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ANALYSIS OF A TYPE II THERMAL PROBLEM INVOLVING A VISCOELASTIC BEAM

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Abstract. In this work, we will study, from both analytical and numerical points of view, a nonlocal problem involving a thermoviscoelastic beam which has been modeled by using the type II thermal law. In the first part, we will show that this problem has a unique solution and that the solutions decay exponentially by using the theory of linear semigroups. We will also prove that the semigroup of contractions is not differentiable and the impossibility of localization, that is, we will obtain that the unique solution which can vanish in an open nonempty set is the null solution. In the second part, we will focus on the numerical approximation of a variational formulation of the thermomechanical problem. By using the finite element method and the implicit Euler scheme to approximate the spatial variable and to discretrize the time derivatives, respectively, a fully discrete scheme will be introduced. Then, we will prove a discrete stability property and we will provide an a priori error analysis. The linear convergence of the approximations will be deduced whenever the continuous solution is regular enough. Finally, some numerical results will be presented to demonstrate the numerical convergence and the exponential decay of the discrete energy.

Key words. Viscoelastic beam, type II thermoelasticity, finite elements, a priori error estimates, numerical simulations.

1. Introduction

The study of thermoelastic materials has received a large number of contributions dealing with both quantitative or qualitative aspects. In these studies, we can find results about the existence, uniqueness and stability as well as the numerical behavior of the solutions. In this sense, it is suitable to recall several contributions concerning plate thermoelastic problems [1, 2, 3, 4, 10, 12, 13, 17, 18, 20, 22]. In these references, the conservative component is mechanical and the dissipative aspect is thermal. The objective of this work is to follow these lines but it is worth noting that, here, we consider a nonlocal thermoviscoelastic bar formulated with a conservative heat conduction model.

Therefore, it is adequate to recall that the idea of non-locality in elasticity was introduced by Eringen [8, 9] and that this mechanism suggests a regularization of the solutions [11]. From the physical point of view, this is introduced to take into account the effects at long distances.On the other hand, from the thermal point of view, it is worth noting that Green and Naghdi proposed three thermoelastic theories depending on the type of heat conduction (see, for details, [14, 15, 16]). In this paper, we consider the so-called type II thermal law, which does not allow the energy dissipation, leading to a conservative behavior. That is, in this paper we change the role of the mechanical and thermal aspects and we consider the nonlocal effect.

The paper is structured as follows. In the next section we will describe the problem that we will study in this work. Then, in Sections 3 and 4 the existence

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and the energy decay of the solutions will be obtained by means of the theory of linear semigroups, although we will also prove that the semigroup is not differentiable. In Section 5, we will see that the unique solution which vanishes for every time $t \geq t_0 \geq 0$ is the null solution. Then, in Section 6 we will focus on the numerical approximation of a variational formulation of the above thermomechanical problem. The fully discrete scheme will be provided by using the classical finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. A discrete stability property and an a priori error analysis will be shown by using a discrete version of Gronwall's inequality. The linear convergence of the approximations will be deduced under some additional regularity of the continuous solution. Finally, in Section 7 we will present two numerical tests: the first one will demonstrate the convergence of the discrete solution when both spatial parameter and time step tend to zero, leading to the theoretical linear convergence. The second example will show the exponential decay of the discrete energy.

2. Preliminaries

In this paper, we will consider a thermoviscoelastic bar which occupies the domain $(0, \pi)$, and modeled by using the Euler-Bernoulli theory when the heat conduction is determined by the type II Green-Naghdi theory (see [16]).

Therefore, we will study the system:

$$\left.\begin{array}{l} \rho u_{tt} - \tau u_{ttxx} + \mu u_{xxxx} + \mu^* u_{txxxx} - \beta \alpha_{txx} = 0, \\ c\alpha_{tt} - \kappa \alpha_{xx} + \beta u_{txx} = 0 \end{array}\right\} \quad \text{in } (0,\pi) \times (0,T). \tag{1}$$

In the previous equations, T is the final time, u is the mechanical displacement and α is the thermal displacement which satisfies $\alpha_t = \theta$ (the temperature). We will also consider the boundary conditions:

$$\left.\begin{array}{l} u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 \\ \alpha(0,t) = \alpha(\pi,t) = 0 \end{array}\right\} \quad \text{for a.e. } t \in (0,T), \tag{2}$$

and the initial conditions:

$$\left.\begin{array}{ll} u(x,0) = u_0(x), & u_t(x,0) = v_0(x) \\ \alpha(x,0) = \alpha_0(x), & \alpha_t(x,0) = \theta_0(x) \end{array}\right\} \quad \text{for a.e. } x \in (0,\pi). \tag{3}$$

In this work, we will assume that ρ, τ, μ, μ^*, c, κ and β are constants and, in general, that ρ, τ, μ, μ^*, c and κ are positive. Moreover, when we study the energy decay of the solutions to our problem, we will also assume that $\beta \neq 0$.

We can recall that our problem is similar to the one studied analytically in [21]. However, here we consider the inertial effects determined by the parameter τ. Furthermore, in this work we emphasize in the numerical aspect of the problem, which was not considered in [21].

3. Existence of solutions

In this section, we will show the existence and uniqueness of solutions to the problem determined by system (1), boundary conditions (2) and initial conditions (3). First, we will write our problem as a Cauchy problem in an adequate Hilbert space. Let us denote $U = (u, v, \alpha, \theta)$ and consider the Hilbert space:

$$\mathcal{H} = H_0^1(0,\pi) \cap H^2(0,\pi) \times H_0^1(0,\pi) \times H_0^1(0,\pi) \times L^2(0,\pi),$$

where $L^2(0,\pi)$, $H_0^1(0,\pi)$ and $H^2(0,\pi)$ represent the usual Sobolev spaces.

We can define the scalar product in $\mathcal{H}$ associated to the norm:

$$\|U\|_{\mathcal{H}}^2 = \mu\|u_{xx}\|^2 + \rho\|v\|^2 + \tau\|v_x\|^2 + \kappa\|\alpha_x\|^2 + c\|\theta\|^2.$$

If we define the operator:

$$\mathcal{A}U = \begin{pmatrix} v \\ -(\rho - \tau\partial_{xx})^{-1}\partial_{xx}(\mu u_{xx} + \mu^* v_{xx} - \beta\theta) \\ \theta \\ c^{-1}\partial_{xx}(\kappa\alpha - \beta v) \end{pmatrix}, \tag{4}$$

we can write our problem in the following abstract form:

$$U_t - \mathcal{A}U = 0, \quad U(0) = (u_0, v_0, \alpha_0, \theta_0). \tag{5}$$

We observe that the domain of the operator $\mathcal{A}$ is defined as

$$\begin{aligned}\mathcal{D}(\mathcal{A}) =&\{U = (u, v, \alpha, \theta) \in \mathcal{H}\,;\, v \in H^2(0, \pi),\, \theta \in H_0^1(0, \pi),\\ &\mu u_{xx} + \mu^* v_{xx} - \beta\theta \in H^{-1}(0, \pi),\, \kappa\alpha - \beta v \in H^2(0, \pi),\\ &\mu u_{xx} + \mu^* v_{xx} \text{ vanishes at } x = 0, \pi\}.\end{aligned}$$

Clearly, this set is dense in $\mathcal{H}$.

