined as follows:

$$\mathcal{L}^2_h = \Bigg\{ v_h = v_h(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i_1=1}^{l_1-1} \cdots \sum_{i_d=1}^{l_d-1} v^{i_1 \cdots i_d}(\boldsymbol{y}) \psi_{i_1 \cdots i_d}(\boldsymbol{x}) \ \Big| \quad v^{i_1 \cdots i_d} \in L^2(\Omega_{\boldsymbol{y}}),$$

$$i_j = 1, \cdots, l_j - 1, \ j = 1, \cdots, d \Bigg\}, \tag{33}$$

with the semi-discrete ℓ^2_h-inner product $(\cdot, \cdot)_{\ell^2_h}$ given by (**??**) with $Y = L^2(\Omega_{\boldsymbol{y}})$.

As an application of **Theorem ??** to the case of $Y = L^2(\Omega_{\boldsymbol{y}})$, the following result holds:

Theorem 3.1. *Let the semi-discrete subspace $\mathcal{L}^2_h$ and ℓ^2_h-norm be defined as in* (**??**) *and* (**??**) *with $Y = L^2(\Omega_{\boldsymbol{y}})$. Then, the following inequalities hold:*

$$\|v_h\|^2_{L^2(\Omega)} \leqslant \|v_h\|^2_{\ell^2_h} \leqslant 3^d \|v_h\|^2_{L^2(\Omega)} \quad \forall v_h \in \mathcal{L}^2_h, \tag{34}$$

where the bounding constants are independent of mesh size h.

Let us proceed to define a semi-discrete subspace $\mathcal{H}_h^1 \subset H_0^1(\Omega)$ in $\boldsymbol{x}$-directions by

$$\begin{aligned}\mathcal{H}_h^1 = \Bigg\{ v_h =& \sum_{i_1=1}^{l_1-1} \cdots \sum_{i_d=1}^{l_d-1} v^{i_1\cdots i_d}(\boldsymbol{y}) \psi_{i_1\cdots i_d}(\boldsymbol{x}) \\ =& \sum_{i_1=0}^{l_1} \cdots \sum_{i_d=0}^{l_d} v^{i_1\cdots i_d}(\boldsymbol{y}) \psi_{i_1\cdots i_d}(\boldsymbol{x}) \ \mid \\ & v^{i_1\cdots i_d} \in H_0^1(\Omega_{\boldsymbol{y}}), i_j = 0, \cdots, l_j,\ j = 1, \cdots, d \Bigg\}. \qquad (35)\end{aligned}$$

We employ the following notation for the backward finite difference operators:

$$\begin{aligned}\mathrm{D}_{h_j}^- u^{i_1\cdots i_j \cdots i_d} =& \frac{1}{h_j^{(i_j)}} (u^{i_1\cdots i_j\cdots i_d} - u^{i_1,\cdots,i_j-1,\cdots,i_d}), \\ & i_j = 1, \cdots, l_j,\ j = 1, \cdots, d. \qquad (36)\end{aligned}$$

Then $\mathbf{D}_h^-$ is defined on $\mathcal{H}_h^1$ as follows: for $v_h \in \mathcal{H}_h^1$,

$$\begin{aligned}\mathbf{D}_h^- v_h =& \begin{pmatrix} \mathrm{D}_{h_1}^- \\ \vdots \\ \mathrm{D}_{h_d}^- \end{pmatrix} \sum_{i_1=1}^{l_1-1} \cdots \sum_{i_d=1}^{l_d-1} v^{i_1\cdots i_d}(\boldsymbol{y}) \psi_{i_1\cdots i_d}(\boldsymbol{x}) \\ =& \sum_{i_1=1}^{l_1-1} \cdots \sum_{i_d=1}^{l_d-1} \begin{pmatrix} \mathrm{D}_{h_1}^- v^{i_1\cdots i_d} \\ \vdots \\ \mathrm{D}_{h_d}^- v^{i_1\cdots i_d} \end{pmatrix} \psi_{i_1\cdots i_d}(\boldsymbol{x}) \in \mathcal{L}_h^2.\end{aligned}$$

Remark 3.2. For any $j = 1, \cdots, d$, let $v_h(x_j) = \sum_{i_j=1}^{l_j-1} v^{i_j} \phi_{i_j}(x_j)$. Then, for each interval $(x_j^{(i_j-1)}, x_j^{(i_j)})$, we have

$$\frac{\partial v_h}{\partial x_j}(x_j) = \frac{v^{i_j} - v^{i_j-1}}{h_j^{(i_j)}} = \mathrm{D}_{h_j}^- v_h \ \forall x_j \in (x_j^{(i_j-1)}, x_j^{(i_j)}).$$

