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A FULLY DISCRETE ULTRA-WEAK DISCONTINUOUS GALERKIN METHOD FOR SOLVING THE DRIFT-DIFFUSION MODEL OF SEMICONDUCTOR DEVICES

XINYU DING, FUZHENG GAO*, AND YUNXIAN LIU*

Abstract. In this paper, we study an ultra-weak discontinuous Galerkin (UWDG) method for spatial discretization to solve the drift-diffusion (DD) model of one-dimensional semiconductor devices. Optimal error estimates are obtained using a special projection for both the semi-discrete and fully discrete UWDG schemes with smooth solutions. In the fully discrete UWDG scheme, we use an explicit third-order total variation diminishing Runge-Kutta method, which ensures stability under general temporal-spatial conditions. Numerical simulations are also performed to verify the analysis.

Key words. Drift-diffusion model, ultra-weak discontinuous Galerkin method, Runge-Kutta, error estimates.

1. Introduction

In this paper, we develop a UWDG method in space for solving the DD model of one-dimensional semiconductor devices, which is coupled with an explicit total variation diminishing Rung-Kutta (TVDRK) time-marching algorithm. Thereafter, we refer to the method as RK-UWDG. The DD model[2] is described by the following equations

$$n_t - \nabla \cdot (\mu E n) = \tau\theta\Delta n, \tag{1}$$

$$\Delta\phi = \frac{e}{\epsilon}(n - n_d), \tag{2}$$

$$E = -\nabla\phi, \tag{3}$$

where the unknown variables are the electron concentration n and the electric potential ϕ. (1) is the electron concentration equation and (2) is the electric potential equation. E represents the electric field.

The DD model is derived from the classical Boltzmann-Poisson system [1] that describes electron transport in semiconductor devices. Many numerical methods have been applied to solve the DD model, such as finite volume method [14, 15, 16], finite element method [17, 18] and also some other types of numerical methods [19]. In [3, 4], Squeff et al. analyzed the P^1 continuous finite element method for solving the DD model coupled with $P^0 - P^1$ mixed finite element method for the poisson equation. Liu and Shu[5] used a local discontinuous Galerkin (LDG) method to solve the DD model and gave suboptimal error estimates and numerical simulations. Later in [6], Liu developed the LDG method for the DD model with the optimal error estimates. Here, we present a RK-UWDG to solve the DD model of one-dimensional semiconductor devices.

The first dicontinuous Galerkin (DG) method was introduced by Reed and Hill[20] within the context of neutron linear transport in 1973. Subsequently, Cockburn

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*Corresponding author.

and Shu et al. developed the RKDG method for the hyperbolic conservation laws, which employs the DG method in space coupled with an explicit TVDRK time-marching[21, 22, 23, 24]. In 1998, Cockburn and Shu[26] proposed the LDG method as an extension to general convection-diffusion problems of the numerical scheme for the compressible Navier-Stokes equation. This method rewrites the equation into a first-order system, then apply the DG method on the system.

Meanwhile, the UWDG method was first proposed in [7], unlike the LDG method, it does not need to introduce any auxiliary variables. The principle of the UWDG method is to use integration by parts repeatedly and to move all the spatial derivatives from the trial function to the test function in the weak formulations. Cheng and Shu[8] proposed the UWDG method for general time-dependent problems with higher order spatial derivatives in 2008. In 2013, Bona et al. proposed a UWDG method for the generalised KdV equation with error estimation in [9]. Chen and Cheng[10] applied the UWDG method to the nonlinear Schrödinger equation and used a special projection to obtain optimal error estimates in 2019. In 2022, Wang and Xu applied the UWDG method to the convection diffusion equation in [11]. The UWDG method has the advantages of the classic DG method. Firstly, it can be easily designed for any order of accuracy that can be locally determined in each cell (p-adaptivity). Secondly, it can be used on arbitrary triangulations, even those with hanging nodes (h-adaptivity). Thirdly, it is extremely local in data communications. The evolution of the solution in each cell needs to communicate only with the immediate neighbors, regardless of the order of accuracy (parallel implementations). Furthermore, it is more compact than the LDG scheme and is simpler in formulation and coding.

In this work, we will first present the optimal error estimate of the semi-discrete UWDG scheme for solving one-dimensional drift-diffusion model with periodic boundary condition. The main technique is a special projection to be defined following from [11]. The projection can eliminate the projection errors involved in the diffusion part, but not eliminate the projection errors involved in the nonlinearity part. To avoid losing accuracy, we use a priori assumption to obtain optimal error estimate for the semi-discrete UWDG scheme. In practice, the DD model of semiconductor devices is described with Dirichlet boundary condition, so we also perform the error estimate of UWDG method with Dirichlet types. Furthermore, the error estimate for the third-order fully discrete explicit TVD RK-UWDG format will be analysed. The interaction of different intermediate time layers in the explicit TVDRK method renders the theoretical analysis of full discretisation considerably more challenging than that of semi-discretisation.

The organization of the paper is as follows. In Section 2, we present the discrete format of the UWDG method for solving the one-dimensional DD model. In Section 3, we describe the corresponding projection. Optimal L^2 error estimation of the semi-discrete UWDG scheme is derived in Section 4. In Section 5, we obtain the error estimate of the UWDG scheme for the DD model with Dirichlet boundary conditions. In Section 6, we present a fully discrete third-order RK-UWDG format and derive optimal L^2 error estimate. Simulation results are presented in Section 7. Concluding remarks and a plan for future work are given in Section 8.

2. The DD model and the semi-discrete UWDG scheme

In this section, we consider the following DD model

$$n_t - (\mu E n)_x - \tau\theta n_{xx} = 0, \tag{4}$$

$$\phi_{xx} = \frac{e}{\epsilon}(n - n_d), \tag{5a}$$

$$E = -\phi_x, \tag{5b}$$

where $x \in (0,1)$, e, μ and ϵ are the electron charge, the mobility and the dielectric permittivity, respectively. $\tau = \frac{m\mu}{e}$ is the relaxation parameter, $\theta = \frac{k}{m}T_0$, m and k are the effective electron mass and the Boltzmann constant, respectively. T_0 is the lattice temperature and n_d is the doping function as the initial value. The electron concentration n has a periodic boundary condition and satisfies the given initial conditions $n(x,0) = n_0(x)$. We will also consider Dirichlet boundary condition for (4) in Section 5. The electric field E and the electric potential ϕ has the Dirichlet boundary conditions:

$$\phi(0,t) = 0, \quad \phi(1,t) = v_{bias}.$$

For computational domain $I = (0,1)$, the grid is divided into N parts consisting of cells $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, for $j = 1, ..., N$, where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < ... < x_{N+\frac{1}{2}} = 1.$$

We denote

$$\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \quad h = \sup_j \Delta x_j, \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}).$$

We define a finite element space consisting of piecewise polynomials

$$V_h^k = \{v : v|_{I_j} \in P^k(I_j), j = 1, 2, ..., N\},$$

where $P^k(I_j)$ denotes the set of polynomials of degree up to k defined on the cell I_j. Both the numerical solution and the test functions will come from V_h^k. Note that in V_h^k, the functions are allowed to have jumps at the interfaces $x_{j+\frac{1}{2}}$. Moreover, both the mesh sizes Δx_j and the degree of polynomials k can be changed from element to element freely, thus allowing for h-p adaptivity easily.

For $\forall v_h \in V_h^k$, we denote

$$v^{\pm}_{h,j+\frac{1}{2}} = v_h(x^{\pm}_{j+\frac{1}{2}}), \quad [v_h] = v_h^+ - v_h^-, \quad \overline{v}_h = \frac{1}{2}(v_h^+ + v_h^-),$$

where $[v_h]$ and $\overline{v}_h$ are the jump and the mean value, respectively. We denote by C a generic positive constant independent of h, which may depend on the exact solution and its derivatives. The value of C may differ in each case.

Since the boundary condition of ϕ is not periodic, we treat it as following. Let $\widetilde{\phi}$ be the solution of

$$\begin{cases} \widetilde{\phi}_{xx} = \phi_{xx} = \frac{e}{\epsilon}(n - n_d), \\ \widetilde{\phi}(0,t) = 0, \quad \widetilde{\phi} \quad \text{is periodic on the boundary.} \end{cases}$$

We can easily check that $\phi = \widetilde{\phi} + v_{bias}x$, $E = \widetilde{E} - v_{bias} = -\widetilde{\phi}_x - v_{bias}$. Since $\widetilde{\phi}$ is periodic, we have $\widetilde{E}$ is periodic, and then E is periodic.

The UWDG method is used exclusively for spatial discretisation of the drift-diffusion equation (4). In order to facilitate the analysis of (5a)-(5b), we make use of continuous methods, such as the finite element method and the direct integration method, etc. Here, we use the direct integration method.

Multiplying equation (4) by test function $v \in V_h^k$, and integrating by parts for all terms involving a spatial derivative arrive at

$$\begin{aligned}&\int_{I_j} n_t v \,\mathrm{d}x + \int_{I_j} \mu E n v_x \,\mathrm{d}x - \mu(En)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \mu(En)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}\\ &- \int_{I_j} \tau\theta n v_{xx} \,\mathrm{d}x + \tau\theta n_{j+\frac{1}{2}} v^-_{x,j+\frac{1}{2}} - \tau\theta n_{j-\frac{1}{2}} v^+_{x,j-\frac{1}{2}}\\ &- \tau\theta n_{x,j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \tau\theta n_{x,j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0, \end{aligned} \tag{6}$$

$$E_x = -\frac{e}{\epsilon}(n - n_d). \tag{7}$$

Replacing the exact solution n and E in the above equations by their numerical approximations n_h and E_h in V_h^k, noticing that the numerical solutions n_h is not continuous on the cell boundaries, then replacing terms on the cell boundaries by suitable numerical fluxes, we obtain the UWDG scheme:

$$\begin{aligned}&\int_{I_j} (n_h)_t v \,\mathrm{d}x + \int_{I_j} \mu E_h n_h v_x \,\mathrm{d}x - \mu(\widehat{E_h n_h})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \mu(\widehat{E_h n_h})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}\\ &- \int_{I_j} \tau\theta n_h v_{xx} \,\mathrm{d}x + \tau\theta \widehat{n}_{h,j+\frac{1}{2}} v^-_{x,j+\frac{1}{2}} - \tau\theta \widehat{n}_{h,j-\frac{1}{2}} v^+_{x,j-\frac{1}{2}}\\ &- \tau\theta \widetilde{n}_{h,x,j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \tau\theta \widetilde{n}_{h,x,j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0, \end{aligned} \tag{8}$$

$$E_{h,x} = \widetilde{E}_{h,x} = -\frac{e}{\epsilon}(n_h - n_d), \tag{9}$$

$$E_h = \widetilde{E}_h - v_{bias} = \int_0^x -\frac{e}{\epsilon}(n_h - n_d)\,\mathrm{d}s + E_0 - v_{bias}, \tag{10}$$

where $E_0 = E_h(0) = \int_0^1 (\int_0^x -\frac{e}{\epsilon}(n_h - n_d)\,\mathrm{d}s)\,\mathrm{d}x$. The "hat" and "tilde" terms are the numerical fluxes. We choose the flux, i.e.,

$$\widehat{n_h} = n_h^+, \quad \widetilde{n_{h,x}} = n_{h,x}^- + \lambda[n_h], \tag{11}$$

$$\widehat{E_h n_h} = min(E_h, 0)(n_h^-) + max(E_h, 0)(n_h^+), \tag{12}$$

where $\lambda = \frac{C_0}{h}$, C_0 is a positive constant independent h, see numerical experiments for more details.

Remark 2.1. *The "$-$" and "$+$" of (11) can be reversed as follows:*

$$\widehat{n_h} = n_h^-, \quad \widetilde{n_{h,x}} = n_{h,x}^+ + \lambda[n_h]. \tag{13}$$

3. Preliminaries

In this paper, we use the norms in the Sobolev space as usual. Let p be a function. The L^2 norm is $\|p\| = (\int_I |p(x)|^2\,\mathrm{d}x)^{\frac{1}{2}}$, and L^∞ norm is $\|p\|_\infty = \max_{x\in I}|p(x)|$. Denote $\|p\|_{L^\infty(0,T;L^2)} = max_{0\leqslant t\leqslant T}\|p\|$. Γ_h denotes the set of boundary points of all elements I_j and $\|p\|_{\Gamma_h} = [\sum_{j=1}^N((p(x^+_{j+\frac{1}{2}}))^2 + (p(x^-_{j+\frac{1}{2}}))^2)]^{\frac{1}{2}}$.

3.1. Projection and interpolation properties. In what follows, we will consider the special projection of a function n with $k+1$ continuous derivatives into space V_h^k, denoted by $\mathcal{P}$, i.e., for each j,

$$\int_{I_j} (\mathcal{P}n(x) - n(x))v(x)\,\mathrm{d}x = 0, \quad \forall v \in P^{k-2}(I_j), \tag{14a}$$

$$\widehat{\mathcal{P}n}(x_{j-\frac{1}{2}}) = n(x_{j-\frac{1}{2}}), \tag{14b}$$

$$\widetilde{(\mathcal{P}n)}_x(x_{j+\frac{1}{2}}) = n_x(x_{j+\frac{1}{2}}), \tag{14c}$$

where $\widehat{\mathcal{P}n}$ and $\widetilde{\mathcal{P}n_x}$ are defined in the same manner as the definition of numerical flux in (11):

$$\widehat{\mathcal{P}n}(x_{j-\frac{1}{2}}) = \mathcal{P}n(x^+_{j-\frac{1}{2}}), \widetilde{(\mathcal{P}n)}_x(x_{j+\frac{1}{2}}) = (\mathcal{P}n)_x(x^-_{j+\frac{1}{2}}) + \lambda[\mathcal{P}n(x_{j+\frac{1}{2}})].$$

Lemma 3.1. *$\mathcal{P}$ is a local projection.*

Proof. We rewrite (14b) as

$$n(x_{j-\frac{1}{2}}) = \mathcal{P}n(x^+_{j-\frac{1}{2}}). \tag{15}$$

Using the result of (15), we rewrite (14c) as

$$n(x_{j+\frac{1}{2}}) - \frac{1}{\lambda}n_x(x_{j+\frac{1}{2}}) = \mathcal{P}n(x^-_{j+\frac{1}{2}}) - \frac{1}{\lambda}(\mathcal{P}n)_x(x^-_{j+\frac{1}{2}}), \tag{16}$$

which implies (14b)-(14c) can be locally decoupled, so $\mathcal{P}$ is a local projection[10]. □

Lemma 3.2. [11] *The projection $\mathcal{P}$ exists uniquely and has the following approximation property:*

$$\|\eta\| + h^{\frac{1}{2}}\|\eta\|_{\Gamma_h} \leqslant Ch^{k+1}, \tag{17}$$

where $\eta = \mathcal{P}n - n$. The positive constant C, solely depending on u, is independent of h.