Now, we have

$$\begin{aligned}\langle \mathcal{A}U, U\rangle =&\mu\langle v_{xx}, u_{xx}\rangle + \rho\langle -(\rho - \tau\partial_{xx})^{-1}\partial_{xx}[\mu u_{xx} + \mu^* v_{xx} - \beta\theta], v\rangle\\ &+ \tau\langle -\partial_x(\rho - \tau\partial_{xx})^{-1}\partial_{xx}[\mu u_{xx} + \mu^* v_{xx} - \beta\theta], v_x\rangle + \kappa\langle \theta_x, \alpha_x\rangle\\ &+ c\langle c^{-1}\partial_{xx}(\kappa\alpha - \beta v), \theta\rangle.\end{aligned}$$

We note that

$$\begin{aligned}&\langle -\partial_x(\rho - \tau\partial_{xx})^{-1}\partial_{xx}[\mu u_{xx} + \mu^* v_{xx} - \beta\theta], v_x\rangle\\ =&\langle \partial_{xx}(\rho - \tau\partial_{xx})^{-1}\partial_{xx}[\mu u_{xx} + \mu^* v_{xx} - \beta\theta], v\rangle,\end{aligned}$$

and therefore,

$$\langle \mathcal{A}U, U\rangle = \mu\langle v_{xx}, u_{xx}\rangle - \langle \partial_{xx}[\mu u_{xx} + \mu^* v_{xx} - \beta\theta], v\rangle + \kappa\langle \theta_x, \alpha_x\rangle + \langle \partial_{xx}(\kappa\alpha - \beta v), \theta\rangle.$$

After the use of the integration by parts, we find that

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle = -\mu^* \int_0^\pi |v_{xx}|^2\, dx \le 0.$$

Now, our aim is to show that operator $\mathcal{A}$ generates a semigroup of contractions. Therefore, it will be enough to show that zero belongs to the resolvent of the operator. So, we consider the element $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$. We must prove that there exists $(u, v, \alpha, \theta) \in \mathcal{D}(\mathcal{A})$ such that

$$\begin{aligned}&v = f_1,\\ &\partial_{xx}(\mu u_{xx} + \mu^* v_{xx} - \beta\theta) = -(\rho - \tau\partial_{xx})f_2,\\ &\theta = f_3,\\ &\partial_{xx}(\kappa\alpha - \beta v) = cf_4.\end{aligned} \tag{6}$$

Of course, it is trivial to obtain v and θ. Therefore, we arrive to the following system:

$$\begin{aligned}&\mu\partial_{xxxx}u = -(\rho - \tau\partial_{xx})f_2 - \mu^*\partial_{xxxx}f_1 + \beta\partial_{xx}f_3,\\ &\kappa\partial_{xx}\alpha = cf_4 + \beta\partial_{xx}f_1.\end{aligned} \tag{7}$$

It is obvious that we can obtain the solutions to system (7) in an easy way and that they belong to the domain $\mathcal{D}(\mathcal{A})$. Moreover, we find that there exists a positive constant K such that

$$\|U\| \le K\|F\|, \quad K > 0.$$

Therefore, we have proved the following theorem.

Theorem 1. *The operator $\mathcal{A}$ defined in* (4) *is the infinitesimal generator of a C^0-semigroup of contractions on $\mathcal{H}$. Thus, for any initial data $U(0) \in \mathcal{D}(\mathcal{A})$ there exists only one solution to Cauchy problem* (5) *verifying*

$$U \in C^1([0,\infty);\mathcal{H}) \cap C([0,\infty);\mathcal{D}(\mathcal{A})).$$

Remark 2. *A similar result can be obtained if we change the boundary conditions $u_{xx}(0,t) = u_{xx}(\pi,t) = 0$ by $u_x(0,t) = u_x(\pi,t) = 0$.*

4. Energy decay of the solutions

In this section, we will show that the solutions obtained in the previous section decay in an exponential way; that is, there exist two positive constants M and ω such that

$$\|U(t)\|_{\mathcal{H}} \leq M e^{-\omega t} \|U(0)\|_{\mathcal{H}}, \tag{8}$$

whenever we assume that $\beta \neq 0$.

In order to show this property, we will use the well-known characterization of the semigroups which are exponentially stable (see [19]). This is equivalent to show that the imaginary axis is contained at the resolvent of the operator and that the asymptotic condition

$$\overline{\lim_{|\gamma|\to\infty}} \|(i\gamma\mathcal{I} - \mathcal{A})^{-1}\| < \infty \tag{9}$$

holds.

We will proceed by contradiction. Thus, assume that there exists a real number $\gamma_0 \neq 0$ such that $i\gamma_0$ does not belong to the resolvent. We can guarantee the existence of a sequence of real numbers $\gamma_n \to \gamma_0$ and a sequence of elements $U_n \in \mathcal{D}(\mathcal{A})$, with $\|U_n\|_{\mathcal{H}} = 1$, such that

$$i\gamma_n u_n - v_n \to 0 \quad \text{in} \quad H^2(0,\pi), \tag{10}$$

$$\begin{aligned} &i(\rho\gamma_n v_n - \tau\gamma_n \partial_{xx} v_n) + \mu\partial_{xxxx} u_n + \mu^* \partial_{xxxx} v_n \\ &\qquad -\beta\partial_{xx}\theta_n \to 0 \quad \text{in} \quad H^{-1}(0,\pi), \end{aligned} \tag{11}$$

$$i\gamma_n \alpha_n - \theta_n \to 0 \quad \text{in} \quad H^1(0,\pi), \tag{12}$$

$$ic\gamma_n\theta_n - \kappa\partial_{xx}\alpha_n + \beta\partial_{xx} v_n \to 0 \quad \text{in} \quad L^2_*(0,\pi). \tag{13}$$

If we use the dissipation inequality we find that $\|\partial_{xx} v_n\| \to 0$ and so, we have that $\|\gamma_n \partial_{xx} u_n\|$ also tends to zero. If we multiply convergence (11) by $\gamma_n^{-1}\alpha_n \in H^1_0(0,\pi)$ it leads

$$\mu\langle \partial_{xx} u_n, \gamma_n^{-1}\partial_{xx}\alpha_n\rangle + \mu^*\langle \partial_{xx} v_n, \gamma_n^{-1}\partial_{xx}\alpha_n\rangle + \beta\langle \partial_x\theta_n, \gamma_n^{-1}\partial_x\alpha_n\rangle \to 0.$$

We can observe that $\gamma_n^{-1}\partial_{xx}\alpha_n$ is bounded. Therefore, the first two summands tend to zero.

Since we assume $\beta \neq 0$ we conclude that $\alpha_n \to 0$ in $H^1(0,\pi)$. If we multiply convergence (13) by α_n, we obtain that $\theta_n \to 0$ in $L^2(0,\pi)$.

The asymptotic condition (9) can be proved following a similar argument since the key point is that γ_n does not tend to zero.

Now, we are going to show that the semigroup generated by the solutions to our problem is not differentiable (and so, it is not analytic). Following [7, p. 112,114], we recall that, if a semigroup is differentiable, then the following condition

$$\lim_{\gamma\to\pm\infty} \|(i\gamma\mathcal{I} - \mathcal{A})^{-1}\| = 0 \tag{14}$$

holds.

Therefore, if we are able to prove that condition (14) is not satisfied, we conclude that the semigroup is not differentiable.

