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A SPLITTING EXTRAPOLATION BASED ON ISO-PARAMETRIC QUADRATIC FINITE ELEMENTS FOR SOLVING SECOND ORDER NONLINEAR PARABOLIC EQUATIONS WITH CURVED BOUNDARIES

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Abstract. This manuscript proposes, analyzes, and illustrates an iso-parametric finite element splitting extrapolation method for accurately and efficiently solving the second-order semi-linear and quasi-linear parabolic equations with curved boundaries. The design of multiple grid size parameters is based on an appropriate domain decomposition of the original problem domain and iso-parametric mapping, hence provides more flexibility to form the grid and constructs the extrapolation schemes. To reach the same level of accuracy of a globally refined grid (or even better accuracy) with less expenses, we only need to solve a group of smaller discrete problems on a set of locally refined grids, instead of solving a much larger discrete problem on the globally refined grid. To develop such an accurate and efficient scheme, multi-parameter expansions for the semi-discrete and fully discrete iso-parametric finite element errors are first proved. Then the extrapolation idea can be utilized to construct the splitting extrapolation schemes based on the designed multiple grid size parameters. A posterior error estimates are provided for the splitting extrapolation solutions. Numerical examples are also provided to illustrate the obvious accuracy improvement from the splitting extrapolation schemes.

Key words. Splitting extrapolation, asymptotic expansion, domain decomposition, parallel algorithm

1. Introduction

The extrapolation techniques for approximations with higher order of accuracy have been developed for a long time. Particularly, Richardson extrapolation and its variations, which are still a popular extrapolation technique, have been developed for various types of methods and problems, including the finite difference methods [18, 19, 20, 47, 59, 66], the finite element methods [5, 17, 38, 55], the numerical integrations and integral equations [16, 24, 39, 50], data science [2, 22], and others [3, 4, 11, 12, 14, 23, 34, 44, 49, 51, 65]. There also exist many other types of extrapolation methods, see [1, 6, 7, 13, 15, 21, 29, 30, 32, 33, 45, 53, 56, 61, 63] and references therein. Moreover, extrapolation methods have also been applied in multiple fields [27, 31, 52, 57, 58, 60, 64].

The splitting extrapolation method was originally proposed by Q. Lin and T. Lü [37]. Then it was extended to various numerical methods [9, 28, 42, 46, 48, 62]. Two monographs [36, 43] were also published to summarize the early works of splitting extrapolation. Particularly, the splitting extrapolation was developed for improving the accuracy and efficiency of the finite element methods [8, 25, 26, 35, 40, 41]. The main advantages of the splitting extrapolation method include the high accuracy with only piecewise smoothness required, the less computational complexity than Richardson extrapolation, the flexibility for meshing based on appropriate domain decomposition, and the natural parallelism.

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In this manuscript, we will propose a finite element splitting extrapolation algorithm for accurately and efficiently solving the second-order nonlinear parabolic equations with curved boundaries. By introducing appropriate domain decomposition, multiple grid parameters, and local grid refinement strategies for the regular d-quadratic iso-parametric finite element method, the splitting extrapolation algorithm significantly improves both the accuracy and efficiency. We will establish multi-parameter error expansions for both semi-discrete and fully discrete finite element solution errors. Then we can construct the finite element splitting extrapolation algorithm while providing a posteriori error estimates. Numerical experiments will also be provided to validate the proposed method.

This manuscript is organized as follows: Section 2 introduces the domain decomposition, the iso-parametric mapping, the weak formulation, the semi-discrete finite element scheme, and the fully-discrete finite element scheme. Section 3 and Section 4 prove the multi-parameter asymptotic error expansion for the semi-discrete solutions and the fully-discrete solutions, respectively. Based on these theoretical foundations, Section 5 constructs the splitting extrapolation algorithm. Section 6 presents a posteriori error estimates, followed by Section 7 which validates the proposed method through numerical experiments. Finally, Section 8 presents the conclusions.

2. Second order nonlinear parabolic equation and d-quadratic iso-parame -tric transform

In this section, we will follow the framework in [25] to present the iso-parametric mapping, the weak formulation of the original problem, the semi-discrete iso-parametric finite element scheme, and the fully discrete iso-parametric finite element scheme. Note that the purpose of d-quadratic iso-parametric mapping is to handle the curved boundaries with high accuracy, as one of the preparations to reach the high accuracy of the splitting extrapolation method.

Consider the semi-linear and quasi-linear second order parabolic equations:

$$(1)\quad \begin{cases} u_t - \sum\limits_{i,j=1}^{d} D_i(a_{ij}(t,x)D_j u) + qu = f(t,x,u) \ , & \text{on} \quad Q_T = [0,T]\times\Omega, \\ u = 0 \ , & \text{on} \quad \Sigma_T = [0,T]\times\partial\Omega, \\ u(0,x) = u_0(x) \ , & \text{on} \quad \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d (d = 2,3)$, $a_{ij}(t,x)$, $q(t,x) \in L^\infty(Q_T)$, $x = (x_1, \cdots, x_d)$ and $D_i = \frac{\partial}{\partial x_i}$.

$$(2)\quad \begin{cases} u_t - \sum\limits_{i,j=1}^{d} D_i(a_{ij}(t,x,u)D_j u) = f(t,x,u) \ , & \text{on} \quad Q_T = [0,T]\times\Omega, \\ u = 0 \ , & \text{on} \quad \Sigma_T = [0,T]\times\partial\Omega, \\ u(0,x) = u_0(x) \ , & \text{on} \quad \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d (d = 2,3)$, $q(t,x) \in L^\infty(Q_T)$, $x = (x_1, \cdots, x_d)$ and $D_i = \frac{\partial}{\partial x_i}$.

Based on the idea of domain decomposition, we construct a group of non-overlapping subdomains whose union is the closure of the original problem domain: $\bar{\Omega} = \bigcup\limits_{k=1}^{m} \bar{\Omega}_k$. Using the regular d-quadratic iso-parametric mapping [10, 25, 41], there exist the translated unit cubes $\hat{\Omega}_k (k = 1, \cdots, m) \subset \mathbb{R}^d$ and one-to-one d-quadratic iso-parametric mappings $\Psi_k : \Omega_k \to \hat{\Omega}_k$ where $\{\Psi_k^{-1}\}$ are sufficiently smooth. Consider an open set $\hat{\Omega}$ with $\bar{\hat{\Omega}} = \bigcup_{k=1}^{m} \bar{\hat{\Omega}}_k$. Define $\hat{\Im}_k^h$ $(k = 1, \cdots, m)$ to be a uniform cuboid partition on $\hat{\Omega}_k$ with appropriately designed grid parameters $\hat{h}_{kj}$ $(j = 1, \cdots, d)$ so that $\hat{\Im}^h = \bigcup_{k=1}^{m} \hat{\Im}_k^h$ can form a piecewise uniform cuboid partition on the whole domain $\hat{\Omega}$ without hanging nodes. Therefore, there may be only l $(l < md)$ independent grid parameters such that there are no hanging nodes on the boundaries of $\hat{\Omega}_k (k = 1, \cdots, m)$. Meanwhile, we choose a temporal step size τ. Then we have $l + 1$ independent grid parameters, denoted by $\hat{h}_1, \cdots, \hat{h}_{l+1}$, with $\hat{h}_{l+1} = \tau$.

By the d-quadratic iso-parametric mapping, the original nonlinear parabolic equations (1) and (2) are converted to the following problems.

$$
(3) \quad \begin{cases} \hat{u}_t - \sum\limits_{i,j=1}^{d} \hat{D}_i(\hat{a}_{ij}(t,\hat{x})\hat{D}_j\hat{u}) + \hat{q}\hat{u} = \hat{f}(t,\hat{x},\hat{u}) \ , & \text{on} \quad \hat{Q}_T = [0,T] \times \hat{\Omega}, \\ \hat{u} = 0 \ , & \text{on} \quad \hat{\Sigma}_T = [0,T] \times \partial\hat{\Omega}, \\ \hat{u}(0,\hat{x}) = \hat{u}_0(\hat{x}) \ , & \text{on} \quad \hat{\Omega}, \end{cases}
$$

$$
(4) \quad \begin{cases} \hat{u}_t - \sum\limits_{i,j=1}^{d} \hat{D}_i(\hat{a}_{ij}(t,\hat{x},\hat{u})\hat{D}_j\hat{u}) = \hat{f}(t,\hat{x},\hat{u}) \ , & \text{on} \quad \hat{Q}_T = [0,T] \times \hat{\Omega}, \\ \hat{u} = 0 \ , & \text{on} \quad \hat{\Sigma}_T = [0,T] \times \partial\hat{\Omega}, \\ \hat{u}(0,\hat{x}) = \hat{u}_0(\hat{x}) \ , & \text{on} \quad \hat{\Omega}. \end{cases}
$$

We also define the following notations:

$$
L^p(0,T;W_p^k(\hat{\Omega})) := \{\hat{u}(t,\cdot)\colon [0,T] \to W_p^k(\hat{\Omega})\} \text{ with norm } \|\hat{u}\| := (\int_0^T \|\hat{u}(t,\cdot)\|_{k,p,\hat{\Omega}}^p \,\mathrm{d}t)^{\frac{1}{p}},
$$

$$
H^m(0,T;B) := \{\hat{u}(t,\cdot)\colon [0,T] \to B\colon \frac{\partial^i \hat{u}}{\partial t^i} \in L^2(0,T;B), i = 0,\cdots,m\},
$$

$$
C^k(0,T;B) := \{\hat{u}(t,\cdot)\colon [0,T] \to B\colon \hat{u}(t) \text{ has up to } k^{th} \text{ order continous derivative}\},
$$

$$
\hat{h} := (\hat{h}_1,\cdots,\hat{h}_l),\ \hat{h}_0 := \max_{1\le i\le l} \hat{h}_i,\ \hat{h}_{00} := \max_{1\le i\le l+1} \hat{h}_i.
$$

Define

$$
(5) \qquad \hat{A}(t;\hat{u},\hat{v}) = \int_{\hat{\Omega}} (\sum_{i,j=1}^{d} \hat{a}_{ij}(t,\hat{x})\hat{D}_i\hat{u}\hat{D}_j\hat{v} + \hat{q}\hat{u}\hat{v})\,\mathrm{d}\hat{x},
$$

for the semi-linear problem and

$$
(6) \qquad \hat{A}(t;\hat{u},\hat{v}) = \int_{\hat{\Omega}} (\sum_{i,j=1}^{d} \hat{a}_{ij}(t,\hat{x},\hat{u})\hat{D}_i\hat{u}\hat{D}_j\hat{v}\,\mathrm{d}\hat{x},
$$

for the quasi-linear problem.