3.2. Inverse properties. We list some inverse properties (see [12]) of the finite element space V_h^k that will be used in our error analysis. For any $v \in V_h^k$, there exists positive constant C_1 independent of v and h, such that

$$(i)\|v_x\| \leqslant C_1h^{-1}\|v\|, \quad (ii)\|v\|_{\Gamma_h} \leqslant C_1h^{-\frac{1}{2}}\|v\|, \quad (iii)\|v\|_\infty \leqslant C_1h^{-\frac{d}{2}}\|v\|, \tag{18}$$

where d is the spatial dimension. In our case $d = 1$.

4. Error estimate

In this section, we will obtain the optimal error estimate of the semi-discrete UWDG scheme by the special projection and the a priori assumption.

Theorem 4.1. *Let n and n_h be the exact solution to (6) and the numerical solution to the semi-discrete UWDG scheme (8), respectively. If the finite element space V_h^k is the piecewise polynomials of degree $k \geqslant 1$, then for small enough h there holds the following error estimate:*

$$\|n - n_h\|_{L^\infty(0,T;L^2)} \leqslant Ch^{k+1}, \tag{19}$$

where the constant C depends on the final time T, k, $\|n\|_{L^\infty(0,T;L^2)}$, $\|n_x\|_\infty$, $\|E\|_\infty$ and $\|E_x\|_\infty$.

Proof. Taking the difference of (6) and (8), we have the following error equation

$$\begin{aligned}\int_{I_j}(n-n_h)_t v\,\mathrm{d}x &+ \int_{I_j}\mu(En - E_hn_h)v_x\,\mathrm{d}x - \mu(En - \widehat{E_hn_h})_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}}\\ &+ \mu(En - \widehat{E_hn_h})_{j-\frac{1}{2}}v^+_{j-\frac{1}{2}} - \int_{I_j}\tau\theta(n-n_h)v_{xx}\,\mathrm{d}x\\ &+ \tau\theta(n - \widehat{n_h})_{j+\frac{1}{2}}v^-_{x,j+\frac{1}{2}} - \tau\theta(n-\widehat{n_h})_{j-\frac{1}{2}}v^+_{x,j-\frac{1}{2}}\\ &- \tau\theta(n_x - \widetilde{n_{h,x}})_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}} + \tau\theta(n_x - \widetilde{n_{h,x}})_{j-\frac{1}{2}}v^+_{j-\frac{1}{2}} = 0.\end{aligned} \tag{20}$$

We write the error $e = n - n_h$, $\xi = \mathcal{P}n - n_h$, $\eta = \mathcal{P}n - n$, obviously, $e = \xi - \eta$, $(\xi-\eta)^+ = \xi^+ - \eta^+$, $(\xi-\eta)_x^- = \xi_x^- - \eta_x^-$. Choosing $v = \xi \in V_h^k$ in the error equation (20), we have

$$\begin{aligned}
&\int_{I_j} (\xi-\eta)_t \xi \,\mathrm{d}x + \int_{I_j} \mu(En - E_h n_h)\xi_x \,\mathrm{d}x - \mu(En - \widehat{E_h n_h})_{j+\frac12}\xi^-_{j+\frac12} \\
&\quad + \mu(En - \widehat{E_h n_h})_{j-\frac12}\xi^+_{j-\frac12} - \int_{I_j} \tau\theta(\xi-\eta)\xi_{xx}\,\mathrm{d}x \\
&\quad + \tau\theta(\xi-\eta)^+_{j+\frac12}\xi^-_{x,j+\frac12} - \tau\theta(\xi-\eta)^+_{j-\frac12}\xi^+_{x,j-\frac12} - \tau\theta(\xi-\eta)^-_{x,j+\frac12}\xi^-_{j+\frac12} \\
&\quad - \tau\theta\lambda[\xi-\eta]_{j+\frac12}\xi^-_{j+\frac12} + \tau\theta(\xi-\eta)^-_{x,j-\frac12}\xi^+_{j-\frac12} + \tau\theta\lambda[\xi-\eta]_{j-\frac12}\xi^+_{j-\frac12} = 0. \qquad (21)
\end{aligned}$$

Using integration by parts and the special projection (14a)-(14c), to obtain

$$\begin{aligned}
&\int_{I_j} (\xi-\eta)_t \xi \,\mathrm{d}x + \int_{I_j} \mu(En - E_h n_h)\xi_x \,\mathrm{d}x - \mu(En - \widehat{E_h n_h})_{j+\frac12}\xi^-_{j+\frac12} \\
&\quad + \mu(En - \widehat{E_h n_h})_{j-\frac12}\xi^+_{j-\frac12} + \int_{I_j} \tau\theta(\xi_x)^2\,\mathrm{d}x \\
&\quad - \tau\theta\xi^-_{j+\frac12}\xi^-_{x,j+\frac12} + \tau\theta\xi^+_{j-\frac12}\xi^+_{x,j-\frac12} + \tau\theta\xi^+_{j+\frac12}\xi^-_{x,j+\frac12} - \tau\theta\xi^+_{j-\frac12}\xi^+_{x,j-\frac12} \\
&\quad - \tau\theta\xi^-_{x,j+\frac12}\xi^-_{j+\frac12} - \tau\theta\lambda[\xi]_{j+\frac12}\xi^-_{j+\frac12} + \tau\theta\xi^-_{x,j-\frac12}\xi^+_{j-\frac12} + \tau\theta\lambda[\xi]_{j-\frac12}\xi^+_{j-\frac12} = 0. \qquad (22)
\end{aligned}$$

Summing over j, and using the periodic boundary condition, we get

$$\begin{aligned}
\int_I \xi_t\xi\,\mathrm{d}x =& \int_I \eta_t\xi\,\mathrm{d}x + \sum_{j=1}^N \Big(-\int_{I_j} \mu(En - E_h n_h)\xi_x\,\mathrm{d}x \\
&+ \mu(En - \widehat{E_h n_h})_{j+\frac12}\xi^-_{j+\frac12} - \mu(En - \widehat{E_h n_h})_{j-\frac12}\xi^+_{j-\frac12}\Big) + \Big(-\tau\theta\int_I (\xi_x)^2\,\mathrm{d}x \\
&- 2\sum_{j=1}^N \tau\theta[\xi]_{j+\frac12}\xi^-_{x,j+\frac12} - \sum_{j=1}^N \tau\theta\lambda[\xi]^2_{j+\frac12}\Big) \\
=:& \sum_{i=1}^3 T_i. \qquad (23)
\end{aligned}$$

Next, we estimate $T_i (i = 1, 2, 3)$ term by term. From the property (17) of the projection and the Schwartz inequality, we can get

$$\begin{aligned}
T_1 &= \int_I \eta_t\xi\,\mathrm{d}x \\
&\leqslant C\int_I (\eta_t)^2\,\mathrm{d}x + C\int_I \xi^2\,\mathrm{d}x \\
&\leqslant Ch^{2k+2} + C\|\xi\|^2. \qquad (24)
\end{aligned}$$

About the term T_2 of (23), we have

$$\begin{aligned}
T_2 =& \sum_{j=1}^N \Big(-\int_{I_j} \mu(En - E_h n_h)\xi_x\,\mathrm{d}x + \mu(En - \widehat{E_h n_h})_{j+\frac12}\xi^-_{j+\frac12} \\
&- \mu(En - \widehat{E_h n_h})_{j-\frac12}\xi^+_{j-\frac12}\Big) \\
=& \sum_{j=1}^N \int_{I_j} \mu(E_h n_h - En)\xi_x\,\mathrm{d}x + \sum_{j=1}^N \mu(\widehat{E_h n_h} - En)_{j+\frac12}[\xi]_{j+\frac12}. \qquad (25)
\end{aligned}$$

For convenience and noticing that E_h is continuous at cell interfaces, we denote

$$E_h\widehat{\widehat{n_h}} = \widehat{E_h n_h} = min(E_h, 0)(n_h^-) + max(E_h, 0)(n_h^+). \tag{26}$$

Therefore, T_2 can be expressed as

$$\begin{aligned}
T_2 =& \sum_{j=1}^{N} \int_{I_j} \mu(E_h n_h - En)\xi_x \, \mathrm{d}x + \sum_{j=1}^{N} \mu(\widehat{E_h n_h} - En)_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}} \\
=& \sum_{j=1}^{N} \int_{I_j} \mu(E_h n_h - E_h n + E_h n - En)\xi_x \, \mathrm{d}x \\
& + \sum_{j=1}^{N} \mu(E_h\widehat{\widehat{n_h}} - E_h n + E_h n - En)_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}} \\
=& \sum_{j=1}^{N} \int_{I_j} \mu E_h(n_h - n)\xi_x \, \mathrm{d}x + \sum_{j=1}^{N} \int_{I_j} \mu(E_h - E)n\xi_x \, \mathrm{d}x \\
& + \sum_{j=1}^{N} \mu E_{h,j+\frac{1}{2}}(\widehat{\widehat{n_h}} - n)_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}} + \sum_{j=1}^{N} \mu(E_h - E)_{j+\frac{1}{2}} n_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}} \\
=& \sum_{j=1}^{N} \int_{I_j} \mu E_h(\eta - \xi)\xi_x \, \mathrm{d}x + \sum_{j=1}^{N} \mu E_{h,j+\frac{1}{2}}(\widehat{\widehat{\eta}} - \widehat{\widehat{\xi}})_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}} \\
& + \sum_{j=1}^{N} \int_{I_j} \mu(E_h - E)n\xi_x \, \mathrm{d}x + \sum_{j=1}^{N} \mu(E_h - E)_{j+\frac{1}{2}} n_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}} \\
=& (-\sum_{j=1}^{N} \int_{I_j} \mu E_h \xi\xi_x \, \mathrm{d}x - \sum_{j=1}^{N} \mu E_{h,j+\frac{1}{2}} \widehat{\widehat{\xi}}_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}}) \\
& + \sum_{j=1}^{N} \mu E_{h,j+\frac{1}{2}} \widehat{\widehat{\eta}}_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}} + \sum_{j=1}^{N} \int_{I_j} \mu E_h \eta\xi_x \, \mathrm{d}x \\
& + (\sum_{j=1}^{N} \int_{I_j} \mu(E_h - E)n\xi_x \, \mathrm{d}x + \sum_{j=1}^{N} \mu(E_h - E)_{j+\frac{1}{2}} n_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}}) \\
=& \sum_{i=1}^{4} T_{2i},
\end{aligned} \tag{27}$$

where $\widehat{\widehat{\xi}} = \xi^+$ and $\widehat{\widehat{\eta}} = \eta^+ (E_h>0)$, $\widehat{\widehat{\xi}} = \xi^-$, $\widehat{\widehat{\eta}} = \eta^- (otherwise)$. Next we estimate the terms $T_{2i} (i = 1, 2, 3, 4)$.

First we make the a priori assumption

$$\|n - n_h\| \leqslant h, \tag{28}$$

which implies that $\|n_h\|\|_\infty \leqslant C$, $\|E_{h,x}\|_\infty \leqslant C$ and $\|E_h\|_\infty \leqslant C$. We will justify (28) later. Using integration by parts, noting the denotation of $\widehat{\widehat{\xi}}$, we have

$$T_{21} = -\sum_{j=1}^{N} \int_{I_j} \mu E_h \xi\xi_x \, \mathrm{d}x - \sum_{j=1}^{N} \mu E_{h,j+\frac{1}{2}} \widehat{\widehat{\xi}}_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}\mu E_h(\xi^2)_x\,\mathrm{d}x - \sum_{j=1}^{N}\mu E_{h,j+\frac{1}{2}}\widehat{\widehat{\xi}}_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}}\\
&=\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}\mu E_{h,x}\xi^2\,\mathrm{d}x + \frac{1}{2}\sum_{j=1}^{N}\mu E_{h,j+\frac{1}{2}}[\xi^2]_{j+\frac{1}{2}}\\
&\quad - \sum_{j=1}^{N}\mu E_{h,j+\frac{1}{2}}\widehat{\widehat{\xi}}_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}}\\
&=\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}\mu E_{h,x}\xi^2\,\mathrm{d}x - \frac{1}{2}\sum_{j=1}^{N}\mu|E_{h,j+\frac{1}{2}}|[\xi]^2_{j+\frac{1}{2}}.
\end{aligned}\tag{29}
$$

Noting that $\frac{1}{2}\sum_{j=1}^{N}\mu|E_{h,j+\frac{1}{2}}|[\xi]^2_{j+\frac{1}{2}} \geqslant 0$, we have

$$
\begin{aligned}
T_{21} &\leqslant \frac{1}{2}\sum_{j=1}^{N}\int_{I_j}\mu E_{h,x}\xi^2\,\mathrm{d}x\\
&\leqslant \frac{1}{2}\mu C\|\xi\|^2,\\
&\leqslant C\|\xi\|^2,
\end{aligned}\tag{30}
$$

where C is dependent on $\|n_x\|_\infty,\ \|n\|_\infty,\ \|E_x\|_\infty$ and $\|E\|_\infty$.