To show this, we will consider the sequence $(0,0,0,\sin nx)$ and also the sequence $(u_n, v_n, \alpha_n, \theta_n)$ satisfying

$$\begin{aligned}
&i\gamma_n u_n - v_n = 0,\\
&i(\rho\gamma_n v_n - \tau\gamma_n v_{n,xx}) + \mu u_{n,xxxx} + \mu^* v_{n,xxxx} - \beta\theta_{n,xx} = 0,\\
&i\gamma_n\alpha_n - \theta_n = 0,\\
&ic\gamma_n\theta_n - \kappa\alpha_{n,xx} + \beta v_{n,xx} = \sin nx.
\end{aligned}$$

This system admits solutions of the form

$$u_n = C_n \sin nx, \quad \alpha_n = D_n \sin nx, \quad v_n = i\gamma_n C_n \sin nx, \quad \theta_n = i\gamma_n D_n \sin nx.$$

If we substitute them into the above system, we find that

$$\begin{aligned}
&(-\rho\gamma_n^2 - \tau n^2\gamma_n^2 + \mu n^4 + i\mu^*\gamma_n n^4)C_n + i\beta n^2\gamma_n D_n = 0,\\
&-i\beta\gamma_n n^2 C_n + (\kappa n^2 - c\gamma_n^2) = 1,
\end{aligned}$$

and so, we have

$$D_n = \frac{i\mu^*\gamma_n n^4 - \rho\gamma_n^2 - \tau n^2\gamma_n^2 + \mu n^4}{-\beta^2\gamma_n^2 n^4 + (i\mu^*\gamma_n n^4 - \rho\gamma_n^2 - \rho\gamma_n^2 - \tau n^2\gamma_n^2 + \mu n^4)(\kappa n^2 - c\gamma_n^2)}.$$

Now, we take $\kappa n^2 - c\gamma_n^2 = -\dfrac{\beta}{\mu^*}$. That is,

$$\gamma_n = \sqrt{\frac{\kappa}{c}n^2 + \frac{\beta}{c\mu^*}}.$$

Therefore, it follows that

$$D_n = \frac{i\mu^*\gamma_n n^4 - \rho\gamma_n^2 - \tau n^2\gamma_n^2 + \mu n^4}{-\beta^2\gamma_n^2 n^4 - \frac{\beta}{\mu^*}(i\mu^*\gamma_n n^4 - \rho\gamma_n^2 - \rho\gamma_n^2 - \tau n^2\gamma_n^2 + \mu n^4)},$$

and so, we can conclude that

$$\lim_{n\to\infty} i\gamma_n D_n = \frac{\mu^*}{\beta^2}$$

and

$$\lim_{n\to\infty} \|U_n\| > 0.$$

Since the norm of the solution to our problem does not tend to zero, it follows that the semigroup is not differentiable (neither analytic).

The results shown in this section are summarized in the following.

Theorem 3. *The solutions generated by the semigroup associated to the operator $\mathcal{A}$ are exponentially stable; that is, there exist two positive constants M and ω such that the inequality* (8) *holds. However, the semigroup is not differentiable.*

5. Impossibility of localization

In this section, we will prove that the unique solution which can vanish in an open (not empty) set is the null solution. Therefore, it will be sufficient to show that the backward in time problem has a unique solution. We recall that our system of equations is written as

$$\begin{aligned}
&\rho u_{tt} - \tau u_{ttxx} + \mu u_{xxxx} - \mu^* u_{txxxx} - \beta^*\alpha_{txx} = 0,\\
&c\alpha_{tt} - \kappa\alpha_{xx} + \beta^* u_{txx} = 0,
\end{aligned}$$

with $\beta^* = -\beta$.

We will study this system together with the boundary conditions (2) and the initial conditions, for a.e. $x \in (0, \pi)$,

$$u_0(x) = v_0(x) = \alpha_0(x) = \theta_0(x) = 0.$$

We will show that this problem only admits the null solution.

First, let us consider the functions

$$F_1(t) = \frac{1}{2} \int_0^\pi \Big(\rho u_t^2 + \tau u_{tx}^2 + \mu u_{xx}^2 + c\theta^2 + \kappa \alpha_x^2 \Big)\, dx, \tag{15}$$

$$F_2(t) = \frac{1}{2} \int_0^\pi \Big(c\theta^2 + \kappa \alpha_x^2 - \rho u_t^2 - \tau u_{tx}^2 - \mu u_{xx}^2 \Big)\, dx, \tag{16}$$

and so we have

$$\begin{aligned} F_{1t}(t) &= \mu^* \int_0^\pi u_{txx}^2\, dx, \\ F_{2t}(t) &= \int_0^\pi (2\beta^* u_{txx}\theta - \mu^* u_{txx}^2)\, dx. \end{aligned}$$

We can use the Lagrange identities to write function $F_2(t)$ in an alternative form. For each $t \in (0, T)$ (with a given time $T > 0$), we find that

$$\begin{aligned}
&\int_0^t \int_0^\pi c\alpha_{tt}(s)\alpha_t(2t-s)\, dxds - \int_0^t \int_0^\pi \kappa\alpha_{xx}(s)\alpha_t(2t-s)\, dxds \\
&\qquad = \int_0^t \int_0^\pi \beta^* u_{txx}(s)\alpha_t(2t-s)\, dxds, \\
&\int_0^t \int_0^\pi c\alpha_{tt}(2t-s)\alpha_t(s)\, dxds - \int_0^t \int_0^\pi \kappa\alpha_{xx}(2t-s)\alpha_t(s)\, dxds \\
&\qquad = \int_0^t \int_0^\pi \beta^* u_{txx}(2t-s)\alpha_t(s)\, dxds, \\
&\int_0^t \int_0^\pi \rho u_{tt}(2t-s)u_t(s)\, dxds + \int_0^t \int_0^\pi \tau u_{ttx}(2t-s)u_{tx}(s)\, dxds \\
&\quad + \int_0^t \int_0^\pi \mu u_{xx}(2t-s)u_{txx}(s) dxds - \int_0^t \int_0^\pi \mu^* u_{txx}(2t-s)u_{txx}(s)\, dxds \\
&\qquad = -\int_0^t \int_0^\pi \beta^* u_{txx}(s)\alpha_t(2t-s)\, dxds, \\
&\int_0^t \int_0^\pi \rho u_{tt}(s)u_t(2t-s)\, dxds + \int_0^t \int_0^\pi \tau u_{ttx}(s)u_{tx}(2t-s)\, dxds \\
&\quad + \int_0^t \int_0^\pi \mu u_{xx}(s)u_{txx}(2t-s) dxds - \int_0^t \int_0^\pi \mu^* u_{txx}(s)u_{txx}(2t-s)\, dxds \\
&\qquad = -\int_0^t \int_0^\pi \beta^* u_{txx}(2t-s)\alpha_t(s)\, dxds.
\end{aligned}$$

If we combine these equalities, after the time integration and using the initial conditions, we obtain

$$\int_0^\pi \Big(c\theta^2 + \mu u_{xx}^2 \Big)\, dx = \int_0^\pi \Big(\rho u_t^2 + \tau u_{xt}^2 + \kappa \alpha_x^2 \Big)\, dx.$$

Therefore, taking into account the previous equalities and by using integration by parts it follows that

$$F_2(t) = \int_0^\pi \Big(\kappa \alpha_x^2 - \mu u_{xx}^2 \Big)\, dx. \tag{17}$$