Then the weak formulation for both the semi-linear and quasi-linear problems can be obtained as follows: find $\hat{u} \in H^1(0,T;H_0^1(\hat{\Omega}))$ satisfying

$$
(7) \qquad \begin{cases} (\hat{u}_t,\hat{v}) + \hat{A}(t;\hat{u},\hat{v}) = (\hat{f}(t,\hat{x},\hat{u}),\hat{v}), \quad \forall \hat{v} \in H_0^1(\hat{\Omega}), \\ \hat{u}(0,\hat{x}) = \hat{u}_0(\hat{x}). \end{cases}
$$

Let $\hat{P}_h$ be the orthogonal projection operator mapping $L_2(\hat{\Omega})$ to $\hat{S}_0^h$, then the semi-discrete scheme for both the semi-linear and quasi-linear problems can be obtained as follows: find $\hat{u}_h \in H^1(0,T;\hat{S}_0^h)$ to satisfy

$$
\left\{\begin{array}{l}(\hat{u}_{t,h},\hat{v}) + \hat{A}(t;\hat{u}_h,\hat{v}) = (\hat{f}(t,\hat{x},\hat{u}_h),\hat{v}), \quad \forall \hat{v}\in\hat{S}_0^h,\\ \hat{u}_h(0,\hat{x}) = \hat{P}_h\hat{u}_0(\hat{x}).\end{array}\right. \tag{8}
$$

Let $\tau = \frac{T}{N}$ denote the temporal step size and $t_n = n\tau,\ n = 1,\cdots,N$. The fully discrete scheme for the semi-linear problem can be obtained by using the linearized Crank-Nicolson-Galerkin method: find $\hat{U}^n \in \hat{S}_0^h,\ n = 1,\cdots,N$ to satisfy

$$
\left\{\begin{array}{l}(\partial_t\hat{U}^n,\hat{v}) + \hat{A}(t_{n-\frac{1}{2}};\hat{U}^{n-\frac{1}{2}},\hat{v})\\ = (\hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{U}^{n-1}),\hat{v}) + \frac{1}{2}\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{U}^{n-1})(\hat{U}^n-\hat{U}^{n-1},\hat{v}), \quad \forall\hat{v}\in\hat{S}_0^h,\\ \hat{U}^0 = \hat{P}_h\hat{u}_0,\end{array}\right. \tag{9}
$$

where $\hat{U}^{n-\frac{1}{2}} = \frac{\hat{U}^n+\hat{U}^{n-1}}{2}$, $\partial_t\hat{U}^n = \frac{\hat{U}^n-\hat{U}^{n-1}}{\tau}$ and $\hat{U}^n$ is the numerical solution to (3) at time t_n.

The fully discrete scheme for the quasi-linear problem can be obtained by using the backward Euler-Galerkin method: find $\hat{U}^n \in \hat{S}_0^h,\ n = 1,\cdots,N$ to satisfy

$$
\left\{\begin{array}{l}(\partial_t\hat{U}^n,\hat{v}) + \int_{\hat{\Omega}}(\sum\limits_{i,j=1}^{d}\hat{a}_{ij}(t_n,\hat{x},\hat{U}^{n-1})\hat{D}_i\hat{U}^n\hat{D}_j\hat{v})\,\mathrm{d}\hat{x} = (\hat{f}(t_n,\hat{x},\hat{U}^{n-1}),\hat{v}), \quad \forall\hat{v}\in\hat{S}_0^h,\\ \hat{U}^0 = \hat{P}_h\hat{u}_0,\end{array}\right. \tag{10}
$$

where $\hat{U}^n$ is the numerical solution to (4) at time t_n.

3. Multi-parameter asymptotic expansion of the semi-discrete d-quadratic iso-parametric finite element error

In this section we will prove the multi-parameter asymptotic expansion of the error of the above semi-discrete finite element scheme, as a preparation for the final conclusion in Section 4.

First, we present the following two lemmas which were proved in [41].

Lemma 3.1. *Consider a linear elliptic weak form*

$$
\sum_{i,j=1}^{d}(\hat{e}_{ij}(\hat{x})\hat{D}_i\hat{w},\hat{D}_i\hat{v}),\hat{v}) + (\hat{p}\hat{w},\hat{v}) = (\hat{f},\hat{v}), \forall\hat{v}\in H_0^1(\hat{\Omega}), \tag{11}
$$

and the corresponding d-quadratic iso-parametric finite element discrete scheme

$$
\sum_{i,j=1}^{d}(\hat{e}_{ij}(\hat{x})\hat{D}_i\hat{w}_h,\hat{D}_i\hat{v}_h) + (\hat{p}\hat{w}_h,\hat{v}_h) = (\hat{f},\hat{v}_h), \forall\hat{v}_h\in\hat{S}_0^h. \tag{12}
$$

Assume that $\hat{e}_{ij},\ \hat{p}\in(\prod_{s=1}^{m}W_\infty^4(\hat{\Omega}_s))\bigcap L_\infty(\hat{\Omega})$ *and* $\hat{w}\in(\prod_{s=1}^{m}H^7(\hat{\Omega}_s))\bigcap H_0^1(\hat{\Omega})$, *then there exist functions* $\hat{\phi}_i \in (\prod_{s=1}^{m}H^r(\hat{\Omega}_s))\bigcap L^\infty(\hat{\Omega})(i = 1,\ \cdots,\ l)$ *independent of* $\hat{h}$ *such that*

$$
\hat{w}^h - \hat{w}^I = \sum_{i=1}^{l}\hat{h}_i^4\hat{\phi}_i^I + \varepsilon, \tag{13}
$$

$$
\|\varepsilon\|_{0,\infty,\hat{\Omega}} = \mathcal{O}(\hat{h}_0^{4+\alpha}|\ln\hat{h}_0|^{\frac{d-1}{d}}), \quad \alpha = min(r,2)-\frac{d}{2} > 0. \tag{14}
$$

Lemma 3.2. *If $\hat{u} \in W_p^6(e), \hat{q} \in W_\infty^4(e), \hat{\phi} \in Q_2(e)$, then*

$$\int_e \hat{q}(\hat{u}-\hat{u}^I)\hat{\phi}\,\mathrm{d}\hat{x} = \sum_{i=1}^{d} \hat{h}_{i,e}^4 \int_e [\frac{1}{480}\hat{q}\hat{\phi}\hat{D}_i^4\hat{u} - \frac{1}{45}\hat{D}_i(\hat{q}\hat{\phi})\hat{D}_i^3\hat{u}]\,\mathrm{d}\hat{x} + R, \tag{15}$$

where

$$|R| \leq C(q)\hat{h}_{00}^6 \|\hat{u}\|_{6,p,e}\|\hat{\phi}\|_{2,p',e}, \ \hat{h}_{00} = \max_{1\leq i \leq d} \hat{h}_{i,e}, \ \frac{1}{p}+\frac{1}{p'} = 1, \tag{16}$$

and $Q_2(e)$ is the set of all double-quadratic polynomials.

Second, we present the following two lemmas which were proved in [25, 36, 43].

Lemma 3.3. *Let $\hat{E}(t;\hat{w},\hat{v}) = \int_{\hat{\Omega}}(\sum_{i,j=1}^{d} \hat{e}_{ij}(t,\hat{x})\hat{D}_i\hat{w}\hat{D}_j\hat{v} + \hat{p}\hat{w}\hat{v})\,\mathrm{d}\hat{x}$, and $\hat{R}_h^t$ denote the Ritz projection operator with respect to $\hat{E}(t;\cdot,\cdot)$, i.e. , $\hat{E}(t;\hat{R}_h^t\hat{u},\hat{v}) = \hat{E}(t;\hat{u},\hat{v})$, $\forall \hat{v} \in \hat{S}_0^h$. Consider a linear parabolic weak form*

$$\begin{cases} (\hat{w}_t,\hat{v}) + \hat{E}(t;\hat{w},\hat{v}) = (\hat{r}(t,\hat{x}),\hat{v}), \quad \forall \hat{v} \in H_0^1(\hat{\Omega}), \\ \hat{w}(0,\hat{x}) = \hat{w}_0(\hat{x}). \end{cases} \tag{17}$$

Assume that $\hat{e}_{ij}$, $\hat{p} \in (\prod_{s=1}^{m} W_\infty^4(\hat{\Omega}_s)) \bigcap L_\infty(\hat{\Omega})$ and $\hat{w} \in (\prod_{s=1}^{m} H^7(\hat{\Omega}_s)) \bigcap H_0^1(\hat{\Omega})$, then there exist functions
$\hat{W}_i \in \prod_{s=1}^{m}(H^r(\hat{\Omega}_s)\bigcap L^\infty(\hat{\Omega}_s))(i = 1, \cdots, l)$ and a constant C independent of $\hat{h}$ such that

$$\|\hat{R}_h^t\hat{w} - \hat{P}_h\hat{w} - \sum_{i=1}^{l}\hat{h}_i^4\hat{P}_h\hat{W}_i\|_{0,\infty,\hat{\Omega}} \leq C\hat{h}_0^{4+\beta_0}, \tag{18}$$

where $\beta_0 = min(r,2) - \frac{d}{2} > 0$.

Lemma 3.4. *Assume the semi-discrete scheme to (17) is*

$$\begin{cases} (\hat{w}_{t,h},\hat{v}) + E(t;\hat{w}_h,\hat{v}) = (\hat{r}(t,\hat{x}),\hat{v}), \quad \forall \hat{v} \in \hat{S}_0^h, \\ \hat{w}_h(0,\hat{x}) = \hat{P}_h\hat{w}_0(\hat{x}). \end{cases} \tag{19}$$

Assume that $\hat{e}_{ij}, \hat{D}_t\hat{e}_{ij}, \hat{p} \in L^\infty(\hat{Q}_T), \hat{r} \in L^2(\hat{Q}_T)$, and $\|\hat{e}_{ij}\|_{0,\infty,\hat{Q}_T}, \|\hat{D}_t\hat{e}_{ij}\|_{0,\infty,\hat{Q}_T}$, $\|\hat{p}\|_{0,\infty,\hat{Q}_T} \leq M < \infty$, then there exist constant C_0 and C_1 independent of $\hat{h}$ but depend on M and μ,such that

$$\|\hat{w}_h(t)\|_{1,\hat{\Omega}}^2 \leq C_0 e^{C_1 t}(\|\hat{w}_0\|_{1,\hat{\Omega}}^2 + \int_0^t \|\hat{r}(\tau,\cdot)\|_{0,\hat{\Omega}}^2\,\mathrm{d}\tau). \tag{20}$$

Remark 3.1. *Since the semi-discrete scheme for the semi-linear problem is just a special case of the semi-discrete scheme for the quasi-linear problem, we only need to prove the multi-parameter asymptotic expansion of the semi-discrete d-quadratic iso-parametric finite element error for the quasi-linear problem.*

Theorem 3.1. *Along with the same assumptions of Lemma 3.3, there exist functions $\hat{\psi}_i \in H^1(0,T;H_0(\hat{\Omega}) \bigcap C(\hat{\Omega}))$ $(i = 1,\cdots,l)$ independent of $\hat{h}$ such that the errors of the solutions to (8) satisfy the following multi-parameter asymptotic expansion:*

$$\hat{u}_h - \hat{u}^I = \sum_{i=1}^{l}\hat{h}_i^4\hat{\psi}_i^I + \hat{\varepsilon}, \tag{21}$$

where

$$\|\hat{\varepsilon}\|_{0,\infty,\hat{Q}_T} = \mathcal{O}(\hat{h}_0^{4+\beta}|\ln \hat{h}_0|^{\frac{d-1}{d}}), \quad \beta > 0. \tag{22}$$

Hence,

$$\hat{u}_h(t,\hat{X}) - \hat{u}(t,\hat{X}) = \sum_{i=1}^{l} \hat{h}_i^4 \hat{\psi}_i(t,\hat{X}) + \hat{\varepsilon}(t,\hat{X}), \ \forall \hat{X} \in \hat{\Omega}_0^h. \tag{23}$$

Proof. Without loss of generality, we assume $\hat{v} \in \hat{S}_0^h$ in the whole proof. Let $\tilde{\hat{u}} = \tilde{R}_h^t \hat{u}$ be the Ritz projection of $\hat{u}$ with respect to

$$\tilde{A}(t;\hat{w},\hat{v}) = (\sum_{i,j=1}^{d} \hat{a}_{ij}(t,\hat{x},\hat{u})\hat{D}_i\hat{w}, \hat{D}_j\hat{v}) + (\hat{q}\tilde{\hat{u}},\hat{v}). \tag{24}$$