Therefore, $\frac{\partial v_h}{\partial x_j}$ can be represented as $\frac{\partial v_h}{\partial x_j} = \sum_{i_j=1}^{l_j-1} \mathrm{D}_{h_j}^- v_h \chi_{(x_j^{(i_j-1)}, x_j^{(i_j)})}$ in $L^2(0, L_j)$. In order to represent $\frac{\partial v_h}{\partial x_j}$ as a function in $\mathcal{L}_h^2$, we can reinterpret it as follows:

$$\mathcal{L}_h^2 \ni \widetilde{\frac{\partial v_h}{\partial x_j}}(x_j) = \sum_{i_j=1}^{l_j-1} \frac{v^{i_j} - v^{i_j-1}}{h_j^{(i_j)}} \phi_{i_j}(x_j) = \sum_{i_j=1}^{l_j-1} \mathrm{D}_{h_j}^- v^{i_j} \phi_{i_j}(x_j),$$

Therefore, for $v_h \in \mathcal{H}_h^1$, we can interpret $\frac{\partial v_h}{\partial x_j}(\boldsymbol{x}) = \partial_{x_j} v_h(\boldsymbol{x}) = \mathrm{D}_{h_j}^- v_h(\boldsymbol{x})$.

Now we are ready to endow $\mathcal{H}_h^1$ with the following semi-discrete h_h^1-inner product (utilizing the Poincaré lemma):

$$(u_h, v_h)_{h_h^1} = (\mathbf{D}_h^- u_h, \mathbf{D}_h^- v_h)_{\ell_h^2} + (\boldsymbol{\nabla}_{\boldsymbol{y}} u_h, \boldsymbol{\nabla}_{\boldsymbol{y}} v_h)_{\ell_h^2} \quad \forall u_h, v_h \in \mathcal{H}_h^1. \tag{37}$$

Remark 3.3. If $\mathcal{H}_h^1$ is a subset of $H^1(\Omega)$ instead of $H_0^1(\Omega)$, the h_h^1-inner product may be replaced by

$$(u_h, v_h)_{h_h^1} = (u_h, v_h)_{\ell_h^2} + \big(\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v_h\big)_{\ell_h^2}. \tag{38}$$

As an application of **Theorem ??**, we establish the equivalence between the continuous H^1-norm and the semi-discrete h_h^1-norm on the semi-discrete space $\mathcal{H}_h^1$. The following theorem was proved in [**?**, Lemma 3.1] for the special case of $d = 2$ under the assumption of uniform partitions. The following theorem is a generalization to the case of uniform or nonuniform partitions in any finite dimension d.

Theorem 3.4. *Let $\mathcal{H}_h^1$ be defined as in (**??**) for $d \geqslant 1$. Then, the following inequalities hold.*

$$\|\boldsymbol{\nabla} v_h\|_{0,\Omega}^2 \leqslant \|\boldsymbol{\nabla} v_h\|_{\ell_h^2}^2 \leqslant 3^d \|\boldsymbol{\nabla} v_h\|_{0,\Omega}^2 \ \forall v_h \in \mathcal{H}_h^1, \tag{39}$$

where the bounding constants are independent of mesh size h.

Proof. Let $v_h = \sum_{i_1=1}^{l_1-1} \cdots \sum_{i_d=1}^{l_d-1} v^{i_1\cdots i_d}(\boldsymbol{y}) \psi_{i_1\cdots i_d}(\boldsymbol{x}) \in \mathcal{H}_h^1$ be arbitrary. Assume the notations $v^{i_1\cdots i_d} = 0$ for $i_j = 0$ or $i_j = l_j$, $j = 1, \cdots, d$. Then, from **Theorem ??** it is obvious that

$$\|\boldsymbol{\nabla}_{\boldsymbol{y}} v_h\|_{L^2(\Omega)}^2 \leqslant \|\boldsymbol{\nabla}_{\boldsymbol{y}} v_h\|_{\ell_h^2}^2 \leqslant 3^d \|\boldsymbol{\nabla}_{\boldsymbol{y}} v_h\|_{L^2(\Omega)}^2. \tag{40}$$

