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ON *A PRIORI* ERROR ANALYSIS OF DISCONTINUOUS GALERKIN METHOD FOR THE VLASOV-NONSTATIONARY STOKES SYSTEM

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Abstract. In the first part of this paper, the uniqueness of a strong solution is established for the Vlasov-unsteady Stokes problem in 3D. The second part deals with a semi discrete scheme, which is derived as a result of spatial discretization of the coupled system of Vlasov and Stokes equations for the 2D problem by discontinuous Galerkin methods, while keeping temporal variable continuous. The proposed semi-discrete scheme preserves both mass and momentum conservation properties. Based on the orthogonal L^2 as well as the Stokes projections, error estimates in the case of smooth compactly supported initial data are derived by employing a variant of nonlinear Grönwall's lemma in a crucial way. Moreover, the generalization of error estimates to 3D problem is also briefly discussed. Finally, using a time splitting algorithm as the phase space is four dimensional, some numerical experiments are conducted, whose results confirm our theoretical findings.

Key words. Vlasov- nonstationary Stokes, uniqueness in 3D, discontinuous Galerkin method, conservation properties, error estimates, nonlinear version of Grönwall's lemma, time-splitting scheme and numerical experiments.

1. Introduction

This paper develops and analyses a discontinuous Galerkin method for the following coupled system of Vlasov-nonstationary Stokes equations:

$$(1)\qquad \begin{cases} \partial_t f + v\cdot\nabla_x f + \nabla_v\cdot\Big((\boldsymbol{u}-v)\,f\Big) = 0 & \text{in } (0,T)\times\Omega_x\times\mathbb{R}^d, \\ \qquad\qquad f(0,x,v) = f_0(x,v) & \text{in } \Omega_x\times\mathbb{R}^d. \end{cases}$$

$$(2)\qquad \begin{cases} \partial_t \boldsymbol{u} - \Delta_x \boldsymbol{u} + \nabla_x p = \displaystyle\int_{\mathbb{R}^d} (v-\boldsymbol{u})\,f\,\mathrm{d}v & \text{in } (0,T)\times\Omega_x, \\ \qquad \nabla_x\cdot\boldsymbol{u} = 0 & \text{in } \Omega_x, \\ \qquad \boldsymbol{u}(0,x) = \boldsymbol{u_0}(x) & \text{in } \Omega_x. \end{cases}$$

Here, Ω_x is a d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ for $d = 2, 3$, $\boldsymbol{u}(t,x)$ denotes the fluid velocity, $p(t,x)$ is the fluid pressure, and $f(t,x,v)$ represents the droplet distribution function. In this paper, we take $\mathbb{T}^d = [0,M]^d, M > 0$ for $d = 2,3$, with periodic boundary conditions. This system arises in the mathematical model of the thin sprays of droplets in a gas medium, which is dispersed into a background fluid medium. The dispersed phase is modelled by a kinetic equation called Vlasov equation, where as, the background fluid medium is described by the nonstationary Stokes equation. The coupling in Vlasov part occurs via a drag force and in Stokes system by a Brinkman force which acts as a forcing term. Study of this type of models comprising of a kinetic equation for the dispersed phase and a fluid equation for the gas goes back to the works of O'Rourke [26] and Williams [32] (also, see [10]).

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Earlier, Hamdache [19] has discussed the global existence of weak solutions to the system (1)-(2) with homogeneous Dirichlet boundary condition for the fluid velocity and with specular reflection boundary condition for the droplet distribution. Various kinetic fluid equations have been discussed in the literature, say, for example Vlasov-Burgers equations [16, 17]; Vlasov-Euler equations [4]; Vlasov-Navier-Stokes equations [5, 9, 34, 22] and references, therein.

This paper deals with some qualitative and quantitative properties of solutions to our system (1)-(2). Note that a proof of the global-in-time existence of strong solution to the system (1)-(2) in 3D is established in [9], but uniqueness is missing there, therefore, our first attempt is to prove uniqueness result in 3D.

Since our major emphasis in the subsequent part of this paper is on developing and analysing an appropriate numerical scheme for the system (1)-(2), which is motivated by [12, 20, 13, 14] for the Vlasov-Poisson system, [25] for the Vlasov-steady Stokes and [7, 33] for the Vlasov-Maxwell system, we apply discontinous Galerkin methods for both kinetic and nonstationary Stokes equations to discretize in both x and v variables keeping time variable continuous. Thereby, we derive a semi-discrete scheme. Note that the kinetic equation is a transport problem which conserves total mass. As discontinuous Galerkin(dG) methods have the property to preserve the conservation property, dG turns out to be a method of choice for the kinetic equation. For the time dependent Stokes system, we apply dG, but other methods like mixed finite element method, the local discontinuous Galerkin or any conforming or nonconforming numerical method can also be employed. Our main contributions in this paper are as follows:

- Uniqueness of strong solution in three dimensions is proved for the Vlasov-unsteady Stokes system. Earlier in [24], there is an error in the proof of uniqueness and a correct proof of uniqueness is given in Section 2 of this paper. This can be achieved after proving some higher moment estimates for the droplet distribution and regularity results for the fluid velocity vector.
- DG methods are proposed for numerical approximations and are shown to conserve mass and the total momentum, which confirm similar conservation properties for the continuous system.
- Error estimates are derived for the fluid velocity and the fluid pressure approximations using Stokes projection in 2D setup, while in 3D case some remarks are given in the subsection 4.4. Moreover, error estimate for the approximation of the droplet distribution is obtained using the orthogonal L^2-projection. The error analysis uses a variant of the nonlinear Grönwall's inequality in a crucial way.
- Since phase space is four or six dimensions depending on $d = 2$ or $d = 3$, a Lie splitting in time marching combined with dimension splitting in the phase space is proposed for computational experiments, whose numerical results confirm our theoretically findings.

This paper is organized as follows. In Section 2, we deal with certain qualitative and quantitative properties of the solutions to the continuum model. We further prove uniqueness result for strong solutions to (1)-(2). Section 3 introduces a semi-discrete dG-dG numerical scheme and analyses some of its properties. *A priori* error estimates for the semi-discrete method are established in Section 4. Further, some comments on 3D problems are presented. Section 5 deals with some numerical experiments based on a time-splitting schemes combined with dG methods, whose results confirm our theoretical findings. Finally in Section 6, some concluding remarks are given.

2. Continuous problem: Some properties and uniqueness of strong solution

Let us denote by the local density ρ and the local macroscopic velocity of the droplets V, respectively, as

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v \quad \text{and} \quad V(t,x) = \frac{1}{\rho}\int_{\mathbb{R}^d} f(t,x,v)v\,\mathrm{d}v.$$

Define the l^{th} order velocity moments of distribution function as

$$m_l f(t,x) = \int_{\mathbb{R}^d} |v|^l f(t,x,v)\,\mathrm{d}v, \quad \text{for} \quad l \in \mathbb{N}\cup\{0\}.$$

Now, we use the standard notations for Sobolev spaces with their norms, say, the space $W^{k,p}$ denotes the k^{th} order L^p-Sobolev spaces. Further, we denote by $\boldsymbol{W}^{k,p} = \left(W^{k,p}(\Omega_x)\right)^2$, for all $k \geq 0$, and for $1 \leq p \leq \infty$. When $p = \infty$, it is defined in the usual manner.

For our subsequent use, set $\boldsymbol{J}_0$ and $\boldsymbol{J}_1$ as a class of divergence free (in the sense of distributions) vector fields defined by

$$\boldsymbol{J}_0 = \left\{\boldsymbol{w} \in \boldsymbol{L}^2 : \nabla_x \cdot \boldsymbol{w} = 0, \boldsymbol{w} \text{ is periodic}\right\},$$

$$\boldsymbol{J}_1 = \left\{\boldsymbol{w} \in \boldsymbol{W}^{1,2} : \nabla_x \cdot \boldsymbol{w} = 0, \boldsymbol{w} \text{ is periodic}\right\}.$$

Throughout this paper, it is assumed that any function defined on Ω_x is periodic in the x-variable and C is a generic constant which is independent on the discretizing parameter h.

2.1. Some properties of the model problem. We begin this subsection by stating a result on L^∞-estimate for the local density.

Lemma 2.1. [22, Prop.4.6, p.44] *Assume that $\boldsymbol{u} \in L^1(0,T;\boldsymbol{L}^\infty)$, and $\sup_{C^r_{t,v}} f_0 \in L^\infty_{loc}\left(\mathbb{R}_+; L^1(\mathbb{R}^d)\right)$, where $C^r_{t,v} := \Omega_x \times B(e^t v, r)$, $\forall\, r > 0$ and $B(e^t v, r)$ denotes the ball centered at $e^t v$ having radius r. Then, there holds:*

$$\|\rho\|_{L^\infty((0,T]\times\Omega_x)} \leq e^{dT} \sup_{t\in[0,T]} \|\sup_{C^r_{t,v}} f_0\|_{L^1(\mathbb{R}^d)}. \tag{3}$$

In the following Lemma, we gather certain conservation properties of solutions $(f,\boldsymbol{u})$ to the system (1)-(2). For its proof, see [19].

Lemma 2.2. *The following properties are satisfied by the solutions $(f,\boldsymbol{u})$ of the Vlasov-Stokes system:*

(1) ***(Positivity preserving)*** *If the initial condition $f_0 \geq 0$, then the solution $f \geq 0$.*

(2) ***(Mass conservation)*** *The total mass of the solution f is conserved in the following sense :*

$$\int_{\mathbb{R}^d}\int_{\Omega_x} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^d}\int_{\Omega_x} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v, \quad t \in [0,T].$$

(3) ***(Total momentum conservation)*** *The total momentum is preserved in the sense that for all $t \in [0,T]$,*

$$\int_{\mathbb{R}^d}\int_{\Omega_x} v f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v + \int_{\Omega_x} \boldsymbol{u}(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\Omega_x} v f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v + \int_{\Omega_x} \boldsymbol{u_0}(x)\,\mathrm{d}x.$$

The following Lemma gives the important energy-dissipation identity satisfied by the solutions to the Vlasov-Stokes system.

Lemma 2.3. ***(Energy dissipation)*** *The total energy of the Vlasov-Stokes system* (1)-(2) *dissipates in time, i.e.,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d}\int_{\Omega_x}|v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v+\int_{\Omega_x}\boldsymbol{u}^2\,\mathrm{d}x\right)\le 0,$$

provided f is non-negative.

Proof. Multiplying (1) by $\frac{|v|^2}{2}$ and (2) by $\boldsymbol{u}\in\boldsymbol{J}_1$, an application of the integration by parts shows

$$\int_{\mathbb{R}^d}\int_{\Omega_x}\partial_t\left(\frac{|v|^2}{2}f\right)\mathrm{d}x\,\mathrm{d}v+\frac{1}{2}\int_{\Omega_x}\partial_t\boldsymbol{u}^2\,\mathrm{d}x+\int_{\Omega_x}|\nabla_x\boldsymbol{u}|^2\,\mathrm{d}x-2\int_{\mathbb{R}^d}\int_{\Omega_x}v\cdot\boldsymbol{u}f\,\mathrm{d}x\,\mathrm{d}v$$
$$+\int_{\mathbb{R}^d}\int_{\Omega_x}|v|^2 f\,\mathrm{d}x\,\mathrm{d}v+\int_{\mathbb{R}^d}\int_{\Omega_x}|\boldsymbol{u}|^2 f\,\mathrm{d}x\,\mathrm{d}v=0.$$

The above equality is the same as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d}\int_{\Omega_x}|v|^2\,f\,\mathrm{d}x\,\mathrm{d}v+\int_{\Omega_x}\boldsymbol{u}^2\mathrm{d}x\right)+2\int_{\Omega_x}|\nabla_x\boldsymbol{u}|^2\,\mathrm{d}x$$
$$+2\int_{\mathbb{R}^d}\int_{\Omega_x}|\boldsymbol{u}-v|^2 f\,\mathrm{d}x\,\mathrm{d}v=0.$$

As the third term is non-negative and as the last term is non-negative because f is non-negative, we drop them to derive the result and this completes the proof. ☐

As a consequence of above lemma, we also obtain the following identity:

$$\begin{aligned}&\frac{1}{2}\left(\int_{\mathbb{R}^d}\int_{\Omega_x}|v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v+\int_{\Omega_x}\boldsymbol{u}^2\,\mathrm{d}x\right)+\int_0^t\int_{\Omega_x}|\nabla_x\boldsymbol{u}|^2\,\mathrm{d}x\,\mathrm{d}t\\&+\int_0^t\int_{\mathbb{R}^d}\int_{\Omega_x}|\boldsymbol{u}-v|^2 f\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t=\frac{1}{2}\int_{\mathbb{R}^d}\int_{\Omega_x}|v|^2\,f_0\,\mathrm{d}x\,\mathrm{d}v+\frac{1}{2}\int_{\Omega_x}\boldsymbol{u}_{\mathbf{0}}^2\,\mathrm{d}x.\end{aligned}\tag{4}$$

Hence, we deduce that

$$\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{L}^2)\quad\text{and}\quad\boldsymbol{u}\in L^2(0,T;\boldsymbol{J}_1)\tag{5}$$

provided $|v|^2 f_0\in L^1(\Omega_x\times\mathbb{R}^d)$ and $\boldsymbol{u}_{\mathbf{0}}\in\boldsymbol{L}^2$.
Recall by Sobolev imbedding $H^1(\Omega_x)\hookrightarrow L^p(\Omega_x)$. Here, when $d=2,\ p\in[2,\infty)$ and for $d=3,\ p\in[2,6]$:

$$\boldsymbol{u}\in L^2(0,T;\boldsymbol{L}^p).\tag{6}$$

Below, we state a result on the local density and local momentum for which proof can be found in [19, Lemma 2.2, p.56]. These results are essential for obtaining the regularity of the solutions to the Stokes system.

Lemma 2.4. *For $p\ge 1$, let $\boldsymbol{u}\in L^2(0,T;\boldsymbol{L}^{p+d}), f_0\in L^\infty(\Omega_x\times\mathbb{R}^d)\cap L^1(\Omega_x\times\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d}\int_{\Omega_x}|v|^p f_0\,\mathrm{d}x\mathrm{d}v<\infty.$$

Then, there hold

$$\rho\in L^\infty\left(0,T;L^{\frac{p+d}{d}}(\Omega_x)\right)\quad\textit{and}\quad\rho V\in L^\infty\left(0,T;L^{\frac{p+d}{d+1}}(\Omega_x)\right).\tag{7}$$

Remark 2.5. *By choosing $p=5$ in Lemma 2.4 for $d=2$, we obtain*

$$\rho\in L^\infty\left(0,T;L^{\frac{7}{2}}(\Omega_x)\right)\quad\textit{and}\quad\rho V\in L^\infty\left(0,T;L^{\frac{7}{3}}(\Omega_x)\right).\tag{8}$$

Remark 2.6. *For $d = 3$, $p = 3$ in Lemma 2.4 yields*

$$\rho \in L^\infty\left(0,T;L^2(\Omega_x)\right) \quad \textit{and} \quad \rho V \in L^\infty\left(0,T;L^{\frac{3}{2}}(\Omega_x)\right). \tag{9}$$

An application of the Stokes regularity results [18, 29] *shows*

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{W}^{2,\frac{3}{2}}). \tag{10}$$

From the Sobolev inequality, it follows that

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{L}^p) \quad \textit{for} \quad \frac{3}{2} \le p < \infty. \tag{11}$$

Taking $p = 5$ in Lemma 2.4, we obtain

$$\rho \in L^\infty\left(0,T;L^{\frac{8}{3}}(\Omega_x)\right) \quad \textit{and} \quad \rho V \in L^\infty\left(0,T;L^2(\Omega_x)\right). \tag{12}$$

An application of the Stokes regularity result shows

$$\boldsymbol{u} \in H^1(0,T;\boldsymbol{L}^2) \cap L^2(0,T;\boldsymbol{H}^2) \cap L^\infty(0,T;\boldsymbol{H}^1). \tag{13}$$

Setting $p = 9 + \delta$ with $\delta > 0$ in Lemma 2.4, we arrive at

$$\rho \in L^\infty\left(0,T;L^{\frac{12+\delta}{3}}(\Omega_x)\right) \quad \textit{and} \quad \rho V \in L^\infty\left(0,T;L^{\frac{12+\delta}{4}}(\Omega_x)\right). \tag{14}$$

These properties of ρ and ρV in the Stokes equation (2) guarantees that $\boldsymbol{u} \in L^2(0,T;\boldsymbol{W}^{1,\infty})$ which is important for our proof of existence of a unique strong solution.

Below, we state and prove a result on the propagation of velocity moments.