By the definition of $\tilde{\hat{u}}$, $\tilde{A}(t;\hat{w},\hat{v})$ and $\hat{A}(t;\hat{u},\hat{v})$, we obtain

$$\tilde{A}(t;\tilde{\hat{u}},\hat{v}) = \tilde{A}(t;\tilde{R}_h^t\hat{u},\hat{v}) = \tilde{A}(t;\hat{u},\hat{v}) = \hat{A}(t;\hat{u},\hat{v}), \tag{25}$$

and

$$\tilde{A}(t;\hat{u}_h,\hat{v}) = \hat{A}(t;\hat{u}_h,\hat{v}) + (\sum_{i,j=1}^{d} (\hat{a}_{ij}(t,\hat{x},\hat{u}) - \hat{a}_{ij}(t,\hat{x},\bar{\hat{u}}))\hat{D}_i\hat{u}_h, \hat{D}_j\hat{v}). \tag{26}$$

Let $\hat{\theta} = \hat{u}_h - \tilde{\hat{u}}, \hat{\rho}_1 = \tilde{\hat{u}} - \hat{u}^I$, then we have

$$\hat{u}_h - \hat{u}^I = \hat{\theta} + \hat{\rho}_1. \tag{27}$$

By Lemma 3.1, we obtain

$$\hat{\rho}_1 = \sum_{k=1}^{l} \hat{h}_k^4 \hat{\phi}_k^I + \hat{\varepsilon}_1, \tag{28}$$

where

$$\|\hat{\varepsilon}_1\|_{0,\infty,\hat{\Omega}} = \mathcal{O}(\hat{h}_0^{4+\alpha}|\ln \hat{h}_0|^{\frac{d-1}{d}}), \quad \alpha = \min(r,2) - \frac{d}{2} > 0. \tag{29}$$

Now we discuss the expansion of $\hat{\theta}$ in detail as follows. Let $\hat{f}'$, $\hat{f}''$, $\hat{a}'_{ij}$ and $\hat{a}''_{ij}$ denote the partial derivatives of functions $\hat{f}$ and $\hat{a}_{ij}$ for $\hat{u}$. By the definitions of $\hat{\theta}$, $\hat{P}_h$, $\tilde{R}_h^t$ and $\tilde{\hat{u}}$, (7), (8), (25), (26) and Taylor expansion, $\forall \hat{v} \in \hat{S}_0^h$, we obtain

$$\begin{aligned}
(\hat{\theta}_t,\hat{v}) + \tilde{A}(t;\hat{\theta},\hat{v}) &= \Big[(\hat{u}_{t,h},\hat{v}) + \tilde{A}(t;\hat{u}_h,\hat{v})\Big] - \Big[(\tilde{\hat{u}}_t,\hat{v}) + \tilde{A}(t;\tilde{\hat{u}},\hat{v})\Big] \\
&= \Big[(\hat{u}_{t,h},\hat{v}) + \hat{A}(t;\hat{u}_h,\hat{v})\Big] - \Big[(\hat{u}_t,\hat{v}) + \hat{A}(t;\hat{u},\hat{v})\Big] \\
&\quad + (\hat{u}_t - \tilde{\hat{u}}_t,\hat{v}) + (\sum_{i,j=1}^{d} (\hat{a}_{ij}(t,\hat{x},\hat{u}) - \hat{a}_{ij}(t,\hat{x},\hat{u}_h))\hat{D}_i\hat{u}_h, \hat{D}_j\hat{v}) \\
&=(\hat{f}(t,\hat{x},\hat{u}_h),\hat{v}) - (\hat{f}(t,\hat{x},\hat{u}),\hat{v}) + (\hat{P}_h\hat{u}_t - \tilde{R}_h^t\hat{u}_t,\hat{v}) \\
&\quad - \sum_{i,j=1}^{d} ((\hat{a}_{ij}(t,\hat{x},\hat{u}_h) - \hat{a}_{ij}(t,\hat{x},\hat{u}))\hat{D}_i\hat{u}_h, \hat{D}_j\hat{v}) \\
&=(\hat{f}'(t,\hat{x},\hat{u})(\hat{u}_h - \hat{u}),\hat{v}) + (\hat{D}_t(\hat{P}_h\hat{u} - \tilde{R}_h^t\hat{u}),\hat{v}) \\
&\quad - \sum_{i,j=1}^{d} (\hat{a}'_{ij}(t,\hat{x},\hat{u})(\hat{u}_h - \hat{u})\hat{D}_i\hat{u}, \hat{D}_j\hat{v}) + \varepsilon_2(\hat{v}),
\end{aligned} \tag{30}$$

where

$$\begin{aligned}\varepsilon_2(\hat{v}) =&(\frac{1}{2}\hat{f}''(t,\hat{x},\xi_1)(\hat{u}_h-\hat{u})^2,\hat{v}) - \sum_{i,j=1}^{d}(\frac{1}{2}\hat{a}''_{ij}(t,\hat{x},\xi_2)(\hat{u}_h-\hat{u})^2\hat{D}_i\hat{u}_h,\hat{D}_j\hat{v})\\ &-\sum_{i,j=1}^{d}(\hat{a}'_{ij}(t,\hat{x},\hat{u})(\hat{u}_h-\hat{u})\hat{D}_i(\hat{u}_h-\hat{u}),\hat{D}_j\hat{v}),\end{aligned} \tag{31}$$

ξ_1 and ξ_2 are between $\hat{u}_h$ and $\hat{u}$. By the Hölder's inequality, the error estimates of d-quadratic finite element approximation and (31), we obtain

$$\begin{aligned}|\hat{\varepsilon}_2(\hat{v})| \le& C_1\|\hat{u}_h-\hat{u}\|^2_{0,\hat{\Omega}}\|\hat{v}\|_{0,\hat{\Omega}} + \sum_{i,j=1}^{d} C_2\|\hat{u}_h-\hat{u}\|^2_{0,\hat{\Omega}}\|\hat{D}_j\hat{v}\|_{0,\hat{\Omega}}\\ &+\sum_{i,j=1}^{d} C_3\|\hat{u}_h-\hat{u}\|_{0,\hat{\Omega}}\|\hat{D}_i(\hat{u}_h-\hat{u})\|_{0,\hat{\Omega}}\|\hat{D}_j\hat{v}\|_{0,\hat{\Omega}}\\ \le& C_4\hat{h}_0^5\|\hat{v}\|_{1,\hat{\Omega}}.\end{aligned} \tag{32}$$

Let $\hat{\rho} = \tilde{\hat{u}} - \hat{u}$, then we have

$$\hat{u}_h - \hat{u} = \hat{\theta} + \hat{\rho}. \tag{33}$$

By plugging (33) into (30) and moving all the terms about $\hat{\theta}$ to the left hand side of the equation, we obtain

$$\begin{aligned}(\hat{\theta}_t,\hat{v}) + \hat{B}(t;\hat{\theta},\hat{v}) =&(\hat{f}'(t,\hat{x},\hat{u})\hat{\rho},\hat{v}) + (\hat{D}_t(\hat{P}_h\hat{u}-\tilde{R}_h^t\hat{u}),\hat{v})\\ &-\sum_{i,j=1}^{d}(\hat{a}'_{ij}(t,\hat{x},\hat{u})\hat{\rho}\hat{D}_i\hat{u},\hat{D}_j\hat{v}) + \varepsilon_2(\hat{v}),\end{aligned} \tag{34}$$

where

$$\hat{B}(t;\hat{\theta},\hat{v}) = \tilde{A}(t;\hat{\theta},\hat{v}) - (\hat{f}'(t,\hat{x},\hat{u})\hat{\theta},\hat{v}) + \sum_{i,j=1}^{d}(\hat{a}'_{ij}(t,\hat{x},\hat{u})\hat{\theta}\hat{D}_i\hat{u},\hat{D}_j\hat{v}). \tag{35}$$

Let $\hat{\rho}_2 = \hat{u}^I - \hat{u}$, then

$$\hat{\rho} = \hat{\rho}_1 + \hat{\rho}_2. \tag{36}$$

By (28), we obtain

$$(\hat{f}'(\hat{u},\hat{x},t)\hat{\rho}_1,\hat{v}) = \sum_{k=1}^{l}\hat{h}_k^4(\hat{f}'(\hat{u},\hat{x},t)\hat{\phi}_k^I,\hat{v}) + \hat{\varepsilon}_3(\hat{v}), \tag{37}$$

$$\sum_{i,j=1}^{d}(\hat{a}'_{ij}(\hat{u},\hat{x},t)\hat{\rho}_1\hat{D}_i\hat{u},\hat{D}_j\hat{v}) = \sum_{k=1}^{l}\hat{h}_k^4(\sum_{i,j=1}^{d}\hat{a}'_{ij}(\hat{u},\hat{x},t)\hat{\phi}_k^I\hat{D}_i\hat{u},\hat{D}_j\hat{v}) + \hat{\varepsilon}_4(\hat{v}), \tag{38}$$

where

$$|\hat{\varepsilon}_3(\hat{v})| =|(\hat{f}'(\hat{u},\hat{x},t)\hat{\varepsilon}_1,\hat{v})|, \tag{39}$$

$$|\hat{\varepsilon}_4(\hat{v})| =|(\sum_{i,j=1}^{d}\hat{a}'_{ij}(\hat{u},\hat{x},t)\hat{\varepsilon}_1\hat{D}_i\hat{u},\hat{D}_j\hat{v})|. \tag{40}$$

By Hölder's inequality, (29) and inverse estimate, we obtain

$$|\hat{\varepsilon}_3(\hat{v})| \leq C_5\hat{h}_0^{4+\alpha+\frac{d}{2}}\|\hat{v}\|_{0,\hat{\Omega}},\ \alpha=\min(r,2)-\frac{d}{2}>0, \tag{41}$$

$$|\hat{\varepsilon}_4(\hat{v})| \leq C_6\hat{h}_0^{4+\alpha+\frac{d}{2}}\|\hat{D}_j\hat{v}\|_{0,\hat{\Omega}} \leq C_6\hat{h}_0^{4+\alpha+\frac{d}{2}}\|\hat{v}\|_{1,\hat{\Omega}},\ \alpha=\min(r,2)-\frac{d}{2}>0. \tag{42}$$

By the definition of $\hat{\rho}_2$ and Lemma 3.2, we have

$$\begin{aligned}(\hat{f}'(\hat{u},\hat{x},t)\hat{\rho}_2,\hat{v}) = &-\sum_{k=1}^{d}\sum_{s=1}^{m}\hat{h}_{ks}^4\int_{\Omega_s}[\frac{1}{480}\hat{f}'(\hat{u},\hat{x},t)\hat{v}\hat{D}_k^4\hat{u}\\ &-\frac{1}{45}\hat{D}_i(\hat{f}'(\hat{u},\hat{x},t)\hat{v})\hat{D}_k^3\hat{u}]\,\mathrm{d}\hat{x}+\hat{\varepsilon}_5(\hat{v}),\end{aligned} \tag{43}$$

$$\begin{aligned}\sum_{i,j=1}^{d}(\hat{a}_{ij}'(\hat{u},\hat{x},t)\hat{\rho}_2\hat{D}_i\hat{u},\hat{D}_j\hat{v}) = &-\sum_{k=1}^{d}\sum_{s=1}^{m}\hat{h}_{ks}^4\int_{\Omega_s}\sum_{i,j=1}^{d}[\frac{1}{480}\hat{a}_{ij}'(\hat{u},\hat{x},t)\hat{D}_i\hat{u}\hat{D}_j\hat{v}\hat{D}_k^4\hat{u}\\ &-\frac{1}{45}\hat{D}_k(\hat{a}_{ij}'(\hat{u},\hat{x},t)\hat{D}_i\hat{u}\hat{D}_j\hat{v})\hat{D}_k^3\hat{u}]\,\mathrm{d}\hat{x}+\hat{\varepsilon}_6(\hat{v}),\end{aligned} \tag{44}$$

where

$$|\hat{\varepsilon}_j(\hat{v})| = |\sum_{s=1}^{m}R_s| \leq \sum_{s=1}^{m}C^s\hat{h}_{s0}^6\|\hat{v}\|_{2,\hat{\Omega}_s} \leq C_j\hat{h}_0^6\|\hat{v}\|_{2,\hat{\Omega}}, j=5,6, \tag{45}$$

R_s is the error for $\hat{\Omega}_s$ in Lemma 3.2, $\hat{h}_{s0}$ is the max step size of $\hat{\Omega}_s$.