Using Young's inequality and (17), we get

$$
\begin{aligned}
T_{22} &=\sum_{j=1}^{N}\mu E_{h,j+\frac{1}{2}}\widehat{\widehat{\eta}}_{j+\frac{1}{2}}[\xi]_{j+\frac{1}{2}}\\
&\leqslant Ch\|\eta\|^2_{\Gamma_h} + \frac{\widetilde{\epsilon}}{h}\sum_{j=1}^{N}\mu|E_{h,j+\frac{1}{2}}|[\xi]^2_{j+\frac{1}{2}}\\
&\leqslant Ch^{2k+2} + \frac{\widetilde{\epsilon}}{h}\sum_{j=1}^{N}\mu|E_{h,j+\frac{1}{2}}|[\xi]^2_{j+\frac{1}{2}}.
\end{aligned}\tag{31}
$$

Let $\widetilde{\epsilon} = \frac{\tau\theta}{\mu\|E_h\|_\infty}$, we have

$$
T_{22} \leqslant Ch^{2k+2} + \frac{\tau\theta}{h}\sum_{j=1}^{N}[\xi]^2_{j+\frac{1}{2}}.\tag{32}
$$

Next, using Young's inequality, (17) and $\|E_h\|_\infty \leqslant C$, to obtain

$$
\begin{aligned}
T_{23} &=\sum_{j=1}^{N}\int_{I_j}\mu E_h\eta\xi_x\,\mathrm{d}x\\
&\leqslant\Big|\sum_{j=1}^{N}\int_{I_j}\mu E_h\eta\xi_x\,\mathrm{d}x\Big|\\
&\leqslant\frac{C}{2}\|\eta\|^2 + \frac{1}{2}\|\xi_x\|^2\\
&\leqslant Ch^{2k+2} + \frac{1}{2}\|\xi_x\|^2.
\end{aligned}\tag{33}
$$

Then, we estimate T_{24}. Using integration by parts, we obtain

$$\begin{aligned}
T_{24} =&(\sum_{j=1}^{N}\int_{I_j}\mu(E_h-E)n\xi_x\,\mathrm{d}x+\sum_{j=1}^{N}\mu(E_h-E)_{j+\frac12}n_{j+\frac12}[\xi]_{j+\frac12})\\
=&-\sum_{j=1}^{N}\int_{I_j}\mu((E_h-E)n)_x\xi\,\mathrm{d}x+\sum_{j=1}^{N}\mu((E_h-E)n)_{j+\frac12}\xi^-_{j+\frac12}\\
&-\sum_{j=1}^{N}\mu((E_h-E)n)_{j-\frac12}\xi^+_{j-\frac12}+\sum_{j=1}^{N}\mu(E_h-E)_{j+\frac12}n_{j+\frac12}[\xi]_{j+\frac12}\\
=&-\sum_{j=1}^{N}\int_{I_j}\mu((E_h-E)n)_x\xi\,\mathrm{d}x\\
&-\sum_{j=1}^{N}\mu(E_h-E)_{j+\frac12}n_{j+\frac12}[\xi]_{j+\frac12}+\sum_{j=1}^{N}\mu(E_h-E)_{j+\frac12}n_{j+\frac12}[\xi]_{j+\frac12}\\
=&-\sum_{j=1}^{N}\int_{I_j}\mu((E_h-E)n)_x\xi\,\mathrm{d}x. \qquad (34)
\end{aligned}$$

Using Schwartz inequality and Young's inequality, to get

$$\begin{aligned}
T_{24} =&-\sum_{j=1}^{N}\int_{I_j}\mu((E_h-E)n)_x\xi\,\mathrm{d}x\\
=&-\sum_{j=1}^{N}\int_{I_j}\mu(E_{h,x}-E_x)n\xi\,\mathrm{d}x-\sum_{j=1}^{N}\int_{I_j}\mu(E_h-E)n_x\xi\,\mathrm{d}x\\
=&\sum_{j=1}^{N}\frac{e\mu}{\epsilon}\int_{I_j}(n-n_h)n\xi\,\mathrm{d}x+\sum_{j=1}^{N}\frac{e\mu}{\epsilon}\int_{I_j}(\int_0^x(n_h-n)\,\mathrm{d}s)n_x\xi\,\mathrm{d}x\\
=&\sum_{j=1}^{N}\frac{e\mu}{\epsilon}\int_{I_j}(\eta-\xi)n\xi\,\mathrm{d}x+\sum_{j=1}^{N}\frac{e\mu}{\epsilon}\int_{I_j}(\int_0^x(\eta-\xi)\,\mathrm{d}s)n_x\xi\,\mathrm{d}x\\
\leqslant&C\|\eta\|^2+C\|\xi\|^2\\
&+C\sum_{j=1}^{N}\frac{e\mu}{\epsilon}\int_{I_j}(\int_0^x(\eta)^2\,\mathrm{d}s)^{\frac12}n_x\xi\,\mathrm{d}x+C\sum_{j=1}^{N}\frac{e\mu}{\epsilon}\int_{I_j}(\int_0^x(\xi)^2\,\mathrm{d}s)^{\frac12}n_x\xi\,\mathrm{d}x\\
\leqslant&C\|\eta\|^2+C\|\xi\|^2\leqslant Ch^{2k+2}+C\|\xi\|^2. \qquad (35)
\end{aligned}$$

Substituting (30), (32), (33) and (35) into (27), we obtain

$$T_2\leqslant Ch^{2k+2}+C\|\xi\|^2+\frac12\|\xi_x\|^2+\frac{\tau\theta}{h}\sum_{j=1}^{N}[\xi]^2_{j+\frac12}. \qquad (36)$$

Finally, we estimate T_3. Using Young's inequality, we get

$$\begin{aligned}
T_3 =&-\tau\theta\int_I(\xi_x)^2\,\mathrm{d}x-2\sum_{j=1}^{N}\tau\theta[\xi]_{j+\frac12}\xi^-_{x,j+\frac12}-\sum_{j=1}^{N}\tau\theta\lambda[\xi]^2_{j+\frac12}\\
\leqslant&-\tau\theta\int_I(\xi_x)^2\,\mathrm{d}x+\sum_{j=1}^{N}\tau\theta(\sigma[\xi]^2_{j+\frac12}+\frac{(\xi_x^-)^2_{j+\frac12}}{\sigma})-\sum_{j=1}^{N}\tau\theta\lambda[\xi]^2_{j+\frac12}
\end{aligned}$$

$$\leqslant -\tau\theta\int_I(\xi_x)^2\,\mathrm{d}x+\sum_{j=1}^{N}\tau\theta\frac{(\xi_x^-)^2_{j+\frac{1}{2}}}{\sigma}-\sum_{j=1}^{N}\tau\theta(\lambda-\sigma)[\xi]^2_{j+\frac{1}{2}}. \tag{37}$$

Using the definition of boundary norm and the inverse property, we obtain

$$\begin{aligned}\sum_{j=1}^{N}(\xi_x^-)^2_{j+\frac{1}{2}}\leqslant&\|\xi_x\|^2_{\Gamma_h}\\ \leqslant&\frac{C_1}{h}\|\xi_x\|^2=\frac{C_1}{h}\int_I(\xi_x)^2\,\mathrm{d}x.\end{aligned} \tag{38}$$

Then we have

$$T_3\leqslant\tau\theta(\frac{C_1}{h\sigma}-1)\int_I(\xi_x)^2\,\mathrm{d}x-\sum_{j=1}^{N}\tau\theta(\lambda-\sigma)[\xi]^2_{j+\frac{1}{2}}. \tag{39}$$

Combining (36) and (39) gets

$$\begin{aligned}T_2+T_3\leqslant&\tau\theta(\frac{C_1}{h\sigma}-1+\frac{1}{2})\int_I(\xi_x)^2\,\mathrm{d}x-\sum_{j=1}^{N}\tau\theta(\lambda-\sigma-\frac{1}{h})[\xi]^2_{j+\frac{1}{2}}\\ &+Ch^{2k+2}+C\|\xi\|^2.\end{aligned} \tag{40}$$

Let $\sigma=\frac{2C_1}{h}$ to obtain

$$T_2+T_3\leqslant Ch^{2k+2}+C\|\xi\|^2-\sum_{j=1}^{N}\tau\theta(\lambda-\frac{2C_1+1}{h})[\xi]^2_{j+\frac{1}{2}}. \tag{41}$$

Letting $C_0=2C_1+1$, we get

$$T_2+T_3\leqslant Ch^{2k+2}+C\|\xi\|^2. \tag{42}$$

Substituting (24) and (42) into (23), we have

$$\frac{d}{dt}\int_I\xi^2\,\mathrm{d}x\leqslant Ch^{2k+2}+C\|\xi\|^2. \tag{43}$$

Using the Gronwall's inequality, we obtain

$$\|\xi\|_{L^\infty(0,T;L^2)}\leqslant Ch^{k+1}. \tag{44}$$

From the above inequality (44) and the property of the projection (17), we get the error estimate (19).

The following is to verify the a priori assumption (28) to complete the proof. For $k\geqslant 1$, we consider h small enough so that $Ch^{k+1}<\frac{1}{2}h$, where C is the constant in (19) determined by the final time T. Then if $t^*=sup\{t:\|n(t)-n_h(t)\|\leqslant h\}$, we should have $\|n(t^*)-n_h(t^*)\|=h$ by continuity if t^* is finite. On the other hand, our proof implies that (19) holds for $t\leqslant t^*$, in particular $\|n(t^*)-n_h(t^*)\|\leqslant Ch^{k+1}<\frac{1}{2}h<h$. This is a contradiction if $t^*<T$. Hence $t^*\geqslant T$ and the assumption (28) is correct. □

5. Error estimates of the UWDG method with Dirichlet boundary conditions

In practice, the DD model of semiconductor devices is described with Dirichlet boundary condition. Therefore, we will discuss the error estimate of the UWDG method with Dirichlet types in this section.

The Dirichlet boundary condition is

$$n(0,t)=n_l,\quad n(1,t)=n_r, \tag{45}$$

$$\phi(0,t)=0, \quad \phi(1,t)=v_{bias}. \tag{46}$$

The semi-discrete UWDG scheme is the same as (8), except that the fluxes $\widehat{n^h}$ and $\widetilde{n^h_x}$ should be changed at one of the boundaries to take care of the Dirichlet boundary condition. We choose the fluxes for $\widehat{n^h}$ and $\widetilde{n^h_x}$ similarly to (11) as the following

$$\begin{aligned}
&(\widehat{n_h})_{\frac{1}{2}}=(n_h^-)_{\frac{1}{2}}=n_l, \quad (\widehat{n_h})_{j-\frac{1}{2}}=(n_h^+)_{j-\frac{1}{2}},\ j=2,...,N,\\
&(\widehat{n_h})_{N+\frac{1}{2}}=(n_h^+)_{N+\frac{1}{2}}=n_r,\\
&(\widetilde{n_{h,x}})_{\frac{1}{2}}=(n_{h,x}^+)_{\frac{1}{2}}+\lambda[n_h]_{\frac{1}{2}},\\
&(\widetilde{n_{h,x}})_{j-\frac{1}{2}}=(n_{h,x}^-)_{j-\frac{1}{2}}+\lambda[n_h]_{j-\frac{1}{2}},\ j=2,...,N+1.
\end{aligned} \tag{47}$$

The flux $\widehat{E_hn_h}$ is the same as before. Then we get the following error estimate.

Theorem 5.1. *Let n be the exact solution to (6). Let n_h be the numerical solution to the semi-discrete UWDG scheme (8), and choose the fluxes of the $\widehat{n_h}$ and $\widetilde{n_{h,x}}$ as (47). If the finite element space V_h^k is the piecewise polynomials of degree $k\geqslant 1$, then for small enough h there holds the following error estimate:*

$$\|n-n_h\|_{L^\infty(0,T;L^2)}\leqslant Ch^{k+1}, \tag{48}$$

where the constant C depends on the final time T, k, $\|n\|_{L^\infty(0,T;L^2)}$, $\|n_x\|_\infty$, $\|E\|_\infty$ and $\|E_x\|_\infty$.

Proof. Taking the difference of (6) and (8), we get the same error equation (20). Taking the flux (47) and still choosing $v=\xi$, by the projection (14a)-(14c) and $(n-\widehat{n_h})_{\frac{1}{2}}=0$, $(n-\widehat{n_h})_{N+\frac{1}{2}}=0$, we have

$$\begin{aligned}
&\int_{I_j}(\xi-\eta)_t\xi\,\mathrm{d}x+\int_{I_j}\mu(En-E_hn_h)\xi_x\,\mathrm{d}x-\mu(En-\widehat{E_hn_h})_{j+\frac{1}{2}}\xi^-_{j+\frac{1}{2}}\\
&\quad+\mu(En-\widehat{E_hn_h})_{j-\frac{1}{2}}\xi^+_{j-\frac{1}{2}}-\int_{I_j}\tau\theta\xi\xi_{xx}\,\mathrm{d}x+\tau\theta\xi^+_{j+\frac{1}{2}}\xi^-_{x,j+\frac{1}{2}}-\tau\theta\xi^+_{j-\frac{1}{2}}\xi^+_{x,j-\frac{1}{2}}\\
&\quad-\tau\theta\xi^-_{x,j+\frac{1}{2}}\xi^-_{j+\frac{1}{2}}-\tau\theta\lambda[\xi]_{j+\frac{1}{2}}\xi^-_{j+\frac{1}{2}}+\tau\theta\xi^-_{x,j-\frac{1}{2}}\xi^+_{j-\frac{1}{2}}+\tau\theta\lambda[\xi]_{j-\frac{1}{2}}\xi^+_{j-\frac{1}{2}}=0,
\end{aligned} \tag{49}$$

for $j=2,...,N-1$, and

$$\begin{aligned}
&\int_{I_1}(\xi-\eta)_t\xi\,\mathrm{d}x+\int_{I_1}\mu(En-E_hn_h)\xi_x\,\mathrm{d}x-\mu(En-\widehat{E_hn_h})_{\frac{3}{2}}\xi^-_{\frac{3}{2}}\\
&\quad+\mu(En-\widehat{E_hn_h})_{\frac{1}{2}}\xi^+_{\frac{1}{2}}-\int_{I_1}\tau\theta\xi\xi_{xx}\,\mathrm{d}x+\tau\theta\xi^+_{\frac{3}{2}}\xi^-_{x,\frac{3}{2}}\\
&\quad-\tau\theta\xi^-_{x,\frac{3}{2}}\xi^-_{\frac{3}{2}}-\tau\theta\lambda[\xi]_{\frac{3}{2}}\xi^-_{\frac{3}{2}}+\tau\theta(\xi-\eta)^+_{x,\frac{1}{2}}\xi^+_{\frac{1}{2}}+\tau\theta\lambda[\xi-\eta]_{\frac{1}{2}}\xi^+_{\frac{1}{2}}=0,
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
&\int_{I_N}(\xi-\eta)_t\xi\,\mathrm{d}x+\int_{I_N}\mu(En-E_hn_h)\xi_x\,\mathrm{d}x-\mu(En-\widehat{E_hn_h})_{N+\frac{1}{2}}\xi^-_{N+\frac{1}{2}}\\
&\quad+\mu(En-\widehat{E_hn_h})_{N-\frac{1}{2}}\xi^+_{N-\frac{1}{2}}-\int_{I_N}\tau\theta\xi\xi_{xx}\,\mathrm{d}x-\tau\theta\xi^+_{N-\frac{1}{2}}\xi^+_{x,N-\frac{1}{2}}-\tau\theta\xi^-_{x,N+\frac{1}{2}}\xi^-_{N+\frac{1}{2}}\\
&\quad-\tau\theta\lambda[\xi]_{N+\frac{1}{2}}\xi^-_{N+\frac{1}{2}}+\tau\theta\xi^-_{x,N-\frac{1}{2}}\xi^+_{N-\frac{1}{2}}+\tau\theta\lambda[\xi]_{N-\frac{1}{2}}\xi^+_{N-\frac{1}{2}}=0.
\end{aligned} \tag{51}$$