Lemma 2.7. *Let $\boldsymbol{u} \in L^1(0,T;\boldsymbol{W}^{1,\infty})$ and let $f_0 \ge 0$ be such that for $k \ge 0$ and $\langle v \rangle = \left(1+|v|^2\right)^{\frac{1}{2}}$*

$$\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v \rangle^{kp}\{|f_0|^p + |\nabla_x f_0|^p + |\nabla_v f_0|^p\}\,\mathrm{d}v\,\mathrm{d}x \le C.$$

Then, the solution f of the kinetic equation for $k \ge 0$ satisfies

$$\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v \rangle^{kp}\{|f|^p + |\nabla_x f|^p + |\nabla_v f|^p\}\,\mathrm{d}v\,\mathrm{d}x \le C \quad \forall\ t > 0.$$

Proof. Consider the equation for $\frac{\partial f}{\partial x_i}$:

$$\partial_t \frac{\partial f}{\partial x_i} + v \cdot \nabla_x \frac{\partial f}{\partial x_i} + \nabla_v \cdot \left(\frac{\partial \boldsymbol{u}}{\partial x_i} f\right) + \nabla_v \cdot \left((\boldsymbol{u} - v)\,\frac{\partial f}{\partial x_i}\right) = 0,$$

for $i = 1,2,3$. Multiplying the above vector equation by $\langle v \rangle^{kp}|\nabla_x f|^{p-1}$, an integration with respect to x, v yields

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v \rangle^{kp}|\nabla_x f|^p\,\mathrm{d}v\,\mathrm{d}x = I_1 + I_2 + I_3,$$

where

$$I_1 = -\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v \rangle^{kp}|\nabla_x f|^{p-1}\nabla_x \boldsymbol{u} \cdot \nabla_v f\,\mathrm{d}v\,\mathrm{d}x,$$

$$I_2 = 3\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v \rangle^{kp}|\nabla_x f|^p\,\mathrm{d}v\,\mathrm{d}x,$$

$$I_3 = -\frac{1}{p}\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v \rangle^{kp}\,(\boldsymbol{u} - v) \cdot \nabla_v\left(|\nabla_x f|^p\right)\,\mathrm{d}v\,\mathrm{d}x.$$

An application of the Young's inequality in I_1 shows

$$I_1 \leq \|\nabla_x \boldsymbol{u}\|_{\boldsymbol{L}^\infty} \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} \left(\frac{p-1}{p} |\nabla_x f|^p + \frac{1}{p} |\nabla_v f|^p \right) \mathrm{d}v \, \mathrm{d}x.$$

On using integration by parts to I_3, we arrive at

$$I_3 = -\frac{3}{p} \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} |\nabla_x f|^p \, \mathrm{d}v \, \mathrm{d}x + I_4$$

with

$$I_4 = k \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp-2} v \cdot (\boldsymbol{u} - v) \, |\nabla_x f|^p \, \mathrm{d}v \, \mathrm{d}x.$$

Again, we apply the Young's inequality to obtain

$$\begin{aligned} I_4 &\leq k \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty} \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp-1} |\nabla_x f|^p \, \mathrm{d}v \, \mathrm{d}x + k \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} |\nabla_x f|^p \, \mathrm{d}v \, \mathrm{d}x \\ &\leq C \left(1 + \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}\right) \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} |\nabla_x f|^p \, \mathrm{d}v \, \mathrm{d}x. \end{aligned}$$

Similar computation involving the equation for $\nabla_v f$ shows

$$\begin{aligned} \frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} |\nabla_v f|^p \, \mathrm{d}v \, \mathrm{d}x &\leq C \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} \left(|\nabla_x f|^p + |\nabla_v f|^p \right) \mathrm{d}v \, \mathrm{d}x \\ &\quad + C \left(1 + \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}\right) \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} |\nabla_v f|^p \, \mathrm{d}v \, \mathrm{d}x. \end{aligned}$$

Altogether, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} \left(|\nabla_x f|^p + |\nabla_v f|^p \right) \mathrm{d}v \, \mathrm{d}x \right) \\ &\leq C \left(1 + \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\infty}}\right) \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} \left(|\nabla_x f|^p + |\nabla_v f|^p \right) \mathrm{d}v \, \mathrm{d}x, \end{aligned}$$

then an application of the Grönwall's lemma concludes the rest of the proof. □

2.2. Existence and Uniqueness result in 3D. This subsection discusses briefly the existence and uniqueness result for the strong solution to (1)-(2) the continuum model.

Below, we stated the result on the existence of strong solution to the Vlasov-Stokes equation whose proof can be found in [9, Theorem 5, p. 2435].

Theorem 2.8. ***(Existence of strong solution)*** *Let the initial data $(f_0, \boldsymbol{u_0})$ be such that*

$$f_0 \in L^1(\Omega_x \times \mathbb{R}^3) \cap L^\infty(\Omega_x \times \mathbb{R}^3), \quad f_0 \geq 0,$$

$$\int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} \left\{ |f_0|^p + |\nabla_x f_0|^p + |\nabla_v f_0|^p \right\} \mathrm{d}v \, \mathrm{d}x \leq C,$$

for $p \in (3, \infty)$, $\quad k > 4 - \frac{3}{p}$ and $\boldsymbol{u_0} \in \boldsymbol{W}^{1,p} \cap \boldsymbol{J}_1$. Then, there exists a global strong solution $(f, \boldsymbol{u}, p)$ to the Vlasov-Stokes system (1)-(2). Moreover,

$$\int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{kp} \left\{ |f|^p + |\nabla_x f|^p + |\nabla_v f|^p \right\} \mathrm{d}v \, \mathrm{d}x \leq C,$$

and

$$\boldsymbol{u} \in \boldsymbol{L}^r(0, T; \boldsymbol{W}^{2,q}) \cap H^1(0, T; \boldsymbol{L}^q)$$

for $q < p$ and $r \in (1, \infty)$. Here, $\langle v \rangle = \left(1 + |v|^2\right)^{\frac{1}{2}}$.

The following theorem is on the uniqueness result for the strong solution of (1)-(2).

Theorem 2.9. ***(Uniqueness of strong solution)*** *Under the hypothesis of Theorem 2.8 there is atmost one strong solution $(f, \boldsymbol{u}, p)$ to (1)-(2).*

Proof. Suppose the solution is not unique, that is, $(f_1, \boldsymbol{u}_1, p_1)$ and $(f_2, \boldsymbol{u}_2, p_2)$ are two distinct strong solutions of (1)-(2) with say $\boldsymbol{u}_1 \neq \boldsymbol{u}_2, p_1 \neq p_2$ and $f_1 \neq f_2$. Let $\bar{\boldsymbol{u}} = \boldsymbol{u}_1 - \boldsymbol{u}_2, \bar{p} = p_1 - p_2$ and $\bar{f} = f_1 - f_2$, then $\bar{\boldsymbol{u}}, \bar{p}$ and $\bar{f}$ satisfies the following equations:

$$\bar{f}_t + v \cdot \nabla_x \bar{f} + \nabla_v \cdot \left(\bar{\boldsymbol{u}} f_1 + \boldsymbol{u}_2 \bar{f} - v \bar{f} \right) = 0, \tag{15}$$

and

$$\begin{cases} \bar{\boldsymbol{u}}_t - \Delta_x \bar{\boldsymbol{u}} + \nabla_x \bar{p} = \displaystyle\int_{\mathbb{R}^3} \left(v \bar{f} - \boldsymbol{u}_2 \bar{f} - \bar{\boldsymbol{u}} f_1 \right) \, \mathrm{d}v, \\ \nabla_x \cdot \bar{\boldsymbol{u}} = 0 \end{cases} \tag{16}$$

with $\bar{f}(0, x, v) = 0$ and $\bar{\boldsymbol{u}}(0, x) = 0$.

Now, multiply the equation (16) by $\bar{\boldsymbol{u}}$ and then integrate in x to obtain

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \|\nabla_x \bar{\boldsymbol{u}}\|_{\boldsymbol{L}^2} \\ &= \int_{\Omega_x} \bar{\boldsymbol{u}} \left(\int_{\mathbb{R}^3} v \bar{f} \, \mathrm{d}v \right) \mathrm{d}x - \int_{\Omega_x} |\bar{\boldsymbol{u}}|^2 \left(\int_{\mathbb{R}^3} f_1 \, \mathrm{d}v \right) \mathrm{d}x - \int_{\Omega_x} \boldsymbol{u}_2 \bar{\boldsymbol{u}} \left(\int_{\mathbb{R}^3} \bar{f} \, \mathrm{d}v \right) \mathrm{d}x \\ &\leq \left(\int_{\Omega_x} |\bar{\boldsymbol{u}}|^2 \left(\int_{\mathbb{R}^3} \frac{1}{\langle v \rangle^{2k-2}} \, \mathrm{d}v \right) \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} |\bar{f}|^2 \, \mathrm{d}v \, \mathrm{d}x \right)^{\frac{1}{2}} \\ &\quad + \|\boldsymbol{u}_2\|_{\boldsymbol{L}^\infty} \left(|\bar{\boldsymbol{u}}|^2 \left(\int_{\mathbb{R}^3} \frac{1}{\langle v \rangle^{2k}} \, \mathrm{d}v \right) \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} |\bar{f}|^2 \, \mathrm{d}v \, \mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq C \|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \|\langle v \rangle^k \bar{f}\|^2_{L^2(\Omega_x \times \mathbb{R}^3)}. \end{aligned} \tag{17}$$

Note that in the second step the Hölder inequality and in the last step, the Young's inequality with assumption that $k > \frac{5}{2}$ are used.

Now, multiply equation (15) by $\langle v \rangle^{2k} \bar{f}$ and integrate in x, v-variables, to obtain

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\langle v \rangle^k \bar{f}\|^2_{L^2(\Omega_x \times \mathbb{R}^2)} + \frac{1}{2} \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} v \cdot \nabla_x \bar{f}^2 \, \mathrm{d}v \, \mathrm{d}x \\ &= - \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} \bar{f} \, \bar{\boldsymbol{u}} \cdot \nabla_v f_1 \, \mathrm{d}v \, \mathrm{d}x - \frac{1}{2} \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} \bar{\boldsymbol{u}}_2 \cdot \nabla_v \bar{f}^2 \, \mathrm{d}v \, \mathrm{d}x \\ &\quad + 3 \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} \bar{f}^2 \, \mathrm{d}v \, \mathrm{d}x + \frac{1}{2} \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} v \cdot \nabla_v \bar{f}^2 \, \mathrm{d}v \, \mathrm{d}x. \end{aligned} \tag{18}$$

A use of integration by parts yields

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\langle v \rangle^k \bar{f}\|^2_{L^2(\Omega_x \times \mathbb{R}^3)} = \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3 \tag{19}$$

where

$$\tilde{T}_1 = - \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} \bar{f} \, \bar{\boldsymbol{u}} \cdot \nabla_v f_1 \, \mathrm{d}v \, \mathrm{d}x$$

$$\tilde{T}_2 = -\frac{1}{2} \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} \bar{\boldsymbol{u}}_2 \cdot \nabla_v \bar{f}^2 \, \mathrm{d}v \, \mathrm{d}x$$

$$\tilde{T}_3 = 3 \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} \bar{f}^2 \, \mathrm{d}v \, \mathrm{d}x + \frac{1}{2} \int_{\Omega_x} \int_{\mathbb{R}^3} \langle v \rangle^{2k} v \cdot \nabla_v \bar{f}^2 \, \mathrm{d}v \, \mathrm{d}x.$$

For $\tilde{T}_1$ term

$$
\begin{aligned}
\tilde{T}_1 &\le \left(\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v\rangle^{2k} \bar{f}^2 \,\mathrm{d}v\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\Omega_x} |\bar{\boldsymbol{u}}|^5 \left(\int_{\mathbb{R}^3} \frac{1}{\langle v\rangle^{5\alpha}}\,\mathrm{d}v\right)\mathrm{d}x\right)^{\frac{1}{5}} \cdot \\
&\quad \left(\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v\rangle^{\frac{10(k+\alpha)}{3}} (\nabla_v f_1)^{\frac{10}{3}} \,\mathrm{d}v\,\mathrm{d}x\right)^{\frac{3}{10}} \\
&\le \|\langle v\rangle^k \bar{f}\|_{L^2(\Omega_x\times\mathbb{R}^3)} \|\bar{\boldsymbol{u}}\|_{\boldsymbol{L}^5} \|\langle v\rangle^{k+\alpha}\nabla_v f_1\|_{L^{\frac{10}{3}}(\Omega_x\times\mathbb{R}^3)} \\
&\le \|\langle v\rangle^k \bar{f}\|_{L^2(\Omega_x\times\mathbb{R}^3)} \left(\|\bar{\boldsymbol{u}}\|_{\boldsymbol{L}^2}^{\frac{1}{10}} \|\nabla\bar{\boldsymbol{u}}\|_{\boldsymbol{L}^2}^{\frac{9}{10}} + \|\bar{\boldsymbol{u}}\|_{\boldsymbol{L}^2}\right) \|\langle v\rangle^{k+\alpha}\nabla_v f_1\|_{L^{\frac{10}{3}}(\Omega_x\times\mathbb{R}^3)} \qquad (20)\\
&\le \frac{1}{2}\|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)} \|\langle v\rangle^{k+\alpha}\nabla_v f_1\|^2_{L^{\frac{10}{3}}(\Omega_x\times\mathbb{R}^3)} + \frac{1}{20}\|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \frac{9}{20}\|\nabla\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} \\
&\qquad + \frac{1}{2}\|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)} \|\langle v\rangle^{k+\alpha}\nabla_v f_1\|^2_{L^{\frac{10}{3}}(\Omega_x\times\mathbb{R}^3)} + \frac{1}{2}\|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} \\
&\le \|\langle v\rangle^{k+\alpha}\nabla_v f_1\|^2_{L^{\frac{10}{3}}(\Omega_x\times\mathbb{R}^3)} \|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)} + 2\|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \frac{9}{20}\|\nabla\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} \\
&\le C\|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)} + 2\|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \frac{9}{20}\|\nabla\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2}.
\end{aligned}
$$

Here, in the second step we have used the Hölder inequality with the assumption that $\alpha > \frac{3}{5}$, in the fourth step the Gagliardo - Nirenberg inequality [21, Theorem 9.9, p. 1317] and in the fifth step, we have applied the Young's inequality. Here, in the second last step we have used estimate from Theorem 2.8.

For the estimate of $\tilde{T}_2$ term, a use of integration by parts with the Hölder inequality yields

$$
\begin{aligned}
\tilde{T}_2 &= k\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v\rangle^{2k-2} v\cdot \boldsymbol{u}_2 \bar{f}^2 \,\mathrm{d}v\,\mathrm{d}x \\
&\le k\|\boldsymbol{u}_2\|_{\boldsymbol{L}^\infty} \|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)}.
\end{aligned} \tag{21}
$$

Apply the integration by parts, then the $\tilde{T}_3$ term can be estimated as

$$
\begin{aligned}
\tilde{T}_3 &= 3\|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)} - k\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v\rangle^{2k} \bar{f}^2 \,\mathrm{d}v\,\mathrm{d}x - \frac{3}{2}\int_{\Omega_x}\int_{\mathbb{R}^3} \langle v\rangle^{2k} \bar{f}^2 \,\mathrm{d}v\,\mathrm{d}x \\
&= -\left(k - \frac{3}{2}\right)\|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)}.
\end{aligned} \tag{22}
$$

Putting the estimates (20)-(22) into (19) shows

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)} \le C\|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)} + 2\|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \frac{9}{20}\|\nabla\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2}. \tag{23}
$$

Adding equations (17) and (23), we obtain

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)}\right) + \frac{11}{20}\|\nabla\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} \le C\left(\|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)} + \|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2}\right). \tag{24}
$$

A use of the Grönwall's lemma yields

$$
\|\bar{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \|\langle v\rangle^k \bar{f}\|^2_{L^2(\Omega_x\times\mathbb{R}^3)} \le 0. \tag{25}
$$

This leads to a contradiction and $\boldsymbol{u}_1 = \boldsymbol{u}_2, f_1 = f_2$ and this concludes the rest of the proof. □

3. Discontinuous Galerkin approximations

This section is on the discontinuous Galerkin method and certain properties of the discrete system for two dimensional problem (1)-(2). However, a remark on the three dimensional problem is given in subsection 4.4. It is clear that a compactly supported (in the velocity variable v) initial datum f_0 yields a compactly supported solution $f(t,x,v)$ in the velocity variable v. Restricting ourselves to compactly supported initial data, it can be assumed without loss of generality that there is some $L>0$ such that for $v\in[-L,L]^2=\Omega_v$ and $t\in(0,T]$; $\mathrm{supp} f(t,x,v)\subset\Omega=\Omega_x\times\Omega_v$.