When we construct the partition in Section 2, there are only l ($l < md$) independent grid parameters $\hat{h}_1,\cdots,\hat{h}_l$ because of compatibility requirement between subdomains to ensure no hanging nodes, then

$$(\hat{f}'(\hat{u},\hat{x},t)\hat{\rho}_2,\hat{v}) = \sum_{k=1}^{l}\hat{h}_k^4M_k+\hat{\varepsilon}_5(\hat{v}), \tag{46}$$

$$\sum_{i,j=1}^{d}(\hat{a}_{ij}'(\hat{u},\hat{x},t)\hat{\rho}_2\hat{D}_i\hat{u},\hat{D}_j\hat{v}) = \sum_{k=1}^{l}\hat{h}_k^4N_k+\hat{\varepsilon}_6(\hat{v}), \tag{47}$$

where M_k is the sum of some integrations like

$$-\int_{\Omega_s}[\frac{1}{480}(\hat{f}'(\hat{u},\hat{x},t)\hat{v}\hat{D}_i^4\hat{u}-\frac{1}{45}\hat{D}_i(\hat{f}'(\hat{u},\hat{x},t)\hat{v})\hat{D}_i^3\hat{u}]\,\mathrm{d}\hat{x},$$

and N_k is the sum of some integrations like

$$-\int_{\Omega_s}\sum_{i,j=1}^{d}[\frac{1}{480}\hat{a}_{ij}'(\hat{u},\hat{x},t)\hat{D}_i\hat{u}\hat{D}_j\hat{v}\hat{D}_k^4\hat{u}-\frac{1}{45}\hat{D}_k(\hat{a}_{ij}'(\hat{u},\hat{x},t)\hat{D}_i\hat{u}\hat{D}_j\hat{v})\hat{D}_k^3\hat{u}]\,\mathrm{d}\hat{x}.$$

By Lemma 3.3, Hölder's inequality and finite element inverse estimate, we obtain

$$(\hat{D}_t(\hat{P}_h\hat{u}-\tilde{R}_h^t\hat{u}),\hat{v}) = \sum_{k=1}^{l}\hat{h}_k^4(\hat{P}_h\hat{D}_t\hat{W}_k(t,\hat{x}),\hat{v})+\hat{\varepsilon}_7(\hat{v}), \tag{48}$$

where

$$|\hat{\varepsilon}_7(\hat{v})| \leq C_9\hat{h}_0^{4+\beta_0+\frac{d}{2}}\|\hat{v}\|_{0,\hat{\Omega}},\ \beta_0=\min(2,r)-\frac{d}{2}>0. \tag{49}$$

By (34), (36), (37), (38), (46), (47) and (48), we obtain

$$(\hat{\theta}_t,\hat{v})+\hat{B}(t;\hat{\theta},\hat{v}) = \sum_{k=1}^{l}\hat{h}_k^4F_k(\hat{u},\hat{v})+\hat{\varepsilon}_8(\hat{v}), \tag{50}$$

where

$$F_k(\hat u,\hat v) =(\hat f'(\hat u,\hat x,t)\hat\phi_k^I,\hat v) + (\sum_{i,j=1}^{d}\hat a'_{ij}(\hat u,\hat x,t)\hat\phi_k^I\hat D_i\hat u,\hat D_j\hat v) + M_k - N_k + (\hat P_h\hat D_t\hat W_k(t,\hat x),\hat v), \tag{51}$$

and

$$\hat\varepsilon_8(\hat v) = \hat\varepsilon_2(\hat v) + \hat\varepsilon_3(\hat v) + \hat\varepsilon_4(\hat v) + \hat\varepsilon_5(\hat v) + \hat\varepsilon_6(\hat v) + \hat\varepsilon_7(\hat v). \tag{52}$$

By (32), (41), (42), (45) and (49), we obtain

$$|\hat\varepsilon_8(\hat v)| \le C_{10}\hat h_0^{4+\beta_1}\|\hat v\|_{2,\hat\Omega},\ \beta_1 = \min(1,r) > 0. \tag{53}$$

By Lemma 3.3, the definition of $\bar\theta$, $\hat u^h$, (8), Hölder's inequality and finite element inverse estimate, we obtain

$$(\hat\theta(0,\cdot),\hat v) = (\hat P_h\hat u_0 - \tilde R_h^0\hat u_0,\hat v) = \sum_{i=1}^{l}\hat h_i^4(W_i(0,\cdot),\hat v) + \hat\varepsilon_9(\hat v), \tag{54}$$

where

$$|\hat\varepsilon_9(\hat v)| \le C_{11}\hat h_0^{4+\beta_0+\frac{d}{2}}\|\hat v\|_{0,\hat\Omega},\ \beta_0 = \min(2,r) - \frac{d}{2} > 0. \tag{55}$$

We construct the following auxiliary problem: find $\hat\varphi_i \in H^1(0,T;H_0^1(\hat\Omega))(i = 1,\cdots,l)$, such that

$$\begin{cases} (D_t\hat\varphi_i,\hat v) + B(t;\hat\varphi_i,\hat v) = F_i(\hat u,\hat v), & \forall\hat v\in H_0^1(\hat\Omega),\\ (\hat\varphi_i(0,\cdot),\hat v) = (\hat W_i(0,\cdot),\hat v), & \forall\hat v\in H_0^1(\hat\Omega).\end{cases}$$

The semi-discrete finite element scheme can be obtained as follows: find $\hat\varphi_{i,h} \in H^1(0,T;\hat S_0^h), i = 1,\cdots,l$, such that

$$\begin{cases} (D_t\hat\varphi_{i,h},\hat v) + B(t;\hat\varphi_{i,h},\hat v) = F_i(\hat u,\hat v), & \forall\hat v\in \hat S_0^h,\\ (\hat\varphi_{i,h}(0,\cdot),\hat v) = (\hat W_i(0,\cdot),\hat v), & \forall\hat v\in \hat S_0^h.\end{cases}$$

Let $\hat\psi = \hat\theta + \sum\limits_{i=1}^{l}\hat h_i^4\hat\varphi_{i,h}$, we obtain

$$\begin{cases} (D_t\hat\psi,\hat v) + B(t;\hat\psi,\hat v) = \hat\varepsilon_8(\hat v), & \forall\hat v\in \hat S_0^h,\\ (\hat\psi(0,\cdot),\hat v) = \hat\varepsilon_9(\hat v), & \forall\hat v\in \hat S_0^h.\end{cases} \tag{56}$$

By applying Lemma 3.4 on (56) and using (53) and (55), we obtain

$$\|\hat\psi\|_{1,\hat\Omega} \le Ch_0^{4+\beta_2}, \beta_2 > \frac{d}{2} - 1. \tag{57}$$

If $1 \ge \gamma > \frac{d}{2} - 1$ and $\hat\varphi_i(t,\cdot) \in H^2(0,T;H^{1+\gamma}(\hat\Omega)\bigcap H_0^1(\hat\Omega))$, then we have [54]

$$\|\hat\varphi_i - \hat\varphi_{i,h}\|_{1,\hat\Omega} \le C\hat h^{\gamma}. \tag{58}$$

By replacing $\hat\varphi_{i,h}$ by $\hat\varphi_i^I$ and using (57), (58) and the inverse inequality, we obtain

$$\|\hat\theta + \sum_{i=1}^{l}\hat h_i^4\hat\varphi_i^I\|_{0,\infty,\hat\Omega} \le C|\ln\hat h_0|^{\frac{d-1}{d}}\hat h_0^{4+\beta_3}, \tag{59}$$

where

$$\beta_3 = \min\left(\beta_2, \gamma + 1 - \frac{d}{2}\right) > 0. \tag{60}$$

Therefore

$$\hat{\theta} = -\sum_{i=1}^{l} \hat{h}_i^4 \hat{\varphi}_i^I + \hat{\varepsilon}_{10}, \tag{61}$$

where

$$\|\hat{\varepsilon}_{10}\|_{0,\infty,\hat{\Omega}} = |\ln \hat{h}_0|^{\frac{d-1}{d}} \hat{h}_0^{4+\beta_3}. \tag{62}$$

Let $\hat{\psi}_i = -\hat{\varphi}_i + \hat{\phi}_i$, then $\hat{\psi}_i^I = -\hat{\varphi}_i^I + \hat{\phi}_i^I$. By (27), (28) and (61), we obtain

$$\hat{u}_h - \hat{u}^I = \sum_{i=1}^{l} \hat{h}_i^4 \hat{\psi}_i^I + \hat{\varepsilon}, \tag{63}$$

where

$$\hat{\varepsilon} = \hat{\varepsilon}_1 + \hat{\varepsilon}_{10}. \tag{64}$$

From (29) and (62), we obtain

$$\|\hat{\varepsilon}\|_{0,\infty,\hat{Q}_T} = \mathcal{O}(\hat{h}_0^{4+\beta} |\ln \hat{h}_0|^{\frac{d-1}{d}}), \tag{65}$$

where

$$\beta_1 = \min(\beta_3, \alpha) > 0. \tag{66}$$

Because $\hat{u}^I(t,\hat{X}) = \hat{u}(t,\hat{X})$ and $\hat{\psi}_i^I(t,\hat{X}) = \hat{\psi}_i(t,\hat{X})$, $\forall \hat{X} \in \hat{\Omega}_0^h$, $\forall t \in [0,T]$, we obtain

$$\hat{u}_h(t,\hat{X}) - \hat{u}(t,\hat{X}) = \sum_{i=1}^{l} \hat{h}_i^4 \hat{\psi}_i(t,\hat{X}) + \hat{\varepsilon}.$$

Hence the proof is completed.

■

4. Multi-parameter asymptotic expansion of the fully discrete d-quadratic iso-parametric finite element error

For the fully discrete finite element schemes presented in Section 2, we will prove the multi-parameter asymptotic error expansions, which will be the foundation to construct the splitting extrapolation schemes in Section 5. Note that the linearized Crank-Nicolson-Galerkin method in (9) has $\mathcal{O}(\tau^2)$ order of accuracy and the backward Euler-Galerkin method in (10) has $\mathcal{O}(\tau)$ order of accuracy. Therefore, we can follow the ideas and arguments in [36, 43] to obtain the following two lemmas.