We note that we also have

$$\int_0^\pi \Big(-\frac{d}{dt}(c\alpha\theta)+c\theta^2\Big)\,dx=\int_0^\pi \Big(\kappa\alpha_x^2+\beta^* u_{txx}\alpha\Big)\,dx,$$

and

$$\int_0^\pi \Big(\frac{d}{dt}(\rho u u_t+\tau u_x u_{xt})-\rho u_t^2-\tau u_{xt}^2\Big)\,dx$$
$$=-\int_0^\pi \Big(\beta^*\theta_{xx}u+\mu u_{xx}^2\Big)\,dx+\frac{1}{2}\frac{d}{dt}\int_0^\pi \mu^* u_{xx}^2\,dx.$$

Therefore, we find that

$$\frac{d}{dt}\int_0^\pi \Big(\frac{\mu^*}{2}u_{xx}^2-\rho u u_t-\tau u_x u_{xt}+c\alpha\theta\Big)\,dx$$
$$=-\int_0^\pi \Big(\beta^* u_{txx}\alpha-\beta^*\theta u_{xx}\Big)\,dx+\int_0^\pi \Big(\mu u_{xx}^2-\kappa\alpha_x^2+c\theta^2-\rho u_t^2-\tau u_{tx}^2\Big)\,dx.$$

Now, we define the function

$$F_0(t)=\varepsilon F_1(t)+F_2(t)+\lambda F_3(t),$$

where functions F_1 and F_2 were defined previously in (15) and (17), respectively, ε is less than 1, λ is a positive constant assumed large enough and

$$F_3(t)=\int_0^\pi \Big(\frac{\mu^*}{2}u_{xx}^2-\rho u u_t-\tau u_x u_{xt}+c\alpha\theta\Big)\,dx.$$

If we define the Lyapunov function

$$G(t)=\int_0^t F_0(s)\,ds,$$

we can observe that

$$G(t)\geq m_1\int_0^t\int_0^\pi \Big(\rho u_t^2+\tau u_{tx}^2+\mu^* u_{xx}^2+\kappa\alpha_x^2+c\theta^2\Big)\,dx,$$

whenever $t\leq t_0$, for $t_0>0$, is small enough and after the use of the Poincaré inequality.

We can immediately obtain that

$$G_t(t)=\int_0^t\int_0^\pi \Big(2\beta^* u_{txx}\theta-(1-\varepsilon)\mu^* u_{txx}^2\Big)\,dxds+\int_0^t\int_0^\pi \lambda\beta^*(\theta u_{xx}-\alpha u_{txx})\,dxds$$
$$+\lambda\int_0^t\int_0^\pi \Big(\mu u_{xx}^2-\kappa\alpha_x^2+c\theta^2-\rho u_t^2-\tau u_{xt}^2\Big)\,dxds.$$

Therefore, we conclude that

$$G_t(t)\leq m_2\int_0^t\int_0^\pi \Big(\rho u_t^2+c\theta^2+\mu^* u_{xx}^2+\kappa\alpha_x^2+\tau u_{xt}^2\Big)\,dxds.$$

In view of the Gronwall inequality it follows that

$$G(t)\leq G(0)e^{kt}=0 \quad \text{for } k>0.$$

Since $G(0)=0$ we have that $u(x,t)=\alpha(x,t)=0$ for every $t\leq t_0$. Now, we can repeat the argument from $\frac{3}{4}t_0$ and we can prove that $u(x,t)=\alpha(x,t)=0$ for every $t\leq\frac{7}{4}t_0$. Repeating again the argument, it leads to the proposed result.

6. A fully discrete scheme: stability and an a priori error analysis

In this section, we will study, from the numerical point of view, the thermomechanical problem studied in the previous section. For the sake of generality, let us assume that the spatial interval is $(0, \ell)$, $\ell > 0$ being the length of the beam and denote by $(0, T)$ the time interval of interest, where $T > 0$ is the final time.

As usual in the numerical analysis of thermoelastic problems involving beams, we need to modify boundary conditions (2) as follows,

$$\left.\begin{array}{l} u(0,t) = u(\ell,t) = u_x(0,t) = u_x(\ell,t) = 0 \\ \alpha(0,t) = \alpha(\ell,t) = 0 \end{array}\right\} \quad \text{for a.e. } t \in (0,T). \tag{18}$$

The reason to do it is that, when we provide the weak form of the thermomechanical problem determined by system (1) with initial conditions (3) and boundary conditions (2), its solution will be in the space $H^1(0, \ell)$ instead of $H^2(0, \ell)$.

In order to obtain the variational formulation, let $Y = L^2(0, \ell)$, $E = H_0^1(0, \ell)$ and $V = H_0^2(0, \ell)$. Moreover, let us denote $(\cdot, \cdot)$ and $\| \cdot \|$ the inner product and the norm defined in $L^2(0, \ell)$, respectively.

Integrating by parts equations (1) and using initial conditions (3) and the new boundary conditions (18), we obtain the following weak formulation written using the velocity $v = u_t$ and the temperature $\theta = \alpha_t$.

Find the velocity $v : [0, T] \to V$ and the temperature $\theta : [0, T] \to E$ such that $v(0) = v_0$ and $\theta(0) = \theta_0$ and, for a.e. $t \in (0, T)$ and for all $w \in V$ and $r \in E$,

$$\rho(v_t(t), w) + \tau(v_{tx}(t), w_x) + \mu(u_{xx}(t), w_{xx}) + \mu^*(v_{xx}(t), w_{xx}) \\ +\beta(\theta_x(t), w_x) = \rho(F_1(t), w), \tag{19}$$

$$c(\theta_t(t), r) + \kappa(\alpha_x(t), r_x) - \beta(v_x(t), r_x) = \rho(F_2(t), r), \tag{20}$$

where the transverse displacement and the thermal displacement are then recovered from the relations:

$$u(t) = \int_0^t v(s)\, ds + u_0, \quad \alpha(t) = \int_0^t \theta(s)\, ds + \alpha_0. \tag{21}$$

We note that we have introduced two supply terms F_1 and F_2 to make the problem more general, and because they will be used in the numerical simulations.

Now, a fully discrete scheme to approximate problem (19)-(21) is introduced. In order to provide the spatial approximation, let us divide the interval $[0, \ell]$ into M subintervals denoted by $a_0 = 0 < a_1 < \ldots < a_M = \ell$. The mesh size is assumed uniform with length $h = a_{i+1} - a_i = \ell/M$. So, the variational spaces E and V are then approximated by the finite dimensional spaces $E^h \subset E$ and $V^h \subset V$ given by

$$E^h = \{r^h \in C([0,\ell]) \,;\, r^h_{[a_i,a_{i+1}]} \in P_1([a_i, a_{i+1}]) \;\; i = 0, \ldots, M-1, \\ r^h(0) = r^h(\ell) = 0\}, \tag{22}$$

$$V^h = \{w^h \in C^1([0,\ell]) \cap H^2(0,\ell) \,;\, w^h_{[a_i,a_{i+1}]} \in P_3([a_i, a_{i+1}]) \\ i = 0, \ldots, M-1,\; w^h_x(0) = w^h_x(\ell) = w^h(0) = w^h(\ell) = 0\}, \tag{23}$$

where the set $P_r([a_i, a_{i+1}])$ denotes the space of polynomials of degree less or equal to r for each subinterval $[a_i, a_{i+1}]$, i.e. the finite element space E^h is made of continuous and piecewise affine functions and the finite element space V^h by C^1 and piecewise cubic functions. Moreover, we construct the discrete initial conditions u_0^h, v_0^h, α_0^h and θ_0^h in the following form:

$$u_0^h = \mathcal{P}_2^h u_0, \quad v_0^h = \mathcal{P}_2^h v_0, \quad \alpha_0^h = \mathcal{P}_1^h \alpha_0, \quad \theta_0^h = \mathcal{P}_1^h \theta_0, \tag{24}$$

where $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ are the finite element projection operators over E^h and V^h, which can be found in [6].