Using integration by parts and summing over j from 1 and N, we get

$$
\begin{aligned}
\int_I \xi_t \xi \, \mathrm{d}x = & \int_I \eta_t \xi \, \mathrm{d}x \\
& + \sum_{j=1}^{N} (- \int_{I_j} \mu(En - E_h n_h)\xi_x \, \mathrm{d}x \\
& + \mu(En - \widehat{E_h n_h})_{j+\frac{1}{2}} \xi^-_{j+\frac{1}{2}} - \mu(En - \widehat{E_h n_h})_{j-\frac{1}{2}} \xi^+_{j-\frac{1}{2}}) \\
& + \sum_{j=1}^{N} (- \int_{I_j} \tau\theta(\xi_x)^2 \, \mathrm{d}x + \tau\theta \xi^-_{j+\frac{1}{2}} \xi^-_{x,j+\frac{1}{2}} - \tau\theta \xi^+_{j+\frac{1}{2}} \xi^+_{x,j+\frac{1}{2}} \\
& + \tau\theta \xi^-_{x,j+\frac{1}{2}} \xi^-_{j+\frac{1}{2}} + \tau\theta\lambda[\xi]_{j+\frac{1}{2}} \xi^-_{j+\frac{1}{2}} - \tau\theta \xi^-_{x,j-\frac{1}{2}} \xi^+_{j-\frac{1}{2}} - \tau\theta\lambda[\xi]_{j-\frac{1}{2}} \xi^+_{j-\frac{1}{2}}) \\
& + \tau\theta \xi^-_{x,\frac{1}{2}} \xi^+_{\frac{1}{2}} - \tau\theta \xi^+_{x,\frac{1}{2}} \xi^+_{\frac{1}{2}} + \tau\theta \eta^+_{x,\frac{1}{2}} \xi^+_{\frac{1}{2}} + \tau\theta\lambda[\eta]_{\frac{1}{2}} \xi^+_{\frac{1}{2}} \\
& + \sum_{j=2}^{N-1} (-\tau\theta \xi^+_{j+\frac{1}{2}} \xi^-_{x,j+\frac{1}{2}} + \tau\theta \xi^+_{j-\frac{1}{2}} \xi^+_{x,j-\frac{1}{2}}) - \tau\theta \xi^+_{\frac{3}{2}} \xi^-_{x,\frac{3}{2}} + \tau\theta \xi^+_{N-\frac{1}{2}} \xi^+_{x,N-\frac{1}{2}} \\
& =: T_1 + T_2 + T_3. \qquad (52)
\end{aligned}
$$

We analyze T_1 and T_2 as before. For T_3, to obtain

$$
\begin{aligned}
T_3 = & \sum_{j=1}^{N} (- \int_{I_j} \tau\theta(\xi_x)^2 \, \mathrm{d}x + \tau\theta \xi^-_{j+\frac{1}{2}} \xi^-_{x,j+\frac{1}{2}} - \tau\theta \xi^+_{j+\frac{1}{2}} \xi^+_{x,j+\frac{1}{2}} \\
& + \tau\theta \xi^-_{x,j+\frac{1}{2}} \xi^-_{j+\frac{1}{2}} + \tau\theta\lambda[\xi]_{j+\frac{1}{2}} \xi^-_{j+\frac{1}{2}} - \tau\theta \xi^-_{x,j-\frac{1}{2}} \xi^+_{j-\frac{1}{2}} - \tau\theta\lambda[\xi]_{j-\frac{1}{2}} \xi^+_{j-\frac{1}{2}}) \\
& + \tau\theta \xi^-_{x,\frac{1}{2}} \xi^+_{\frac{1}{2}} - \tau\theta \xi^+_{x,\frac{1}{2}} \xi^+_{\frac{1}{2}} + \tau\theta \eta^+_{x,\frac{1}{2}} \xi^+_{\frac{1}{2}} + \tau\theta\lambda[\eta]_{\frac{1}{2}} \xi^+_{\frac{1}{2}} \\
& + \sum_{j=2}^{N-1} (-\tau\theta \xi^+_{j+\frac{1}{2}} \xi^-_{x,j+\frac{1}{2}} + \tau\theta \xi^+_{j-\frac{1}{2}} \xi^+_{x,j-\frac{1}{2}}) - \tau\theta \xi^+_{\frac{3}{2}} \xi^-_{x,\frac{3}{2}} + \tau\theta \xi^+_{N-\frac{1}{2}} \xi^+_{x,N-\frac{1}{2}} \\
= & - \int_I \tau\theta(\xi_x)^2 \, \mathrm{d}x + \sum_{j=1}^{N} (-2\tau\theta[\xi]_{j+\frac{1}{2}} \xi^-_{x,j+\frac{1}{2}} - \tau\theta\lambda[\xi]^2_{j+\frac{1}{2}}) \\
& - 2\tau\theta \xi^+_{x,\frac{1}{2}} \xi^+_{\frac{1}{2}} + \tau\theta \eta^+_{x,\frac{1}{2}} \xi^+_{\frac{1}{2}} - \tau\theta\lambda(\xi^+_{\frac{1}{2}})^2. \qquad (53)
\end{aligned}
$$

Using Schwartz inequality and Young's inequality, we get

$$
\begin{aligned}
T_3 \leqslant & - \tau\theta \int_I (\xi_x)^2 \, \mathrm{d}x + \sum_{j=1}^{N} \tau\theta(\sigma[\xi]^2_{j+\frac{1}{2}} + \frac{(\xi^-_x)^2_{j+\frac{1}{2}}}{\sigma}) - \sum_{j=1}^{N} \tau\theta\lambda[\xi]^2_{j+\frac{1}{2}} \\
& + \tau\theta(\frac{(\xi^+_{x,\frac{1}{2}})^2}{\sigma} + \sigma(\xi^+_{\frac{1}{2}})^2) + Ch^{2k+2} + \frac{1}{2}\tau\theta(\xi^+_{\frac{1}{2}})^2 - \tau\theta\lambda(\xi^+_{\frac{1}{2}})^2 \\
\leqslant & - \tau\theta \int_I (\xi_x)^2 \, \mathrm{d}x + \sum_{j=1}^{N} \tau\theta \frac{(\xi^-_x)^2_{j+\frac{1}{2}}}{\sigma} - \sum_{j=1}^{N} \tau\theta(\lambda - \sigma)[\xi]^2_{j+\frac{1}{2}} \\
& + \tau\theta(\frac{(\xi^+_{x,\frac{1}{2}})^2}{\sigma} + Ch^{2k+2} - \tau\theta(\lambda - \sigma - \frac{1}{2})(\xi^+_{\frac{1}{2}})^2. \qquad (54)
\end{aligned}
$$

Using the definition of boundary norm and the inverse property, to obtain

$$
\sum_{j=1}^{N} (\xi^-_x)^2_{j+\frac{1}{2}} + (\xi^+_x)^2_{\frac{1}{2}} \leqslant \|\xi_x\|^2_{\Gamma_h}
$$

$$\leqslant \frac{C_1}{h}\|\xi_x\|^2 = \frac{C_1}{h}\int_I (\xi_x)^2\,\mathrm{d}x. \tag{55}$$

Next, we get

$$\begin{aligned} T_3 \leqslant & \tau\theta(\frac{C_1}{h\sigma}-1)\int_I (\xi_x)^2\,\mathrm{d}x - \sum_{j=1}^{N}\tau\theta(\lambda-\sigma)[\xi]^2_{j+\frac{1}{2}} \\ & + Ch^{2k+2} - \tau\theta(\lambda-\sigma-\frac{1}{2})(\xi^+_{\frac{1}{2}})^2. \end{aligned} \tag{56}$$

Combining (36) and (56) obtains

$$\begin{aligned} T_2+T_3 \leqslant & \tau\theta(\frac{C_1}{h\sigma}-1+\frac{1}{2})\int_I (\xi_x)^2\,\mathrm{d}x - \sum_{j=1}^{N}\tau\theta(\lambda-\sigma-\frac{1}{h})[\xi]^2_{j+\frac{1}{2}} \\ & + Ch^{2k+2} + C_3\|\xi\|^2 - \tau\theta(\lambda-\sigma-\frac{1}{2})(\xi^+_{\frac{1}{2}})^2. \end{aligned} \tag{57}$$

Let $\sigma = \frac{2C_1}{h}$, then

$$T_2+T_3 \leqslant Ch^{2k+2} + C\|\xi\|^2 - \sum_{j=1}^{N}\tau\theta(\lambda-\frac{2C_1+1}{h})[\xi]^2_{j+\frac{1}{2}} - \tau\theta(\lambda-\sigma-\frac{1}{2})(\xi^+_{\frac{1}{2}})^2. \tag{58}$$

Letting $C_0 = 2C_1+1$, we get

$$T_2+T_3 \leqslant Ch^{2k+2} + C\|\xi\|^2. \tag{59}$$

Substituting (24) and (59) into (52), we have

$$\frac{d}{dt}\int_I \xi^2\,\mathrm{d}x \leqslant Ch^{2k+2} + C\|\xi\|^2. \tag{60}$$

Using the Gronwall's inequality, we obtain

$$\|\xi\|_{L^\infty(0,T;L^2)} \leqslant Ch^{k+1}. \tag{61}$$

From the above inequality (61) and the property of the projection (17), we get the error estimate (48).

□

6. TVD Rung-Kutta fully discrete UWDG schemes and error estimate

First, we rewrite (8) in the following variation form:

$$((n_h)_t, v)_j = \mathcal{H}_j(E_h n_h, v) + \mathcal{L}_j(n_h, v), \tag{62}$$

where

$$\mathcal{H}_j(E_h n_h, v) = -\int_{I_j}\mu E_h n_h v_x\,\mathrm{d}x + \mu(\widehat{E_h n_h})_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}} - \mu(\widehat{E_h n_h})_{j-\frac{1}{2}}v^+_{j-\frac{1}{2}}, \tag{63}$$

$$\begin{aligned} \mathcal{L}_j(n_h, v) = & \int_{I_j}\tau\theta n_h v_{xx}\,\mathrm{d}x - \tau\theta\widehat{n}_{h,j+\frac{1}{2}}v^-_{x,j+\frac{1}{2}} + \tau\theta\widehat{n}_{h,j-\frac{1}{2}}v^+_{x,j-\frac{1}{2}} \\ & + \tau\theta\widetilde{n}_{h,x,j+\frac{1}{2}}v^-_{j+\frac{1}{2}} - \tau\theta\widetilde{n}_{h,x,j-\frac{1}{2}}v^+_{j-\frac{1}{2}}. \end{aligned} \tag{64}$$

For convenience of analysis, we denote

$$(u,v) = \sum_{j=1}^{N}(u,v)_j,$$

which is the inner product in $L^2(I)$. Let $\mathcal{H}(\cdot,\cdot) = \sum_{j=1}^{N} \mathcal{H}_j(\cdot,\cdot)$, $\mathcal{L}(\cdot,\cdot) = \sum_{j=1}^{N} \mathcal{L}_j(\cdot,\cdot)$. After summing over $j = 1, 2, ..., N$ in the variation formulation (62), we get the semi-discrete UWDG scheme in the form:

$$((n_h)_t, v) = \mathcal{H}(E_h n_h, v) + \mathcal{L}(n_h, v). \tag{65}$$

6.1. TVD Rung-Kutta fully discrete UWDG scheme. In this section, we would like to adopt the third-order TVDRK method [13] to update the semi-discrete UWDG scheme (62).

Let $\{t^m = m\Delta t\}_{m=0}^{M}$ be the uniform partition of the time interval $[0, T]$, here Δt is time step. Given n_h^m, we can get E_h^m and ϕ_h^m, then we find the numerical solution at the next level t^{m+1}, through several intermediate stages $t^{m,l}$, by the following TVDRK methods.

The explicit third order TVD RK-UWDG scheme is: Find the numerical solution $n_h^{m+1} \in V_h^k$, such that

$$\begin{aligned}
(n_h^{m,1}, v_1)_{I_j} =& (n_h^m, v_1)_{I_j} + \Delta t \mathcal{H}_j(E_h^m n_h^m, v_1) + \Delta t \mathcal{L}_j(n_h^m, v_1),\\
(n_h^{m,2}, v_2)_{I_j} =& \frac{3}{4}(n_h^m, v_2)_{I_j} + \frac{1}{4}(n_h^{m,1}, v_2)_{I_j}\\
&+ \frac{1}{4}\Delta t \mathcal{H}_j(E_h^{m,1} n_h^{m,1}, v_2) + \frac{1}{4}\Delta t \mathcal{L}_j(n_h^{m,1}, v_2),\\
(n_h^{m+1}, v_3)_{I_j} =& \frac{1}{3}(n_h^m, v_3)_{I_j} + \frac{2}{3}(n_h^{m,2}, v_3)_{I_j}\\
&+ \frac{2}{3}\Delta t \mathcal{H}_j(E_h^{m,2} n_h^{m,2}, v_3) + \frac{2}{3}\Delta t \mathcal{L}_j(n_h^{m,2}, v_3),
\end{aligned} \tag{66}$$

for $j = 1, ..., N$ and arbitrary $v_1, v_2, v_3 \in V_h^k$.

The direct integration method of the electric potential equation is: Find $E_h^{m,l}, \phi_h^{m,l} \in V_h^k$, such that

$$\begin{aligned}
E_{h,x}^{m,l} &= \widetilde{E}_{h,x}^{m,l} = -\frac{e}{\epsilon}(n_h^{m,l} - n_d)\\
E_h^{m,l} &= \widetilde{E}_h^{m,l} - v_{bias} = \int_0^x -\frac{e}{\epsilon}(n_h^{m,l} - n_d)\,\mathrm{d}s + E_0^{m,l} - v_{bias},
\end{aligned} \tag{67}$$

where $E_0^{m,l} = E_h^{m,l}(0) = \int_0^1 (\int_0^x -\frac{e}{\epsilon}(n_h^{m,l} - n_d)\,\mathrm{d}s)\,\mathrm{d}x$, for $j = 1, ..., N$ and $l = 0, 1, 2$, $u^{m,0} = u^m$ ($u = E_h$ or n_h).

6.2. Error estimate.

Lemma 6.1. [11] *For any $u, v \in V_h^k$, there holds*

$$\mathcal{L}(u, v) = \mathcal{L}(v, u). \tag{68}$$

Lemma 6.2. [25] *For any $u \in V_h^k$, if $\lambda \geqslant \frac{C_1}{h}$, there holds*

$$\mathcal{L}(u, u) \leqslant 0. \tag{69}$$

Lemma 6.3. [11] *For any $u \in V_h^k$, if $C_0 \geqslant \frac{1}{2} + 2C_1$, there hold*

$$\mathcal{L}(u, u) \leqslant -\frac{\tau\theta}{2}(\|u_x\|^2 + \sum_{j=1}^{N} h^{-1}[u]_{j+\frac{1}{2}}^2), \tag{70}$$

$$\|u_x\|^2 + \sum_{j=1}^{N} h^{-1}[u]_{j+\frac{1}{2}}^2 \leqslant -\frac{2}{\tau\theta}\mathcal{L}(u, u). \tag{71}$$

Theorem 6.1. *Let n^m be the exact solution to (6) at time level m, which is smooth with bounded derivatives. Let n_h^m be the numerical solution to the explicit third order TVD RK-UWDG scheme (66). If the finite element space V_h^k is the piecewise polynomials of degree $k \geqslant 1$, then for small enough h there holds the following error estimates:*

$$\|n - n_h\|_{L^\infty(0,T;L^2)} \leqslant C(h^{k+1} + \Delta t^3), \tag{72}$$

where the constant C depends on the final time T, k, $\|n\|_{L^\infty(0,T;L^2)}$, $\|n_x\|_\infty$, $\|E\|_\infty$ and $\|E_x\|_\infty$.