Lemma 4.1. *Along with the same assumptions of Theorem 3.1, assume that the solution to (8) satisfies $\hat{u}_h \in C^5(0,T;S_0^h)$ and $\hat{f}(t,\hat{x},\hat{u})$ is third order differentiable for $\hat{u}$. Then for the solution $\hat{U}^n$ to (9), there exists a function $\hat{\psi}_{l+1}(t,\cdot) \in H^1(0,T;\hat{S}_0^h)$ independent of τ and $\hat{h}$ such that*

$$\hat{U}^n - \hat{u}_h^n = \tau^2 \hat{\psi}_{l+1}^n + \hat{r}_h^n, 1 \le n \le N, \tag{67}$$

where

$$\max_{1\le n\le N} \|\hat{r}_h^n\|_{0,\infty,\hat{Q}_T} = \mathcal{O}(\tau^2 \hat{h}_{00}^{\beta_4} |\ln \hat{h}_0|^{\frac{d-1}{d}}),\ \beta_4 = min(1+\gamma-\frac{d}{2}, 1). \tag{68}$$

Proof. We use the undetermined coefficient method for the proof. Assume

$$\hat{U}^n - \hat{u}_h^n = \tau^2\hat{\psi}_{l+1}^n + \hat{r}_h^n, \tag{69}$$

where $\hat{\psi}_{l+1}^n$ and $\hat{r}_h^n$ are undetermined. By (8) and (9), we obtain

$$\begin{aligned}(\bar{\partial}_t\hat{U}^n - \hat{u}_{t,h}(t_{n-\frac{1}{2}}),\hat{v}) + A(t_{n-\frac{1}{2}};\hat{u}_h^{n-\frac{1}{2}} - \hat{u}_h(t_{n-\frac{1}{2}}),\hat{v})\\ =(\hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{U}^{n-1}),\hat{v}) - (\hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h(t_{n-\frac{1}{2}})),\hat{v})\\ +\frac{1}{2}[\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{U}^{n-1})(\hat{U}^n-\hat{U}^{n-1},\hat{v})],\ \forall\hat{v}\in\hat{S}_0^h.\end{aligned} \tag{70}$$

By the Taylor expansions of $\hat{u}_h^n$ and $\hat{u}_h^{n-1}$ at $t_{n-\frac{1}{2}} = (n-\frac{1}{2})\tau$, we obtain

$$\begin{aligned}\bar{\partial}_t\hat{u}_h^n &= \frac{\hat{u}_h^n - \hat{u}_h^{n-1}}{\tau} = \hat{u}_{t,h}(t_{n-\frac{1}{2}}) + \frac{\tau^2}{24}D_t^3\hat{u}_h(t_{n-\frac{1}{2}}) + \eta_1^n,\\ \eta_1^n &= \frac{\tau^4}{3840}(D_t^5\hat{u}_h(\theta_1,\hat{x}) + D_t^5\hat{u}_h(\theta_2,\hat{x})),\end{aligned} \tag{71}$$

where $(n-\frac{1}{2})\tau \le \theta_1 \le n\tau$ and $(n-1)\tau \le \theta_2 \le (n-\frac{1}{2})\tau$.

By the Taylor expansions of $\hat{\psi}_{l+1}^n$ and $\hat{\psi}_{l+1}^{n-1}$ at $t_{n-\frac{1}{2}}$, we obtain

$$\begin{aligned}\bar{\partial}_t\hat{\psi}_{l+1}^n &= \frac{\hat{\psi}_{l+1}^n - \hat{\psi}_{l+1}^{n-1}}{\tau} = D_t\hat{\psi}_{l+1}(t_{n-\frac{1}{2}}) + \eta_2^n,\\ \eta_2^n &= \frac{\tau^2}{48}(D_t^3\hat{u}_h(\theta_3,\hat{x}) + D_t^3\hat{u}_h(\theta_4,\hat{x})),\end{aligned} \tag{72}$$

where $(n-\frac{1}{2})\tau \le \theta_3 \le n\tau$ and $(n-1)\tau \le \theta_4 \le (n-\frac{1}{2})\tau$.

Therefore, by (69), (71) and (72), we obtain

$$\begin{aligned}\bar{\partial}_t\hat{U}^n =&\bar{\partial}_t\hat{u}_h^n + \tau^2\bar{\partial}_t\hat{\psi}_{l+1}^n + \bar{\partial}_t\hat{r}_h^n\\ =&\hat{u}_{t,h}(t_{n-\frac{1}{2}}) + \frac{\tau^2}{24}D_t^3\hat{u}_h(t_{n-\frac{1}{2}}) + \tau^2 D_t\hat{\psi}_{l+1}(t_{n-\frac{1}{2}}) + \bar{\partial}_t\hat{r}_h^n + \varepsilon_1^n, \quad (73)\\ \varepsilon_1^n =&\eta_1^n + \tau^2\eta_2^2. \quad (74)\end{aligned}$$

By (71) and (72), we obtain

$$\|\varepsilon_1^n\|_{0,\infty,\hat{\Omega}} = \mathcal{O}(\tau^4). \tag{75}$$

By the Taylor expansions of $\hat{u}_h^n$ and $\hat{u}_h^{n-1}$ at $t_{n-\frac{1}{2}}$, we obtain

$$\begin{aligned}\hat{u}_h^{n-\frac{1}{2}} &= \frac{\hat{u}_h^n + \hat{u}_h^{n-1}}{2} = \hat{u}_h(t_{n-\frac{1}{2}}) + \frac{\tau^2}{8}D_t^2\hat{u}_h(t_{n-\frac{1}{2}}) + \eta_3^n,\\ \eta_3^n &= \frac{\tau^4}{768}(D_t^4\hat{u}_h(\theta_5,\hat{x}) + D_t^4\hat{u}_h(\theta_6,\hat{x})),\end{aligned} \tag{76}$$

where $(n-\frac{1}{2})\tau \le \theta_5 \le n\tau$ and $(n-1)\tau \le \theta_6 \le (n-\frac{1}{2})\tau$.

By the Taylor expansions of $\hat{\psi}_{l+1}^n$ and $\hat{\psi}_{l+1}^{n-1}$ at $t_{n-\frac{1}{2}}$, we obtain

(77)

$$\hat{\psi}_{l+1}^{n-\frac{1}{2}} = \frac{\hat{\psi}_{l+1}^n + \hat{\psi}_{l+1}^{n-1}}{2} = \hat{\psi}_{l+1}(t_{n-\frac{1}{2}}) + \eta_4^n,\ \eta_4^n = \frac{\tau^2}{16}(D_t^2\hat{u}_h(\theta_7,\hat{x}) + D_t^2\hat{u}_h(\theta_8,\hat{x})),$$

where $(n-\frac{1}{2})\tau \le \theta_7 \le n\tau$ and $(n-1)\tau \le \theta_8 \le (n-\frac{1}{2})\tau$.

Therefore, by (69), (76) and (77), we obtain

$$\hat{U}^{n-\frac{1}{2}} =\hat{u}_h^{n-\frac{1}{2}} + \tau^2\hat{\psi}_{l+1}^{n-\frac{1}{2}} + \hat{r}_h^{n-\frac{1}{2}}$$

$$=\hat{u}_h(t_{n-\frac{1}{2}}) + \frac{\tau^2}{8}D_t^2\hat{u}_h(t_{n-\frac{1}{2}}) + \tau^2\hat{\psi}_{l+1}(t_{n-\frac{1}{2}}) + \hat{r}_h^{n-\frac{1}{2}} + \varepsilon_2^n, \tag{78}$$

$$\varepsilon_2^n =\eta_3^n + \tau^2\eta_4^n. \tag{79}$$

By (76) and (77), we obtain

$$\|\varepsilon_2^n\|_{0,\infty,\hat{\Omega}} = \mathcal{O}(\tau^4). \tag{80}$$

By (70), (73) and (78), we obtain

(81)

$$\begin{aligned}
&(\frac{\tau^2}{24}D_t^3\hat{u}_h(t_{n-\frac{1}{2}}) + \tau^2 D_t\hat{\psi}_{l+1}(t_{n-\frac{1}{2}}) + \bar{\partial}_t\hat{r}_h^n + \varepsilon_1^n, \hat{v}) \\
&+ A(t_{n-\frac{1}{2}}; \frac{\tau^2}{8}D_t^2\hat{u}_h(t_{n-\frac{1}{2}}) + \tau^2\hat{\psi}_{l+1}(t_{n-\frac{1}{2}}) + \hat{r}_h^{n-\frac{1}{2}} + \varepsilon_2^n, \hat{v}) \\
=&(\hat{f}(t_{n-\frac{1}{2}}, \hat{x}, \hat{U}^{n-1}), \hat{v}) \\
&- (\hat{f}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h(t_{n-\frac{1}{2}})), \hat{v}) + \frac{1}{2}[\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}}, \hat{x}, \hat{U}^{n-1})(\hat{U}^n - \hat{U}^{n-1}, \hat{v})], \ \forall\hat{v}\in\hat{S}_0^h.
\end{aligned}$$

By the Taylor expansions, we obtain

$$\begin{aligned}
\hat{f}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h(t_{n-\frac{1}{2}})) =&\hat{f}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h^{n-1}) + \hat{f}_{\hat{u}}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h^{n-1})[\hat{u}_h(t_{n-\frac{1}{2}}) - \hat{u}_h^{n-1}] \\
&+ \frac{1}{2}\hat{f}_{\hat{u}\hat{u}}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h^{n-1})[\hat{u}_h(t_{n-\frac{1}{2}}) - \hat{u}_h^{n-1}]^2 \\
&+ \frac{1}{6}\hat{f}_{\hat{u}\hat{u}\hat{u}}(t_{n-\frac{1}{2}}, \hat{x}, \theta_9)[\hat{u}_h(t_{n-\frac{1}{2}}) - \hat{u}_h^{n-1}]^3,
\end{aligned} \tag{82}$$

$$\hat{u}_h(t_{n-\frac{1}{2}}) =\hat{u}_h^{n-1} + \frac{\tau}{2}\hat{u}_{t,h}^{n-1} + \frac{\tau^2}{8}\hat{u}_{tt,h}(\theta_{10}). \tag{83}$$

Therefore

$$\begin{aligned}
\hat{f}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h(t_{n-\frac{1}{2}})) =&\hat{f}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h^{n-1}) + \hat{f}_{\hat{u}}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h^{n-1})[\hat{u}_h(t_{n-\frac{1}{2}}) - \hat{u}_h^{n-1}] \\
&+ \frac{\tau^2}{8}\hat{f}_{\hat{u}\hat{u}}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h^{n-1})\hat{u}_{t,h}^2(t_{n-1}) + \eta_5^n,
\end{aligned} \tag{84}$$

where

(85)

$$\begin{aligned}
\eta_5^n =&\frac{1}{6}\hat{f}_{\hat{u}\hat{u}\hat{u}}(t_{n-\frac{1}{2}}, \hat{x}, \theta_9)[\frac{\tau}{2}\hat{u}_{t,h}^{n-1} + \frac{\tau^2}{8}\hat{u}_{tt,h}(\theta_{10})]^3 + \frac{\tau^4}{128}\hat{f}_{\hat{u}\hat{u}}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h^{n-1})\hat{u}_{tt,h}^2(\theta_{10}) \\
&+ \frac{\tau^3}{16}\hat{f}_{\hat{u}\hat{u}}(t_{n-\frac{1}{2}}, \hat{x}, \hat{u}_h^{n-1})\hat{u}_{tt,h}(\theta_{10})\hat{u}_{t,h}(t_{n-1}).
\end{aligned}$$

By (76), we obtain

$$\hat{u}_h(t_{n-\frac{1}{2}}) - \frac{\hat{u}_h^n + \hat{u}_h^{n-1}}{2} = -\frac{\tau^2}{8}\hat{u}_{tt,h}(t_{n-\frac{1}{2}}) - \eta_3^n, \tag{86}$$

Therefore,

$$\begin{aligned}
\hat{u}_h(t_{n-\frac{1}{2}}) - \hat{u}_h^{n-1} =&(\hat{u}_h(t_{n-\frac{1}{2}}) - \frac{\hat{u}_h^n + \hat{u}_h^{n-1}}{2}) + (\frac{\hat{u}_h^n + \hat{u}_h^{n-1}}{2} - \hat{u}_h^{n-1}) \\
=&-\frac{\tau^2}{8}\hat{u}_{tt,h}(t_{n-\frac{1}{2}}) - \eta_3^n + \frac{\hat{u}_h^n - \hat{u}_h^{n-1}}{2}.
\end{aligned} \tag{87}$$