Let $\mathcal{T}_h^x$ and $\mathcal{T}_h^v$ be the Cartesian partitions of Ω_x and Ω_v, respectively, which are shape regular and quasi-uniform. Let $\mathcal{T}_h$ be denoted by

$$\mathcal{T}_h=\left\{R=T^x\times T^v : T^x\in\mathcal{T}_h^x,\, T^v\in\mathcal{T}_h^v\right\}.$$

Here, the mesh sizes h_x, h_v and h corresponding to these partitions are defined as

$$h_x:=\max_{T^x\in\mathcal{T}_h^x}\mathrm{diam}(T^x);\quad h_v:=\max_{T^v\in\mathcal{T}_h^v}\mathrm{diam}(T^v);\quad h:=\max\left(h_x,h_v\right).$$

We use Γ_x and Γ_v to denote the set of all edges of the partitions $\mathcal{T}_h^x$ and $\mathcal{T}_h^v$, respectively and $\Gamma=\Gamma_x\times\Gamma_v$. Further, let Γ_x^0 (respectively, Γ_v^0) and Γ_x^∂ (respectively, Γ_v^∂) denote the sets of interior and boundary edges of partition $\mathcal{T}_h^x$ (respectively, $\mathcal{T}_h^v$), so that $\Gamma_x=\Gamma_x^0\cup\Gamma_x^\partial$ (respectively, $\Gamma_v=\Gamma_v^0\cup\Gamma_v^\partial$).
We define the discontinuous finite element spaces as

$$\begin{aligned}
\boldsymbol{H}_h &:= \left\{\boldsymbol{\phi}\in L^2(\Omega_x):\boldsymbol{\phi}|_{T^x}\in\left(\mathbb{P}^k(T^x)\right)^2,\ \forall T^x\in\mathcal{T}_h^x\right\},\\
L_h &:= \left\{\phi\in L_0^2(\Omega_x):\phi|_{T^x}\in\mathbb{P}^k(T^x),\ \forall T^x\in\mathcal{T}_h^x\right\},\\
X_h &:= \{\phi\in L^2(\Omega_x):\phi|_{T^x}\in\mathbb{P}^m(T^x),\ \forall T^x\in\mathcal{T}_h^x\},\\
V_h &:= \{\phi\in L^2(\Omega_v):\phi|_{T^v}\in\mathbb{P}^m(T^v),\ \forall T^v\in\mathcal{T}_h^v\},\\
\mathcal{Z}_h &:= \left\{\psi\in L^2(\Omega_x\times\Omega_v):\psi|_R\in\mathbb{P}^m(T^x)\times\mathbb{P}^m(T^v),\ \forall R=T^x\times T^v\in\mathcal{T}_h\right\}
\end{aligned}$$

where $\mathbb{P}^k(T)$ and $\mathbb{P}^m(T)$ denote the spaces of polynomials of degree at most k amd m in each variable.

Below, we define the jump and average value on the mesh. Let the inward and outward unit normal vectors on the element $T^r, r=x$ or v denoted by $\boldsymbol{n}^-$ and $\boldsymbol{n}^+$, respectively. Following [1, 31], we define the average and jump of a scalar function ϕ and a vector-valued function $\boldsymbol{\phi}$ at the edges as follows:

$$\{\phi\}=\frac{1}{2}(\phi^-+\phi^+),\qquad [\![\phi]\!]=\phi^-\boldsymbol{n}^-+\phi^+\boldsymbol{n}^+\quad\text{on }\Gamma_r^0,\quad r=x\text{ or }v$$

$$\{\boldsymbol{\phi}\}=\frac{1}{2}(\boldsymbol{\phi}^-+\boldsymbol{\phi}^+),\qquad [\![\boldsymbol{\phi}]\!]=\boldsymbol{\phi}^-\cdot\boldsymbol{n}^-+\boldsymbol{\phi}^+\cdot\boldsymbol{n}^+\quad\text{on }\Gamma_r^0,\quad r=x\text{ or }v,$$

where,

$$\phi_{T^x}^{\pm}(x,.)=\lim_{\epsilon\to0}\phi_{T^x}(x\pm\epsilon\boldsymbol{n}^-,.)\qquad\forall\,x\in\partial T^x.$$

For a vector valued function $\boldsymbol{\phi}$ the weighted average is denoted as

$$\{\boldsymbol{\phi}\}_\delta:=\delta\boldsymbol{\phi}^++(1-\delta)\,\boldsymbol{\phi}^-\quad\text{for}\quad 0\le\delta\le1.$$

Now for a fixed edge $\boldsymbol{n}^-=-\boldsymbol{n}^+$ and for the boundary edges, the jump and the average are defined, respectively, as $[\![\phi]\!]=\phi\boldsymbol{n}$ and $\{\phi\}=\phi$.
The discrete spaces $W^{k,p}(\mathcal{T}_h)$ are defined as

$$W^{k,p}(\mathcal{T}_h)=\{\phi\in L^p(\Omega):\phi\,|_R\in W^{k,p}(R),\ \forall\,R\in\mathcal{T}_h\}\quad k\ge0,\ 1\le p\le\infty.$$

Further, use $H^k(\mathcal{T}_h)$ to denote $W^{k,2}(\mathcal{T}_h)$ for $k \geq 1$. The discrete norms are defined by

$$\|z\|_{m,\mathcal{T}_h} = \left(\sum_{R\in\mathcal{T}_h} \|z\|^2_{m,R}\right)^{\frac{1}{2}}, \ \forall\ z \in H^m(\mathcal{T}_h),\ m \geq 0,$$

$$\|z\|^p_{L^p(\mathcal{T}_h)} = \sum_{R\in\mathcal{T}_h} \|z\|_{L^p(R)},\ \forall\, z \in L^p(\mathcal{T}_h),$$

for all $1 \leq p < \infty$ and $\|z\|_{L^\infty(\mathcal{T}_h)} = \text{esssup}_{z\in\mathcal{T}_h}|z|$. For $F \in \Gamma_x, z_h \in X_h$

$$\|z_h\|^2_{0,F} = \int_F [\![z_h]\!]_F \cdot [\![z_h]\!]_F \,\mathrm{d}s(x).$$

For all $(\boldsymbol{w}_h, q_h) \in \boldsymbol{H}_h \times L_h$,

$$|\!|\!|\boldsymbol{w}_h|\!|\!|^2 = \|\nabla \boldsymbol{w}_h\|^2_{\boldsymbol{L}^2} + \sum_{F\in\Gamma_x} h_x^{-1}\|[\boldsymbol{w}_h]\|^2_{\boldsymbol{L}^2(F)},$$

$$|\!|\!|(\boldsymbol{w}_h, q_h)|\!|\!|^2 = |\!|\!|\boldsymbol{w}_h|\!|\!|^2 + \|q_h\|^2_{L^2(\mathcal{T}_h^x)} + \sum_{F\in\Gamma_x} h_x\|[q_h]\|^2_{L^2(F)}.$$

For our subsequent use, we recall some standard estimates.

- **Trace inequality:** (see [15, Lemma 1.46, p. 27]) Let $\phi_h \in X_h$, then for all $F \in \Gamma_x$ and $T^x \in \mathcal{T}_h^x$ it holds

$$\|\phi_h\|_{0,F} \leq C h_x^{-\frac{1}{2}} \|\phi_h\|_{0,T^x}. \tag{26}$$

- **Inverse inequality:** (see [15, Lemma 1.44, p. 26]) If $\phi_h \in X_h$. Then

$$\|\nabla_x \phi_h\|_{0,T^x} \leq C h_x^{-1} \|\phi_h\|_{0,T^x} \quad \forall\ T^x \in \mathcal{T}_h^x, \tag{27}$$

  and if $1 \leq p, q \leq \infty$, then (see [15, Lemma 1.50, p. 29])

$$\|\phi_h\|_{L^p(T^x)} \leq C h_x^{\frac{2}{p}-\frac{2}{q}} \|\phi_h\|_{L^q(T^x)} \quad \forall \quad T^x \in \mathcal{T}_h^x. \tag{28}$$

- In [3, Lemma 2.1, p. 744], the author proved a **Poincaré-Friedrichs** type inequality which says that

$$\|w_h\|_{0,\Omega_x} \leq C|\!|\!|w_h|\!|\!|, \quad \forall \quad w_h \in H^1(\Omega_x).$$

For $k \geq 0$, define the standard L^2-projections as fellows: $\mathcal{P}_x : L^2(\Omega_x) \to X_h, \mathcal{P}_v : L^2(\Omega_v) \to V_h$, $\boldsymbol{\mathcal{P}_x} : \boldsymbol{L^2} \to \boldsymbol{H}_h$, and $\mathcal{P}_h : L^2(\Omega) \to \mathcal{Z}_h$. Note that $\mathcal{P}_h = \mathcal{P}_x \otimes \mathcal{P}_v$, (see [8, 2]).
From the definition, $\mathcal{P}_h$ is stable in L^2-norm and also it is also stable in all L^p-norms (see, [11]). Let $1 \leq p \leq \infty$, then for $w \in L^p(\Omega)$

$$\|\mathcal{P}_h w\|_{L^p(\mathcal{T}_h)} \leq C\|w\|_{L^p(\Omega)}. \tag{29}$$

Approximation properties of projection operator: For $m \geq 0$ and $\phi \in H^{m+1}(R)$ for $R \in \mathcal{T}_h$ there exist a positive constant independent of h, such that
(30)

$$\|\phi - \mathcal{P}_h\phi\|_{0,\mathcal{T}_h} + h^{\frac{1}{2}}\|\phi - \mathcal{P}_h\phi\|_{0,\Gamma_x\times\mathcal{T}_h^v} + h^{\frac{1}{2}}\|\phi - \mathcal{P}_h\phi\|_{0,\mathcal{T}_h^x\times\Gamma_v} \leq Ch^{m+1}\|\phi\|_{m+1,\Omega}.$$

3.1. Semi-discrete dG method. The semi-discrete dG method is to seek $(f_h, \boldsymbol{u}_h, p_h)(t) \in \mathcal{Z}_h \times \boldsymbol{H}_h \times L_h$, for $t \in [0,T]$ such that

$$\left(\frac{\partial f_h}{\partial t}, \phi_h\right) + \mathcal{B}_h(\boldsymbol{u}_h; f_h, \phi_h) = 0 \;\; \forall\, \phi_h \in \mathcal{Z}_h \tag{31}$$

coupled with semi-discrete dG scheme for the Stokes system as follows:

$$\left(\frac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{\psi}_h\right) + a_h(\boldsymbol{u}_h, \boldsymbol{\psi}_h) + b_h(\boldsymbol{\psi}_h, p_h) + (\rho_h \boldsymbol{u}_h, \boldsymbol{\psi}_h) = (\rho_h V_h, \boldsymbol{\psi}_h) \;\; \forall\, \boldsymbol{\psi}_h \in \boldsymbol{H}_h, \tag{32}$$

$$-b_h(\boldsymbol{u}_h, w_h) + s_h(p_h, w_h) = 0 \quad \forall\, w_h \in L_h, \tag{33}$$

with $f_h(0) = f_{0h} \in \mathcal{Z}_h$ and $\boldsymbol{u}_h(0) = \boldsymbol{u_{0h}} \in \boldsymbol{H}_h$ to be defined later, where

$$\mathcal{B}_h(\boldsymbol{u}_h; f_h, \phi_h) := \sum_{R \in \mathcal{T}_h} \mathcal{B}_{h,R}(\boldsymbol{u}_h; f_h, \phi_h), \tag{34}$$

with

$$\begin{aligned} \mathcal{B}_{h,R}(\boldsymbol{u}_h; f_h, \phi_h) := &-\int_R f_h v \cdot \nabla_x \phi_h \,\mathrm{d}v\,\mathrm{d}x + \int_{T^v}\int_{\partial T^x} \widehat{v \cdot \boldsymbol{n} f_h}\, \phi_h \,\mathrm{d}s(x)\,\mathrm{d}v \\ &- \int_R f_h(\boldsymbol{u_h} - v) \cdot \nabla_v \phi_h \,\mathrm{d}v\,\mathrm{d}x + \int_{T^x}\int_{\partial T^v} \widehat{(\boldsymbol{u}_h - v) \cdot \boldsymbol{n} f_h}\, \phi_h \,\mathrm{d}s(v)\,\mathrm{d}x, \end{aligned} \tag{35}$$

where the numerical fluxes are defined as

$$\begin{cases} \widehat{v \cdot \boldsymbol{n} f_h} = \{v f_h\}_\beta \cdot \boldsymbol{n} := \left(\{v f_h\} + \dfrac{|v \cdot \boldsymbol{n}|}{2}[\![f_h]\!]\right) \cdot \boldsymbol{n} \quad \text{on } \Gamma_x^0 \times T^v, \\ \widehat{(\boldsymbol{u}_h - v) \cdot \boldsymbol{n} f_h} = \{(\boldsymbol{u}_h - v) f_h\}_\alpha \cdot \boldsymbol{n} := \\ \qquad \left(\{(\boldsymbol{u}_h - v) f_h\} + \dfrac{|(\boldsymbol{u}_h - v) \cdot \boldsymbol{n}|}{2}[\![f_h]\!]\right) \cdot \boldsymbol{n} \quad \text{on } T^x \times \Gamma_v^0, \end{cases} \tag{36}$$

with $\boldsymbol{n} = \boldsymbol{n}^-$, $\beta = \frac{1}{2}(1 \pm \mathrm{sign}(v \cdot \boldsymbol{n}^\pm))$ and $\alpha = \frac{1}{2}(1 \pm \mathrm{sign}((\boldsymbol{u}_h - v) \cdot \boldsymbol{n}^\pm))$. For the more details about weighted average refer [6]. On the boundary edges $e \in \Gamma_r^\partial, r = x, v$, we impose periodic boundary condition for $\widehat{v \cdot \boldsymbol{n} f_h}$ and the compact support condition for $\widehat{(\boldsymbol{u}_h - v) \cdot \boldsymbol{n} f_h}$. Thus, we rewrite (34) as

$$\begin{aligned} \mathcal{B}_h(\boldsymbol{u}_h; f_h, \phi_h) := &\sum_{R \in \mathcal{T}_h} \left(-\int_R f_h v \cdot \nabla_x \phi_h \,\mathrm{d}v\,\mathrm{d}x - \int_R f_h\,(\boldsymbol{u}_h - v) \cdot \nabla_v \phi_h \,\mathrm{d}v\,\mathrm{d}x\right) \\ &+ \sum_{T^v \in \mathcal{T}_h^v} \int_{T^v}\int_{\Gamma_x} \{v f_h\}_\beta \cdot [\![\phi_h]\!] \,\mathrm{d}s(x)\,\mathrm{d}v \\ &+ \sum_{T^x \in \mathcal{T}_h^x} \int_{T^x}\int_{\Gamma_v} \{(\boldsymbol{u}_h - v) f_h\}_\alpha \cdot [\![\phi_h]\!] \,\mathrm{d}s(v)\,\mathrm{d}x. \end{aligned}$$

In (32), the bilinear form $a_h(\boldsymbol{u}_h, \boldsymbol{\psi}_h)$ is given by

$$a_h(\boldsymbol{u}_h, \boldsymbol{\psi}_h) = \sum_{i=1}^{2} a_{h,i}(u_{h,i}, \psi_{h,i}) \tag{37}$$

where, $u_{h,1}, u_{h,2}$ and $\psi_{h,1}, \psi_{h,2}$ are defined as the Cartesian components of $\boldsymbol{u}_h$ and $\boldsymbol{\psi}_h$, respectively, with

$$a_{h,i}(u_{h,i}, \psi_{h,i}) = \sum_{T^x \in \mathcal{T}_h^x} \int_{T^x} \nabla u_{h,i} \cdot \nabla \psi_{h,i} \, \mathrm{d}x + \sum_{F \in \Gamma_x} \int_F \frac{\vartheta}{h_x} [\![u_{h,i}]\!] \cdot [\![\psi_{h,i}]\!] \, \mathrm{d}s(x) - \sum_{F \in \Gamma_x} \int_F \left(\{\nabla u_{h,i}\} \cdot [\![\psi_{h,i}]\!] + [\![u_{h,i}]\!] \cdot \{\nabla \psi_{h,i}\} \right) \mathrm{d}s(x)$$

and the penalty parameter $\vartheta > 0$.
The $b_h(\cdot, \cdot)$ stands for

$$b_h(\boldsymbol{u}_h, w_h) = -\sum_{T^x \in \mathcal{T}_h^x} \int_{T^x} w_h \nabla \cdot \boldsymbol{u}_h \, \mathrm{d}x + \sum_{F \in \Gamma_x} \int_F [\![\boldsymbol{u}_h]\!] \{w_h\} \, \mathrm{d}s(x) \,, \tag{38}$$

the stabilization term

$$s_h(p_h, w_h) = \sum_{F \in \ \Gamma_x^0} h_x \int_F [\![p_h]\!] \cdot [\![w_h]\!] \, \mathrm{d}s(x). \tag{39}$$