Proof. The proof of this Theorem can be found in the Appendix A.1. □

7. Numerical experiments for the DD model

In this section, we combine the third-order TVDRK method[13] for the time discretization with the UWDG method for the spatial discretisation to solve a one-dimensional DD model. We choose locally orthogonal Legendre polynomial basis over $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$,

$$v_0^{(j)}(x) = 1, \quad v_1^{(j)}(x) = x - x_j, \quad v_2^{(j)}(x) = (x - x_j)^2 - \frac{1}{12}\Delta x_j^2, \quad ...$$

The numerical solution can be written as

$$n_h(x,t) = \sum_{l=0}^{k} u_j^{(l)}(t) v_l^{(j)}(x), \quad x \in I_j.$$

We use the L^2 projection for initial value discretisation. Additionally, for $\lambda = \frac{C_0}{h}$ in (11), we take $C_0 = 10$ in our numerical experiments. Here, we take $\Delta t = 0.06h^2$ for numerical stability.

Example 1 (The accuracy test). Let $I = [0, 2\pi]$, $J \in [0, 1]$, we consider the following coupled equations with periodic boundary condition

$$\begin{aligned} &n_t - (En)_x - n_{xx} = f(x,t), \\ &\phi_{xx} = n - g(x,t), \end{aligned}$$

where $f(x,t)$ and $g(x,t)$ are suitably chosen such that the exact solutions are

$$\begin{aligned} &n = (2 - e^t)\cos x, \quad x \in I, t \in J, \\ &\phi = e^t \cos x. \end{aligned}$$

Table 1 shows that the convergence rates of errors in L^∞-norm and L^2-norm attain k+1 order of accuracy.

Example 2 (Convection-dominated Problem). Let $I = [0, 2\pi]$, $J \in [0, 1]$, we consider the following coupled equations with Dirichlet boundary condition

$$\begin{aligned} &n_t - (En)_x - \varepsilon n_{xx} = f(x,t), \\ &\phi_{xx} = n - g(x,t), \end{aligned}$$

where $f(x,t)$ and $g(x,t)$ are suitably chosen such that the exact solutions are

$$\begin{aligned} &n = \sin x \cos t, \quad x \in I, t \in J, \\ &\phi = \cos x \sin t. \end{aligned}$$

Table 2 and 3 show that the convergence rates of errors in L^∞-norm and L^2-norm attain k+1 order of accuracy. This example illustrates that the UWDG method can solve the convection-dominated problem efficiently.

Example 3 (Semiconductor Device Simulation). We simulate the DD model

TABLE 1. The errors and orders of L^∞ and L^2 norms for Example 1.

	N	L^∞-error	order	L^2-error	order
P^1	10	4.4205E-02	-	7.0326E-02	-
	20	1.2009E-02	1.88	1.8155E-02	1.95
	40	3.0501E-03	1.98	4.6321E-03	1.97
	80	7.6952E-04	1.99	1.1710E-03	1.98
	160	1.9331E-04	1.99	2.9447E-04	1.99
P^2	10	2.4962E-03	-	3.8005E-03	-
	20	3.1027E-04	3.01	5.0752E-04	2.91
	40	3.8928E-05	2.99	6.5908E-05	2.95
	80	4.8549E-06	3.00	8.4005E-06	2.97
	160	6.0565E-07	3.00	1.0604E-06	2.99

TABLE 2. The errors and orders of L^∞ and L^2 norms for Example 2 with $\varepsilon = 10^{-6}$.

	N	L^∞-error	order	L^2-error	order
P^1	10	4.9630E-02	-	7.1574E-02	-
	20	9.8705E-03	2.33	1.5197E-02	2.24
	40	2.3071E-03	2.10	3.3619E-03	2.18
	80	5.3977E-04	2.10	7.8711E-04	2.09
	160	1.3282E-04	2.02	1.9050E-04	2.05
P^2	10	2.5942E-03	-	4.1789E-03	-
	20	3.0507E-04	3.09	3.6859E-04	3.50
	40	3.8902E-05	2.97	3.8588E-05	3.26
	80	4.9019E-06	2.99	4.3548E-06	3.15
	160	6.1851E-07	2.99	5.1454E-07	3.08

with a length of $0.6\mu m$ and a doping in Figure 1 defined by $n_d = 5 \times 10^{17} cm^{-3}$ in $[0, 0.1]$ and $[0.5, 0.6]$ and $n_d = 2 \times 10^{15} cm^{-3}$ in $[0.15, 0.45]$, and a smooth transition in between. The lattice temperature is taken as $T_0 = 300°K$. The electron effective mass $m = 0.26 \times 0.9109 \times 10^{-31} kg$, the electron charge $e = 0.1602$, the mobility $\mu = 0.75$, the dielectric constant $\epsilon = 11.7 \times 8.85418$, and Boltzmann constant $k = 0.138 \times 10^{-4}$. The boundary conditions are given as follows: $\phi = \phi_0 = \frac{kT}{e} ln(\frac{n_d}{n_i})$ at the left boundary, with $n_i = 1.4 \times 10^{10} cm^{-3}$, $\phi = \phi_0 + v_{bias}$ with the voltage drop $v_{bias} = 1.5$ at the right boundary for the potential; $T = 300°K$ at both boundaries for the temperature; and $n = 5 \times 10^{17} cm^{-3}$ at both boundaries for the concentration. The results are compared with those obtained by the LDG method [5].

Figure 2 shows numerical results for the electron concentration n, the electric

TABLE 3. The errors and orders of L^∞ and L^2 norms for Example 2 with $\varepsilon = 10^{-9}$.

	N	L^∞-error	order	L^2-error	order
$\boldsymbol{P^1}$	10	4.9630E-02	-	7.1575E-02	-
	20	9.8707E-03	2.33	1.5197E-02	2.24
	40	2.3072E-03	2.10	3.3620E-03	2.18
	80	5.3978E-04	2.10	7.8713E-04	2.09
	160	1.3282E-04	2.02	1.9051E-04	2.05
$\boldsymbol{P^2}$	10	2.5943E-03	-	4.1894E-03	-
	20	3.0505E-04	3.09	3.6882E-04	3.51
	40	3.8882E-05	2.97	3.8597E-05	3.26
	80	4.8907E-06	2.99	4.3553E-06	3.15
	160	6.1278E-07	3.00	5.1431E-07	3.08

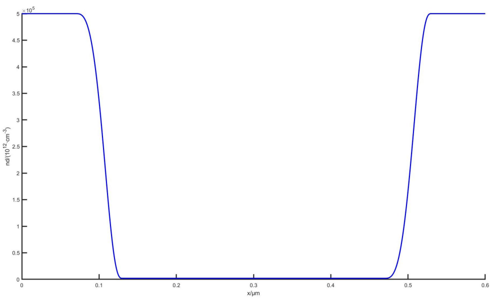

FIGURE 1. Doping n_d.

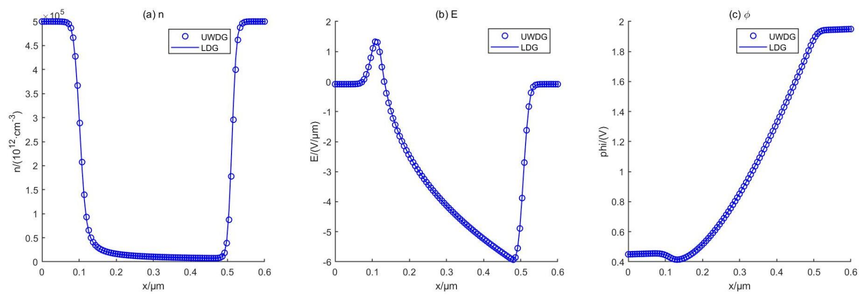

FIGURE 2. [0,0.6] with 100 mesh cell. (a) density n; (b) electric field E; (c) potential ϕ.

field E and the electric potential ϕ under UWDG and LDG discretization. At the point of reaching the steady state at $t = 0.4945s$, Figure 1 (a) shows the numerical result of the electron concentration n using UWDG and LDG discretization, while Figure 1 (b) and (c) show the numerical results of the electric field E and the electric potential ϕ, respectively. The results from both methods are consistent and

TABLE 4. The CPU time of UWDG method and LDG method.

N	40	80	100	160	200
UWDG Time/s	2.12	13.10	22.48	116.04	149.64
LDG Time/s	2.51	15.00	31.06	150.48	186.45

as expected. Table 4 shows the CPU time for both methods. The UWDG method requires a CPU time comparable to that of the LDG method, but has the advantage of not requiring additional variables and having a simpler structure. Therefore, it is a reliable tool for studying the numerical simulation of various semiconductor models.

8. Concluding remarks and future work

In this paper, we utilize the UWDG method to solve the one-dimensional DD model, in which both the first derivative convection term and the second derivative diffusion term exist. For the Dirichlet and periodic boundary conditions, we obtain optimal error estimates in L^2-norm for the electron concentration n by means of a special projection and a priori assumption. We also present optimal error estimate for the third order fully discrete RK-UWDG scheme. Numerical experiments are presented to demonstrate the accuracy of the RK-UWDG method and confirm our theoretical analysis. In particular, for the DD model with Dirichlet boundary condition, the practical numerical simulation is performed and compared with that of the LDG method. This method does not require the introduction of auxiliary variable and can get similar results to the LDG method. In future work, we aim to apply this method to other semiconductor models, such as the energy transport model and the hydrodynamic model.

Appendix A. Appendix

A.1. Proof of Theorem 6.1. First, we rewrite the scheme (66) as

$$\begin{aligned}
&(n_h^{m,1} - n_h^m, v_1)_{I_j} = \Delta t\mathcal{H}_j(E_h^m n_h^m, v_1) + \Delta t\mathcal{L}_j(n_h^m, v_1),\\
&(4n_h^{m,2} - 3n_h^m - n_h^{m,1}, v_2)_{I_j} = \Delta t\mathcal{H}_j(E_h^{m,1} n_h^{m,1}, v_2) + \Delta t\mathcal{L}_j(n_h^{m,1}, v_2),\\
&(\frac{3}{2}n_h^{m+1} - \frac{1}{2}n_h^m - n_h^{m,2}, v_3)_{I_j} = \Delta t\mathcal{H}_j(E_h^{m,2} n_h^{m,2}, v_3) + \Delta t\mathcal{L}_j(n_h^{m,2}, v_3),
\end{aligned} \tag{A.1}$$

To get the error equation of the explicit third order TVD RK-UWDG scheme, we rewrite (6) at the time level m as the following:

$$\begin{aligned}
&(n^{m,1} - n^m, v_1)_{I_j} = \Delta t\mathcal{H}_j(E^m n^m, v_1) + \Delta t\mathcal{L}_j(n^m, v_1),\\
&(4n^{m,2} - 3n^m - n^{m,1}, v_2)_{I_j} = \Delta t\mathcal{H}_j(E^{m,1} n^{m,1}, v_2) + \Delta t\mathcal{L}_j(n^{m,1}, v_2),\\
&(\frac{3}{2}n^{m+1} - \frac{1}{2}n^m - n^{m,2}, v_3)_{I_j} = \Delta t\mathcal{H}_j(E^{m,2} n^{m,2}, v_3) + \Delta t\mathcal{L}_j(n^{m,2}, v_3)\\
&+ (\zeta^m, v_3),
\end{aligned} \tag{A.2}$$

where ζ^m is the truncation error and $\|\zeta^m\| \leqslant C(\Delta t)^4$.

We rewrite the error $e^m = n^m - n_h^m = \xi^m - \eta^m$, where $\xi^m = \mathcal{P}n^m - n_h^m$, $\eta^m = \mathcal{P}n^m - n^m$. Taking the difference of (A.2) and (A.1), we get the following error equations

$$\begin{aligned}
&(\xi^{m,1} - \xi^m, v_1)_{I_j} = (\eta^{m,1} - \eta^m, v_1)_{I_j} + \Delta t\mathcal{H}_j(E^m n^m - E_h^m n_h^m, v_1)\\
&+ \Delta t\mathcal{L}_j(\xi^m - \eta^m, v_1),
\end{aligned}$$

$$(4\xi^{m,2}-3\xi^m-\xi^{m,1},v_2)_{I_j}=(4\eta^{m,2}-3\eta^m-\eta^{m,1},v_2)_{I_j}$$
$$+\Delta t\mathcal{H}_j(E^{m,1}n^{m,1}-E_h^{m,1}n_h^{m,1},v_2)+\Delta t\mathcal{L}_j(\xi^{m,1}-\eta^{m,1},v_2),$$
$$(\frac{3}{2}\xi^{m+1}-\frac{1}{2}\xi^m-\xi^{m,2},v_3)_{I_j}=(\frac{3}{2}\eta^{m+1}-\eta^m-\frac{1}{2}\eta^{m,2},v_3)_{I_j}$$
$$+\Delta t\mathcal{H}_j(E^{m,2}n^{m,2}-E_h^{m,2}n_h^{m,2},v_3)+\Delta t\mathcal{L}_j(\xi^{m,2}-\eta^{m,2},v_3)+(\zeta^m,v_3), \tag{A.3}$$

Using the projection (14a)-(14c), choosing $v_1=\xi^m$, $v_2=\xi^{m,1}$, $v_3=4\xi^{m,2}$ in (A.3), we have

$$(\xi^{m,1}-\xi^m,\xi^m)_{I_j}=(\eta^{m,1}-\eta^m,\xi^m)_{I_j}+\Delta t\mathcal{H}_j(E^mn^m-E_h^mn_h^m,\xi^m)$$
$$+\Delta t\mathcal{L}_j(\xi^m,\xi^m),$$
$$(4\xi^{m,2}-3\xi^m-\xi^{m,1},\xi^{m,1})_{I_j}=(4\eta^{m,2}-3\eta^m-\eta^{m,1},\xi^{m,1})_{I_j}$$
$$+\Delta t\mathcal{H}_j(E^{m,1}n^{m,1}-E_h^{m,1}n_h^{m,1},\xi^{m,1})+\Delta t\mathcal{L}_j(\xi^{m,1},\xi^{m,1}),$$
$$(\frac{3}{2}\xi^{m+1}-\frac{1}{2}\xi^m-\xi^{m,2},4\xi^{m,2})_{I_j}=(\frac{3}{2}\eta^{m+1}-\eta^m-\frac{1}{2}\eta^{m,2},4\xi^{m,2})_{I_j}$$
$$+\Delta t\mathcal{H}_j(E^{m,2}n^{m,2}-E_h^{m,2}n_h^{m,2},4\xi^{m,2})+\Delta t\mathcal{L}_j(\xi^{m,2},4\xi^{m,2})+(\zeta^m,4\xi^{m,2}), \tag{A.4}$$