By (84) and (87), we obtain

$$\begin{aligned}\hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h(t_{n-\frac{1}{2}})) =&\hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1}) + \frac{1}{2}\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})(\hat{u}_h^n - \hat{u}_h^{n-1}) \\ &- \frac{\tau^2}{8}\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{u}_{tt,h}(t_{n-\frac{1}{2}}) \\ &+ \frac{\tau^2}{8}\hat{f}_{\hat{u}\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{u}_{t,h}^2(t_{n-1}) + \varepsilon_3^n,\end{aligned} \tag{88}$$

where

$$\varepsilon_3^n = \eta_5^n - \eta_3^n\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1}). \tag{89}$$

By (85) and (76), we obtain

$$\|\varepsilon_3^n\|_{0,\infty,\hat{\Omega}} = \mathcal{O}(\tau^3). \tag{90}$$

By plugging (88) into (81), we obtain

$$\begin{aligned}&(\frac{\tau^2}{24}D_t^3\hat{u}_h(t_{n-\frac{1}{2}}) + \tau^2 D_t\hat{\psi}_{l+1}(t_{n-\frac{1}{2}}) + \bar{\partial}_t\hat{r}_h^n + \varepsilon_1^n, \hat{v}) \\ &+ A(t_{n-\frac{1}{2}}; \frac{\tau^2}{8}D_t^2\hat{u}_h(t_{n-\frac{1}{2}}) + \tau^2\hat{\psi}_{l+1}(t_{n-\frac{1}{2}}) + \hat{r}_h^{n-\frac{1}{2}} + \varepsilon_2^n, \hat{v}) \\ =&(\hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{U}^{n-1}) - \hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1}), \hat{v}) \\ &+ \frac{1}{2}(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{U}^{n-1})(\hat{U}^n - \hat{U}^{n-1}), \hat{v}) - \frac{1}{2}(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})(\hat{u}_h^n - \hat{u}_h^{n-1}), \hat{v}) \\ &\frac{\tau^2}{8}(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{u}_{tt,h}(t_{n-\frac{1}{2}}) - \hat{f}_{\hat{u}\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{u}_{t,h}^2(t_{n-1}) - \varepsilon_3^n, \hat{v}),\ \forall\hat{v}\in\hat{S}_0^h.\end{aligned} \tag{91}$$

By the Taylor expansions and (69), we obtain

$$\begin{aligned}\hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{U}^{n-1}) =&\hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1}) + \hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})(\hat{U}^{n-1} - \hat{u}_h^{n-1}) + \eta_6^n \\ =&\hat{f}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1}) + \hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})(\tau^2\hat{\psi}_{l+1}^{n-1} + \hat{r}_h^{n-1}) + \eta_6^n,\end{aligned} \tag{92}$$

where

$$\eta_6^n = \frac{1}{2}\hat{f}_{\hat{u}\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})(\hat{U}^{n-1} - \hat{u}_h^{n-1})^2. \tag{93}$$

By the finite element error estimate, we obtain

$$\|\eta_6^n\|_{0,\infty,\hat{\Omega}} = \mathcal{O}(\tau^4). \tag{94}$$

By (69), we obtain

$$\hat{U}^n - \hat{U}^{n-1} = (\hat{u}_h^n - \hat{u}_h^{n-1}) + \tau^2(\hat{\psi}_{l+1}^n - \hat{\psi}_{l+1}^{n-1}) + (\hat{r}_h^n - \hat{r}_h^{n-1}). \tag{95}$$

By plugging (92) and (95) into (91), we obtain

(96)
$$
\begin{aligned}
&(\frac{\tau^2}{24}D_t^3\hat{u}_h(t_{n-\frac{1}{2}})+\tau^2 D_t\hat{\psi}_{l+1}(t_{n-\frac{1}{2}})+\bar{\partial}_t\hat{r}_h^n+\varepsilon_1^n,\hat{v})\\
&+A(t_{n-\frac{1}{2}};\frac{\tau^2}{8}D_t^2\hat{u}_h(t_{n-\frac{1}{2}})+\tau^2\hat{\psi}_{l+1}(t_{n-\frac{1}{2}})+\hat{r}_h^{n-\frac{1}{2}}+\varepsilon_2^n,\hat{v})\\
=&\tau^2(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{\psi}_{l+1}^{n-1},\hat{v})+(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{r}_h^{n-1},\hat{v})\\
&+\frac{1}{2}(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})(\hat{r}_h^n-\hat{r}_h^{n-1}),\hat{v})\\
&\frac{\tau^2}{8}(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{u}_{tt,h}(t_{n-\frac{1}{2}})-\hat{f}_{\hat{u}\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{u}_{t,h}^2(t_{n-1})+(\eta_7^n,\hat{v}),\ \forall\hat{v}\in\hat{S}_0^h,
\end{aligned}
$$

where

$$
\begin{aligned}
\eta_7^n=&-\varepsilon_3^n+\eta_6^n+\frac{\tau^2}{2}\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{U}^{n-1})(\hat{\psi}_{l+1}^n-\hat{\psi}_{l+1}^{n-1})\\
&+\frac{1}{2}[\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{U}^{n-1})-\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})][(\hat{u}_h^n-\hat{u}_h^{n-1})+(\hat{r}_h^n-\hat{r}_h^{n-1})].
\end{aligned}
\tag{97}
$$

By (90), (94), Taylor expansion and the finite element error estimate, we obtain

$$
\|\eta_7^n(\hat{v})\|_{0,\infty,\hat{\Omega}}=\mathcal{O}(\tau^3).
\tag{98}
$$

We choose $\hat{\psi}_{l+1}\in H^1(0,T;\hat{S}_0^h)$ such that

(99)
$$
\begin{cases}
 & (D_t\hat{\psi}_{l+1}(t_{n-\frac{1}{2}}),\hat{v})+A(t_{n-\frac{1}{2}};\hat{\psi}_{l+1}(t_{n-\frac{1}{2}}),\hat{v})-(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{\psi}_{l+1}^{n-1},\hat{v})\\
= & -(\frac{1}{24}D_t^3\hat{u}_h(t_{n-\frac{1}{2}}),\hat{v})-\frac{1}{8}A(t_{n-\frac{1}{2}};D_t^2\hat{u}_h(t_{n-\frac{1}{2}}),\hat{v})\\
 & +\frac{1}{8}(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{u}_{tt,h}(t_{n-\frac{1}{2}})-\hat{f}_{\hat{u}\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{u}_{t,h}^2(t_{n-1}),\hat{v}),\ \forall\hat{v}\in\hat{S}_0^h,\\
 & \hat{\psi}_{l+1}(0,\cdot)=0.
\end{cases}
$$

Then by (69), (96) and (99), we obtain

$$
\begin{cases}
(\bar{\partial}_t\hat{r}_h^n,\hat{v})+A(t_{n-\frac{1}{2}};\hat{r}_h^{n-\frac{1}{2}},\hat{v})-(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})\hat{r}_h^{n-1},\hat{v})\\
\quad-\frac{1}{2}(\hat{f}_{\hat{u}}(t_{n-\frac{1}{2}},\hat{x},\hat{u}_h^{n-1})(\hat{r}_h^n-\hat{r}_h^{n-1}),\hat{v})=\varepsilon_4^n(\hat{v}),\ \forall\hat{v}\in\hat{S}_0^h,\\
\hat{r}_h^0=0,
\end{cases}
\tag{100}
$$

where

$$
\varepsilon_4^n(\hat{v})=(\eta_7^n-\varepsilon_1^n,\hat{v})-A(t_{n-\frac{1}{2}};\varepsilon_2^n,\hat{v}).
\tag{101}
$$

Then by (75), (80) and (98), we obtain

$$
|\varepsilon_4^n(\hat{v})|=\mathcal{O}(\tau^3).
\tag{102}
$$

In summary. (69), (99), and (100) together complete the proof of (67).

Let $\hat{v}=\bar{\partial}_t\hat{r}_h^n$ in (100). Then by the boundedness of $\hat{f}_{\hat{u}}(t,\hat{x},\hat{u}_h)$ in $\hat{Q}_T$, the intermediate value theorem, the Cauchy-Schwarz inequality, the Gronwall inequality, and the inverse inequality, we can follow the corresponding arguments in [36, 43] to obtain (68).

■

Following the framework and arguments of the above proof and the ideas in [36, 43], one can similarly obtain the following conclusion.

Lemma 4.2. *Along with the same assumptions of Theorem 3.1, assume that the solution to (8) satisfies* $\hat{u}_h \in C^3(0,T;S_0^h)$, $\hat{a}_{ij}(t,\hat{x},\hat{u})$ *and* $\hat{f}(t,\hat{x},\hat{u})$ *are third order differentiable for* $\hat{u}$. *Then for the solution* $\hat{U}^n$ *to (10), there exists a function* $\hat{\psi}_{l+1}(t,\cdot) \in H^1(0,T;\hat{S}_0^h)$ *independent of* τ *and* $\hat{h}$ *such that*

$$\hat{U}^n - \hat{u}_h^n = \tau\hat{\psi}_{l+1}^n + \hat{r}_h^n, 1 \le n \le N, \tag{103}$$

where

$$\max_{1\le n\le N} \|\hat{r}_h^n\|_{0,\infty,\hat{Q}_T} = \mathcal{O}((\tau^{\frac{3}{2}}\hat{h}_0^{1-\frac{d}{2}} + \tau\hat{h}_0^{\alpha_1})|\ln\hat{h}_0|^{\frac{d-1}{d}}),\ \alpha_1 = 1+\gamma-\frac{d}{2}. \tag{104}$$

Combining the above results with Theorem 3.1, we can obtain the following two theorems for the multi-parameter asymptotic expansion of the fully discrete d-quadratic iso-parametric finite element errors.

Theorem 4.1. *Along with the same assumptions of Theorem 3.1 and Lemma 4.1, for the solution to (9), there exist functions* $\hat{\psi}_i(t,x)(i=1,\cdots,l+1)$ *independent of* $\hat{h}$ *such that*

$$\hat{U}^n(x) - \hat{u}^n(x) = \sum_{i=1}^{l} \hat{h}_i^4\hat{\psi}_i^n(x) + \hat{h}_{l+1}^2\hat{\psi}_{l+1}^n(x) + \hat{\varepsilon}^n(x), \forall x \in \Omega_0^h, 1 \le n \le N, \tag{105}$$

where

$$\|\hat{\varepsilon}^n\|_{0,\infty,\hat{Q}_T} = \mathcal{O}((\hat{h}_0^{4+\beta} + \hat{h}_{l+1}^2\hat{h}_{00}^{\beta_4})|\ln\hat{h}_0|^{\frac{d-1}{d}}). \tag{106}$$

Proof. By combining Theorem 3.1 and Lemma 4.1, we obtain

$$\begin{aligned}\hat{U}^n(\hat{X}) - \hat{u}^n(\hat{X}) =& (\hat{U}^n(\hat{X}) - \hat{u}_h^n(\hat{X})) + (\hat{u}_h^n(\hat{X}) - \hat{u}^n(\hat{X}))\\ =& \sum_{i=1}^{l}\hat{h}_i^4\hat{\psi}_i^n(\hat{X}) + \hat{h}_{l+1}^2\hat{\psi}_{l+1}^n(\hat{X}) + \hat{\varepsilon}^n. \end{aligned}\tag{107}$$

By (22) and (68), we obtain the estimate for $\hat{\varepsilon}^n$.