Secondly, to provide the discretization of the first-order time derivatives, we consider the classical implicit Euler scheme and so, we use a uniform partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, with time step $k = T/N$ and nodes $t_n = n\,k$ for $n = 0,1,\ldots,N$.

Hence, the fully discrete approximation of problem (19)-(21) is written as follows.

Find the discrete velocity $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$ *and the discrete temperature* $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset E^h$ *such that* $v_0^{hk} = v_0^h$, $\theta_0^{hk} = \theta_0^h$ *and, for all* $w^h \in V^h$ *and* $r^h \in E^h$, *and* $n = 1,\ldots,N$,

$$\begin{aligned}&\rho(\delta v_n^{hk}, w^h) + \tau(\delta(v_n^{hk})_x, w_x^h) + \mu((u_n^{hk})_{xx}, w_{xx}^h) + \mu^*((v_n^{hk})_{xx}, w_{xx}^h)\\ &\qquad +\beta((\theta_n^{hk})_x, w_x^h) = \rho(F_{1n}, w^h),\end{aligned} \tag{25}$$

$$c(\delta\theta_n^{hk}, r^h) + \kappa((\alpha_n^{hk})_x, r_x^h) - \beta((v_n^{hk})_x, r_x^h) = \rho(F_{2n}, r^h), \tag{26}$$

where we used the notations $z_n = z(t_n)$ and $\delta z_n = (z_n - z_{n-1})/k$ for a given continuous function $z(t)$ and for a sequence $\{z_n\}_{n=0}^N$, respectively. Moreover, the discrete transverse displacement u_n^{hk} and the thermal displacement α_n^{hk} are now obtained as

$$u_n^{hk} = k\sum_{j=1}^n v_j^{hk} + u_0^h, \quad \alpha_n^{hk} = k\sum_{j=1}^n \theta_j^{hk} + \alpha_0^h. \tag{27}$$

Applying the classical Lax Milgram lemma, we can easily deduce that problem (25)-(27) admits a unique solution under the assumptions required in Section 2 on the constitutive coefficients.

In the rest of this section, we will show the discrete stability and we will provide an a priori error analysis and so, for the sake of simplicity in the calculations, we assume that the supply terms F_1 and F_2 vanish.

First, we will prove a discrete stability property. This is summarized in the following result.

Lemma 4. *If we assume that* ρ, τ, μ, μ^*, c *and* κ *are positive, then there exists a positive constant* C, *which is independent of the discretization parameters* h *and* k, *such that*

$$\|v_n^{hk}\|_E + \|u_n^{hk}\|_V + \|\theta_n^{hk}\| + \|\alpha_n^{hk}\|_E \le C \quad \text{for } n = 1,\ldots,N,$$

where, here and in what follows, we denote by $\|\cdot\|_X$ *the norm in the Hilbert space* X.

Proof. First, we take as a test function in equation (25) $w^h = v_n^{hk}$ to obtain

$$\begin{aligned}&\rho(\delta v_n^{hk}, v_n^{hk}) + \tau(\delta(v_n^{hk})_x, (v_n^{hk})_x) + \mu((u_n^{hk})_{xx}, (v_n^{hk})_{xx}) + \mu^*((v_n^{hk})_{xx}, (v_n^{hk})_{xx})\\ &\qquad + \beta((\theta_n^{hk})_x, (v_n^{hk})_x) = \rho(f_n, v_n^{hk}).\end{aligned}$$

Taking into account that

$$\begin{aligned}(\delta v_n^{hk}, v_n^{hk}) &\ge \frac{1}{2k}\Big\{\|v_n^{hk}\|^2 - \|v_{n-1}^{hk}\|^2\Big\},\\ (\delta(v_n^{hk})_x, (v_n^{hk})_x) &\ge \frac{1}{2k}\Big\{\|(v_n^{hk})_x\|^2 - \|(v_{n-1}^{hk})_x\|^2\Big\},\\ ((u_n^{hk})_{xx}, (v_n^{hk})_{xx}) &\ge \frac{1}{2k}\Big\{\|(u_n^{hk})_{xx}\|^2 - \|(u_{n-1}^{hk})_{xx}\|^2\Big\},\end{aligned}$$

by using the Cauchy-Schwarz inequality and the Young's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \tag{28}$$

it follows that

$$\begin{aligned} &\frac{\rho}{2k}\Big\{\|v_n^{hk}\|^2 - \|v_{n-1}^{hk}\|^2\Big\} + \frac{\tau}{2k}\Big\{\|(v_n^{hk})_x\|^2 - \|(v_{n-1}^{hk})_x\|^2\Big\} + \beta((\theta_n^{hk})_x, (v_n^{hk})_x) \\ &\quad + \frac{\mu}{2k}\Big\{\|(u_n^{hk})_{xx}\|^2 - \|(u_{n-1}^{hk})_{xx}\|^2\Big\} \leq C\Big(1 + \|v_n^{hk}\|^2\Big). \end{aligned} \tag{29}$$

Now, we obtain the estimates for the discrete temperature. If we take as a test function $r^h = \theta_n^{hk}$ in discrete equation (26) we have

$$c(\delta\theta_n^{hk}, \theta_n^{hk}) + \kappa((\alpha_n^{hk})_x, (\theta_n^{hk})_x) - \beta((v_n^{hk})_x, (\theta_n^{hk})_x) = \rho(s_n, \theta_n^{hk}).$$

Keeping in mind that

$$\begin{aligned} (\delta\theta_n^{hk}, \theta_n^{hk}) &\geq \frac{2}{k}\Big\{\|\theta_n^{hk}\|^2 - \|\theta_{n-1}^{hk}\|^2\Big\}, \\ ((\alpha_n^{hk})_x, (\theta_n^{hk})_x) &\geq \frac{2}{k}\Big\{\|(\alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1}^{hk})_x\|^2\Big\}, \end{aligned}$$

by using again the above inequalities we find that

$$\begin{aligned} &\frac{c}{2k}\Big\{\|\theta_n^{hk}\|^2 - \|\theta_{n-1}^{hk}\|^2\Big\} + \frac{\kappa}{2k}\Big\{\|(\alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1}^{hk})_x\|^2\Big\} \\ &\quad - \beta((v_n^{hk})_x, (\theta_n^{hk})_x) \leq C\Big(1 + \|\theta_n^{hk}\|^2\Big). \end{aligned} \tag{30}$$

Combining estimates (29) and (30) it follows that

$$\begin{aligned} &\frac{1}{2k}\Big\{\|v_n^{hk}\|^2 - \|v_{n-1}^{hk}\|^2\Big\} + \frac{1}{2k}\Big\{\|(v_n^{hk})_x\|^2 - \|(v_{n-1}^{hk})_x\|^2\Big\} \\ &+ \frac{1}{2k}\Big\{\|(u_n^{hk})_{xx}\|^2 - \|(u_{n-1}^{hk})_{xx}\|^2\Big\} + \frac{1}{2k}\Big\{\|\theta_n^{hk}\|^2 - \|\theta_{n-1}^{hk}\|^2\Big\} \\ &+ \frac{1}{2k}\Big\{\|(\alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1}^{hk})_x\|^2\Big\} \\ \leq& C\Big(1 + \|v_n^{hk}\|^2 + \|\theta_n^{hk}\|^2\Big). \end{aligned}$$