Next, we consider $\|\partial_{x_j} v_h\|_{L^2(\Omega)}^2$ for $j = 1, \cdots, d$. By definition and Remark **??**,

$$\begin{aligned} \|\partial_{x_j} v_h\|_{L^2(\Omega)}^2 &= \sum_{i_1=1}^{l_1} \cdots \sum_{i_d=1}^{l_d} \int\limits_{K_{i_1\cdots i_d}} \Big\| \sum_{\boldsymbol{\alpha} \in \mathcal{A}_{K_{i_1\cdots i_d}}} (\partial_{x_j} v^{\boldsymbol{\alpha}}) \psi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \Big\|_{L^2(\Omega_{\boldsymbol{y}})}^2 \mathrm{d}\boldsymbol{x} \\ &= \sum_{i_1=1}^{l_1} \cdots \sum_{i_d=1}^{l_d} \int\limits_{K_{i_1\cdots i_d}} \Big\| \sum_{\boldsymbol{\alpha} \in \mathcal{A}_{K_{i_1\cdots i_d}}} (\mathrm{D}_{h_j}^- v^{\boldsymbol{\alpha}}) \psi_{\boldsymbol{\alpha}}(\boldsymbol{x}) \Big\|_{L^2(\Omega_{\boldsymbol{y}})}^2 \mathrm{d}\boldsymbol{x}. \end{aligned}$$

Hence, the arguments leading to (**??**) and (**??**) yield

$$\|\partial_{x_j} v_h\|_{L^2(\Omega)}^2 \leqslant \|\mathrm{D}_{h_j}^- v_h\|_{\ell_h^2}^2 \leqslant 3^d \|\partial_{x_j} v_h\|_{L^2(\Omega)}^2 \quad \forall v_h \in \mathcal{H}_h^1 \tag{41}$$

for all $j = 1, \cdots, d$. Therefore, we have

$$\|\boldsymbol{\nabla}_{\boldsymbol{x}} v_h\|_{L^2(\Omega)}^2 \leqslant \|\mathbf{D}_h^- v_h\|_{\ell_h^2}^2 \leqslant 3^d \|\boldsymbol{\nabla}_{\boldsymbol{x}} v_h\|_{L^2(\Omega)}^2 \quad \forall v_h \in \mathcal{H}_h^1. \tag{42}$$

The combination of (**??**) and (**??**) gives (**??**). This completes the proof. □

4. Numerical examples

In this section, functions of random values and of checkerboard pattern will be used to confirm the sharpness of the norm equivalence (**??**) and (**??**) on uniform and non-uniform meshes. Our numerical experiments are confined to 2D scenarios.

Example 4.1. Set $\Omega_{\boldsymbol{x}} = (0,1)^2$ and consider a regular family of triangulation $(\mathcal{T}_h)_{h>0}$ by $l_1 \times l_2$ rectangles of $\Omega_{\boldsymbol{x}}$. Let V_h be the Q_1 finite element subspace of $H_0^1(\Omega_{\boldsymbol{x}})$ associated with $\mathcal{T}_h$. Throughout the section, the representation formulae (**??**) and (**??**) will be employed for $\|\cdot\|^2_{L^2(\Omega_{\boldsymbol{x}};Y)}$ and $\|\cdot\|^2_{\ell_h^2}$ in the computation of norms of $v_h(\boldsymbol{x}) = \sum_{i_1=1}^{l_1-1}\sum_{i_2=1}^{l_2-1} v^{i_1 i_2}\psi_{i_1 i_2}(\boldsymbol{x}) \in V_h$.

(1) **Uniform meshes.** We first take uniform meshes with $h_j = 1/l_j, j = 1,2$. For random coefficients γ_{kl}, consider

(43)
$$f_1(x_1,x_2) = \sum_{k=0}^{K}\sum_{l=0}^{L} \gamma_{kl}\sin(2\pi k x_1)\sin(2\pi l x_2) \quad \text{with } K = L = 10,$$

where $\gamma_{kl} = \left(r_{kl} - \frac{1}{2}\right)\mathrm{e}^{\min\left\{\frac{1}{r_{kl}},10\right\}}$ with random numbers $r_{kl} \in (0,1)$. The coefficients $v^{i_1 i_2}$ are chosen such that $v^{i_1 i_2} = f_1(x_1^{(i_1)}, x_2^{(i_2)}), i_j = 1,\cdots,l_j, j = 1,2$. The numerical results are presented in Table **??**. We observe from Table **??** that the inequalities

$$\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})} \leqslant \|v_h\|^2_{\ell_h^2} \leqslant 3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$$

hold for such $v_h \in V_h$. These confirm partially the inequalities in **Theorem ??**.