■

Theorem 4.2. *Along with the same assumptions of Theorem 3.1 and Lemma 4.2, for the solution to (10), there exist functions* $\hat{\psi}_i(t,x)(i=1,\cdots,l+1)$ *independent of* $\hat{h}$ *such that*

$$\hat{U}^n(x) - \hat{u}^n(x) = \sum_{i=1}^{l} \hat{h}_i^4\hat{\psi}_i^n(x) + \hat{h}_{l+1}\hat{\psi}_{l+1}^n(x) + \hat{\varepsilon}^n(x), \forall x \in \Omega_0^h, 1 \le n \le N, \tag{108}$$

where

$$\|\hat{\varepsilon}^n\|_{0,\infty,\hat{Q}_T} = \mathcal{O}((\hat{h}_0^{4+\beta} + \hat{h}_{l+1}^{\frac{3}{2}}\hat{h}_0^{1-\frac{d}{2}} + \hat{h}_{l+1}\hat{h}_0^{\alpha_1})|\ln\hat{h}_0|^{\frac{d-1}{d}}). \tag{109}$$

Proof. By combining Theorem 3.1 and Lemma 4.2, we obtain

$$\begin{aligned}\hat{U}^n(\hat{X}) - \hat{u}^n(\hat{X}) =& (\hat{U}^n(\hat{X}) - \hat{u}_h^n(\hat{X})) + (\hat{u}_h^n(\hat{X}) - \hat{u}^n(\hat{X}))\\ =& \sum_{i=1}^{l}\hat{h}_i^4\hat{\psi}_i^n(\hat{X}) + \hat{h}_{l+1}\hat{\psi}_{l+1}^n(\hat{X}) + \hat{\varepsilon}^n. \end{aligned}\tag{110}$$

By (22) and (104), we obtain the estimate for $\hat{\varepsilon}^n$.

■

Remark 4.1. *Similar to the Remark 4 in* [41]*, the expansions are true for all the original coarse grid nodes in* $\hat{\Omega}_0^h$*, as well as all the edge midpoints and centers in* $\hat{\Omega}_0^h$*. Furthermore, the multi-parameter expansions only require the local solution smoothness in each sub-domain* $\hat{\Omega}_s$*. Hence it is efficient to utilize the splitting extrapolation methods to solve interface problems, by using the original problem interface for the domain decomposition steps.*

5. Splitting extrapolation formulas at globally fine grid points

Based on the above multi-parameter asymptotic error expansions and the basic idea of splitting extrapolation, in this section we will develop the splitting extrapolation schemes for all the nodes in the globally refined grid, not only on the coarse grid or the locally refined grids. Here we first explain the basic idea of splitting extrapolation based on the multi-parameter asymptotic error expansions, before we present the detailed formulations. First, on each of the coarse grid and the locally refined grids, we can apply the multi-parameter asymptotic error expansion with its grid parameters. Then all of the low order error terms in these multi-parameter asymptotic error expansions can be canceled out by taking an appropriate linear combination of these expansions, which is the key of the splitting extrapolation. Eventually this linear combination will lead to a new numerical solution with only the higher order error terms survived. Hence the new solution achieves higher order accuracy.

Because the derivation is similar to that of [25, 26], we only show the conclusions without proof here. Let $\hat{\Omega}_0^h$ denotes the set of grid points obtained from the initial grid parameter $\hat{h}_1, \cdots, \hat{h}_{l+1}$, $\hat{\Omega}_i^h$ denote the set of grid points obtained from $\hat{h}^{(i)} = (\hat{h}_1, \cdots, \frac{\hat{h}_i}{2}, \cdots, \hat{h}_{l+1}, \hat{\Omega}_i^h)$, $i = 1, \cdots, l+1$, $\hat{U}_0^n$ denote the fully discrete approximation at time t_n on $\hat{\Omega}_0^h$, and $\hat{U}_i^n$ denote the fully discrete approximation at time t_n on $\hat{\Omega}_i^h$, $i = 1, \cdots, l+1$.

First, consider the splitting extrapolation formulas for the semi-linear parabolic equation.

(1) type 0: grid points in $\hat{\Omega}_0^h$. Suppose A is a grid point in $\hat{\Omega}_0^h$, then the splitting extrapolation formula for A is

$$U_0(A) = \frac{16}{15}\sum_{i=1}^{l}\hat{U}_i^n(A) + \frac{4}{3}\hat{U}_{(l+1)}^n(A) + \left[-\frac{16}{15}l - \frac{1}{3}\right]\hat{U}_0^n(A). \tag{111}$$

(2) type 1: grid points in $\bigcup_{i=1}^{l+1}\hat{\Omega}_i^h \backslash \hat{\Omega}_0^h$. Let A_1 and A_2 be the two neighboring coarse grid points. Suppose B is the midpoint of A_1A_2 and $B \in \hat{\Omega}_i^h \backslash \hat{\Omega}_0^h$. Then the splitting extrapolation formula for B is

$$\begin{aligned} U_1(B) =& \hat{U}_i^n(B) - \frac{1}{30}\sum_{k=1}^{2}\left[\hat{U}_0^n(A_k) - \hat{U}_i^n(A_k)\right] \\ & - \frac{8}{15}\sum_{\substack{j=1\\ j\neq i}}^{l}\sum_{k=1}^{2}\left[\hat{U}_0^n(A_k) - \hat{U}_j^n(A_k)\right] - \frac{2}{3}\sum_{k=1}^{2}\left[\hat{U}_0^n(A_k) - \hat{U}_{l+1}^n(A_k)\right]. \end{aligned} \tag{112}$$

(3) type 2: Centers of rectangular elements. Suppose C is the center of a rectangular element, A_k $(k = 1, \cdots, 4)$ are the four vertices and B_k $(k = 1, \cdots, 4)$ are the midpoints of the four edges. First, $U_0(A_k)$ and $U_1(B_k)$ are computed according to (111) and (112). Then by using an incomplete bi-quadratic interpolation without

term x^2y^2 [46, 40], we obtain

$$U_2(C) = \frac{1}{2}\sum_{k=1}^{4} U_1(B_k) - \frac{1}{4}\sum_{k=1}^{4} U_0(A_k). \tag{113}$$

(4) type 3: Centers of rectangular parallelepiped elements. Suppose D is the center of a rectangular parallelepiped element, A_k ($k = 1, \cdots, 8$) are the eight vertices and B_k ($k = 1, \cdots, 12$) are the midpoints of the twelve edges. First, $U_0(A_k)$ and $U_1(B_k)$ are computed according to (111) and (112). Then by using an incomplete tri-quadratic interpolation without term $x^2y^2z^2,\ x^2y^2z,\ x^2yz^2,\ xy^2z^2,\ x^2y^2,$ $x^2z^2,\ y^2z^2$ [46, 40], we obtain

$$U_3(D) = \frac{1}{4}\sum_{k=1}^{12} U_1(B_k) - \frac{1}{4}\sum_{k=1}^{8} U_0(A_k). \tag{114}$$

Second, consider the splitting extrapolation formulas for the quasi-linear parabolic equation.

(1) type 0: grid points in $\hat{\Omega}_0^h$. Suppose A is a grid point in $\hat{\Omega}_0^h$. Then the splitting extrapolation formula for A is

$$U_0(A) = \frac{16}{15}\sum_{i=1}^{l} \hat{U}_i^n(A) + 2\hat{U}_{(l+1)}^n(A) + \left[-\frac{16}{15}l - 1\right]\hat{U}_0^n(A). \tag{115}$$

(2) type 1: grid points in $\bigcup_{i=1}^{l+1} \hat{\Omega}_i^h \backslash \hat{\Omega}_0^h$. Let A_1 and A_2 be the two neighboring coarse grid points. Suppose B is the midpoint of A_1A_2 and $B \in \hat{\Omega}_i^h \backslash \hat{\Omega}_0^h$. Then the splitting extrapolation formula for B is

$$\begin{aligned} U_1(B) =& \hat{U}_i^n(B) - \frac{1}{30}\sum_{k=1}^{2}\left[\hat{U}_0^n(A_k) - \hat{U}_i^n(A_k)\right] \\ & - \frac{8}{15}\sum_{\substack{j=1 \\ j\neq i}}^{l}\sum_{k=1}^{2}\left[\hat{U}_0^n(A_k) - \hat{U}_j^n(A_k)\right] - \sum_{k=1}^{2}\left[\hat{U}_0^n(A_k) - \hat{U}_{l+1}^n(A_k)\right]. \end{aligned} \tag{116}$$

(3) type 2 and type 3: The formulas are the same as (113) and (113) with computing $U_0(A_k)$ and $U_1(B_k)$ are computed according to (115) and (116).

6. A Posteriori error estimate

In this section, we also present some a posteriori error estimates. Because the techniques for the proof are the same as in [25, 26], we only show the conclusions without proof here. Suppose A is a grid point in $\hat{\Omega}_0^h$.

First, consider some a posteriori error estimates for the semi-linear parabolic equation.

Theorem 6.1. *For the semi-linear parabolic equation* (1)*, let $\hat{U}_0^n(A)$ and $\hat{U}_j^n(A)$ ($j = 1, \dots, l+1$) be the finite element solutions on the coarse and locally refined grids, respectively, at a grid point $A \in \hat{\Omega}_0^h$. Then the following estimates hold:*

$$\begin{aligned} \left|\hat{U}_0^n(A) - \hat{u}^n(A)\right| \leq& \frac{16}{15}\sum_{j=1}^{l}\left|\hat{U}_0^n(A) - \hat{U}_j^n(A)\right| + \frac{4}{3}\left|\hat{U}_0^n(A) - \hat{U}_{l+1}^n(A)\right| \\ & + \mathcal{O}((\hat{h}_0^{4+\beta} + \hat{h}_{l+1}^2\hat{h}_{00}^{\beta_4})|\ln \hat{h}_0|^{\frac{d-1}{d}}), \end{aligned} \tag{117}$$

$$\begin{aligned}\left|\hat{U}_k^n(A)-\hat{u}^n(A)\right| \leq & \frac{16}{15}\sum_{j=1}^{l}\left|\hat{U}_0^n(A)-\hat{U}_j^n(A)\right| \\ & +\frac{4}{3}\left|\hat{U}_0^n(A)-\hat{U}_{l+1}^n(A)\right|+\left|U_0^n(A)-U_k^n(A)\right| \\ & +\mathcal{O}((\hat{h}_0^{4+\beta}+\hat{h}_{l+1}^2\hat{h}_{00}^{\beta_4})|\ln\hat{h}_0|^{\frac{d-1}{d}}),\ k=1,\cdots,l,\end{aligned} \tag{118}$$

$$\begin{aligned}\left|\hat{U}_{l+1}^n(A)-\hat{u}^n(A)\right| \leq & \frac{16}{15}\sum_{j=1}^{l}\left|\hat{U}_0^n(A)-\hat{U}_j^n(A)\right|+\frac{1}{3}\left|\hat{U}_0^n(A)-\hat{U}_{l+1}^n(A)\right| \\ & +\mathcal{O}((\hat{h}_0^{4+\beta}+\hat{h}_{l+1}^2\hat{h}_{00}^{\beta_4})|\ln\hat{h}_0|^{\frac{d-1}{d}}).\end{aligned} \tag{119}$$

Theorem 6.2. *For the semi-linear parabolic equation* (1)*, the error of the averaged solution at a grid point* $A \in \hat{\Omega}_0^h$ *satisfies:*

$$\begin{aligned}\left|\frac{1}{l+1}\sum_{j=1}^{l+1}\hat{U}_j^n(A)-\hat{u}^n(A)\right| \leq & \left|\frac{1}{l+1}\sum_{j=1}^{l+1}\hat{U}_j^n(A)-\hat{U}_0^n(A)\right| \\ & +\frac{16}{15}\sum_{j=1}^{l}\left|\hat{U}_0^n(A)-\hat{U}_j^n(A)\right|+\frac{4}{3}\left|\hat{U}_0^n(A)-\hat{U}_{l+1}^n(A)\right| \\ & +\mathcal{O}((\hat{h}_0^{4+\beta}+\hat{h}_{l+1}^2\hat{h}_{00}^{\beta_4})|\ln\hat{h}_0|^{\frac{d-1}{d}}).\end{aligned} \tag{120}$$

Second, consider some a posteriori error estimates for the quasi-linear parabolic equation.