Like in continuous case, set discrete local density ρ_h and discrete local macroscopic velocity V_h as

$$\rho_h = \sum_{T^v \in \mathcal{T}_h^v} \int_{T^v} f_h \, \mathrm{d}v \quad \text{and} \quad V_h = \frac{1}{\rho_h} \left(\sum_{T^v \in \mathcal{T}_h^v} \int_{T^v} v f_h \, \mathrm{d}v \right). \tag{40}$$

Note that equations (32) and (33) are equivalent to
(41)

$$\left(\frac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{\psi}_h \right) + \tilde{\mathcal{A}}((\boldsymbol{u}_h, p_h), (\boldsymbol{\psi}_h, w_h)) + (\rho_h \boldsymbol{u}_h, \boldsymbol{\psi}_h) = (\rho_h V_h, \boldsymbol{\psi}_h) \ \forall \, (\boldsymbol{\psi}_h, w_h) \in \boldsymbol{H}_h \times L_h,$$

where

$$\tilde{\mathcal{A}}((\boldsymbol{u}_h, p_h), (\boldsymbol{\psi}_h, w_h)) = a_h(\boldsymbol{u}_h, \boldsymbol{\psi}_h) + b_h(\boldsymbol{\psi}_h, p_h) - b_h(\boldsymbol{u}_h, w_h) + s_h(p_h, w_h). \tag{42}$$

Note that equation (38) is equivalent to

$$b_h(\boldsymbol{u}_h, w_h) = \sum_{T^x \in \mathcal{T}_h^x} \int_{T^x} \boldsymbol{u}_h \cdot \nabla w_h \, \mathrm{d}x - \sum_{F \in \Gamma_x^0} \int_F [\![w_h]\!] \{\boldsymbol{u}_h\} \cdot \boldsymbol{n}_F \, \mathrm{d}s(x).$$

The bilinear form $a_h(\boldsymbol{u}_h, \boldsymbol{\psi}_h)$ is coercive with respect to $|\!|\!|\cdot|\!|\!|$-norm (for proof, see [15, Lemma 4.12, p. 129]), i.e. there exists non-negative β_0 independent of h such that

$$\beta_0 |\!|\!| \boldsymbol{u}_h |\!|\!|^2 \le a_h(\boldsymbol{u}_h, \boldsymbol{u}_h). \tag{43}$$

The bilinear form $\tilde{\mathcal{A}}((\boldsymbol{u}_h, p_h), (\boldsymbol{\psi}_h, w_h))$ satisfy the following properties in the $|\!|\!|(\cdot,\cdot)|\!|\!|$-norm:

- discrete inf-sup stability condition, i.e. for $\alpha > 0$ (for proof, refer [15, Lemma 6.13, p. 253])

$$\alpha |\!|\!| (\boldsymbol{u}_h, p_h) |\!|\!| \le \sup_{(\boldsymbol{\psi}_h, w_h) \in \boldsymbol{H}_h \times L_h \setminus \{(0,0)\}} \frac{\tilde{\mathcal{A}}((\boldsymbol{u}_h, p_h), (\boldsymbol{\psi}_h, w_h))}{|\!|\!| (\boldsymbol{\psi}_h, w_h) |\!|\!|} \tag{44}$$

- boundedness property

$$|\tilde{\mathcal{A}}((\boldsymbol{u}_h, p_h)(\boldsymbol{\psi}_h, w_h))| \le C |\!|\!| (\boldsymbol{u}_h, p_h) |\!|\!| |\!|\!| (\boldsymbol{\psi}_h, w_h) |\!|\!|, \tag{45}$$

  where, the non-negative constant C is independent of h.

Since $\mathcal{Z}_h \times \boldsymbol{H}_h \times L_h$ is finite dimensional, the discrete problem (31) and (41) gives rise to a system of non-linear ODEs. Then, an appeal to the Picard's theorem ensures an existence of a local-in-time unique solution $(f_h, \boldsymbol{u}_h, p_h)$. In order to continue the discrete solution for all $t \in [0, T]$, we need the boundedness of the discrete solution which we shall comment it as a Remark 4.14 later on in this paper.

3.2. Some properties of the discrete solution. Below, we stated some properties satisfied by the discrete system. The proofs of which are similar to the proof of [23, Lemma 3.3 - 3.6]. So we are skipping the details.

Lemma 3.1. *Let* $(f_h, \boldsymbol{u}_h, p_h) \in C^1(0, T; \mathcal{Z}_h \times \boldsymbol{H}_h) \times C^0(0, T; L_h)$ *be the dG-dG approximation obtained as a solution to* (31)-(33) *with initial datum* $f_h(0) = \mathcal{P}_h f_0$ *and* $\boldsymbol{u}_h(0) = \boldsymbol{\mathcal{P}_x u_0}$. *Then, for all* $t \geq 0$,

(i) Mass conservation

$$\int_\Omega f_h(t, x, v)\,\mathrm{d}v\,\mathrm{d}x = \int_\Omega f_0(x, v)\,\mathrm{d}v\,\mathrm{d}x \quad \forall\ m \geq 0.$$

(ii) Momentum conservation

$$\int_\Omega v f_h(t, x, v)\,\mathrm{d}v\,\mathrm{d}x + \int_{\Omega_x} \boldsymbol{u}_h\,\mathrm{d}x = \int_\Omega v f_0(x, v)\,\mathrm{d}v\,\mathrm{d}x + \int_{\Omega_x} \boldsymbol{u_0}\,\mathrm{d}x \quad \textit{for} \quad m \geq 1.$$

(iii) For $m \geq 0$

$$\max_{t\in[0,T]} \|f_h\|_{0,\mathcal{T}_h} \leq e^T \|f_0\|_{0,\mathcal{T}_h}. \tag{46}$$

(iv) Energy identity

$$\begin{aligned} &\frac{1}{2}\left(\int_\Omega |v|^2 f_h\,\mathrm{d}x\,\mathrm{d}v + \int_{\Omega_x} |\boldsymbol{u}_h|^2\,\mathrm{d}x\right) + \int_0^t a_h(\boldsymbol{u}_h(s), \boldsymbol{u}_h(s))\,\mathrm{d}s \\ &+ \int_0^t s_h(p_h(s), p_h(s))\,\mathrm{d}s + \int_0^t \int_\Omega |\boldsymbol{u}_h(s) - v|^2 f_h(s)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s \\ &= \frac{1}{2}\left(\int_\Omega |v|^2 f_0\,\mathrm{d}x\,\mathrm{d}v + \int_{\Omega_x} |\boldsymbol{u}_0|^2\,\mathrm{d}x\right), \quad \textit{when} \quad m \geq 2. \end{aligned}$$

As a result of the Lemma 3.1(i), for a given non-negative initial data, we obtain

$$\int_\Omega f_h(t, x, v)\mathrm{d}x\,\mathrm{d}v \geq 0 \quad \text{and} \quad \int_{\Omega_x} \rho_h(t, x)\mathrm{d}x \geq 0.$$

Since it is difficult to prove non-negative property of f_h, that is, $f_h \geq 0$, therefore, we do not have the discrete version of the energy dissipation as given in Lemma 2.3.

We need the following lemma for our subsequent use.

Lemma 3.2. *Let* ρ *and* ρ_h *be the continuum and the discrete local density associated to the* f *and* f_h, *respectively. Then,*

$$\|\rho - \rho_h\|_{0,\mathcal{T}_h^x} \leq 2L\|f - f_h\|_{0,\mathcal{T}_h} \quad \textit{and} \quad \|\rho - \rho_h\|_{\infty,\mathcal{T}_h^x} \leq 4L^2\|f - f_h\|_{\infty,\mathcal{T}_h}.$$

Moreover,

$$\|\rho V - \rho_h V_h\|_{0,\mathcal{T}_h^x} \leq 4L^2\|f - f_h\|_{0,\mathcal{T}_h}.$$

For details of the proof of the above lemma, refer [25, Lemma 6]. From equation (46), it follows

$$\|\rho_h\|_{0,\mathcal{T}_h^x} \leq C(T)L\|f_0\|_{0,\mathcal{T}_h}. \tag{47}$$

4. Rate of convergence

In this section, some a priori error estimates for the discrete solution are derived.

4.1. Error estimates for nonstationary Stokes system. This subsection focusses on error estimates for the time dependent Stokes system.

Stokes projection: Define the Stokes projection $(\boldsymbol{\Pi_u u}(t), \Pi_p p(t))$ of $(\boldsymbol{u}(t), p(t))$ for all $t \in [0,T]$ satisfying

$$a_h(\boldsymbol{u} - \boldsymbol{\Pi_u u}, \boldsymbol{\psi}_h) + b_h(\boldsymbol{\psi}_h, p - \Pi_p p) = 0 \quad \forall\ \boldsymbol{\psi}_h \in \boldsymbol{H}_h, \tag{48}$$

$$-b_h(\boldsymbol{u} - \boldsymbol{\Pi_u u}, w_h) + s_h(p - \Pi_p p, w_h) = 0 \quad \forall\ w_h \in L_h. \tag{49}$$

Systems (48) and (49) can be written as

$$\tilde{\mathcal{A}}\left((\boldsymbol{\Pi_u u} - \boldsymbol{u}, \Pi_p p - p), (\boldsymbol{\psi}_h, w_h)\right) = 0, \quad \forall\ (\boldsymbol{\psi}_h, w_h) \in \boldsymbol{H}_h \times L_h \tag{50}$$

where, $\tilde{\mathcal{A}}\left((\cdot,\cdot),(\cdot,\cdot)\right)$ is defined in (42).

From equation (44), $\tilde{\mathcal{A}}\left((\boldsymbol{\Pi_u u}, \Pi_p p), (\boldsymbol{\psi}_h, w_h)\right)$ satisfies the discrete inf-sup condition in the $|||(\cdot,\cdot)|||$-norm and by equation (45), it is bounded from above in the $|||(\cdot,\cdot)|||$-norm. Therefore, a use of the Lax-Milgram lemma shows existence of a unique pair of discrete solutions $(\boldsymbol{\Pi_u u}, \Pi_p p)(t) \in \boldsymbol{H}_h \times L_h$, for $t \in (0,T]$.

The following lemma shows the approximation properties of the Stokes projection, for a proof, refer [15, Corollary 6.26, p. 260].

Lemma 4.1. *Let* $(\boldsymbol{\Pi_u u}, \Pi_p p) \in \boldsymbol{H}_h \times L_h$ *solve* (50). *Assume that* $(\boldsymbol{u}, p) \in L^\infty(0,T;\boldsymbol{H}^{k+1}) \times L^\infty(0,T;H^k)$. *Then,*

$$\|\boldsymbol{u} - \boldsymbol{\Pi_u u}\|_{\boldsymbol{L}^2} + h_x|||\boldsymbol{u} - \boldsymbol{\Pi_u u}||| + h_x\|p - \Pi_p p\|_{0,\mathcal{T}_h^x} \le C h_x^{k+1}, \tag{51}$$

where, C is a positive constant which is independent of h_x.

Next, lemma shows a relation between the L^∞ and L^2 bounds while approximating function in the broken polynomial space. For a proof, see [25, Lemma 3, p. 5].

Lemma 4.2. *Let $\boldsymbol{u}_h \in \boldsymbol{H}_h$, an approximation of $\boldsymbol{u}$ be defined by* (32)-(33). *Assume that $\boldsymbol{u} \in \boldsymbol{W}^{1,\infty} \cap \boldsymbol{H}^{k+1}$. Then,*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^\infty} \lesssim h_x\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\infty}} + h_x^k\|\boldsymbol{u}\|_{\boldsymbol{k+1,2}} + h_x^{-1}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^2}.$$

Error equation for Stokes part: Since the scheme (32)-(33) is consistent, (32)-(33) also hold for the solution $(f, \boldsymbol{u}, p)$ to the continuum model. Hence, by taking the difference with $\boldsymbol{e_u} := \boldsymbol{u} - \boldsymbol{u}_h,\ e_p := p - p_h$, we obtain the following equations:

$$\begin{aligned}\left(\frac{\partial \boldsymbol{e_u}}{\partial t}, \boldsymbol{\psi}_h\right) + a_h\left(\boldsymbol{e_u}, \boldsymbol{\psi}_h\right) + b_h\left(\boldsymbol{\psi}_h, e_p\right) + \left(\rho\boldsymbol{u} - \rho_h\boldsymbol{u}_h, \boldsymbol{\psi}_h\right)& \\ = \left(\rho V - \rho_h V_h, \boldsymbol{\psi}_h\right),\ \forall\ \boldsymbol{\psi}_h \in \boldsymbol{H}_h,&\end{aligned} \tag{52}$$

$$-b_h(\boldsymbol{e_u}, w_h) + s_h(e_p, w_h) = 0 \quad \forall\ w_h \in L_h. \tag{53}$$

Using the Stokes projection operator, we rewrite

$$\begin{aligned}\boldsymbol{e_u} &:= \boldsymbol{u} - \boldsymbol{u}_h := (\boldsymbol{\Pi_u u} - \boldsymbol{u}_h) - (\boldsymbol{\Pi_u u} - \boldsymbol{u}) =: \boldsymbol{\theta_u} - \boldsymbol{\eta_u},\\ e_p &:= p - p_h := (\Pi_p p - p_h) - (\Pi_p p - p) =: \theta_p - \eta_p.\end{aligned} \tag{54}$$

Using (54) and (48)-(49), the error equation (52)-(53) becomes

$$
\begin{aligned}
&\left(\frac{\partial \boldsymbol{\theta}_{\boldsymbol{u}}}{\partial t}, \boldsymbol{\psi}_h\right) + a_h\left(\boldsymbol{\theta}_{\boldsymbol{u}}, \boldsymbol{\psi}_h\right) + b_h\left(\boldsymbol{\psi}_h, \theta_p\right) + \left(\rho \boldsymbol{\theta}_{\boldsymbol{u}}, \boldsymbol{\psi}_h\right) = \left(\rho V - \rho_h V_h, \boldsymbol{\psi}_h\right) \\
&\qquad + \left(\frac{\partial \boldsymbol{\eta}_{\boldsymbol{u}}}{\partial t}, \boldsymbol{\psi}_h\right) + \left(\rho \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\psi}_h\right) + \left(\left(\rho - \rho_h\right) \boldsymbol{\theta}_{\boldsymbol{u}}, \boldsymbol{\psi}_h\right) \\
&\qquad - \left(\left(\rho - \rho_h\right) \boldsymbol{\eta}_{\boldsymbol{u}}, \boldsymbol{\psi}_h\right) - \left(\left(\rho - \rho_h\right) \boldsymbol{u}, \boldsymbol{\psi}_h\right), \quad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{H}_h
\end{aligned} \tag{55}
$$

$$
-b_h(\boldsymbol{\theta}_{\boldsymbol{u}}, w_h) + s_h(\theta_p, w_h) = 0 \quad \forall\, w_h \in L_h. \tag{56}
$$

Lemma 4.3. *Let $(\boldsymbol{u}, p) \in L^\infty(0, T; \boldsymbol{H}^{k+1}) \times L^\infty(0, T; H^k)$ be the unique pair of solutions of the time dependent Stokes equation (2). Further, let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times L_h$ be a pair of solutions of the discrete problem (41). Then, for all $t \in (0, T]$, there holds*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2}^2 + \beta |||\boldsymbol{\theta}_{\boldsymbol{u}}(t)|||^2 + s_h(\theta_p, \theta_p) \\
&\qquad \leq C\left(h^{k+1} + \|f - f_h\|_{0,\mathcal{T}_h} + h^{-1}\|f - f_h\|_{0,\mathcal{T}_h}\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2}\right)\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2},
\end{aligned} \tag{57}
$$

where C is a positive constant depends on L and T, but independent on h.