(2) **Checkerboard pattern.** On uniform meshes of $l_1 \times l_2$ rectangles, we consider the following function of checkerboard pattern:

$$f_2(x_1,x_2) = \cos\big((l_1x_1 - 1)\pi\big)\cos\big((l_2x_2 - 1)\pi\big), \quad (x_1,x_2) \in (0,1)^2.$$

We take the coefficients $v^{i_1 i_2} = f_2(x_1^{(i_1)}, x_2^{(i_2)})$ for $i_j = 1,\cdots,l_j - 1, j = 1,2$. Our numerical experiments were performed for $l_1 = l_2 = 20 * 2^{k-1}, k = 1,\cdots,9$. Table **??** displays numerical results which confirm the ratio of $\|v_h\|^2_{\ell_h^2}$ and $\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$ approaches 3^2 as in (**??**). These results also confirm the theoretical analysis given in Example **??**.

TABLE 1. Comparison of $\|v_h\|^2_{\ell^2_h}$ and $\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$ on uniform meshes for a random function.

$l_1 = l_2$	$h_1 = h_2$	$\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$	$\|v_h\|^2_{\ell^2_h}$	$\|v_h\|^2_{\ell^2_h}/\big(3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}\big)$
20	5.00E-02	6.19E-01	1.52E+00	0.27
40	2.50E-02	1.41E+00	1.94E+00	0.15
80	1.25E-02	1.78E+00	1.94E+00	0.12
160	6.25E-03	1.90E+00	1.94E+00	0.11
320	3.13E-03	1.93E+00	1.94E+00	0.11
640	1.56E-03	1.94E+00	1.94E+00	0.11
1280	7.81E-04	1.94E+00	1.94E+00	0.11
2560	3.91E-04	1.94E+00	1.94E+00	0.11
5120	1.95E-04	1.94E+00	1.94E+00	0.11

TABLE 2. Comparison of $\|v_h\|^2_{\ell^2_h}$ and $3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$ on uniform meshes for a function of checkerboard pattern.

$l_1 = l_2$	$h_1 = h_2$	$3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$	$\|v_h\|^2_{\ell^2_h}$	$\|v_h\|^2_{\ell^2_h}/\big(3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}\big)$
20	5.00E-02	1.00E+02	9.03E+01	0.90
40	2.50E-02	1.00E+02	9.52E+01	0.95
80	1.25E-02	1.00E+02	9.75E+01	0.98
160	6.25E-03	1.00E+02	9.88E+01	0.99
320	3.13E-03	1.00E+02	9.94E+01	0.99
640	1.56E-03	1.00E+02	9.97E+01	1.00
1280	7.81E-04	1.00E+02	9.98E+01	1.00
2560	3.91E-04	1.00E+02	9.99E+01	1.00
5120	1.95E-04	1.00E+02	1.00E+02	1.00

(3) **Random function on graded meshes.** Consider the graded mesh defined as follows (see FIGURE **??**): for $j = 1, 2$,

$$
\begin{aligned}
&x_j^{(i_j)} = \frac{1}{2^{l_j - i_j}},\ i_j = 1, \cdots, l_j;\\
&h_j^{(1)} = \frac{1}{2^{l_j-1}}, \quad \bar{h}_j^{(1)} = \frac{1}{2^{l_j-1}};\\
&h_j^{(i_j)} = \frac{1}{2^{l_j-i_j+1}}, \quad \bar{h}_j^{(i_j)} = \frac{3}{2^{l_j-i_j+2}},\ i_j = 2, \cdots, l_j.
\end{aligned}
$$

We employ the random function (**??**) and the corresponding numerical results are presented in TABLE **??**. The inequalities in (**??**) are verified also in this case.

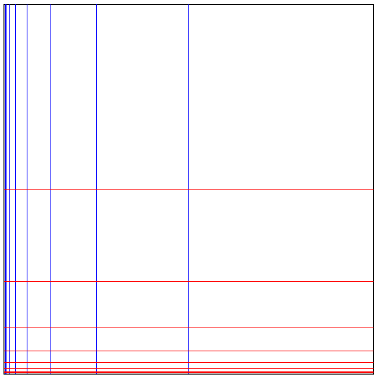


FIGURE 3. The graded mesh for grading factor $n = 8$.

TABLE 3. Comparison of $\|v_h\|^2_{\ell^2_h}$ and $\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$ on a graded mesh for a random function.