Multiplying these estimates by k and summing them until n, we find that

$$\begin{aligned} &\|v_n^{hk}\|^2 + \|(v_n^{hk})_x\|^2 + \|(u_n^{hk})_{xx}\|^2 + \|\theta_n^{hk}\|^2 + \|(\alpha_n^{hk})_x\|^2 \\ \leq& Ck\sum_{j=1}^{n}\Big(1 + \|v_j^{hk}\|^2 + \|\theta_j^{hk}\|^2\Big) + C\Big(\|v_0^h\|_E^2 + \|u_0^h\|_V^2 + \|\theta_0^h\|^2 + \|\alpha_0^h\|_E^2\Big), \end{aligned}$$

and, applying a discrete version of Gronwall's inequality (see, e.g., [5]) we conclude the desired a priori stability estimates. □

Now, we provide an a priori error analysis in the rest of this section. The main result is the following.

Theorem 5. *Under the assumptions of Lemma 4, if we denote by (u, v, α, θ) the solution to the variational problem* (19)-(21) *and by $\{u_n^{hk}, v_n^{hk}, \alpha_n^{hk}, \theta_n^{hk}\}_{n=0}^N$ the solution to the discrete variational problem* (25)-(27)*, then we have the following a* *priori error estimates, for all* $\{w_n^h\}_{n=0}^N \subset V^h$ *and* $\{r_n^h\}_{n=0}^N \subset E^h$,

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{\|v_n - v_n^{hk}\|_E^2 + \|u_n - u_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|^2 + \|\alpha_n - \alpha_n^{hk}\|_E^2\Big\}\\
&\quad\le Ck\sum_{j=1}^N\Big(\|v_{tj} - \delta v_j\|_E^2 + \|u_{tj} - \delta u_j\|_V^2 + \|v_j - w_j^h\|_V^2 + \|\theta_{tj} - \delta\theta_j\|^2\\
&\qquad+\|\alpha_{tj} - \delta\alpha_j\|_E^2 + \|\theta_j - r_j^h\|_E^2\Big) + C\max_{0\le n\le N}\Big\{\|v_n - w_n^h\|^2 + \|\theta_n - r_n^h\|^2\Big\}\\
&\qquad+\frac{C}{k}\sum_{j=1}^{N-1}\Big\{\|v_j - w_j^h - (v_{j+1} - w_{j+1}^h)\|^2 + \|\theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h)\|^2\Big\}\\
&\qquad+C\Big(\|v_0 - v_0^h\|^2 + \|u_0 - u_0^h\|_V^2 + \|\theta_0 - \theta_0^h\|^2 + \|\alpha_0 - \alpha_0^h\|_E^2\Big).
\end{aligned}$$

Proof. First, we obtain the error estimates on the velocity field. We subtract variational equation (19), at time t_n, and for a test function $w = w^h \in V^h$, and discrete variational equation (25), and we obtain, for all $w^h \in V^h$,

$$\begin{aligned}
&\rho(v_{tn} - \delta v_n^{hk}, w^h) + \tau((v_{tn} - \delta v_n^{hk})_x, w_x^h) + \mu((u_n - u_n^{hk})_{xx}, w_{xx}^h)\\
&\qquad+\mu^*((v_n - v_n^{hk})_{xx}, w_{xx}^h) + \beta((\theta_n - \theta_n^{hk})_x, w_x^h) = 0,
\end{aligned}$$

and so, it follows that, for all $w^h \in V^h$,

$$\begin{aligned}
&\rho(v_{tn} - \delta v_n^{hk}, v_n - v_n^{hk}) + \tau((v_{tn} - \delta v_n^{hk})_x, (v_n - v_n^{hk}))\\
&\quad+\mu((u_n - u_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx})\\
&\quad+\mu^*((v_n - v_n^{hk})_{xx}, (v_n - v_n^{hk})_{xx}) + \beta((\theta_n - \theta_n^{hk})_x, (v_n - v_n^{hk})_x)\\
&=\rho(v_{tn} - \delta v_n^{hk}, v_n - w^h) + \tau((v_{tn} - \delta v_n^{hk})_x, (v_n - w^h)_x)\\
&\quad+\mu((u_n - u_n^{hk})_{xx}, (v_n - w^h)_{xx})\\
&\quad+\mu^*((v_n - v_n^{hk})_{xx}, (v_n - w^h)_{xx}) + \beta((\theta_n - \theta_n^{hk})_x, (v_n - w^h)_x).
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
\rho(\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk}) &\ge \frac{\rho}{2k}\Big\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big\},\\
\tau((\delta v_n - \delta v_n^{hk})_x, (v_n - v_n^{hk})_x) &\ge \frac{\tau}{2k}\Big\{\|(v_n - v_n^{hk})_x\|^2 - \|(v_{n-1} - v_{n-1}^{hk})_x\|^2\Big\},\\
\mu((u_n - u_n^{hk})_{xx}, (\delta u_n - \delta u_n^{hk})_{xx}) &\ge \frac{\mu}{2k}\Big\{\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2\Big\},\\
((\theta_n - \theta_n^{hk})_x, (v_n - w^h)_x) &= -(\theta_n - \theta_n^{hk}, (v_n - w^h)_{xx}),
\end{aligned}$$

after several algebraic manipulations, by using Cauchy-Schwarz inequality and the Young's inequality (28) we find that, for all $w^h \in V^h$,

$$\begin{aligned}
&\frac{\rho}{2k}\Big\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big\}\\
&\quad+\frac{\tau}{2k}\Big\{\|(v_n - v_n^{hk})_x\|^2 - \|(v_{n-1} - v_{n-1}^{hk})_x\|^2\Big\}\\
&\quad+\frac{\mu}{2k}\Big\{\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2\Big\}\\
&\quad+\beta((\theta_n - \theta_n^{hk})_x, (v_n - v_n^{hk})_x)\\
&\le C\Big(\|v_n - v_n^{hk}\|^2 + \|v_{tn} - \delta v_n\|_E^2 + \|u_{tn} - \delta u_n\|_V^2 + \|v_n - w^h\|_V^2\\
&\quad+\|\theta_n - \theta_n^{hk}\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h)\Big). \qquad (31)
\end{aligned}$$

Secondly, we derive the error estimates for the temperature. We subtract variational equation (20), at time t_n and for a test function $r^h \in E^h$, and discrete variational equation (26), and we have, for all $r^h \in E^h$,

$$c(\theta_{tn} - \delta\theta_n^{hk}, r^h) + \kappa((\alpha_n - \alpha_n^{hk})_x, r_x^h) - \beta((v_n - v_n^{hk})_x, r_x^h) = 0.$$