Proof. Choose $\boldsymbol{\psi}_h = \boldsymbol{\theta}_{\boldsymbol{u}}$ and $w_h = \theta_p$ in (55) and (56), respectively. Then add the resulting expressions. Using the Hölder inequality and the coercivity (43) of $a_h(\boldsymbol{\theta}_{\boldsymbol{u}}, \boldsymbol{\theta}_{\boldsymbol{u}})$, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2}^2 + \beta |||\boldsymbol{\theta}_{\boldsymbol{u}}(t)|||^2 + s_h(\theta_p, \theta_p) + \|\rho^{\frac{1}{2}}\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2}^2 &\leq \big(\|\rho V - \rho_h V_h\|_{0,\mathcal{T}_h^x} + \|\partial_t \boldsymbol{\eta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2} \\
&\quad + \|\rho\|_{L^\infty(\Omega_x)}\|\boldsymbol{\eta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2} + \|\rho - \rho_h\|_{0,\mathcal{T}_h^x}\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty}\big)\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2} \\
&\quad + \left(\|\rho - \rho_h\|_{0,\mathcal{T}_h^x}\left(\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2} + \|\boldsymbol{\eta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2}\right)\right)\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^\infty}.
\end{aligned}
$$

Since the last term on the left hand side is non-negative, it can be dropped. An application of the projection estimate along with the Lemma 3.2 and the estimate

$$
\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{L^\infty} \leq C h^{-\frac{d}{2}}\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{L^2} \tag{58}
$$

with $d = 2$ yields (57). □

4.2. Error estimates for the kinetic equation. Since the scheme (31) is consistent, the solution $(\boldsymbol{u}, f)$ to the continuum problem satisfies

$$
\left(\frac{\partial f}{\partial t}, \phi_h\right) + \mathcal{B}(\boldsymbol{u}; f, \phi_h) = 0 \quad \forall\, \phi_h \in \mathcal{Z}_h, \tag{59}
$$

where

$$
\mathcal{B}(\boldsymbol{u}; f, \phi_h) := \sum_{R \in \mathcal{T}_h} \mathcal{B}_R(\boldsymbol{u}; f, \phi_h),
$$

with

$$
\begin{aligned}
\mathcal{B}_R(\boldsymbol{u}; f, \phi_h) &= -\int_R f v \cdot \nabla_x \phi_h \,\mathrm{d}x\,\mathrm{d}v - \int_R f(\boldsymbol{u} - v) \cdot \nabla_v \phi_h \,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_{T^v}\int_{\partial T^x} v \cdot \boldsymbol{n} f \phi_h \,\mathrm{d}s(x)\,\mathrm{d}v + \int_{\partial T^v}\int_{T^x} (\boldsymbol{u} - v) \cdot \boldsymbol{n} f \phi_h \,\mathrm{d}x\,\mathrm{d}s(v).
\end{aligned}
$$

Subtracting equation (31) from (59), we obtain:

$$
\left(\frac{\partial}{\partial t}\left(f - f_h\right), \phi_h\right) + \mathcal{B}(\boldsymbol{u}; f, \phi_h) - \mathcal{B}_h(\boldsymbol{u}_h; f_h, \phi_h) = 0 \quad \forall\, \phi_h \in \mathcal{Z}_h.
$$

By denoting $e_f := f - f_h$, we rewrite the above equation as

$$\left(\frac{\partial e_f}{\partial t}, \phi_h\right) + a_h^0(e_f, \phi_h) + \mathcal{N}(\boldsymbol{u}; f, \phi_h) - \mathcal{N}^h(\boldsymbol{u}_h; f_h, \phi_h) = 0 \quad \forall\, \phi_h \in \mathcal{Z}_h, \tag{60}$$

where

$$a_h^0(e_f, \phi_h) := -\sum_{R\in\mathcal{T}_h} \int_R e_f v \cdot \nabla_x \phi_h \,\mathrm{d}x\,\mathrm{d}v + \sum_{T^v\in\mathcal{T}_h^v} \int_{T^v} \int_{\Gamma_x} \{v e_f\}_\beta \cdot [\![\phi_h]\!] \,\mathrm{d}s(x)\,\mathrm{d}v,$$

$$\mathcal{N}(\boldsymbol{u}; f, \phi_h) := -\sum_{R\in\mathcal{T}_h} \int_R f(\boldsymbol{u}-v)\cdot \nabla_v \phi_h \,\mathrm{d}x\,\mathrm{d}v + \sum_{T^x\in\mathcal{T}_h^x} \int_{\Gamma_v} \int_{T^x} (\boldsymbol{u}-v) f \cdot [\![\phi_h]\!] \,\mathrm{d}x\,\mathrm{d}s(v)\,,$$

and

$$\begin{aligned}\mathcal{N}^h(\boldsymbol{u}_h; f_h, \phi_h) := &-\sum_{R\in\mathcal{T}_h} \int_R f_h(\boldsymbol{u}_h - v) \cdot \nabla_v \phi_h \,\mathrm{d}x\,\mathrm{d}v \\ &+ \sum_{T^x\in\mathcal{T}_h^x} \int_{\Gamma_v} \int_{T^x} \{(\boldsymbol{u}_h - v) f_h\}_\alpha \cdot [\![\phi_h]\!] \,\mathrm{d}x\,\mathrm{d}s(v)\,.\end{aligned}$$

With the help of the L^2-projection $\mathcal{P}_h : L^2(\Omega) \to \mathcal{Z}_h$, split the error $e_f := f - f_h$ as

$$e_f = (\mathcal{P}_h f - f_h) - (\mathcal{P}_h f - f) := \theta_f - \eta_f. \tag{61}$$

Below, we state the error representation of the nonlinear term.

Lemma 4.4. *Let $f \in C^0(\Omega), \boldsymbol{u} \in C^0(\Omega_x)$ and $f_h \in \mathcal{Z}_h$. Then, there holds*

$$\begin{aligned}\mathcal{N}(\boldsymbol{u}; f, \theta_f) - \mathcal{N}^h(\boldsymbol{u}_h; f_h, \theta_f) = &\sum_{T^x\in\mathcal{T}_h^x} \int_{\Gamma_v} \int_{T^x} \frac{|(\boldsymbol{u}_h - v)\cdot \boldsymbol{n}|}{2} |[\![\theta_f]\!]|^2 \,\mathrm{d}x\,\mathrm{d}s(v) \\ &+ \sum_{R\in\mathcal{T}_h} \int_R \left((\boldsymbol{u} - \boldsymbol{u}_h)\cdot \nabla_v f\, \theta_f - \theta_f^2\right) \,\mathrm{d}x\,\mathrm{d}v + \mathcal{K}^2(\boldsymbol{u}_h - v, f, \theta_f),\end{aligned}$$

where

$$\begin{aligned}\mathcal{K}^2(\boldsymbol{u}_h - v, f, \theta_f) = &\sum_{R\in\mathcal{T}_h} \int_R \eta_f(\boldsymbol{u}_h - v)\cdot \nabla_v \theta_f \,\mathrm{d}x\,\mathrm{d}v \\ &- \sum_{T^x\in\mathcal{T}_h^x} \int_{\Gamma_v} \int_{T^x} \{((\boldsymbol{u}_h - v)\eta_f)\}_\alpha \cdot [\![\theta_f]\!] \,\mathrm{d}x\,\mathrm{d}s(v)\,.\end{aligned} \tag{62}$$

Proof. The proof is similar to the [25, Lemma 11, p.13]. □

For the estimate of $e_f := \theta_f - \eta_f$, it is enough to estimate θ_f as the estimate of η_f is already known from equation (30). After taking $\phi_h = \theta_f$ in (60), a use of (61) with Lemma 4.4, yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_f\|_{0,\mathcal{T}_h}^2 + \sum_{T^v\in\mathcal{T}_h^v} \int_{T^v} \int_{\Gamma_x} \frac{|v\cdot \boldsymbol{n}|}{2} |[\![\theta_f]\!]|^2 \,\mathrm{d}s(x)\,\mathrm{d}v \\ &+ \sum_{T^x\in\mathcal{T}_h^x} \int_{T^x} \int_{\Gamma_v} \frac{|(\boldsymbol{u}_h - v)\cdot \boldsymbol{n}|}{2} |[\![\theta_f]\!]|^2 \,\mathrm{d}s(v)\,\mathrm{d}x = (\partial_t \eta_f, \theta_f) - \mathcal{K}^1(v, \eta_f, \theta_f) \\ &- \sum_{R\in\mathcal{T}_h} \int_R \left((\boldsymbol{u} - \boldsymbol{u}_h)\cdot \nabla_v f\, \theta_f - \theta_f^2\right) \,\mathrm{d}v\,\mathrm{d}x - \mathcal{K}^2(\boldsymbol{u}_h - v, f, \theta_f),\end{aligned} \tag{63}$$

where,
(64)
$$\mathcal{K}^1(v,\eta_f,\theta_f) = \sum_{R\in\mathcal{T}_h}\int_R \eta_f v\cdot\nabla_x\theta_f \,\mathrm{d}x\,\mathrm{d}v - \sum_{T^v\in\mathcal{T}_h^v}\int_{T^v}\int_{\Gamma_x}\{v\eta_f\}_\beta\cdot[\![\theta_f]\!]\,\mathrm{d}s(x)\,\mathrm{d}v\,.$$

Inspired by the proof of [14, Lemma 2.1-2.2, p. 4], we provide below a short proof of the estimates of $\mathcal{K}^1$ and $\mathcal{K}^2$.

Lemma 4.5. *Let f and $f_h(t)\in\mathcal{Z}_h$ be the solution of* (1) *and* (31), *respectively. Assume that*
$$f\in C^0(0,T;W^{1,\infty}(\Omega)\cap H^{m+1}(\Omega)).$$
Then, for $m\geq 1$ and $t\in[0,T]$, $\mathcal{K}^1(v,\eta_f,\theta_f)$ as defined in (64), *there holds*
$$|\mathcal{K}^1(v,\eta_f,\theta_f)|\leq Ch^{2m+1}\|f\|^2_{m+1,\Omega}+\|\theta_f\|^2_{0,\mathcal{T}_h}+\frac14\||v\cdot\boldsymbol{n}|^{\frac12}[\![\theta_f]\!]\|^2_{0,\Gamma_x\times\mathcal{T}_h^v}.$$

Proof. We write the expression (64) of $\mathcal{K}^1$ as
$$\mathcal{K}^1(v,\eta_f,\theta_f) = \sum_{R\in\mathcal{T}_h}\int_R \eta_f v\cdot\nabla_x\theta_f \,\mathrm{d}x\,\mathrm{d}v - \sum_{T^v\in\mathcal{T}_h^v}\int_{T^v}\int_{\Gamma_x}\{v\eta_f\}_\beta\cdot[\![\theta_f]\!]\,\mathrm{d}s(x)\,\mathrm{d}v = \mathcal{K}^1_1+\mathcal{K}^1_2.$$

For the first term $\mathcal{K}^1_1$, a use of the definition of the orthogonal L^2-projection as $\mathcal{P}^0_v(v)\cdot\nabla_x\theta_f\in\mathcal{Z}_h$ with the Hölder inequality, the inverse inequality (27), the projection estimate (30) and the Young's inequality shows
$$\begin{aligned}|\mathcal{K}^1_1| &\leq \sum_{R\in\mathcal{T}_h}\int_R\eta_f\left(v-\mathcal{P}^0_v(v)\right)\cdot\nabla_x\theta_f\,\mathrm{d}x\,\mathrm{d}v+\sum_{R\in\mathcal{T}_h}\int_R\eta_f\mathcal{P}^0_v(v)\cdot\nabla_x\theta_f\,\mathrm{d}x\,\mathrm{d}v\\ &\leq \|v-\mathcal{P}^0_v(v)\|_{L^\infty(\mathcal{T}_h)}\|\eta_f\|_{0,\mathcal{T}_h}\|\nabla_x\theta_f\|_{0,\mathcal{T}_h}\leq Ch_vh_x^{-1}\|\eta_f\|_{0,\mathcal{T}_h}\|\theta_f\|_{0,\mathcal{T}_h}\\ &\leq Ch^{2(m+1)}\|f\|^2_{m+1,\Omega}+\|\theta_f\|^2_{0,\mathcal{T}_h}.\end{aligned}$$

Again by using the Hölder inequality, the trace inequality (26), the projection estimae (30) with the Young's inequality, we arrive at the following estimate for the second term $\mathcal{K}^1_2$
$$\begin{aligned}|\mathcal{K}^1_2| &= \Big|\sum_{T^v\in\mathcal{T}_h^v}\int_{T^v}\int_{\Gamma_x}\left(\{v\eta_f\}+\frac{|v\cdot\boldsymbol{n}|}{2}[\![\eta_f]\!]\right)\cdot[\![\theta_f]\!]\,\mathrm{d}s(x)\,\mathrm{d}v\Big|\\ &\leq\left(\sum_{T^v\in\mathcal{T}_h^v}\int_{T^v}\int_{\Gamma_x}8|v\cdot\boldsymbol{n}||\{\eta_f\}|^2\,\mathrm{d}s(x)\,\mathrm{d}v\right)^{\frac12}\left(\sum_{T^v\in\mathcal{T}_h^v}\int_{T^v}\int_{\Gamma_x}\frac{|v\cdot\boldsymbol{n}|}{8}|[\![\theta_f]\!]|^2\,\mathrm{d}s(x)\,\mathrm{d}v\right)^{\frac12}\\ &+\left(\sum_{T^v\in\mathcal{T}_h^v}\int_{T^v}\int_{\Gamma_x}8|v\cdot\boldsymbol{n}||[\![\eta_f]\!]|^2\,\mathrm{d}s(x)\,\mathrm{d}v\right)^{\frac12}\left(\sum_{T^v\in\mathcal{T}_h^v}\int_{T^v}\int_{\Gamma_x}\frac{|v\cdot\boldsymbol{n}|}{8}|[\![\theta_f]\!]|^2\,\mathrm{d}s(x)\,\mathrm{d}v\right)^{\frac12}\\ &\leq C\,\|v\|_{L^\infty(\mathcal{T}_h^v)}\|\eta_f\|^2_{0,\Gamma_x\times\mathcal{T}_h^v}+\frac14\||v\cdot\boldsymbol{n}|^{\frac12}[\![\theta_f]\!]\|^2_{0,\Gamma_x\times\mathcal{T}_h^v}\\ &\leq Ch^{2m+1}L\|f\|^2_{m+1,\Omega}+\frac14\||v\cdot\boldsymbol{n}|^{\frac12}[\![\theta_f]\!]\|^2_{0,\Gamma_x\times\mathcal{T}_h^v}.\end{aligned}$$

This completes the proof. □

Lemma 4.6. *Let $(f,\boldsymbol{u})$ and $(f_h,\boldsymbol{u}_h)\in\mathcal{Z}_h\times\boldsymbol{H}_h$ be the unique solution of* (1) *and* (31), *respectively. Assume that*
$$(\boldsymbol{u},f)\in C^0(0,T;\boldsymbol{W}^{1,\infty}\cap\boldsymbol{H}^{k+1})\times C^0(0,T;W^{1,\infty}(\Omega)\cap H^{m+1}(\Omega)).$$

Then, for $m \geq 1$, $\mathcal{K}^2((\boldsymbol{u}_h - v), f, \theta_f)$ *as defined in* (62) *satisfies the estimate*

$$
\begin{aligned}
|\mathcal{K}^2(\boldsymbol{u}_h - v, f, \theta_f)| \leq C \left(h^m \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^\infty} + h^{m+1}\right) \|f\|_{m+1,\Omega} \|\theta_f\|_{0,\mathcal{T}_h} \\
+ Ch^{2m+1} \left(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty} + 2L\right) \|f\|^2_{m+1,\Omega} \\
+ \frac{1}{4} \| | (\boldsymbol{u}_h - v) \cdot \boldsymbol{n}|^{\frac{1}{2}} [\![\theta_f]\!] \|^2_{0,\mathcal{T}_h^x \times \Gamma_v}.
\end{aligned}
\tag{65}
$$

Proof. From the expression (62) of $\mathcal{K}^2$, it follows that

$$
\begin{aligned}
\mathcal{K}^2(\boldsymbol{u}_h - v, f, \theta_f) = \sum_{R \in \mathcal{T}_h} \int_R \eta_f (\boldsymbol{u}_h - v) \cdot \nabla_v \theta_f \, \mathrm{d}x \, \mathrm{d}v \\
- \sum_{T^x \in \mathcal{T}_h^x} \int_{\Gamma_v} \int_{T^x} \{((\boldsymbol{u}_h - v)\eta_f)\}_\alpha \cdot [\![\theta_f]\!] \, \mathrm{d}x \, \mathrm{d}s(v) = \mathcal{K}^2_1 + \mathcal{K}^2_2.
\end{aligned}
$$

The first term $\mathcal{K}^2_1$ of the above expression can be bounded by

$$
\begin{aligned}
|\mathcal{K}^2_1| &= \Big| \sum_{R \in \mathcal{T}_h} \int_R \eta_f (\boldsymbol{u}_h - v) \cdot \nabla_v \theta_f \, \mathrm{d}v \, \mathrm{d}x \Big| \\
&= \Big| \sum_{R \in \mathcal{T}_h} \int_R \eta_f \left((\boldsymbol{u}_h - \boldsymbol{u}) + (\boldsymbol{u} - \boldsymbol{\mathcal{P}}^0_x \boldsymbol{u}) - (v - \mathcal{P}^0_v v) + (\boldsymbol{\mathcal{P}}^0_x \boldsymbol{u} - \mathcal{P}^0_v v)\right) \cdot \nabla_v \theta_f \, \mathrm{d}v \, \mathrm{d}x \Big| \\
&\leq \left(\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{u} - \boldsymbol{\mathcal{P}}^0_x \boldsymbol{u}\|_{\boldsymbol{L}^\infty} + \|v - \mathcal{P}^0_v v\|_{L^\infty(\mathcal{T}_h^v)}\right) \|\eta_f\|_{0,\mathcal{T}_h} \|\nabla_v \theta_f\|_{0,\mathcal{T}_h} \\
&\leq C h_v^{-1} \left(\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{u} - \boldsymbol{\mathcal{P}}^0_x \boldsymbol{u}\|_{\boldsymbol{L}^\infty} + \|v - \mathcal{P}^0_v v\|_{L^\infty(\mathcal{T}_h^v)}\right) \|\eta_f\|_{0,\mathcal{T}_h} \|\theta_f\|_{0,\mathcal{T}_h} \\
&\leq C \left(h^m \|\boldsymbol{u}_h - \boldsymbol{u}\|_{\boldsymbol{L}^\infty} + h^{m+1}\right) \|f\|_{m+1,\Omega} \|\theta_f\|_{0,\mathcal{T}_h}.
\end{aligned}
$$

Here, in the second step, the last term vanishes by the definition of the orthogonal L^2-projection as $(\boldsymbol{\mathcal{P}}^0_x \boldsymbol{u} - \mathcal{P}^0_v v) \cdot \nabla_v \theta_f \in \mathcal{Z}_h$. In the third step, we use the Hölder inequality with the inverse inequality (27) and in the last step, we apply the projection estimate (30).