Summing the above equations and summing over j, we get

$$\begin{aligned}3\|\xi^{m+1}\|^2-3\|\xi^m\|^2=&\{(\eta^{m,1}-\eta^m,\xi^m)+(4\eta^{m,2}-3\eta^m-\eta^{m,1},\xi^{m,1})\\&+(\frac{3}{2}\eta^{m+1}-\eta^m-\frac{1}{2}\eta^{m,2},4\xi^{m,2})+(\zeta^m,4\xi^{m,2})\}\\&+\{3(\xi^{m+1}-\xi^m,\xi^{m+1}-2\xi^{m,2}+\xi^m)\\&+(2\xi^{m,2}-\xi^{m,1}-\xi^m,2\xi^{m,2}-\xi^{m,1}-\xi^m)\}\\&+\Delta t\{\mathcal{H}(E^mn^m-E_h^mn_h^m,\xi^m)+\mathcal{L}(\xi^m,\xi^m)\\&+\mathcal{H}(E^{m,1}n^{m,1}-E_h^{m,1}n_h^{m,1},\xi^{m,1})+\mathcal{L}(\xi^{m,1},\xi^{m,1})\\&+\mathcal{H}(E^{m,2}n^{m,2}-E_h^{m,2}n_h^{m,2},4\xi^{m,2})+\mathcal{L}(\xi^{m,2},4\xi^{m,2})\}\\=:&T_1+T_2+T_3.\end{aligned} \tag{A.5}$$

Now we estimate T_i term by term. From the property (17) of the projection and the Schwartz inequality, we have

$$T_1\leqslant C\Delta th^{2k+2}+C\Delta t(\|\xi^{m,2}\|^2+\|\xi^{m,1}\|^2+\|\xi^m\|^2)+C(\Delta t)^7. \tag{A.6}$$

For the convenience of analysis, we denote

$$\begin{aligned}G_1u^m&=u^{m,1}-u^m,\\G_2u^m&=2u^{m,2}-u^{m,1}-u^m,\\G_3u^m&=u^{m+1}-2u^{m,2}+u^m,\end{aligned} \tag{A.7}$$

for any function u. With these notations, we rewrite T_2 as

$$\begin{aligned}T_2=&3(\xi^{m+1}-\xi^m,\xi^{m+1}-2\xi^{m,2}+\xi^m)\\&+(2\xi^{m,2}-\xi^{m,1}-\xi^m,2\xi^{m,2}-\xi^{m,1}-\xi^m)\\=&(2\xi^{m,2}-\xi^{m,1}-\xi^m,2\xi^{m,2}-\xi^{m,1}-\xi^m)\\&+3(\xi^{m+1}-2\xi^{m,2}+\xi^m,\xi^{m,1}-\xi^m)\\&+3(\xi^{m+1}-2\xi^{m,2}+\xi^m,2\xi^{m,2}-\xi^{m,1}-\xi^m)\\&+3(\xi^{m+1}-2\xi^{m,2}+\xi^m,\xi^{m+1}-2\xi^{m,2}+\xi^m)\\=&-(G_2\xi^m,G_2\xi^m)+2(G_2\xi^m,G_2\xi^m)+3(G_3\xi^m,G_1\xi^m)\end{aligned}$$

$$+ 3(G_3\xi^m, G_2\xi^m) + 3(G_3\xi^m, G_3\xi^m)$$

$$=: -\|G_2\xi^m\|^2 + S_1 + S_2 + S_3 + S_4. \tag{A.8}$$

Using the projection (14a)-(14c) and summing over j, we rewrite (A.3) as

$$\begin{aligned}(G_1\xi^m, v_1) =&(G_1\eta^m, v_1) + \Delta t\mathcal{H}(E^m n^m - E_h^m n_h^m, v_1) + \Delta t\mathcal{L}(\xi^m, v_1),\\ (G_2\xi^m, v_2) =&(G_2\eta^m, v_2) + \frac{1}{2}\Delta t\mathcal{H}(G_1(E^m n^m - E_h^m n_h^m), v_2)\\ &+ \frac{1}{2}\Delta t\mathcal{L}(G_1\xi^m, v_2),\\ (G_3\xi^m, v_3) =&(G_3\eta^m, v_3) + \frac{1}{3}\Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), v_3)\\ &+ \frac{1}{3}\Delta t\mathcal{L}(G_2\xi^m, v_3) + (\zeta^m, v_3),\end{aligned} \tag{A.9}$$

Choosing $v_2 = G_2\xi^m$ in (A.9), we rewrite S_1 as

$$\begin{aligned}S_1 =&2(G_2\xi^m, G_2\xi^m)\\ =&2(G_2\eta^m, G_2\xi^m) + \Delta t\mathcal{H}(G_1(E^m n^m - E_h^m n_h^m), G_2\xi^m)\\ &+ \Delta t\mathcal{L}(G_1\xi^{m,1}, G_2\xi^m).\end{aligned} \tag{A.10}$$

Similarly, choosing $v_3 = G_1\xi^m, G_2\xi^m, G_3\xi^m$ respectively in (A.9), we rewrite S_2, S_3 and S_4 as

$$\begin{aligned}S_2 =&3(G_3\eta^m, G_1\xi^m) + \Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_1\xi^m)\\ &+ \Delta t\mathcal{L}(G_2\xi^{m,2}, G_1\xi^m) + (\zeta^m, G_1\xi^m)\\ S_3 =&3(G_3\eta^m, G_2\xi^m) + \Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_2\xi^m)\\ &+ \Delta t\mathcal{L}(G_2\xi^{m,2}, G_2\xi^m) + (\zeta^m, G_2\xi^m)\\ S_4 =&3(G_3\eta^m, G_3\xi^m) + \Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_3\xi^m)\\ &+ \Delta t\mathcal{L}(G_2\xi^{m,2}, G_3\xi^m) + (\zeta^m, G_3\xi^m)\end{aligned} \tag{A.11}$$

We denote

$$\begin{aligned}S_1 + S_2 + S_3 =&\{2(G_2\eta^m, G_2\xi^m) + 3(G_3\eta^m, G_1\xi^m) + 3(G_3\eta^m, G_2\xi^m)\\ &+ (\zeta^m, G_1\xi^m) + (\zeta^m, G_2\xi^m)\} + \{\Delta t\mathcal{H}(G_1(E^m n^m - E_h^m n_h^m), G_2\xi^m)\\ &+ \Delta t\mathcal{L}(G_1\xi^{m,1}, G_2\xi^m) + \Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_1\xi^m)\\ &+ \Delta t\mathcal{L}(G_2\xi^{m,2}, G_1\xi^m) + \Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_2\xi^m)\\ &+ \Delta t\mathcal{L}(G_2\xi^{m,2}, G_2\xi^m)\}\\ =:&Q_1 + Q_2.\end{aligned} \tag{A.12}$$

Using the projection (17), Young's inequality and Schwartz's inequality, we get

$$\begin{aligned}Q_1 =&2(G_2\eta^m, G_2\xi^m) + 3(G_3\eta^m, G_1\xi^m) + 3(G_3\eta^m, G_2\xi^m)\\ &+ (\zeta^m, G_1\xi^m) + (\zeta^m, G_2\xi^m)\\ \leqslant&C\Delta th^{2k+2} + C\Delta t(\|\xi^{m,2}\|^2 + \|\xi^{m,1}\|^2 + \|\xi^m\|^2) + C(\Delta t)^7.\end{aligned} \tag{A.13}$$

About the first term of Q_2, by (25)-(26), we have

$$\begin{aligned}&\Delta t\mathcal{H}(G_1(E^m n^m - E_h^m n_h^m), G_2\xi^m)\\ =&\Delta t\int_I \mu G_1(E_h^m n_h^m - E^m n^m)(G_2\xi^m)_x\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\Delta t\sum_{j=1}^{N}\mu G_1(\widehat{E_h^m n_h^m}-E^m n^m)_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}}\\
=&\Delta t\int_I \mu G_1(E_h^m n_h^m-E_h^m n^m+E_h^m n^m-E^m n^m)(G_2\xi^m)_x\,\mathrm{d}x\\
&+\Delta t\sum_{j=1}^{N}\mu G_1(E_h^m\widehat{\widehat{n_h^m}}-E_h^m n^m+E_h^m n^m-E^m n^m)_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}}\\
=&\Delta t\int_I \mu G_1(E_h^m(n_h^m-n^m))(G_2\xi^m)_x\,\mathrm{d}x+\Delta t\int_I \mu G_1((E_h^m-E^m)n^m)(G_2\xi^m)_x\,\mathrm{d}x\\
&+\Delta t\sum_{j=1}^{N}\mu G_1(E_h^m(\widehat{\widehat{n_h^m}}-n^m))_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}}\\
&+\Delta t\sum_{j=1}^{N}\mu G_1((E_h^m-E^m)n^m)_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}}\\
=&\Delta t\int_I \mu G_1(E_h^m(\eta^m-\xi^m))(G_2\xi^m)_x\,\mathrm{d}x+\Delta t\int_I \mu G_1((E_h^m-E^m)n^m)(G_2\xi^m)_x\,\mathrm{d}x\\
&+\Delta t\sum_{j=1}^{N}\mu G_1(E_h^m(\widehat{\widehat{\eta^m}}-\widehat{\widehat{\xi^m}}))_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}}\\
&+\Delta t\sum_{j=1}^{N}\mu G_1((E_h^m-E^m)n^m)_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}}\\
=&(-\Delta t\int_I \mu G_1(E_h^m\xi^m)(G_2\xi^m)_x\,\mathrm{d}x-\Delta t\sum_{j=1}^{N}\mu G_1(E_h^m\widehat{\widehat{\xi^m}})_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}})\\
&+\Delta t\int_I \mu G_1(E_h^m\eta^m)(G_2\xi^m)_x\,\mathrm{d}x+\Delta t\sum_{j=1}^{N}\mu G_1(E_h^m\widehat{\widehat{\eta^m}})_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}}\\
&+(\Delta t\int_I \mu G_1((E_h^m-E^m)n^m)(G_2\xi^m)_x\,\mathrm{d}x\\
&+\Delta t\sum_{j=1}^{N}\mu G_1((E_h^m-E^m)n^m)_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}})
\end{aligned}
$$

(A.14)

$$=:V_1+V_2+V_3+V_4$$

Now, we make the a priori assumption

$$\|n^m-n_h^m\|\leqslant h. \tag{A.15}$$

We will verify the reasonableness of this a priori assumption later. The a priori assumption implies that $\|n_h^m\|_\infty\leqslant C$, $\|E_h^m\|_\infty\leqslant C_M$. To ensure the stability of the RK-UWDG scheme, we request the time step Δt satisfy the following temporal-spatial condition

$$\max\{\mu C_M,\tau\theta\}\frac{\Delta t}{h^2}\leqslant\alpha, \tag{A.16}$$

where $\alpha\leqslant 1$, the CFL number, is suitable constant independent of h and Δt. A sufficient condition will be given in the next analysis process.

Using the Young's inequality, $\|E_h^m\|_\infty \leqslant C_M$ and the inverse property (18), we get

$$\begin{aligned} V_1 =& -\Delta t\int_I \mu G_1(E_h^m\xi^m)(G_2\xi^m)_x\,\mathrm{d}x - \Delta t\sum_{j=1}^{N}\mu G_1(E_h^m\widehat{\xi^m})_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}} \\ \leqslant& \frac{1}{2\widetilde{\epsilon}}\Delta t\|G_1(E_h^m\xi^m)\|^2 + \widetilde{\epsilon}\Delta t\|(G_2\xi^m)_x\|^2 + \widetilde{\epsilon}\Delta t h^{-1}[G_2\xi^m]_{j+\frac{1}{2}}^2 \\ \leqslant& \frac{1}{2\widetilde{\epsilon}}\Delta t\|E_h^{m,1}\xi^{m,1} - E_h^m\xi^m\|^2 + \widetilde{\epsilon}\Delta t(\|(G_2\xi^m)_x\|^2 + h^{-1}[G_2\xi^m]_{j+\frac{1}{2}}^2) \\ \leqslant& \frac{1}{\widetilde{\epsilon}}\Delta t(\|E_h^{m,1}\xi^{m,1}\|^2 + \|E_h^m\xi^m\|^2) + \widetilde{\epsilon}\Delta t(\|(G_2\xi^m)_x\|^2 + h^{-1}[G_2\xi^m]_{j+\frac{1}{2}}^2) \\ \leqslant& \frac{C_M}{\widetilde{\epsilon}}\Delta t(\|\xi^{m,1}\|^2 + \|\xi^m\|^2) + \widetilde{\epsilon}\Delta t(\|(G_2\xi^m)_x\|^2 + h^{-1}[G_2\xi^m]_{j+\frac{1}{2}}^2). \end{aligned} \tag{A.17}$$

By the Young's inequality, $\|E_h^m\|_\infty \leqslant C_M$ and the projection (17), we have

$$\begin{aligned} V_2 =& \Delta t\int_I \mu G_1(E_h^m\eta^m)(G_2\xi^m)_x\,\mathrm{d}x \\ \leqslant& C\Delta t h^{2k+2} + \widetilde{\epsilon}\Delta t\|(G_2\xi^m)_x\|. \end{aligned} \tag{A.18}$$

Using the Young's inequality, $\|E_h^m\|_\infty \leqslant C_M$, the inverse property (18) and the projection(17), we get

$$\begin{aligned} V_3 =& \Delta t\sum_{j=1}^{N}\mu G_1(E_h^m\widehat{\eta^m})_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}} \\ \leqslant& C\Delta t h^{2k+2} + \widetilde{\epsilon}\Delta t h^{-1}\sum_{j=1}^{N}[G_2\xi^m]_{j+\frac{1}{2}}^2. \end{aligned} \tag{A.19}$$

Similarly as the estimate of the term T_{24} in the semi-UWDG scheme, we have

$$\begin{aligned} V_4 =& (\Delta t\int_I \mu G_1((E_h^m - E^m)n^m)(G_2\xi^m)_x\,\mathrm{d}x \\ &+ \Delta t\sum_{j=1}^{N}\mu G_1((E_h^m - E^m)n^m)_{j+\frac{1}{2}}[G_2\xi^m]_{j+\frac{1}{2}}) \\ \leqslant& C\Delta t h^{2k+2} + C\Delta t(\|\xi^{m,1}\|^2 + \|\xi^m\|^2 + \|G_2\xi^m\|^2). \end{aligned} \tag{A.20}$$