$l_1 = l_2$	$h_{\min}$	$\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$	$\|v_h\|^2_{\ell^2_h}$	$\|v_h\|^2_{\ell^2_h}/\big(3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}\big)$
1	5.00E-01	6.66E-61	6.66E-61	0.25
2	2.50E-01	1.68E-04	1.68E-04	0.25
3	1.25E-01	5.42E-02	5.42E-02	0.23
4	6.25E-02	7.64E-02	7.64E-02	0.27
5	3.13E-02	1.27E-01	1.27E-01	0.25
6	1.56E-02	1.40E-01	1.40E-01	0.23
7	7.81E-03	1.40E-01	1.40E-01	0.23
8	3.91E-03	1.40E-01	1.40E-01	0.23
9	1.95E-03	1.40E-01	1.40E-01	0.23
10	9.77E-04	1.40E-01	1.40E-01	0.23
11	4.88E-04	1.40E-01	1.40E-01	0.23
12	2.44E-04	1.40E-01	1.40E-01	0.23

(4) **A function of checkerboard pattern on graded meshes.** On the same graded mesh, we choose the coefficients $v^{i_1 i_2} = f_3(x_1^{(i_1)}, x_2^{(i_2)})$ where

$$f_3(x_1,x_2) = \cos\big((l_1{+}\log_2 x_1{-}1)\pi\big)\cos\big((l_2{+}\log_2 x_2{-}1)\pi\big),\ (x_1,x_2)\in(0,1)^2.$$

Numerical results are presented in TABLE **??** and FIGURE **??**. We stress that the more meshes are used in the graded meshes, the sharper the upper bound in the **Theorem ??** is attained.

TABLE 4. Comparison of $\|v_h\|^2_{\ell^2_h}$ and $3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$ on graded meshes for a function of checkerboard pattern.

$l_1 = l_2$	$h_{\min}$	$3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}$	$\|v_h\|^2_{\ell^2_h}$	$\|v_h\|^2_{\ell^2_h}/\left(3^2\|v_h\|^2_{L^2(\Omega_{\boldsymbol{x}})}\right)$
1	5.00E-02	2.50E+01	6.25E+00	0.25
2	2.50E-02	2.50E+01	1.41E+01	0.56
3	1.25E-02	2.50E+01	1.91E+01	0.77
4	6.25E-03	2.50E+01	2.20E+01	0.88
5	3.13E-03	2.50E+01	2.35E+01	0.94
6	1.56E-03	2.50E+01	2.42E+01	0.97
7	7.81E-04	2.50E+01	2.46E+01	0.98
8	3.91E-04	2.50E+01	2.48E+01	0.99
9	1.95E-04	2.50E+01	2.49E+01	1.00
10	9.77E-05	2.50E+01	2.50E+01	1.00
11	4.88E-05	2.50E+01	2.50E+01	1.00
12	2.44E-05	2.50E+01	2.50E+01	1.00

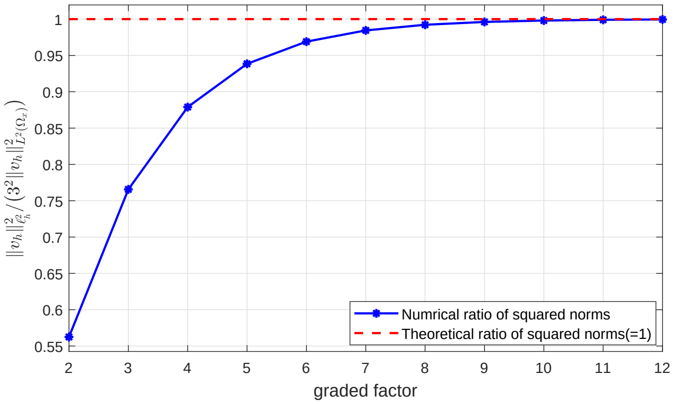


FIGURE 4. Ratio of norms on graded mesh for checkerboard function.

5. Conclusions

In this article, we derive several useful representation formulae for the L^2-norm and the semi-discrete ℓ^2_h-norm that is intrinsic to mass lumping in any dimension. Using these reprentation formulae, accurate bounds between the two norms are obtained. Then we prove such bounds holds for $H^1(\Omega)$-norm and the semi-discrete h^1_h-norm. We emphasize that the bounding constants are independent of mesh size h, which means that the equivalence relationships are uniform in h. However, one of the bounding constants depends on dimension. Specific examples and numerical experiments are provided to validate the theoretical results, confirming their correctness and applicability. The theorems are applicable to analyze finite element methods where the mass lumping schemes used in some directions as well as theoretical analysis of the Finite Difference FEM.

In future work, we will focus on the equivalent relationships between other norms in continuous and discrete spaces in any finite dimensional space.

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