The following bound for the second term $\mathcal{K}^2_2$ is obtained by using the Hölder inequality with the trace inequality (26) and the projection estimate (30)

$$
\begin{aligned}
|\mathcal{K}^2_2| &= \Big| \sum_{T^x \in \mathcal{T}_h^x} \int_{T^x} \int_{\Gamma_v} \left(\{(\boldsymbol{u}_h - v)\eta_f\} + \frac{|(\boldsymbol{u}_h - v) \cdot \boldsymbol{n}_v|}{2} [\![\eta_f]\!]\right) \cdot [\![\theta_f]\!] \, \mathrm{d}s(v) \, \mathrm{d}x \Big| \\
&\leq \left(\sum_{T^x \in \mathcal{T}_h^x} \int_{T^x} \int_{\Gamma_v} 8 |(\boldsymbol{u}_h - v) \cdot \boldsymbol{n}_v| |\{\eta_f\}|^2 \, \mathrm{d}s(v) \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{T^x \in \mathcal{T}_h^x} \int_{T^x} \int_{\Gamma_v} \frac{|(\boldsymbol{u}_h - v) \cdot \boldsymbol{n}_v|}{8} |[\![\theta_f]\!]|^2 \, \mathrm{d}s(v) \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{T^x \in \mathcal{T}_h^x} \int_{T^x} \int_{\Gamma_v} 8 |(\boldsymbol{u}_h - v) \cdot \boldsymbol{n}_v| |[\![\eta_f]\!]|^2 \, \mathrm{d}s(v) \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{T^x \in \mathcal{T}_h^x} \int_{T^x} \int_{\Gamma_v} \frac{|(\boldsymbol{u}_h - v) \cdot \boldsymbol{n}_v|}{8} |[\![\theta_f]\!]|^2 \, \mathrm{d}s(v) \, \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq C \left(\|\boldsymbol{u}_h\|_{\boldsymbol{L}^\infty} + \|v\|_{L^\infty(\mathcal{T}_h^v)}\right) \|\eta_f\|^2_{0,\mathcal{T}_h^x \times \Gamma_v} + \frac{1}{4} \| |(\boldsymbol{u}_h - v) \cdot \boldsymbol{n}_v|^{\frac{1}{2}} [\![\theta_f]\!] \|^2_{0,\mathcal{T}_h^x \times \Gamma_v} \\
&\leq Ch^{2m+1} \left(\|\boldsymbol{u}_h - \boldsymbol{u}\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty} + 2L\right) \|f\|^2_{m+1,\Omega} + \frac{1}{4} \| |(\boldsymbol{u}_h - v) \cdot \boldsymbol{n}_v|^{\frac{1}{2}} [\![\theta_f]\!] \|^2_{0,\mathcal{T}_h^x \times \Gamma_v}.
\end{aligned}
$$

This completes the rest of the proof. □

4.3. Error estimates. This subsection deals with main results on a priori error analysis.

Lemma 4.7. *Let regularity results*

$$\boldsymbol{u} \in C^0(0,T;\boldsymbol{H}^{k+1} \cap \boldsymbol{W}^{1,\infty}),$$

for the fluid velocity and

$$f \in C^1(0,T;H^{m+1}(\Omega)) \cap C^0(0,T;W^{1,\infty}(\Omega))$$

for the distribution function hold. Further, let $(\boldsymbol{u}_h, f_h) \in \boldsymbol{H}_h \times \mathcal{Z}_h$ *be the dG-dG solutions of* (31) *and* (41). *Then for* $k \geq 1$ *and* $m \geq 1$, *there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_f\|^2_{0,\mathcal{T}_h} \leq C\left(h^{2m+1} + \|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\boldsymbol{L}^2}\right) + \|\theta_f\|^2_{0,\mathcal{T}_h}, \tag{66}$$

where C *depends on the polynomial degree* m, *the final time* T, *the shape regularity of the partition and also regularity result of* $(\boldsymbol{u}, f)$, *but is independent of the mesh parameters.*

Proof. From equation (63), there follows

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_f\|^2_{0,\mathcal{T}_h} + \frac{1}{2}\||v\cdot\boldsymbol{n}|^{\frac{1}{2}}[\![\theta_f]\!]\|^2_{\Gamma_x\times\mathcal{T}_h^v} + \frac{1}{2}\||(\boldsymbol{u}_h - v)\cdot\boldsymbol{n}|^{\frac{1}{2}}[\![\theta_f]\!]\|^2_{\mathcal{T}_h^x\times\Gamma_v} \\ = I_1 - I_2 - \mathcal{K}^1 - \mathcal{K}^2 + I_3 \leq |I_1| + |I_2| + |\mathcal{K}^1| + |\mathcal{K}^2| + |I_3|.\end{aligned} \tag{67}$$

Here, we use the standard triangle inequality and

$$I_1 = (\partial_t\eta_f, \theta_f),\ I_2 = \sum_{R\in\mathcal{T}_h}\int_R (\boldsymbol{u}-\boldsymbol{u}_h)\cdot\nabla_v f\,\theta_f\,\mathrm{d}v\,\mathrm{d}x \text{ and } I_3 = \sum_{R\in\mathcal{T}_h}\int_R \theta_f^2\,\mathrm{d}v\,\mathrm{d}x.$$

In order to estimate I_1, a use of the Cauchy-Schwarz inequality with (30) yields

$$\begin{aligned}|I_1| \leq \|\partial_t\eta_f\|_{0,\mathcal{T}_h}\|\theta_f\|_{0,\mathcal{T}_h} &\leq Ch^{m+1}\|f_t\|_{m+1,\mathcal{T}_h}\|\theta_f\|_{0,\mathcal{T}_h} \\ &\leq Ch^{2(m+1)}\|f_t\|^2_{m+1,\mathcal{T}_h} + C\|\theta_f\|^2_{0,\mathcal{T}_h}.\end{aligned} \tag{68}$$

For I_2, there holds

$$|I_2| \leq \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2}\|\nabla_v f\|_{L^\infty(\Omega)}\|\theta_f\|_{0,\mathcal{T}_h} \leq C\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h}. \tag{69}$$

Now, $\mathcal{K}^1$ is estimated in Lemma 4.5,

$$|\mathcal{K}^1(v,\eta_f,\theta_f)| \leq Ch^{2m+1}\|f\|^2_{m+1,\Omega} + \|\theta_f\|^2_{0,\mathcal{T}_h} + \frac{1}{4}\||v\cdot\boldsymbol{n}|^{\frac{1}{2}}[\![\theta_f]\!]\|^2_{0,\Gamma_x\times\mathcal{T}_h^v}. \tag{70}$$

To estimate $\mathcal{K}^2$, we use the bound (65) from Lemma 4.6 with Lemma 4.2 to obtain

$$\begin{aligned}&|\mathcal{K}^2| \leq C\left(h^m\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^\infty} + h^{m+1}\right)\|f\|_{m+1,\Omega}\|\theta_f\|_{0,\mathcal{T}_h} \\ &+ Ch^{2m+1}\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty} + 2L\right)\|f\|^2_{m+1,\Omega} + \frac{1}{4}\||(\boldsymbol{u}_h - v)\cdot\boldsymbol{n}|^{\frac{1}{2}}[\![\theta_f]\!]\|^2_{0,\mathcal{T}_h^x\times\Gamma_v} \\ &\leq C\left(h^m\left(h\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\infty}} + h^k\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}} + h^{-1}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2}\right) + h^{m+1}\right)\|f\|_{m+1,\Omega}\|\theta_f\|_{0,\mathcal{T}_h} \\ &+ Ch^{2m+1}\left(h\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\infty}} + h^k\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}} + h^{-1}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2} + \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty} + 2L\right)\|f\|^2_{m+1,\Omega} \\ &\quad + \frac{1}{4}\||(\boldsymbol{u}_h - v)\cdot\boldsymbol{n}|^{\frac{1}{2}}[\![\theta_f]\!]\|^2_{0,\mathcal{T}_h^x\times\Gamma_v} \\ &\leq C_{\boldsymbol{u},f}h^{2m+1} + \|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h} + \frac{1}{4}\||(\boldsymbol{u}_h - v)\cdot\boldsymbol{n}|^{\frac{1}{2}}[\![\theta_f]\!]\|^2_{0,\mathcal{T}_h^x\times\Gamma_v}.\end{aligned} \tag{71}$$

Now, substituting the estimates (68)-(71) into (67) with a use of kick-back argument and noting that the last two terms on the left hand side of the equation (67) are non-negative, we obtain (66) and this completes the proof. □

Below, we discuss the final theorem of this section.

Theorem 4.8. *Let $k \geq 1, m \geq 1$ and let*

$$f \in C^1(0,T; H^{m+1}(\Omega)) \cap C^0(0,T; W^{1,\infty}(\Omega))$$

$$(\boldsymbol{u}, p) \in \left(C^1(0,T; \boldsymbol{H}^{k+1}) \cap C^0(0,T; \boldsymbol{W}^{1,\infty})\right) \times C^0(0,T; H^k(\Omega_x))$$

denote the solution of the Vlasov-nonstationary Stokes equation (1)-(2)*. Further, let*

$$(f_h, \boldsymbol{u}_h, p_h) \in C^1(0,T; \mathcal{Z}_h) \times C^1(0,T; \boldsymbol{H}_h) \times C^0(0,T; L_h)$$

be the dG-dG approximations of (31) *and*(41)*. Then, there holds*

$$\|f(t) - f_h(t)\|_{L^\infty(0,T;L^2(\Omega))} + \|(\boldsymbol{u} - \boldsymbol{u}_h)(t)\|_{L^\infty(0,T;\boldsymbol{L}^2)} \leq C h^{\min\left(k+1, m+\frac{1}{2}\right)} \quad \forall\, t \in [0,T],$$

where C depends on the polynomial degrees k and m, the final time T, the shape regularity of the partition and also regularity result of $(\boldsymbol{u}, f)$, but is independent of h.

Proof. A use of equations (54) and (61) with triangle inequality implies

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^2} + \|f - f_h\|_{0,\mathcal{T}_h} \leq \|\boldsymbol{\eta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2} + \|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2} + \|\eta_f\|_{0,\mathcal{T}_h} + \|\theta_f\|_{0,\mathcal{T}_h}.$$

Now, from (51) and (30), the estimates of $\|\boldsymbol{\eta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2}$ and $\|\eta_f\|_{0,\mathcal{T}_h}$ are known, therefore, it is sufficient to estimate $\|\boldsymbol{\theta}_{\boldsymbol{u}}\|_{\boldsymbol{L}^2} + \|\theta_f\|_{0,\mathcal{T}_h}$.
Adding (57) and (66) and using equations (54) and (61), triangle inequality with the projection estimates (51) and (30), we obtain
(72)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{\theta}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h}\right) &\lesssim h^{\min(2(k+1),2m+1)} + \|\boldsymbol{\theta}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h} + h^{-2}\|\theta_f\|^2_{0,\mathcal{T}_h}\|\boldsymbol{\theta}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} \\ &\lesssim h^{\min(2(k+1),2m+1)} + \|\boldsymbol{\theta}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h} + h^{-2}\|\theta_f\|^4_{0,\mathcal{T}_h} + h^{-2}\|\boldsymbol{\theta}_{\boldsymbol{u}}\|^4_{\boldsymbol{L}^2}.\end{aligned}$$

Setting

$$\||(\boldsymbol{\theta}_{\boldsymbol{u}}, \theta_f)\||^2 := \|\boldsymbol{\theta}_{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h} \tag{73}$$

and a function Ψ as

$$\Psi(t) = h^{\min(2(k+1),2m+1)} + \int_0^t \left(\||(\boldsymbol{\theta}_{\boldsymbol{u}}, \theta_f)(s)\||^2 + h^{-2}\||(\boldsymbol{\theta}_{\boldsymbol{u}}, \theta_f)(s)\||^4\right) \mathrm{d}s. \tag{74}$$

An integration of equation (72) with respect to time from 0 to t with (73)-(74) yields

$$\||(\boldsymbol{\theta}_{\boldsymbol{u}}, \theta_f)(t)\||^2 \leq C\Psi(t).$$

Without loss of generality, we assume that $\||(\boldsymbol{\theta}_{\boldsymbol{u}}, \theta_f)(t)\|| > 0$, otherwise, we add an arbitrary small quantity say δ and proceed in a similar way as describe below. Then, we pass the limit as $\delta \to 0$ to obtain the desired result. Note that $0 < \Psi(0) \leq \Psi(t)$ and $\Psi(t)$ is differentiable in time. On differentiating $\Psi(t)$ with respect to t, it now follows that

$$\begin{aligned}\partial_t \Psi(t) &= \||(\boldsymbol{\theta}_{\boldsymbol{u}}, \theta_f)\||^2 + h^{-2}\||(\boldsymbol{\theta}_{\boldsymbol{u}}, \theta_f)\||^4 \\ &\leq C\left(\Psi(t) + h^{-2}(\Psi(t))^2\right).\end{aligned}$$

Note that $\partial_t \Psi(t) > 0$ and therefore, the function $\Psi(t)$ is strictly monotonically increasing, which is also positive. On integrating in time, we now arrive at

$$\int_0^t \frac{\partial_s \Psi(s)}{\Psi(s)\left(1 + h^{-2}\Psi(s)\right)}\,\mathrm{d}s \leq \int_0^t C\,\mathrm{d}s \leq CT.$$

On evaluating integral exactly on the left hand side with $\Psi(0) = h^{\min(2(k+1),2m+1)}$ and then taking exponential both side, we find that

$$\Psi(t)\left(1 - h^{\min(2k,2m-1)}\left(e^{CT} - 1\right)\right) \leq e^{CT} h^{\min(2(k+1),2m+1)}.$$

Now, for small $h > 0$, $\left(1 - h^{\min(2k,2m-1)}\left(e^{CT} - 1\right)\right) > 0$ and this implies

$$\Psi(t) \leq C h^{\min(2(k+1),2m+1)}.$$

This completes the rest of the proof. □

The idea to define function Ψ and applying non-linear Grönwall's lemma in the proof of above Theorem is inspired by Lemma 4.2 in [27].