Substituting (A.17)-(A.19) and (A.20) into (A.14), by (71), we obtain

$$\begin{aligned} &\Delta t\mathcal{H}(G_1(E^m n^m - E_h^m n_h^m), G_2\xi^m) \\ \leqslant& C\Delta t h^{2k+2} + C\Delta t(\|\xi^{m,1}\|^2 + \|\xi^m\|^2 + \|G_2\xi^m\|^2) \\ &+ \widetilde{\epsilon}\Delta t(\|(G_2\xi^m)_x\|^2 + h^{-1}\sum_{j=1}^{N}[G_2\xi^m]_{j+\frac{1}{2}}^2) \\ \leqslant& C\Delta t h^{2k+2} + C\Delta t(\|\xi^{m,1}\|^2 + \|\xi^m\|^2 + \|G_2\xi^m\|^2) \\ &- \frac{2\widetilde{\epsilon}}{\tau\theta}\Delta t\mathcal{L}(G_2\xi^m, G_2\xi^m). \end{aligned} \tag{A.21}$$

Similarly, we have

$$\begin{aligned} &\Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_1\xi^m) \\ \leqslant& C\Delta t h^{2k+2} + C\Delta t(\|G_1\xi^m\|^2 + \|\xi^{m,1}\|^2 + \|\xi^m\|^2 + \|\xi^{m,2}\|^2) \end{aligned}$$

$$
\begin{aligned}
&- \frac{2\widetilde{\epsilon}}{\tau\theta}\Delta t\mathcal{L}(G_1\xi^m, G_1\xi^m),\\
&\Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_2\xi^m)\\
\leqslant & C\Delta t h^{2k+2} + C\Delta t(\|G_2\xi^m\|^2 + \|\xi^{m,1}\|^2 + \|\xi^m\|^2 + \|\xi^{m,2}\|^2)\\
&- \frac{2\widetilde{\epsilon}}{\tau\theta}\Delta t\mathcal{L}(G_2\xi^m, G_2\xi^m),\\
&\Delta t\mathcal{H}((E^m n^m - E_h^m n_h^m), \xi^m) \leqslant C\Delta t h^{2k+2} + C\Delta t\|\xi^m\|^2 - \frac{2\widetilde{\epsilon}}{\tau\theta}\Delta t\mathcal{L}(\xi^m, \xi^m),\\
&\Delta t\mathcal{H}((E^{m,1} n^{m,1} - E_h^{m,1} n_h^{m,1}), \xi^{m,1}) \leqslant C\Delta t h^{2k+2} + C\Delta t\|\xi^{m,1}\|^2\\
&- \frac{2\widetilde{\epsilon}}{\tau\theta}\Delta t\mathcal{L}(\xi^{m,1}, \xi^{m,1}),\\
&\Delta t\mathcal{H}((E^{m,2} n^{m,2} - E_h^{m,2} n_h^{m,2}), 4\xi^{m,2}) \leqslant C\Delta t h^{2k+2} + C\Delta t\|\xi^{m,2}\|^2\\
&- \frac{2\widetilde{\epsilon}}{\tau\theta}\Delta t\mathcal{L}(\xi^{m,2}, \xi^{m,2}).
\end{aligned}
\tag{A.22}
$$

By Lemma 5.1 and (A.22), we get

$$
\begin{aligned}
Q_2 =& \Delta t\mathcal{H}(G_1(E^m n^m - E_h^m n_h^m), G_2\xi^m) + \Delta t\mathcal{L}(G_1\xi^m, G_2\xi^m)\\
&+ \Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_1\xi^m) + \Delta t\mathcal{L}(G_2\xi^m, G_1\xi^m)\\
&+ \Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_2\xi^m) + \Delta t\mathcal{L}(G_2\xi^m, G_2\xi^m)\\
\leqslant & C\Delta t h^{2k+2} + C\Delta t(\|G_1\xi^m\|^2 + \|G_2\xi^m\|^2)\\
&+ 2\Delta t\mathcal{L}(G_1\xi^m, G_2\xi^m) + (1 - \frac{4\widetilde{\epsilon}}{\tau\theta})\Delta t\mathcal{L}(G_2\xi^m, G_2\xi^m))\\
&- \frac{2\widetilde{\epsilon}}{\tau\theta}\Delta t\mathcal{L}(G_1\xi^m, G_1\xi^m).
\end{aligned}
\tag{A.23}
$$

Now, we estimate $-\frac{2\widetilde{\epsilon}}{\tau\theta}\Delta t\mathcal{L}(G_1\xi^m, G_1\xi^m)$ in (A.23). By (64), (18) and Lemma 5.1, we have

$$
\begin{aligned}
-\mathcal{L}(G_1\xi^m, G_1\xi^m) =& \tau\theta(\int_I (G_1\xi^m)_x^2 \,\mathrm{d}x + 2\sum_{j=1}^N [G_1\xi^m]_{j+\frac{1}{2}}(G_1\xi^m)^-_{x,j+\frac{1}{2}} + \lambda\sum_{j=1}^N [G_1\xi^m]^2)\\
\leqslant & \tau\theta(\int_I (G_1\xi^m)_x^2 \,\mathrm{d}x + \sum_{j=1}^N h^{-1}[G_1\xi^m]^2_{j+\frac{1}{2}} + \sum_{j=1}^N h((G_1\xi^m)^-_{x,j+\frac{1}{2}})^2\\
&+ \lambda\sum_{j=1}^N [G_1\xi^m]^2)\\
\leqslant & \tau\theta(2\int_I (G_1\xi^m)_x^2 \,\mathrm{d}x + \sum_{j=1}^N (\lambda + h^{-1})[G_1\xi^m]^2_{j+\frac{1}{2}})\\
\leqslant & \tau\theta(2\int_I (\xi^{m,1} - \xi^m)_x^2 \,\mathrm{d}x + (\lambda + h^{-1})\sum_{j=1}^N [\xi^{m,1} - \xi^m]^2)\\
\leqslant & \tau\theta(4\int_I (\xi_x^{m,1})^2 \,\mathrm{d}x + 4\int_I (\xi_x^m)^2 \,\mathrm{d}x\\
&+ 2(C_0 + 1)h^{-1}\sum_{j=1}^N [\xi^{m,1}]^2 + 2(C_0 + 1)h^{-1}\sum_{j=1}^N [\xi^m]^2)
\end{aligned}
$$

$$\leqslant \max\{4, 2C_0+2\}\tau\theta\Big(\int_I (\xi_x^{m,1})^2\,\mathrm{d}x + \int_I (\xi_x^m)^2\,\mathrm{d}x + h^{-1}\sum_{j=1}^N [\xi^{m,1}]^2 + h^{-1}\sum_{j=1}^N [\xi^m]^2\Big)$$

$$\leqslant -\max\{4, 2C_0+2\}(\mathcal{L}(\xi^{m,1},\xi^{m,1}) + \mathcal{L}(\xi^m,\xi^m)), \tag{A.24}$$

then

$$-\frac{2\widetilde{\epsilon}}{\tau\theta}\Delta t\mathcal{L}(G_1\xi^m, G_1\xi^m) \leqslant -\frac{2\widetilde{\epsilon}}{\tau\theta}\max\{4, 2C_0+2\}\Delta t(\mathcal{L}(\xi^{m,1},\xi^{m,1}) + \mathcal{L}(\xi^m,\xi^m)). \tag{A.25}$$

Substituting (A.25) into (A.23), we obtain

$$\begin{aligned} Q_2 \leqslant & C\Delta t h^{2k+2} + C\Delta t(\|G_1\xi^m\|^2 + \|G_2\xi^m\|^2) \\ & + 2\Delta t\mathcal{L}(G_1\xi^m, G_2\xi^m) + (1-\frac{4\widetilde{\epsilon}}{\tau\theta})\Delta t\mathcal{L}(G_2\xi^m, G_2\xi^m)) \\ & - \frac{2\widetilde{\epsilon}}{\tau\theta}\max\{4, 2C_0+2\}\Delta t(\mathcal{L}(\xi^{m,1},\xi^{m,1}) + \mathcal{L}(\xi^m,\xi^m)). \end{aligned} \tag{A.26}$$

Summing Q_1 and Q_2, we get

$$\begin{aligned} S_1+S_2+S_3 \leqslant & C\Delta t h^{2k+2} + C\Delta t(\|\xi^{m,2}\|^2 + \|\xi^{m,1}\|^2 + \|\xi^m\|^2) + C(\Delta t)^7 \\ & + 2\Delta t\mathcal{L}(G_1\xi^m, G_2\xi^m) + (1-\frac{4\widetilde{\epsilon}}{\tau\theta})\Delta t\mathcal{L}(G_2\xi^m, G_2\xi^m)) \\ & - \frac{2\widetilde{\epsilon}}{\tau\theta}\max\{4, 2C_0+2\}\Delta t(\mathcal{L}(\xi^{m,1},\xi^{m,1}) + \mathcal{L}(\xi^m,\xi^m)). \end{aligned} \tag{A.27}$$

Denote

$$\begin{aligned} W_1 = & 2\Delta t\mathcal{L}(G_1\xi^m, G_2\xi^m) + (1-\frac{4\widetilde{\epsilon}}{\tau\theta})\Delta t\mathcal{L}(G_2\xi^m, G_2\xi^m)) \\ & - \frac{2\widetilde{\epsilon}}{\tau\theta}\max\{4, 2C_0+2\}\Delta t(\mathcal{L}(\xi^{m,1},\xi^{m,1}) + \mathcal{L}(\xi^m,\xi^m)), \end{aligned} \tag{A.28}$$

we have

$$\begin{aligned} & S_1+S_2+S_3 \\ \leqslant & C\Delta t h^{2k+2} + C\Delta t(\|\xi^{m,2}\|^2 + \|\xi^{m,1}\|^2 + \|\xi^m\|^2) + C(\Delta t)^7 + W_1. \end{aligned} \tag{A.29}$$

Now we estimate S_4, which need a more analysis than the formers. We denote

$$\begin{aligned} S_4 = & (3(G_3\eta^m, G_3\xi^m) + (\zeta^m, G_3\xi^m)) + \Delta t\mathcal{H}(G_2(E^m n^m - E_h^m n_h^m), G_3\xi^m) \\ & + \Delta t\mathcal{L}(G_2\xi^{m,2}, G_3\xi^m) \\ =: & R_1 + R_2 + R_3. \end{aligned} \tag{A.30}$$

By Schwartz's inequality and (17), we have

$$\begin{aligned} R_1 = & 3(G_3\eta^m, G_3\xi^m) + (\zeta^m, G_3\xi^m) \\ \leqslant & C(\Delta t h^{k+1} + (\Delta t)^4)\|G_3\xi^m\|. \end{aligned} \tag{A.31}$$

By Schwartz's inequality, the inverse property (18), $\|E_h^m\|_\infty \leqslant C_M$ and T_{24} in the semi-UWDG scheme, we get

$$R_2 = \Big(-\Delta t\int_I \mu G_2(E_h^m\xi^m)(G_3\xi^m)_x\,\mathrm{d}x - \Delta t\sum_{j=1}^N \mu G_2(E_h^m\widehat{\widehat{\xi^m}})_{j+\frac12}[G_3\xi^m]_{j+\frac12}\Big)$$

$$
\begin{aligned}
&+ \Delta t \int_I \mu G_2(E_h^m \eta^m)(G_3\xi^m)_x \,\mathrm{d}x + \Delta t \sum_{j=1}^N \mu G_2(E_h^m \widehat{\eta^m})_{j+\frac12}[G_3\xi^m]_{j+\frac12} \\
&+ (\Delta t \int_I \mu G_2((E_h^m - E^m)n^m)(G_3\xi^m)_x \,\mathrm{d}x \\
&+ \Delta t \sum_{j=1}^N \mu G_2((E_h^m - E^m)n^m)_{j+\frac12}[G_3\xi^m]_{j+\frac12}) \\
\leqslant&(\Delta t h^{-1}\mu(2\|G_2(E_h^m\xi^m)\| + \|G_2(E_h^m\eta^m)\| + h^{\frac12}\|G_2(E_h^m\eta^m)\|_{\Gamma_h}) \\
&+ 2\mu\|G_2((E_h^m - E^m)n^m)_x\|)\|G_3\xi^m\| \\
\leqslant&(\Delta t h^{-1}\mu(4\|E_h^m\xi^{m,2}\| + 2\|E_h^m\xi^{m,1}\| + 2\|E_h^m\xi^m\| + 2\|E_h^m\eta^{m,2}\| \\
&+ \|E_h^m\eta^{m,1}\| + \|E_h^m\eta^m\| + 2h^{\frac12}\|E_h^m\eta^{m,2}\|_{\Gamma_h} + h^{\frac12}\|E_h^m\eta^{m,1}\|_{\Gamma_h} \\
&+ h^{\frac12}\|E_h^m\eta^m\|_{\Gamma_h}) + 4\mu\|((E_h^{m,2} - E^{m,2})n^{m,2})_x\| \\
&+ 2\mu\|((E_h^{m,1} - E^{m,1})n^{m,1})_x\| + 2\mu\|((E_h^m - E^m)n^m)_x\|)\|G_3\xi^m\| \\
\leqslant&\mu(C_M\Delta t h^{-1}(h^{k+1} + \|\xi^{m,2}\| + \|\xi^{m,1}\| + \|\xi^m\|) \\
&+ C_M\Delta t(h^{k+1} + \|\xi^{m,2}\| + \|\xi^{m,1}\| + \|\xi^m\|))\|G_3\xi^m\|. \qquad \text{(A.32)}
\end{aligned}
$$

By (A.16) and the inverse property (18), we have

$$
\begin{aligned}
R_3 =& -\Delta t\tau\theta(\int_I (G_2\xi^m)_x(G_3\xi^m)_x \,\mathrm{d}x \\
&+ \sum_{j=1}^N ([G_2\xi^m](G_3\xi^m)_x^- + [G_3\xi^m](G_2\xi^m)_x^- + \lambda[G_2\xi^m][G_3\xi^m])_{j+\frac12}) \\
\leqslant&((3 + C_0)\tau\theta\Delta t h^{-2}\|G_2\xi^m\|)\|G_3\xi^m\| \\
\leqslant&(3 + C_0)\alpha\|G_2\xi^m\|\|G_3\xi^m\|. \qquad \text{(A.33)}
\end{aligned}
$$