Thus, we find that,

$$\begin{aligned}
&c(\theta_{tn} - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}) + \kappa((\alpha_n - \alpha_n^{hk})_x, (\theta_n - \theta_n^{hk})_x) \\
&- \beta((v_n - v_n^{hk})_x, (\theta_n - \theta_n^{hk})_x) \\
=&c(\theta_{tn} - \delta\theta_n^{hk}, \theta_n - r^h) + \kappa((\alpha_n - \alpha_n^{hk})_x, (\theta_n - r^h)_x) \\
&- \beta((v_n - v_n^{hk})_x, (\theta_n - r^h)_x).
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&c(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}) \geq \frac{c}{2k}\Big\{\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big\}, \\
&\kappa((\alpha_n - \alpha_n^{hk})_x, (\delta\alpha_n - \delta\alpha_n^{hk})_x) \geq \frac{\kappa}{2k}\Big\{\|(\alpha_n - \alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_x\|^2\Big\},
\end{aligned}$$

using again several times Cauchy-Schwarz inequality and the Young's inequality (28), it leads

$$\begin{aligned}
&\frac{c}{2k}\Big\{\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big\} \\
&+ \frac{\kappa}{2k}\Big\{\|(\alpha_n - \alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_x\|^2\Big\} \\
&- \beta((v_n - v_n^{hk})_x, (\theta_n - \theta_n^{hk})_x) \\
\leq& C\Big(\|\theta_{tn} - \delta\theta_n\|^2 + \|\alpha_{tn} - \delta\alpha_n\|_E^2 + \|\theta_n - \theta_n^{hk}\|^2 \\
&+ \|(v_n - v_n^{hk})_x\|^2 + \|\theta_n - r^h\|_E^2 \\
&+ \|(\alpha_n - \alpha_n^{hk})_x\|^2 + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h)\Big).
\end{aligned} \tag{32}$$

Combining estimates (31) and (32) we obtain, for all $w^h \in V^h$ and $r^h \in E^h$,

$$\begin{aligned}
&\frac{\rho}{2k}\Big\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\Big\} \\
&+ \frac{\tau}{2k}\Big\{\|(v_n - u_n^{hk})_x\|^2 - \|(v_{n-1} - u_{n-1}^{hk})_x\|^2\Big\} \\
&+ \frac{\mu}{2k}\Big\{\|(u_n - u_n^{hk})_{xx}\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_{xx}\|^2\Big\} \\
&+ \frac{c}{2k}\Big\{\|\theta_n - \theta_n^{hk}\|^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|^2\Big\} \\
&+ \frac{\kappa}{2k}\Big\{\|(\alpha_n - \alpha_n^{hk})_x\|^2 - \|(\alpha_{n-1} - \alpha_{n-1}^{hk})_x\|^2\Big\} \\
\leq& C\Big(\|v_n - v_n^{hk}\|^2 + \|v_{tn} - \delta v_n\|_E^2 + \|u_{tn} - \delta u_n\|_V^2 + \|v_n - w^h\|_V^2 + \|\theta_n - \theta_n^{hk}\|^2 \\
&+ \|(u_n - u_n^{hk})_{xx}\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h) + \|\theta_{tn} - \delta\theta_n\|^2 + \|\alpha_{tn} - \delta\alpha_n\|_E^2 \\
&+ \|(v_n - v_n^{hk})_x\|^2 + \|\theta_n - r^h\|_E^2 + \|(\alpha_n - \alpha_n^{hk})_x\|^2 + (\delta\theta_n - \delta\theta_n^{hk}, \theta_n - r^h)\Big).
\end{aligned}$$

Multiplying these estimates by k and summing up to n we have, for all $\{w_j^h\}_{j=0}^n \subset V^h$ and $\{r_j^h\}_{j=0}^n \subset E^h$,

$$
\begin{aligned}
&\|v_n - v_n^{hk}\|^2 + \|(v_n - v_n^{hk})_x\|^2 + \|(u_n - u_n^{hk})_{xx}\|^2 \\
&\quad + \|\theta_n - \theta_n^{hk}\|^2 + \|(\alpha_n - \alpha_n^{hk})_x\|^2 \\
\leq & Ck \sum_{j=1}^n \Big(\|v_j - v_j^{hk}\|^2 + \|v_{tj} - \delta v_j\|_E^2 + \|u_{tj} - \delta u_j\|_V^2 + \|v_j - w_j^h\|_V^2 + \|\theta_j - \theta_j^{hk}\|^2 \\
&\quad + \|(u_j - u_j^{hk})_{xx}\|^2 + (\delta v_j - \delta v_j^{hk}, v_j - w_j^h) + \|\theta_{tj} - \delta\theta_j\|^2 + \|\alpha_{tj} - \delta\alpha_j\|_E^2 \\
&\quad + \|(v_j - v_j^{hk})_x\|^2 + \|\theta_j - r_j^h\|_E^2 + \|(\alpha_j - \alpha_j^{hk})_x\|^2 + (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - r_j^h) \Big) \\
&\quad + C\Big(\|v_0 - v_0^h\|_E^2 + \|u_0 - u_0^h\|_V^2 + \|\theta_0 - \theta_0^h\|^2 + \|\alpha_0 - \alpha_0^h\|_E^2 \Big).
\end{aligned}
$$

Finally, we take into account that

$$
\begin{aligned}
k\sum_{j=1}^n (\delta v_j - \delta v_j^{hk}, v_j - w_j^h) &= (v_n - v_n^{hk}, v_n - w_n^h) + (v_0^h - v_0, v_1 - w_1^h) \\
&\quad + \sum_{j=1}^{n-1} (v_j - v_j^{hk}, v_j - w_j^h - (v_{j+1} - w_{j+1}^h)), \\
k\sum_{j=1}^n (\delta\theta_j - \delta\theta_j^{hk}, \theta_j - r_j^h) &= (\theta_n - \theta_n^{hk}, \theta_n - r_n^h) + (\theta_0^h - \theta_0, \theta_1 - r_1^h) \\
&\quad + \sum_{j=1}^{n-1} (\theta_j - \theta_j^{hk}, \theta_j - r_j^h - (\theta_{j+1} - r_{j+1}^h)),
\end{aligned}
$$

applying again a discrete version of Gronwall's inequality we deduce the desired a priori error estimates. □

As a particular case of derivation of the convergence order, we obtain the linear convergence under suitable additional regularity conditions on the continuous solution. So, we have the following.

Corollary 6. *Let us assume that the solution to variational problem* (19)-(21) *has the following additional regularity:*

$$
\begin{aligned}
&u \in H^3(0,T;E) \cap H^2(0,T;V) \cap C^1([0,T]; H^3(0,\ell)), \\
&\alpha \in H^3(0,T;Y) \cap H^2(0,T;E) \cap C^1([0,T]; H^2(0,\ell)).
\end{aligned}
$$

Therefore, there exists a positive constant, which is independent of the discretization parameters h and k, such that

$$
\max_{0\leq n\leq N} \Big\{ \|v_n - v_n^{hk}\|_E + \|u_n - u_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\| + \|\alpha_n - \alpha_n^{hk}\|_E \Big\} \leq C(h+k).
$$

7. Numerical results

In this section, we examine two academic cases: one to check if the accuracy of the algorithm we propose is achieved numerically; and a second one to numerically assert that the energy decays exponentially.

The finite element code to solve the problem was implemented using Matlab and solved in a 3.40 GHz computer with 16 GB of RAM. A typical run with 200 elements and 1000 timesteps took about 1.5 seconds.

TABLE 1. Numerical errors obtained for values of h and k (multiplied by 100).