Lemma 4.9. *Let the hypothesis of Theorem 4.8 holds. Then, there exists a positive constant C independent of h, such that for all $t \in (0, T]$*

$$\|\partial_t \boldsymbol{\theta_u}\|^2_{L^2(0,T;\boldsymbol{L}^2)} + \|\!|\boldsymbol{\theta_u}(t)|\!\|^2 + s_h(\theta_p, \theta_p) \leq C h^{\min(2(k+1),2m+1)}.$$

Proof. A differentiation of equation (56) with respect to time with a choice $w_h = \theta_p$ shows

$$-b_h(\partial_t \boldsymbol{\theta_u}, \theta_p) + s_h(\partial_t \theta_p, \theta_p) = 0. \tag{75}$$

By choosing $\boldsymbol{\psi}_h = \partial_t \boldsymbol{\theta_u}$ in equation (55), we obtain

$$\begin{aligned}\|\partial_t \boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2} + \frac{1}{2}\frac{\partial}{\partial t} a_h(\boldsymbol{\theta_u}, \boldsymbol{\theta_u}) + b_h(\partial_t \boldsymbol{\theta_u}, \theta_p) &= (\rho V - \rho_h V_h, \partial_t \boldsymbol{\theta_u}) \\ &+ ((\rho - \rho_h)\, \boldsymbol{\theta_u}, \partial_t \boldsymbol{\theta_u}) - ((\rho - \rho_h)\, \boldsymbol{\eta_u}, \partial_t \boldsymbol{\theta_u}) - ((\rho - \rho_h)\, \boldsymbol{u}, \partial_t \boldsymbol{\theta_u}) \\ &- (\rho \boldsymbol{\theta_u}, \partial_t \boldsymbol{\theta_u}) + (\rho \boldsymbol{\eta_u}, \partial_t \boldsymbol{\theta_u}) + (\partial_t \boldsymbol{\eta_u}, \partial_t \boldsymbol{\theta_u}).\end{aligned}$$

A use of equation (75) with the Hölder inequality, the Young's inequality, projection estimates, Lemma 3.2 and estimate $\|\cdot\|_{L^\infty} \leq h^{-1}\|\cdot\|_{L^2}$ shows

$$\begin{aligned}\|\partial_t \boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2} + \frac{1}{2}\frac{\partial}{\partial t}\left(a_h(\boldsymbol{\theta_u}, \boldsymbol{\theta_u}) + s_h(\theta_p, \theta_p)\right) &\leq C\Big(h^{2(k+1)} + \|f - f_h\|^2_{0,\mathcal{T}_h} \\ &+ h^{-2}\|f - f_h\|^2_{0,\mathcal{T}_h}\|\boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2} + \|\boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2}\Big) + \frac{1}{2}\|\partial_t \boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2}.\end{aligned}$$

A kick-back argument with integration in time with respect to time using coercivity property (43) and estimates from Theorem 4.8 completes the rest of the proof. □

Remark 4.10. *From the Lemma 4.9, we derive a super-convergence result*

$$\|\!|\left(\boldsymbol{\Pi_u u} - \boldsymbol{u}_h\right)(t)|\!\| \leq C h^{\min\left(k+1, m+\frac{1}{2}\right)} \quad \forall \quad t \in (0, T].$$

Since for $1 \leq p < \infty$

$$\|\left(\boldsymbol{\Pi_u u} - \boldsymbol{u}_h\right)(t)\|_{\boldsymbol{L}^p} \leq C\|\!|\left(\boldsymbol{\Pi_u u} - \boldsymbol{u}_h\right)(t)|\!\| \quad \forall \quad t \in (0, T]$$

then

$$\|\left(\boldsymbol{\Pi_u u} - \boldsymbol{u}_h\right)(t)\|_{\boldsymbol{L}^p} \leq C h^{\min\left(k+1, m+\frac{1}{2}\right)} \quad \forall \quad t \in (0, T].$$

Again as $\Omega_x \subset \mathbb{R}^2$, the discrete Sobolev imbedding (refer [30]) implies

$$\begin{aligned}\|\left(\boldsymbol{\Pi_u u} - \boldsymbol{u}_h\right)(t)\|_{\boldsymbol{L}^\infty} &\leq C\left(log\left(\frac{1}{h}\right)\right)\|\!|\left(\boldsymbol{\Pi_u u} - \boldsymbol{u}_h\right)(t)|\!\| \\ &\leq C\left(log\left(\frac{1}{h}\right)\right) h^{\min\left(k+1, m+\frac{1}{2}\right)} \quad \forall \quad t \in (0, T].\end{aligned}$$

Remark 4.11. *If*

$$\|\boldsymbol{u} - \boldsymbol{\Pi_u u}\|_{L^\infty(0,T;\boldsymbol{L}^p)} \leq \begin{cases} Ch^{k+1} & \text{for} \quad 1 \leq p < \infty, \\ C\left(log\left(\frac{1}{h}\right)\right) h^{k+1} & \text{for} \quad p = \infty, \end{cases}$$

then

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty(0,T;\boldsymbol{L}^p)} \leq \begin{cases} Ch^{\min\left(k+1, m+\frac{1}{2}\right)} & \text{for} \quad 1 \leq p < \infty, \\ C\left(log\left(\frac{1}{h}\right)\right) h^{\min\left(k+1, m+\frac{1}{2}\right)} & \text{for} \quad p = \infty. \end{cases}$$

Lemma 4.12. *Under the hypothesis of Theorem 4.8, there exists a positive constant C independent of h such that for all $t \in (0, T]$*

$$\int_0^T |||(\boldsymbol{\theta_u}(s), \theta_p(s))|||^2 \,\mathrm{d}s \leq Ch^{\min(2(k+1), 2m+1)}.$$

Proof. Note that equation (55)-(56) is also equivalent to

$$\left(\frac{\partial \boldsymbol{\theta_u}}{\partial t}, \boldsymbol{\psi}_h\right) + \tilde{\mathcal{A}}\left((\boldsymbol{\theta_u}, \theta_p), (\boldsymbol{\psi}_h, w_h)\right) + (\rho\boldsymbol{\theta_u}, \boldsymbol{\psi}_h) = (\rho V - \rho_h V_h, \boldsymbol{\psi}_h) + (\partial_t \boldsymbol{\eta_u}, \boldsymbol{\psi}_h)$$
$$+ (\rho\boldsymbol{\eta_u}, \boldsymbol{\psi}_h) + ((\rho - \rho_h)\,\boldsymbol{\theta_u}, \boldsymbol{\psi}_h) - ((\rho - \rho_h)\,\boldsymbol{\eta_u}, \boldsymbol{\psi}_h) + ((\rho - \rho_h)\boldsymbol{u}, \boldsymbol{\psi}_h)\,,$$

where, $\tilde{\mathcal{A}}\left((\cdot,\cdot),(\cdot,\cdot)\right)$ is defined in (42), and hence,

$$\tilde{\mathcal{A}}\left((\boldsymbol{\theta_u}, \theta_p), (\boldsymbol{\psi}_h, w_h)\right) = (\rho V - \rho_h V_h, \boldsymbol{\psi}_h) - ((\rho - \rho_h)\,\boldsymbol{\eta_u}, \boldsymbol{\psi}_h) + ((\rho - \rho_h)\,\boldsymbol{\theta_u}, \boldsymbol{\psi}_h)$$
$$- ((\rho - \rho_h)\,\boldsymbol{u}, \boldsymbol{\psi}_h) + (\rho\boldsymbol{\eta_u}, \boldsymbol{\psi}_h) - (\rho\boldsymbol{\theta_u}, \boldsymbol{\psi}_h) + (\partial_t\boldsymbol{\eta_u}, \boldsymbol{\psi}_h) - (\partial_t\boldsymbol{\theta_u}, \boldsymbol{\psi}_h)\,.$$

An application of the discrete inf-sup stability condition (44) with the Hölder inequality, the projection estimate, the Lemma 3.2, the Lemma 4.9 and the Theorem 4.8 concludes the rest of the proof. □

Theorem 4.13. *Let $k \geq 1, m \geq 1$ and let $(f, \boldsymbol{u}, p)$ be the solutions of the Vlasov-nonstationary Stokes equation* (1)-(2) *satisfying*

$$f \in C^1(0,T; H^{m+1}(\Omega)) \cap C^0(0,T; W^{1,\infty}(\Omega))$$

$$(\boldsymbol{u}, p) \in \left(C^1(0,T; \boldsymbol{H}^{k+1}) \cap C^0(0,T; \boldsymbol{W}^{1,\infty})\right) \times C^0(0,T; H^k(\Omega_x)).$$

Further, let

$$(f_h, \boldsymbol{u}_h, p_h) \in C^1(0,T; \mathcal{Z}_h) \times C^1(0,T; \boldsymbol{H}_h) \times C^0(0,T; L_h)$$

be the dG-dG approximations of (31) *and* (41). *Then, there holds*

$$\|p(t) - p_h(t)\|_{L^2(0,T;L^2(\Omega_x))} \leq Ch^{\min\left(k, m+\frac{1}{2}\right)} \quad \forall\, t \in [0,T]$$

where C depends on the polynomial degrees k and m, the final time T, the shape regularity of the partition and also regularity result of $(\boldsymbol{u}, f)$, but is independent of h.

Proof. A use of equation (54) with triangle inequality gives

$$\|p - p_h\|_{0,\mathcal{T}_h^x} \leq \|\eta_p\|_{0,\mathcal{T}_h^x} + \|\theta_p\|_{0,\mathcal{T}_h^x}.$$

Rest of the proof follows from Lemma 51 and Lemma 4.12. □

Remark 4.14. *From equation* (46) *and Theorem 4.8 along with equation* (5)*, it follows that*

$$f_h \in L^\infty(0,T;L^2(\mathcal{T}_h)) \quad \text{and} \quad \boldsymbol{u}_h \in L^\infty(0,T;\boldsymbol{L}^2).$$

Thanks to these bounds, the earlier local-in-time existence result for the discrete problem can be improved to a global-in-time existence result by extending the interval of existence.

4.4. Comment on 3D. In 3D the inequality (28) becomes

$$\|w_h\|_{L^p(T^x)} \le C h_x^{\frac{3}{p}-\frac{3}{q}} \|w_h\|_{L^q(T^x)}. \tag{76}$$

An application of (76) shows

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^\infty} \lesssim h_x\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\infty}} + h_x^{k-\frac{1}{2}}\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}} + h_x^{-\frac{3}{2}}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2}. \tag{77}$$

On using (76) the result of Lemma 4.3 is now modified as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2} &+ \beta|||\boldsymbol{\theta_u}(t)|||^2 + s_h(\theta_p,\theta_p)\\ &\le C\left(h^{k+1} + \|f-f_h\|_{0,\mathcal{T}_h} + h^{-\frac{3}{2}}\|f-f_h\|_{0,\mathcal{T}_h}\|\boldsymbol{\theta_u}\|_{\boldsymbol{L}^2}\right)\|\boldsymbol{\theta_u}\|_{\boldsymbol{L}^2}.\end{aligned} \tag{78}$$

The change in the estimate of $\mathcal{K}^2$ term (71) is as follows:

$$\begin{aligned}|\mathcal{K}^2| &\le C\left(h^m\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^\infty} + h^{m+1}\right)\|f\|_{m+1,\Omega}\|\theta_f\|_{0,\mathcal{T}_h}\\
&+ Ch^{2m+1}\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^\infty} + \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty} + 2L\right)\|f\|^2_{m+1,\Omega} + \frac{1}{4}\||\,(\boldsymbol{u}_h-v)\cdot\boldsymbol{n}|^{\frac{1}{2}}[\![\theta_f]\!]\|^2_{0,\mathcal{T}_h^x\times\Gamma_v}\\
&\le C\left(h^m\left(h\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\infty}} + h^{k-\frac{1}{2}}\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}} + h^{-\frac{3}{2}}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2}\right) + h^{m+1}\right)\|f\|_{m+1,\Omega}\|\theta_f\|_{0,\mathcal{T}_h}\\
&+ Ch^{2m+1}\left(h\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\infty}} + h^{k-\frac{1}{2}}\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}} + h^{-\frac{3}{2}}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2} + \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty} + 2L\right)\|f\|^2_{m+1,\Omega}\\
&\quad + \frac{1}{4}\||\,(\boldsymbol{u}_h-v)\cdot\boldsymbol{n}|^{\frac{1}{2}}[\![\theta_f]\!]\|^2_{0,\mathcal{T}_h^x\times\Gamma_v}\\
&\le C_{\boldsymbol{u},f}h^{2m+1} + h^{2m-3}\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h} + \frac{1}{4}\||\,(\boldsymbol{u}_h-v)\cdot\boldsymbol{n}|^{\frac{1}{2}}[\![\theta_f]\!]\|^2_{0,\mathcal{T}_h^x\times\Gamma_v}.\end{aligned} \tag{79}$$

This modifies the (66) as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_f\|^2_{0,\mathcal{T}_h} \le C_{\boldsymbol{u},f}h^{\min(2(k+1),2m+1)} + Ch^{2(m+k)-1} + Ch^{2m-3}\|\boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h}. \tag{80}$$

By adding (78) and (80), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h}\right) &\lesssim h^{\min(2(k+1),2m+1)} + h^{2(m+k)-1} + h^{2m-3}\|\boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2}\\ &+ \|\theta_f\|^2_{0,\mathcal{T}_h} + h^{-3}\|\boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2}\|\theta_f\|^2_{0,\mathcal{T}_h}.\end{aligned} \tag{81}$$

Thus, for $m \ge 2$, it becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h}\right) &\lesssim h^{\min(2(k+1),2m+1)} + \left(\|\boldsymbol{\theta_u}\|^2_{\boldsymbol{L}^2} + \|\theta_f\|^2_{0,\mathcal{T}_h}\right)\\ &+ h^{-3}\left(\|\boldsymbol{\theta_u}\|^4_{\boldsymbol{L}^2} + \|\theta_f\|^4_{0,\mathcal{T}_h}\right)\\ &\lesssim h^{\min(2(k+1),2m+1)} + |||(\boldsymbol{\theta_u},\theta_f)|||^2 + h^{-3}|||(\boldsymbol{\theta_u},\theta_f)|||^4,\end{aligned} \tag{82}$$

with same definition of $|||(\cdot,\cdot)|||$ as in Theorem 4.7. Set

$$\Psi(t) = h^{\min(2(k+1),2m+1)} + \int_0^t\left(|||(\boldsymbol{\theta_u},\theta_f)(s)|||^2 + h^{-3}|||(\boldsymbol{\theta_u},\theta_f)(s)|||^4\right)\mathrm{d}s.$$

On integration and proceed in a similar manner as in the proof of Theorem 4.7 to obtain

$$\Psi(t)\left(1 - h^{\min(2k-1,2m-2)}(e^{CT}-1)\right) \le e^{CT} h^{\min(2(k+1),2m+1)}.$$

For $m \ge 2$ and with smallness assumption on h

$$(1 - h^{\min(2k-1,2m-2)}(e^{CT}-1)) > 0.$$

Hence, the result of Theorem 4.8 follows for 3D for $m \ge 2$.

5. Numerical Experiments

This section, we report on some numerical simulations based on a splitting algorithm. We have approximated solutions to the following Vlasov-nonstationary Stokes model with source terms:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \nabla_v\cdot\Big((\boldsymbol{u}-v)\,f\Big) = \mathcal{F}(t,x,v) & \text{in } (0,T)\times\Omega_x\times\Omega_v, \\ f(0,x,v) = f_0(x,v) & \text{in } \Omega_x\times\Omega_v. \end{cases} \tag{83}$$

$$\begin{cases} \partial_t \boldsymbol{u} - \Delta_x \boldsymbol{u} + \rho\boldsymbol{u} + \nabla_x p = \rho V + \mathcal{G}(t,x) & \text{in } \Omega_x, \\ \nabla_x\cdot\boldsymbol{u} = 0 & \text{in } \Omega_x. \end{cases} \tag{84}$$

For the computational results, we employed splitting algorithm for the linear kinetic equation (83), using the Lie-Trotter splitting method. To achieve this, firstly, split the equation (83) as:

$$(a)\quad \partial_t f + \nabla_v\cdot((\boldsymbol{u}-v)\,f) = \mathcal{F}(t,x,v), \qquad (b)\quad \partial_t f + v\cdot\nabla_x f = 0. \tag{85}$$

To compute the solution f, we first solve part (a) of (85) over the full time step using f_0 as the initial data to obtain an intermediate solution $\tilde{f}$. Next, part (b) of (85) is solved over the full time step with $\tilde{f}$ as the initial data.