Summing R_1, R_2 and R_3, we get

$$
\begin{aligned}
S_4 \leqslant&(C\Delta t h^{k+1} + C(\Delta t)^4 + \mu C_M\Delta t h^{-1}(h^{k+1} + \|\xi^{m,2}\| + \|\xi^{m,1}\| + \|\xi^m\|) \\
&+ \mu C_M\Delta t(h^{k+1} + \|\xi^{m,2}\| + \|\xi^{m,1}\| + \|\xi^m\|) + (3 + C_0)\alpha\|G_2\xi^m\|)\|\|G_3\xi^m\| \\
\leqslant&(C\Delta t h^{k+1} + C(\Delta t)^4 + (3 + C_0)\alpha\|G_2\xi^m\| + \mu C_M\Delta t h^{-1}(h^{k+1} + \|\xi^{m,2}\| \\
&+ \|\xi^{m,1}\| + \|\xi^m\|) + C\Delta t(\|\xi^{m,2}\| + \|\xi^{m,1}\| + \|\xi^m\|))\|G_3\xi^m\|. \qquad \text{(A.34)}
\end{aligned}
$$

Since $S_4 = (G_3\xi^m, G_3\xi^m) = \|G_3\xi^m\|^2$, we can eliminate $\|G_3\xi^m\|$ at the right side of (A.34), then we get

$$
\begin{aligned}
\|G_3\xi^m\| \leqslant&(C\Delta t h^{k+1} + C(\Delta t)^4 + (3 + C_0)\alpha\|G_2\xi^m\| \\
&+ \mu C_M\Delta t h^{-1}(h^{k+1} + \|\xi^{m,2}\| + \|\xi^{m,1}\| + \|\xi^m\|) \\
&+ C\Delta t(\|\xi^{m,2}\| + \|\xi^{m,1}\| + \|\xi^m\|)). \qquad \text{(A.35)}
\end{aligned}
$$

Squaring both sides of (A.35) and using (A.16), we have

$$
\begin{aligned}
S_4 \leqslant&4(C\Delta t h^{k+1} + C(\Delta t)^4 \\
&+ C\Delta t(\|\xi^{m,2}\| + \|\xi^{m,1}\| + \|\xi^m\|))^2 + 2(3 + C_0)^2\alpha^2\|G_2\xi^m\|^2 \\
&+ 4\mu^2 C_M^2\Delta t^2 h^{-2}(h^{2k+2} + \|\xi^{m,2}\|^2 + \|\xi^{m,1}\|^2 + \|\xi^m\|^2) \\
\leqslant&4(C\Delta t h^{k+1} + C(\Delta t)^4
\end{aligned}
$$

$$+ C\Delta t(\|\xi^{m,2}\| + \|\xi^{m,1}\| + \|\xi^m\|))^2 + 2(3+C_0)^2\alpha^2\|G_2\xi^m\|^2$$
$$+ 4\mu C_M\Delta t\alpha(h^{2k+2} + \|\xi^{m,2}\|^2 + \|\xi^{m,1}\|^2 + \|\xi^m\|^2)$$
$$\leqslant C\Delta t h^{2k+2} + C(\Delta t)^8 + 2(3+C_0)^2\alpha^2\|G_2\xi^m\|^2$$
$$+ C\Delta t(\|\xi^{m,2}\|^2 + \|\xi^{m,1}\|^2 + \|\xi^m\|^2). \tag{A.36}$$

Substituting (A.29) and (A.36) into (A.8), we get

$$T_2 \leqslant C\Delta t h^{2k+2} + C\Delta t(\|\xi^{m,2}\|^2 + \|\xi^{m,1}\|^2 + \|\xi^m\|^2) + C(\Delta t)^7 + W_1$$
$$- (1 - 2(3+C_0)^2\alpha^2)\|G_2\xi^m\|^2. \tag{A.37}$$

We estimate T_3 similarly as Q_2, by Lemma 5.3, we have

$$T_3 = \Delta t\{\mathcal{H}(E^m n^m - E_h^m n_h^m, \xi^m) + \mathcal{L}(\xi^m, \xi^m) + \mathcal{H}(E^{m,1}n^{m,1} - E_h^{m,1}n_h^{m,1}, \xi^{m,1})$$
$$+ \mathcal{L}(\xi^{m,1}, \xi^{m,1}) + \mathcal{H}(E^{m,2}n^{m,2} - E_h^{m,2}n_h^{m,2}, 4\xi^{m,2}) + \mathcal{L}(\xi^{m,2}, 4\xi^{m,2})\}$$
$$\leqslant C\Delta t h^{2k+2} + C\Delta t(\|\xi^m\|^2 + \|\xi^{m,1}\|^2 + \|\xi^{m,2}\|^2)$$
$$+ (1 - \frac{2\widetilde{\epsilon}}{\tau\theta})(\mathcal{L}(\xi^m, \xi^m) + \mathcal{L}(\xi^{m,1}, \xi^{m,1}) + \mathcal{L}(\xi^{m,2}, \xi^{m,2})) \tag{A.38}$$

By Lemma 5.2, let $\widetilde{\epsilon} \leqslant \frac{\tau\theta}{2}$, since $\mathcal{L}(\xi^{m,2}, \xi^{m,2}) \leqslant 0$ we get

$$T_3 \leqslant C\Delta t h^{2k+2} + C\Delta t(\|\xi^m\|^2 + \|\xi^{m,1}\|^2 + \|\xi^{m,2}\|^2)$$
$$+ (1 - \frac{2\widetilde{\epsilon}}{\tau\theta})\Delta t(\mathcal{L}(\xi^m, \xi^m) + \mathcal{L}(\xi^{m,1}, \xi^{m,1})) \tag{A.39}$$

Denote

$$W_2 = (1 - \frac{2\widetilde{\epsilon}}{\tau\theta})\Delta t(\mathcal{L}(\xi^m, \xi^m) + \mathcal{L}(\xi^{m,1}, \xi^{m,1})), \tag{A.40}$$

we have

$$T_3 \leqslant C\Delta t h^{2k+2} + C\Delta t(\|\xi^m\|^2 + \|\xi^{m,1}\|^2 + \|\xi^{m,2}\|^2) + W_2. \tag{A.41}$$

Summing T_2 and T_3 together, we get

$$T_2 + T_3 \leqslant C\Delta t h^{2k+2} + C(\Delta t)^7 + C\Delta t(\|\xi^m\|^2 + \|\xi^{m,1}\|^2 + \|\xi^{m,2}\|^2) + W_1 + W_2$$
$$- (1 - 2(3+C_0)^2\alpha^2)\|G_2\xi^m\|^2. \tag{A.42}$$

Now, we estimate $W_1 + W_2$. By (A.28) and (A.40), we have

$$W_1 + W_2 = \Delta t(2\mathcal{L}(G_1\xi^m, G_2\xi^m) + (1 - \frac{4\widetilde{\epsilon}}{\tau\theta})\mathcal{L}(G_2\xi^m, G_2\xi^m)$$
$$+ (1 - \frac{2\widetilde{\epsilon}}{\tau\theta}(1 + \max\{4, 2C_0 + 2\}))(\mathcal{L}(\xi^{m,1}, \xi^{m,1}) + \mathcal{L}(\xi^m, \xi^m))), \tag{A.43}$$

where $\widetilde{\epsilon}$ is a very small constant. We denote

$$C_\epsilon^2 = 1 - \frac{2\widetilde{\epsilon}}{\tau\theta}(1 + \max\{4, 2C_0 + 2\}). \tag{A.44}$$

By Lemma 5.2 and linearity of the operator, we have

$$W_1 + W_2 = \Delta t(2\mathcal{L}(G_1\xi^m, G_2\xi^m) + (1 - \frac{4\widetilde{\epsilon}}{\tau\theta})\mathcal{L}(G_2\xi^m, G_2\xi^m) + C_\epsilon^2(\mathcal{L}(\xi^{m,1}, \xi^{m,1})$$
$$+ \mathcal{L}(\xi^m, \xi^m)))$$
$$= \Delta t(2\mathcal{L}(\xi^{m,1} - \xi^m, G_2\xi^m) + (1 - \frac{4\widetilde{\epsilon}}{\tau\theta})\mathcal{L}(G_2\xi^m, G_2\xi^m) + C_\epsilon^2(\mathcal{L}(\xi^{m,1}, \xi^{m,1})$$
$$+ \mathcal{L}(\xi^m, \xi^m)))$$

$$\begin{aligned}
=&\Delta t(2\mathcal{L}(\xi^{m,1},G_2\xi^m)-2\mathcal{L}(\xi^m,G_2\xi^m)+(1-\frac{4\widetilde{\epsilon}}{\tau\theta})\mathcal{L}(G_2\xi^m,G_2\xi^m)\\
&+C_\epsilon^2(\mathcal{L}(\xi^{m,1},\xi^{m,1})+\mathcal{L}(\xi^m,\xi^m)))\\
=&\Delta t(\mathcal{L}(C_\epsilon\xi^{m,1}+\frac{1}{C_\epsilon}G_2\xi^m,C_\epsilon\xi^{m,1}+\frac{1}{C_\epsilon}G_2\xi^m)\\
&+\mathcal{L}(C_\epsilon\xi^m-\frac{1}{C_\epsilon}G_2\xi^m,C_\epsilon\xi^m-\frac{1}{C_\epsilon}G_2\xi^m)\\
&+(1-\frac{4\widetilde{\epsilon}}{\tau\theta}-\frac{2}{C_\epsilon^2})\mathcal{L}(G_2\xi^m,G_2\xi^m)).
\end{aligned} \tag{A.45}$$

Since $\widetilde{\epsilon}$ is small enough to approach 0, C_ϵ^2 is close to 1, then $1-\frac{4\widetilde{\epsilon}}{\tau\theta}-\frac{2}{C_\epsilon^2}\leqslant 0$. By Lemma 5.2, we get

$$\begin{aligned}
W_1+W_2\leqslant&(1-\frac{4\widetilde{\epsilon}}{\tau\theta}-\frac{2}{C_\epsilon^2})\Delta t\mathcal{L}(G_2\xi^m,G_2\xi^m)\\
\leqslant&-(\frac{4\widetilde{\epsilon}}{\tau\theta}+\frac{2}{C_\epsilon^2}-1)\Delta t\mathcal{L}(G_2\xi^m,G_2\xi^m).
\end{aligned} \tag{A.46}$$

By (64) and the inverse property (18), we have

$$\begin{aligned}
&-\Delta t\mathcal{L}(G_2\xi^m,G_2\xi^m)\\
&=\Delta t\tau\theta(\int_I(G_2\xi^m)_x^2\,\mathrm{d}x+2\sum_{j=1}^N[G_2\xi^m]_{j+\frac12}(G_2\xi^m)^-_{x,j+\frac12}+\lambda\sum_{j=1}^N[G_2\xi^m]^2)\\
=&\Delta t\tau\theta(\int_I(G_2\xi^m)_x^2\,\mathrm{d}x+2\sum_{j=1}^N[G_2\xi^m]_{j+\frac12}(G_2\xi^m)^-_{x,j+\frac12}+\frac{C_0}{h}\sum_{j=1}^N[G_2\xi^m]^2)\\
\leqslant&(3+C_0)\tau\theta\Delta th^{-2}\|G_2\xi^m\|^2.
\end{aligned} \tag{A.47}$$

By (A.16), we have

$$\begin{aligned}
W_1+W_2\leqslant&(\frac{2}{C_\epsilon^2}+\frac{4\widetilde{\epsilon}}{\tau\theta}-1)(3+C_0)\tau\theta\Delta th^{-2}\|G_2\xi^m\|^2\\
\leqslant&(\frac{2}{C_\epsilon^2}+\frac{4\widetilde{\epsilon}}{\tau\theta}-1)(3+C_0)\alpha\|G_2\xi^m\|^2.
\end{aligned} \tag{A.48}$$

Then we get

$$\begin{aligned}
T_2+T_3\leqslant&C\Delta th^{2k+2}+C(\Delta t)^7+C\Delta t(\|\xi^m\|^2+\|\xi^{m,1}\|^2+\|\xi^{m,2}\|^2)\\
&-(1-(\frac{2}{C_\epsilon^2}+\frac{4\widetilde{\epsilon}}{\tau\theta}-1)(3+C_0)\alpha-2(3+C_0)^2\alpha^2)\|G_2\xi^m\|^2.
\end{aligned} \tag{A.49}$$

Substituting (A.6) and (A.49) into (A.5), we obtain

$$\begin{aligned}
3\|\xi^{m+1}\|^2-3\|\xi^m\|^2\leqslant&C\Delta t^{2k+2}+C\Delta t(\|\xi^{m,2}\|^2+\|\xi^{m,1}\|^2+\|\xi^m\|^2)+C(\Delta t)^7\\
&-(1-(\frac{2}{C_\epsilon^2}+\frac{4\widetilde{\epsilon}}{\tau\theta}-1)(3+C_0)\alpha-2(3+C_0)^2\alpha^2)\|G_2\xi^m\|^2.
\end{aligned} \tag{A.50}$$

Since $\alpha^2\leqslant\alpha$, it yields that

$$\begin{aligned}
3\|\xi^{m+1}\|^2-3\|\xi^m\|^2\leqslant&C\Delta t^{2k+2}+C\Delta t(\|\xi^{m,2}\|^2+\|\xi^{m,1}\|^2+\|\xi^m\|^2)+C(\Delta t)^7\\
&-(1-(\frac{2}{C_\epsilon^2}+\frac{4\widetilde{\epsilon}}{\tau\theta}-1)(3+C_0)\alpha^2-2(3+C_0)^2\alpha^2)\|G_2\xi^m\|^2.
\end{aligned} \tag{A.51}$$

Taking α, such that $((\frac{2}{C_\epsilon^2}+\frac{4\widetilde{\epsilon}}{\tau\theta}-1)(3+C_0)+2(3+C_0)^2)\alpha^2\leqslant 1$, we have

(A.52)
$$3\|\xi^{m+1}\|^2-3\|\xi^m\|^2\leqslant C\Delta t^{2k+2}+C\Delta t(\|\xi^{m,2}\|^2+\|\xi^{m,1}\|^2+\|\xi^m\|^2)+C(\Delta t)^7.$$

Similar argument as the result of (A.52), we get

$$\|\xi^{m,l}\|^2\leqslant C\|\xi^m\|^2+C\Delta th^{2k+2}+C(\Delta t)^7,\quad l=1,2,3. \tag{A.53}$$

Combining (A.52) and (A.53), using the discrete Gronwall's inequality, we obtain

$$\|\xi^M\|\leqslant\|\xi^0\|+Ch^{k+1}+C(\Delta t)^3. \tag{A.54}$$

To complete the proof, let us verify the a-priori assumption (A.15). For $m=0$, we choose n_h^0 as the projection of n^0, so obviously, the assumption (A.15) holds. If (A.15) holds for $m=1,...,M-1$, then for $m=M$, we can get (A.54), i.e., (A.15) holds also for $m=M$.

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School of Mathematics, Shandong University, Jinan, 250100, China

E-mail: 202211828@mail.sdu.edu.cn and fzgao@sdu.edu.cn and yxliu@sdu.edu.cn