$h \downarrow k \rightarrow$	1×10^{-2}	5×10^{-3}	1×10^{-3}	1×10^{-4}	1×10^{-5}	5×10^{-6}	1×10^{-6}
1×10^{-1}	5.28993	5.28354	5.27856	5.27813	5.27806	5.27806	5.27805
5×10^{-2}	2.20091	2.18994	2.18307	2.18254	2.18249	2.18249	2.18248
2×10^{-2}	0.80629	0.77573	0.76027	0.75963	0.75958	0.75958	0.75957
1×10^{-2}	0.43509	0.39174	0.36163	0.36067	0.36062	0.36061	0.36061
5×10^{-3}	0.26236	0.21642	0.17757	0.17557	0.17549	0.17551	0.17548
2×10^{-3}	0.16271	0.11426	0.07350	0.06836	0.07027	0.06831	0.06831
1×10^{-3}	0.13260	0.08093	0.04170	0.04161	0.03415	0.03984	0.03415
2×10^{-4}	0.22676	0.02269	0.03791	0.01431	0.01315	0.01940	0.00972

7.1. Numerical convergence. To be able to compare the numerical solution with a known analytical solution, we use the supply terms introduced in the previous section. Given the expression for the solution and the problem parameters, we are able to manufacture some functions that will give such solution. Then, this analytical solution is compared with the one obtained numerically leading to the exact error, as described in the previous section. Repeating this process for several timesteps and element sizes we achieve the convergence table. The problem was run until a final time of $T = 0.5$ was achieved.

For this case, we consider the following problem parameters:

$$\ell = 1, \quad \rho = 5, \quad \tau = 1, \quad \mu = 0.5, \quad \mu^* = 3, \quad \kappa = 5, \quad \beta = 7, \quad c = 10.$$

We impose the following analytical solutions, for $(x,t) \in (0,1)\times(0,1)$,

$$u(x,t) = (1-x)^3\, x^3\, e^t, \quad \alpha(x,t) = 3\,(1-x)^3\, x^3\, e^t,$$

that possess the required regularity at the boundary to be compatible with the boundary conditions of the problem. In order to obtain the previous solution, the expressions for the supply terms are the following, for $(x,t) \in (0,1)\times(0,1)$,

$$\begin{aligned} F_1(x,t) =& 66\,x^3\,\mathrm{e}^t\,(2\,x-2) - 252\,\mathrm{e}^t\,(x-1)^2 - 252\,x^2\,\mathrm{e}^t + 396\,x^2\,\mathrm{e}^t\,(x-1)^2 \\ & - 5\,x^3\,\mathrm{e}^t\,(x-1)^3 - 378\,x\,\mathrm{e}^t\,(2\,x-2) + 132\,x\,\mathrm{e}^t\,(x-1)^3, \\ F_2(x,t) =& 24\,x^3\,\mathrm{e}^t\,(2\,x-2) + 144\,x^2\,\mathrm{e}^t\,(x-1)^2 - 30\,x^3\,\mathrm{e}^t\,(x-1)^3 \\ & + 48\,x\,\mathrm{e}^t\,(x-1)^3. \end{aligned}$$

Finally, the initial conditions were chosen to be compatible with the analytical solution, they were obtained by evaluating the solution u and α at time $t = 0$.

The numerical errors given by

$$\max_{0\le n\le N} \left\{ \|v_n - v_n^{hk}\|_E + \|u_n - u_n^{hk}\|_V + \|\theta_n - \theta_n^{hk}\| + \|\alpha_n - \alpha_n^{hk}\|_E \right\}$$

and obtained for different timesteps k and element sizes h are listed in Table 1. We can see the convergence of the approximations when parameters h and k decrease and that the error reduces linearly with both parameters. In order to clearly show this effect, the main diagonal of the table is plotted in Figure 1.

7.2. Exponential decay. To check the exponential energy decay, we do not consider for this section any source function and so, $F_1(x,t) = F_2(x,t) = 0$. This case was solved with an element size of $h = 0.01$ and a timestep of $k = 10^{-3}$ until a final time of $T = 100$ was reached; we checked that the mesh was sufficiently refined to avoid affecting the results.

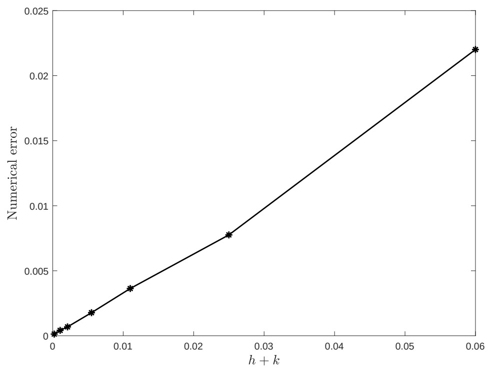

FIGURE 1. Linear convergence of the algorithm. The numerical error is plotted against the mesh size $(h+k)$.

The initial conditions and data for this problem are

$$u^0(x)=\alpha^0(x)=5\,(1-x)^3\,x^3 \quad \text{for all } x\in[0,1],$$
$$\ell=1,\quad \rho=5,\quad \tau=1,\quad \mu=0.5,\quad \mu^*=10,\quad \kappa=1,\quad \beta=7,\quad c=5.$$

In this case, we will focus on the behavior of the discrete energy. Following the definition given in the continuous case, we consider that it has the expression:

$$E_n^{hk}=\frac{1}{2}\Big(\mu\|(u_n^{hk})_{xx}\|^2+\rho\|v_n^{hk}\|^2+\tau\|(v_n^{hk})_x\|^2+\kappa\|(\alpha_n^{hk})_x\|^2+c\|\theta_n^{hk}\|^2\Big).$$

The evolution of this discrete energy with time for that problem is shown in Figure 2. The decay of the energy with time is clearly seen in the left figure. The plot on the right shows the same energy plotted in a semilogarithmic scale; the straight line after a short initial transient confirms that the decay is exponential, as expected.

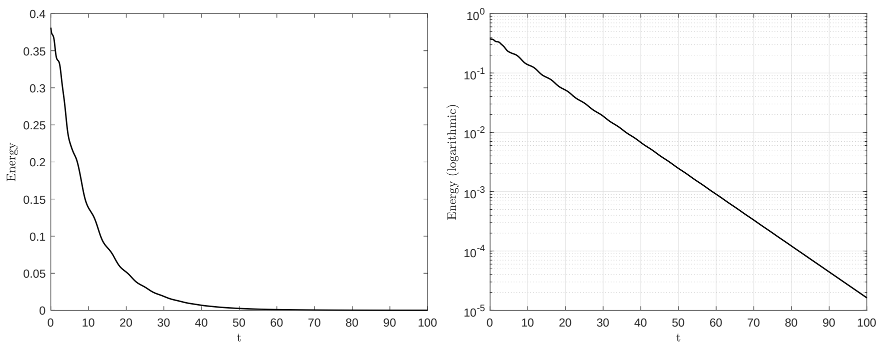

FIGURE 2. Evolution of the energy of the system without source functions in natural (left) and semilogarithmic (right) scales.

8. Conclusions

In this paper, we have studied a thermoelastic problem involving a viscoelastic beam where the heat conduction has been modeled by using the type II Green-Naghdi theory. First, by using the theory of linear semigroups we have proved that this problem has a unique solution and, taking into account the characterization of the exponentially stable semigroups, we have shown the exponential energy decay. We have also obtained that the semigroup is not differentiable and the impossibility of localization; that is, the unique solution which can vanish in an open (not empty) set is the null solution. Secondly, we have provided a fully discrete approximation by using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. Some properties as the discrete stability and a priori error estimates have been proved, leading to the linear convergence of the numerical scheme. Finally, we have presented some numerical simulations to demonstrate the numerical convergence and the behavior of the discrete energy.

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