Theorem 6.3. *For the quasi-linear parabolic equation* (2)*, let* $\hat{U}_0^n(A)$ *and* $\hat{U}_j^n(A)$ *(*$j=1,\ldots,l+1$*) be the finite element solutions on the coarse and locally refined grids, respectively, at a grid point* $A \in \hat{\Omega}_0^h$*. Then the following estimates hold:*

$$\begin{aligned}\left|\hat{U}_0^n(A)-\hat{u}^n(A)\right| \leq & \frac{16}{15}\sum_{j=1}^{l}\left|\hat{U}_0^n(A)-\hat{U}_j^n(A)\right|+2\left|\hat{U}_0^n(A)-\hat{U}_{l+1}^n(A)\right| \\ & +\mathcal{O}((\hat{h}_0^{4+\beta}+\hat{h}_{l+1}^{\frac{3}{2}}\hat{h}_0^{1-\frac{d}{2}}+\hat{h}_{l+1}\hat{h}_0^{\alpha_1})|\ln\hat{h}_0|^{\frac{d-1}{d}}),\end{aligned} \tag{121}$$

$$\begin{aligned}\left|\hat{U}_k^n(A)-\hat{u}^n(A)\right| \leq & \frac{16}{15}\sum_{j=1}^{l}\left|\hat{U}_0^n(A)-\hat{U}_j^n(A)\right| \\ & +2\left|\hat{U}_0^n(A)-\hat{U}_{l+1}^n(A)\right|+\left|U_0^n(A)-U_k^n(A)\right| \\ & +\mathcal{O}((\hat{h}_0^{4+\beta}+\hat{h}_{l+1}^{\frac{3}{2}}\hat{h}_0^{1-\frac{d}{2}}+\hat{h}_{l+1}\hat{h}_0^{\alpha_1})|\ln\hat{h}_0|^{\frac{d-1}{d}}),\ k=1,\cdots,l,\end{aligned} \tag{122}$$

$$\begin{aligned}\left|\hat{U}_{l+1}^n(A)-\hat{u}^n(A)\right| \leq & \frac{16}{15}\sum_{j=1}^{l}\left|\hat{U}_0^n(A)-\hat{U}_j^n(A)\right|+\left|\hat{U}_0^n(A)-\hat{U}_{l+1}^n(A)\right| \\ & +\mathcal{O}((\hat{h}_0^{4+\beta}+\hat{h}_{l+1}^{\frac{3}{2}}\hat{h}_0^{1-\frac{d}{2}}+\hat{h}_{l+1}\hat{h}_0^{\alpha_1})|\ln\hat{h}_0|^{\frac{d-1}{d}}).\end{aligned} \tag{123}$$

Theorem 6.4. *For the quasi-linear parabolic equation* (2)*, the error of the averaged solution at a grid point* $A \in \hat{\Omega}_0^h$ *satisfies:*

$$\begin{aligned}\left|\frac{1}{l+1}\sum_{j=1}^{l+1}\hat{U}_j^n(A)-\hat{u}^n(A)\right| &\le \left|\frac{1}{l+1}\sum_{j=1}^{l+1}\hat{U}_j^n(A)-\hat{U}_0^n(A)\right| \\ &+\frac{16}{15}\sum_{j=1}^{l}\left|\hat{U}_0^n(A)-\hat{U}_j^n(A)\right|+2\left|\hat{U}_0^n(A)-\hat{U}_{l+1}^n(A)\right| \\ &+\mathcal{O}((\hat{h}_0^{4+\beta}+\hat{h}_{l+1}^2\hat{h}_{00}^{\beta_4})|\ln\hat{h}_0|^{\frac{d-1}{d}}). \qquad (124)\end{aligned}$$

7. Numerical experiments

In this section, we will present two numerical examples to illustrate the features of the proposed finite element splitting extrapolation method. We will see that our method is valid for solving interface problems if we utilize the interfaces of the original problems for the domain decomposition.

As explained in [26], in order to obtain the splitting extrapolation solution on the globally fine grid, we only need to compute the regular bi-quadratic finite element solutions on the coarse grid and the locally fine grids. In these computations, we do not need to compute the finite element solutions at the globally fine grid nodes which are not the nodes of either the coarse grid or the locally fine grids. In the tables of this section, let “** denote these errors which are not computed on the coarse grid and the locally fine grids, “Error of FE denote the error of the regular bi-quadratic finite element solutions, “Error of SE denote the error of the splitting extrapolation solution, and “Max error denote the maximum error on all nodes at all time steps.

Example 1: Consider a semi-linear parabolic interface equation

$$\begin{cases} \frac{\partial u}{\partial t} - \bigtriangledown(a(x,y)\bigtriangledown u) = f(x,y,t,u) & \text{on} \quad \Omega\times[0,T], \\ u(x,y,0)=\Psi(x,y) & \text{on} \quad \bar{\Omega}, \\ u(x,y,t)=0 & \text{on} \quad \partial\Omega\times[0,T], \end{cases}$$

where

$$a(x,y)=\begin{cases} r, & x<1, \\ 1, & x\ge 1, \end{cases}$$

and

$$f(x,y,t,u)=\begin{cases} -\tan t\cdot u+15r(r+1)y(y-1)(3x-2)\cos t & \\ \quad -15r[6-(r+1)x(x-1)^2]\cos t, & x<1, \\ -\tan t\cdot u+15(r+1)y(y-1)(3x-2)\cos t & \\ \quad -15[2rx+2-2r-(r+1)x(x-1)^2]\cos t, & x\ge 1. \end{cases}$$

Here Ω is a curved quadrangle, whose bottom boundary is on the straight line connecting $P_1=(0,0)$ and $P_2=(2,0)$, top boundary is on the straight line connecting $P_4=(0,1)$ and $P_3=(2,1)$, left boundary is a parabola connecting P_1, $P_8=(-0.25,0.5)$, and P_4, and right boundary is a parabola connecting P_2, $P_6=(2.25,0.5)$, and P_3. Also, we define $P_5=(1,0), P_7=(1,1), P_9=(1,\frac{1}{2})$. The initial domain decomposition is constructed as $\bar{\Omega}=\bigcup_{s=1}^{2}\bar{\Omega}_s$ where $\Omega_1=\Omega\bigcap\{x<1\}$ and $\Omega_2=\Omega\bigcap\{x>1\}$. With the d-quadratic iso-parametric mapping, Ω, Ω_1, and Ω_2 are mapped to $\hat{\Omega}=(0,2)\times(0,1)$, $\hat{\Omega}_1=(0,1)\times(0,1)$, and $\hat{\Omega}_2=(1,2)\times(0,1)$ separately. Then we design four independent step sizes: define $h_i(i=1,2)$ to be the grid step sizes of $\hat{\Omega}_i(i=1,2)$ in the x-direction, h_3 to be the grid step size in the y-direction, and h_4 to be the temporal step size. The interface of the domain decomposition $\bar{\Omega} = \Omega_1 \bigcup \Omega_2$ is exactly the same as the interface of $a(x, y)$, i.e., $x = 1$. Choose $h_i = \frac{1}{4}(i = 1, 2, 3, 4)$ and $T = 1$. Then the numerical results in Table 1 show the dramatic accuracy improvement.

TABLE 1. Numerical error comparison between the regular finite element (FE) solutions and the splitting extrapolation (SE) solutions for Example 1.

Grid points	Point type	Error of FE	Error of SE
$(-0.0801, 0.3750, T)$	type 0	-1.8850×10^{-4}	1.1282×10^{-6}
$(1.0000, 0.5000, T)$	type 0	-1.8595×10^{-3}	2.4163×10^{-4}
$(0.0889, 0.6875, T)$	type 1	-3.5537×10^{-4}	1.9720×10^{-6}
$(1.0000, 0.5625, T)$	type 1	-1.3739×10^{-3}	7.4213×10^{-5}
$(0.4958, 0.1875, T)$	type 2	**	-2.1941×10^{-5}
$(0.8015, 0.9375, T)$	type 2	**	-4.8336×10^{-7}
Max error on coarse grid		-1.8595×10^{-3}	-5.3819×10^{-4}
Max error on fine grid		**	-5.3819×10^{-4}

Example 2: Consider a quasi-linear parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - \bigtriangledown(u \bigtriangledown u) = f(x, y, t, u), & \text{on} \quad \Omega \times [0, T], \\ u(x, y, 0) = x(x-2)y(y-1), & \text{on} \quad \bar{\Omega}, \\ u(x, y, t) = 0, & \text{on} \quad \partial\Omega \times [0, T], \end{cases}$$

where Ω is the same curved quadrangle as in Example 1 and

$$\begin{aligned} f(x, y, t, u) =& x(x-2)y(y-1)e^t - \frac{2}{x(x-2)}u^2 - 2x^2(x-2)^2e^{2t}y(y-1) \\ &- 4(x-1)^2y^2(y-1)^2e^{2t} - (2y-1)^2x^2(x-2)^2e^{2t}. \end{aligned}$$

The construction of the initial domain decomposition and design of independent step sizes are the same as in Example 1. Choose $h_i = \frac{1}{4}(i = 1, 2, 3, 4)$ and $T = 1$. Then the numerical results in Table 2 show the dramatic accuracy improvement.

8. Conclusion and Future Prospects

In this manuscript, we developed a finite element splitting extrapolation method for more accurately and efficiently solving the nonlinear second order parabolic equations. Based on the idea of domain decomposition, a group of independent grid size parameters were designed for the subdomains to form a grid on the whole domain. And the regular iso-parametric finite element method is presented based on this grid. After the multi-parameter asymptotic expansions of the semi-discrete and fully discrete iso-parametric finite element errors were proved, they are utilized to construct the splitting extrapolation schemes. A posterior error estimates are also presented for the finite element splitting extrapolation schemes. Numerical examples are provided to illustrate the effect of the splitting extrapolation schemes.

Sophisticated interfaces, such as general smooth curves, are indeed more difficult to analyze and compute than the straight-line interfaces. The approximation

TABLE 2. Numerical error comparison between the regular finite element (FE) solutions and the splitting extrapolation (SE) solutions for Example 2.

Grid points	Point type	Error of FE	Error of SE
$(2.0391, 0.7500, T)$	type 0	-1.3048×10^{-2}	1.9629×10^{-5}
$(1.0000, 0.1250, T)$	type 0	1.5548×10^{-2}	3.8575×10^{-4}
$(2.0903, 0.5625, T)$	type 1	-1.0150×10^{-2}	8.7175×10^{-6}
$(1.0000, 0.1875, T)$	type 1	3.1612×10^{-2}	3.1334×10^{-4}
$(1.6482, 0.8125, T)$	type 2	**	8.2897×10^{-4}
$(-0.0125, 0.5625, T)$	type 2	**	1.7171×10^{-4}
Max error on coarse grid		6.1057×10^{-2}	-3.5523×10^{-3}
Max error on fine grid		**	-2.0857×10^{-2}

of curved interfaces would generate geometric discretization errors, for which the iso-parametric d-quadratic mapping is more accurate than the piecewise linear approximation of the interfaces. It is an interesting future work to extend the proposed splitting extrapolation method to more complex interface problems, which clearly have a wider range of application scenarios and more significant challenges.

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