For a temporal discretization, let $\{t_n\}_{n=0}^N$ be a uniform partition of the time interval $[0,T]$, where $t_n = n\Delta t$ with time step $\Delta t > 0$. Let $F^n \in \mathcal{Z}_h, \boldsymbol{U}^n \in \boldsymbol{H}_h, P^n \in L_h, \rho_h^n$ and $\rho_h^n V_h^n$ be the approximations of $f^n = f(t_n), \boldsymbol{u}^n = \boldsymbol{u}(t_n), p^n = p(t_n), \rho^n = \rho(t_n)$ and $\rho^n V^n = \rho(t_n)V(t_n)$, respectively. Our numerical algorithm is to find $\left(\boldsymbol{U}^{n+1}, P^{n+1}, F^{n+1}\right) \in \boldsymbol{H}_h \times L_h \times \mathcal{Z}_h$, for $n = 0,1,\cdots,N-1$ such that

$$\begin{cases} \left(\dfrac{\boldsymbol{U}^{n+1}-\boldsymbol{U}^n}{\Delta t}, \boldsymbol{\psi}_h\right) + a_h(\boldsymbol{U}^{n+1},\boldsymbol{\psi}_h) + b_h(\boldsymbol{\psi}_h, P^{n+1}) + \left(\rho_h^n \boldsymbol{U}^{n+1}, \boldsymbol{\psi}_h\right) \\ \qquad\qquad = \left(\rho_h^n V_h^n + \mathcal{G}^{n+1}, \boldsymbol{\psi}_h\right)\ \forall\ \boldsymbol{\psi}_h \in \boldsymbol{H}_h, \\ -b_h(\boldsymbol{U}^{n+1}, w_h) + s_h(P^{n+1}, w_h) = 0 \quad \forall \quad w_h \in L_h, \\ \left(\dfrac{\tilde{F}-F^n}{\Delta t}, \phi_h\right) + \mathcal{B}_h^x\left(\boldsymbol{U}^{n+1}; F^n, \phi_h\right) = (\mathcal{F}^n, \phi_h) \quad \forall \quad \phi_h \in \mathcal{Z}_h, \\ \left(\dfrac{F^{n+1}-\tilde{F}}{\Delta t}, \psi_h\right) + \mathcal{B}_h^v(\tilde{F}, \psi_h) = 0 \quad \forall \quad \psi_h \in \mathcal{Z}_h, \end{cases} \tag{86}$$

where, $a_h(\boldsymbol{U}^n, \boldsymbol{\psi}_h), b_h(\boldsymbol{\psi}_h, P^n)$ and $s_h(P^n, w_h)$ are defined by (37)-(39) at $t = t_n$. In (86) we have used the following notations:

$$\mathcal{B}_h^v(F^n, \phi_h) := \sum_{R\in\mathcal{T}_h} \mathcal{B}_{h,R}^v(F^n, \phi_h)$$

with

$$\mathcal{B}^v_{h,R}(F^n,\phi_h) := -\int_{T^v}\int_{T^x} F^n\, v.\nabla_x\phi_h \,\mathrm{d}x\,\mathrm{d}v + \int_{T^v}\int_{\partial T^x} \widehat{v\cdot \boldsymbol{n}F^n}\phi_h \,\mathrm{d}s(x)\,\mathrm{d}v$$

and

$$\mathcal{B}^x_h(\boldsymbol{U}^{n+1};F^n,\phi_h) := \sum_{R\in\mathcal{T}_h}\mathcal{B}^x_{h,R}(\boldsymbol{U}^{n+1};F^n,\phi_h),$$

with

$$\begin{aligned}\mathcal{B}^x_{h,R}(\boldsymbol{U}^{n+1};F^n,\phi_h) := &-\int_{T^x}\int_{T^v} F^n\left(\boldsymbol{U}^{n+1}-v\right).\nabla_v\phi_h\,\mathrm{d}v\,\mathrm{d}x\\ &+\int_{T^x}\int_{\partial T^v}\widehat{\left(\boldsymbol{U}^{n+1}-v\right)\cdot\boldsymbol{n}F^n}\phi_h\,\mathrm{d}s(v)\,\mathrm{d}x\end{aligned}$$

where the numerical fluxes are given by (36) at $t=t_n$.

For our computation, let $\boldsymbol{x}_l$ and $\boldsymbol{v}_m$ are the nodes in $\mathcal{T}^x_h$ and $\mathcal{T}^v_h$, respectively, where $l=1,\cdots,N_x$ and $m=1,\cdots,N_v$. Any function in the $\mathcal{Z}_h$ space can be represented as

$$g=\sum_{l,m} g(\boldsymbol{x}_l,\boldsymbol{v}_m)L^l_x(\boldsymbol{x})L^m_v(\boldsymbol{v})$$

on R, where $L^l_x(\boldsymbol{x})$ and $L^m_v(\boldsymbol{v})$ are the l-th and m-th Lagrangian interpolating polynomials in T^x and T^v, respectively.

In this setting, the equations for f in (85) can be solved in the reduced dimensions. For example, first we solve equation (85)(a),

$$\partial_t f(\boldsymbol{x}_l) + \nabla_v\cdot\left(\left(\boldsymbol{u}(\boldsymbol{x}_l)-v\right)f(\boldsymbol{x}_l)\right) = \mathcal{F}(t,\boldsymbol{x}_l,v)$$

in the v-direction for a fixed nodal point in the x-direction, say $\boldsymbol{x}_l$, and obtain updated point values of $f(\boldsymbol{x}_l,\boldsymbol{v}_m)$ for all $\boldsymbol{v}_m\in\mathcal{T}^v_h$.

Similarly, we solve the equation (85)(b)

$$\partial_t f(\boldsymbol{v}_m) + \boldsymbol{v}_m\cdot\nabla_x f(\boldsymbol{v}_m) = 0$$

by a dG method in the x-direction by fixing a nodal point in the v-direction, say $\boldsymbol{v}_m$, and obtain updated point values of $f(\boldsymbol{x}_l,\boldsymbol{v}_m)$ for all $\boldsymbol{x}_l\in\mathcal{T}^x_h$.

For the plots, we take the degree of polynomials m to approximate the distribution function in x and v-variables and k for both the fluid velocity and fluid pressure. The mesh sizes for $\mathcal{T}^x_h$ and $\mathcal{T}^v_h$ are represented by h_x and h_v, respectively. For the numerical experiments, we take $h_x=h_v=h$. We calculate the errors $f-f_h, \boldsymbol{u}-\boldsymbol{u_h}$ and $p-p_h$ in $L^2(\Omega), \boldsymbol{L^2}$ and $L^2(\Omega_x)$-norms, respectively, at final time T and denoted by errL2f, errL2u and errL2p, respectively.

Example 5.1. *Our first example has the following as its exact solution:*

$$\begin{aligned}f(t,x_1,x_2,v_1,v_2) &= \cos(t)\sin(2\pi x_1)\sin(2\pi x_2)e^{(-v_1^2-v_2^2)}(1+v_1)(1+v_2)(1-v_1^2)(1-v_2^2),\\ u_1(t,x_1,x_2) &= \cos(t)\left(-\cos(2\pi x_1)\sin(2\pi x_2)+\sin(2\pi x_2)\right),\\ u_2(t,x_1,x_2) &= \cos(t)\left(\sin(2\pi x_1)\cos(2\pi x_2)-\sin(2\pi x_1)\right),\\ p(t,x_1,x_2) &= 2\pi\cos(t)\left(\cos(2\pi x_2)-\cos(2\pi x_1)\right).\end{aligned}$$

Note that corresponding is the initial data

$$f(0,x_1,x_2,v_1,v_2) = \sin(2\pi x_1)\sin(2\pi x_2)e^{(-v_1^2-v_2^2)}(1+v_1)(1+v_2)(1-v_1^2)(1-v_2^2),$$

$$u_1(0,x_1,x_2) = -\cos(2\pi x_1)\sin(2\pi x_2)+\sin(2\pi x_2),$$

$$u_2(0,x_1,x_2) = \sin(2\pi x_1)\cos(2\pi x_2)-\sin(2\pi x_1).$$

The simulations are performed for the domains $\Omega_x = [0,1]^2$ and $\Omega_v = [-1,1]^2$. The penalty parameter is set to be 10 and for $k = m = 1$ the final time is 0.1, while for $k = m = 2$ the final time is 0.01.

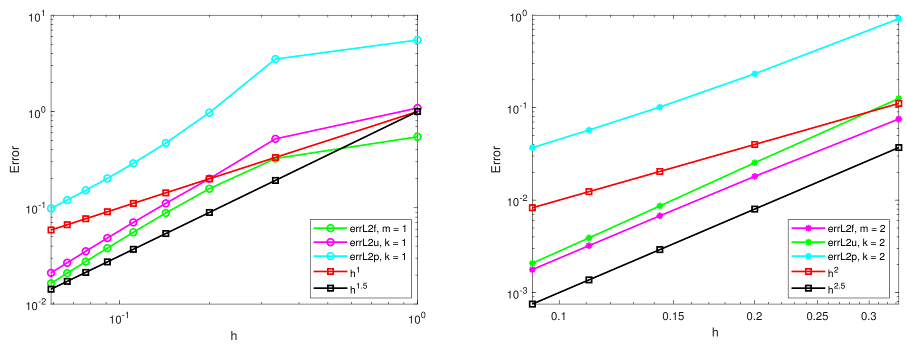

FIGURE 1. Convergence rates for the distribution function f, for the fluid velocity $\boldsymbol{u}$ and for the fluid pressure p in the Example 5.1.

Example 5.2. *The second example takes the following as the exact solutions:*

$$f(t, x_1, x_2, v_1, v_2) = \sin(\pi(x_1 - t)) \sin(\pi(x_2 - t)) e^{(-v_1^2 - v_2^2)} (1 + v_1)(1 + v_2)(1 - v_1^2)(1 - v_2^2),$$
$$u_1(t, x_1, x_2) = -\cos(2\pi x_1 - t) \sin(2\pi x_2 - t),$$
$$u_2(t, x_1, x_2) = \sin(2\pi x_1 - t) \cos(2\pi x_2 - t),$$
$$p(t, x_1, x_2) = 2\pi \left(\cos(2\pi x_2 - t) - \cos(2\pi x_1 - t)\right),$$

with the corresponding initial data:

$$f(0, x_1, x_2, v_1, v_2) = \sin(\pi x_1) \sin(\pi x_2) e^{(-v_1^2 - v_2^2)} (1 + v_1)(1 + v_2)(1 - v_1^2)(1 - v_2^2),$$
$$u_1(0, x_1, x_2) = -\cos(2\pi x_1) \sin(2\pi x_2), \qquad u_2(0, x_1, x_2) = \sin(2\pi x_1) \cos(2\pi x_2).$$

The simulations are performed for the domains $\Omega_x = [0,1]^2$ and $\Omega_v = [-1,1]^2$. The penalty parameter is set to be 10 and for $k = m = 1$ the final time is 0.1, while for $k = m = 2$ the final time is 0.01.

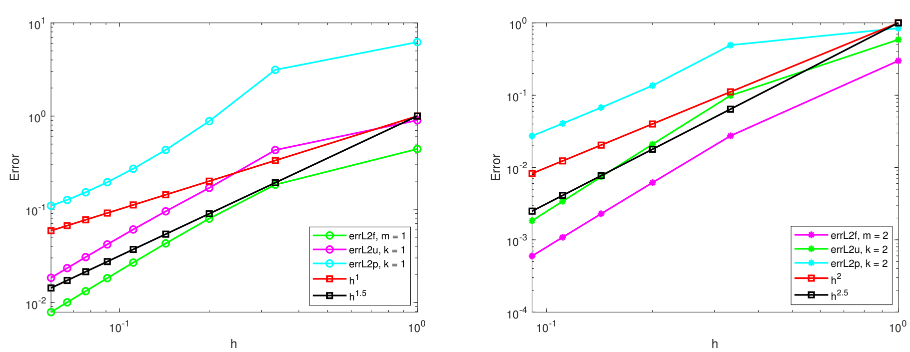

FIGURE 2. Convergence rates for the distribution function f, for the fluid velocity $\boldsymbol{u}$ and for the fluid pressure p in the Example 5.2.

In the next example, we examine the preservation of the mass and momentum conservation property of our scheme.

Example 5.3. *Let $\mathcal{F} = 0$ and $\mathcal{G} = 0$ in* (83)-(84). *Choose the initial data as:*

$$f(0, x_1, x_2, v_1, v_2) = \sin(2\pi x_1)\sin(2\pi x_2)e^{(-v_1^2 - v_2^2)}(1 - v_1^2)(1 - v_2^2),$$

$$u_1(0, x_1, x_2) = -\cos(2\pi x_1)\sin(2\pi x_2) + \sin(2\pi x_2),$$

$$u_2(0, x_1, x_2) = \sin(2\pi x_1)\cos(2\pi x_2) - \sin(2\pi x_1).$$

Let $\Omega_x = [0, 1]^2, \Omega_v = [-1, 1]^2, k_x = 1, k_v = 1, h = h_x = h_v = 1/6, \Delta t = 0.001$ *and penalty parameter* $\vartheta = 8$.

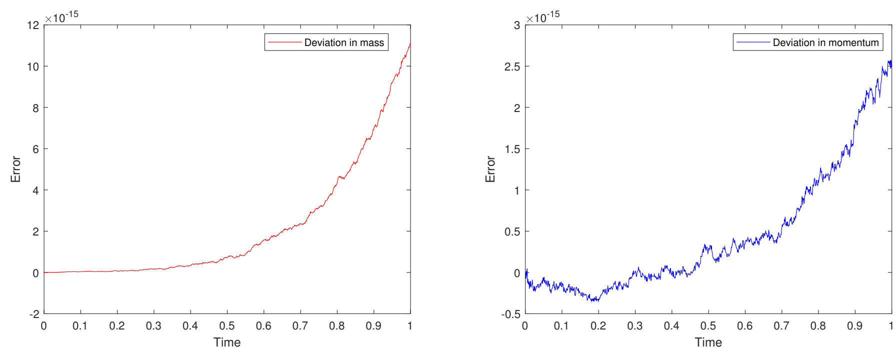

Figure 3. Deviation in mass and Deviation in momentum for Example 5.3.

Observations:

- From Figure 1(Left) and Figure 2(Left), it is easily observed that by choosing degrees of polynomials $k = 1, m = 1$, we achieve an order of convergence for the distribution function f and the fluid velocity $\boldsymbol{u}$ equaling $\min(k + 1, m + \frac{1}{2})$ which is $3/2$ and for pressure p, the order of convergence obtained equals $\min(k, m + \frac{1}{2})$ which is 1.
- By taking degree of polynomials $k = m = 2$, from Figure 1(Right) and Figure 2(Right), we obtain the order of convergence for the distribution function f and the fluid velocity $\boldsymbol{u}$ equals $\min(k + 1, m + \frac{1}{2})$ which is $5/2$, for the fluid pressure p the order of convergence equals $\min(k, m + \frac{1}{2})$ which is 2.
- For the example 5.3, we compute the deviation in mass as the difference between the initial mass and the mass at time t, see Figure 3(Left). Similarly, the deviation in momentum is calculated as the difference between the initial momentum and the momentum at time t, see Figure 3(Right). The deviations in mass and total momentum are up to the machine error, which validate our results of Lemma 3.1.
- In (86), since the kinetic equation is hyperbolic in nature, its approximate solutions $\tilde{F}$ and F^{n+1} are computed at each time level explicitly using splitting method, while Stokes part is computed implicitly. However, for time marching in the computation of the Vlasov-Stokes system, a small time step due to CFL condition for stability of the kinetic part is employed for computational experiments of all examples in this Section. Since the discrete system of the Stokes equation is less stiff than the kinetic part, therefore, for efficient time marching scheme, as in [28], it may be possible to choose unequal time steps, especially a larger time step for the Stokes,

while smaller time step in the kinetic part without loosing overall order of convergence. This will form a part of our future endeavour.

6. Conclusion

In this paper, a semi-discrete numerical method for 2D Vlasov-Stokes equation is introduced and analysed (see (31)-(33) for the discrete problem). This is a dG-dG method for the kinetic and the Stokes equations in phase space. The scheme is mass and momentum conserving. In continuum case, a non-negative initial data ($f_0(x,v) \geq 0$) yields a non-negative solution for all times. At present, we are unable to prove a similar positivity preserving property for our discrete system and hence, it is difficult to prove a discrete energy dissipation property. The rates of convergence, with regards to the degree of polynomials $m \geq 1$ and $k \geq 1$, for the distribution function $f(t,x,v)$ and the fluid velocity $\boldsymbol{u}(t,x)$ are proved in the Theorem 4.8 in $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;\boldsymbol{L}^2)$-norms, respectively. These rates help us deduce super-convergence result for $(\boldsymbol{\Pi_u u} - \boldsymbol{u}_h)$ in $|||\cdot|||$-norm (see Lemma 4.9). This enables us to derive the rate of convergence for the fluid velocity in Ł$^\infty(0,T;\boldsymbol{L}^p)$-norm $1 \leq p \leq \infty$, subject to the availability of certain projection estimates. The rate of convergence for the fluid pressure in $L^2((0,T)\times\Omega_x)$-norm is derived in Theorem 4.13. Finally, in subsection 4.4, we comment on the 3D Vlasov-Stokes equation.

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