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A UNIFIED ANALYSIS FRAMEWORK FOR UNIFORM STABILITY OF DISCRETIZED VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

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Abstract. We provide a unified analysis framework for discretized Volterra integrodifferential equations by considering the ϑ-type convolution quadrature, where different ϑ corresponds to different schemes. We first derive the long-time l^∞ stability of discrete solutions, and then prove a discrete Wiener-Lévy theorem to support the analysis of long-time l^1 stability. The methods we adopt include the integral transforms in the Stieltjes sense, the complex analysis techniques, and a linear algebra approach for an indirect estimate of intricate terms. Meanwhile, we relax the commonly-used regularity assumption of the initial data in the literature by novel treatments. Numerical simulations are performed to substantiate the theoretical findings.

Key words. Volterra integrodifferential equation, ϑ-type convolution quadrature, uniform stability, long-time behaviour, completely monotonic kernels.

1. Introduction

1.1. Problem formulation and motivation. This work considers the unified analysis framework for the temporal discretization of the following Volterra integrodifferential equation [2, 3, 21, 26, 27]

$$\frac{\partial u}{\partial t} + \int_0^t \chi(t-r)Au(r)dr = 0, \quad t > 0;\ u(0) = u_0, \tag{1}$$

where A is a positive self-adjoint linear operator defined in a dense subspace $\mathrm{D}(A)$ of the real Hilbert space $\mathbf{H}$ with a complete eigensystem $\{\gamma_q, \varphi_q\}_{q=1}^{\infty}$; $u_0 \in \mathbf{H}$ and the kernel $\chi(t)$ on $(0, \infty)$ satisfies that

$$\chi \text{ is completely monotonic}, \quad \chi \in L^1_{\mathrm{loc}}(0, \infty), \quad 0 \le \chi(\infty) < \chi(0^+) \le \infty. \tag{2}$$

Problem (1) plays an important role in various fields such as the simple shearing motions or torsion of a rod in viscoelasticity and the dynamic behavior of the velocity field of a 'linear' homogeneous isotropic incompressible viscoelastic fluid [31, 34], and extensive mathematical and numerical analysis can be found in [9, 11, 14–20, 30, 33, 36, 37, 43, 44, 46–48]. In particular, investigating the long-time behavior of the solutions to model (1) is critical and challenging. For the continuous case, it was proved in [4, 5] that the L^1 stability of the solutions to model (1) $\int_0^\infty \|u(r)\|dr \le C\|u_0\|$ holds, in which $C > 0$ is independent of $u(t)$ and $\|\cdot\|$ indicates the norm in $\mathbf{H}$ defined via the inner product $(\cdot, \cdot)$ of $\mathbf{H}$. The uniform L^1 behavior of the exponential decay of the solutions to model (1) was proved in [12]. For the discretized problems, several works have considered the asymptotic l^1 analysis of (1) with a completely monotonic kernel. Xu studied the backward Euler temporal discretization for model (1) based on a first-order convolution quadrature and proved the l^1 stability [39] and the l^1 convergence [40]. A second-order temporal finite difference approach was investigated with the stability [41] and the convergence [42] proved for model

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(1). Harris and Noren [13] utilized the backward-Euler method and deduced the uniform l^1 stability for (1).

This work intends to provide a unified analysis framework for temporal discretization of problem (1) by considering the convolution quadrature method [1, 6, 7, 23] of ϑ-type, where different ϑ corresponds to different discretizations. In particular, $\vartheta = 1$ relates to the backward Euler method, while $\vartheta = 1/2$ corresponds to the Crank-Nicolson scheme. Compared with these classical methods, the ϑ-type methods not only serve as a mathematical generalization, but also unify the analysis of backward Euler and Crank-Nicolson methods. In [10, 45], the corresponding studies for their applications in discretizing integrodifferential equations are far from mature. In particular, how to analyze the long-time stability of the numerical solutions to model (1) under the ϑ-type convolution quadrature method in order to characterize the long-time behaviour remains untreated in the literature due to its complexity, which motivates the current study.

1.2. ϑ-type convolution quadrature. Define the Laplace transform by $\widehat{w}(z) = \int_0^\infty \mathrm{e}^{-zt} w(t)dt$ for $\Re(z) > \lambda$ where $w(t)\mathrm{e}^{-\lambda t} \in L^1(\mathbb{R}^+)$ for all $\lambda > 0$ and $\Re(\cdot)$ denotes the real part of the complex number. We fix the time step size $k > 0$ and $t_n = nk$ with $n \geq 0$ and generate numerical solution $U^n \approx u(t_n)$ such that for $\vartheta \in [1/2, 1]$ (see [24, 25])

$$\widehat{u}\left(\frac{\delta(\xi)}{k}\right) = k\sum_{n=0}^{\infty} U^n \xi^n, \quad \xi \in \mathbb{C}, \quad |\xi| < 1; \quad \delta(\xi) = \frac{1-\xi}{\vartheta + (1-\vartheta)\xi}. \tag{3}$$

Applying the Laplace transform to (1), we obtain $\widehat{u}(z) = \left(\widehat{\chi}(z)A + zI\right)^{-1} u_0$ for $\Re(z) \geq 0$. Combining this equation and (3) we obtain

$$k\sum_{j=0}^{\infty} w_j(k)\xi^j \sum_{n=j}^{\infty} AU^{n-j}\xi^{n-j} + \sum_{n=0}^{\infty} \frac{1-\xi}{\vartheta + (1-\vartheta)\xi} U^n \xi^n = u_0, \tag{4}$$

where the quadrature weight $w_j(k)$ is determined by [25]

$$\widehat{\chi}\left(\frac{\delta(\xi)}{k}\right) = \sum_{j=0}^{\infty} w_j(k)\xi^j, \quad \xi \in \mathbb{C}, \quad |\xi| \leq 1. \tag{5}$$

We swap the summation indices of the first left-hand side term of (4) to get

$$k\sum_{n=0}^{\infty}\sum_{j=0}^{n} w_j(k)AU^{n-j}\xi^n + \sum_{n=0}^{\infty} \frac{1-\xi}{\vartheta + (1-\vartheta)\xi} U^n \xi^n = u_0.$$

Further, we arrange the above formula to obtain

$$\begin{aligned} &\vartheta k\sum_{n=0}^{\infty}\sum_{j=0}^{n} w_j(k)AU^{n-j}\xi^n + (1-\vartheta)k\sum_{n=0}^{\infty}\sum_{j=0}^{n} w_j(k)AU^{n-j}\xi^{n+1} + U^0 \\ &\quad + \sum_{n=1}^{\infty}(U^n - U^{n-1})\xi^n = [\vartheta + (1-\vartheta)\xi]u_0. \end{aligned} \tag{6}$$

By comparing the coefficients of ξ^n ($n \geq 0$) on both sides of (6), we obtain the ϑ-type convolution quadrature scheme of model (1) for $\frac{1}{2} \leq \vartheta \leq 1$,

$$U^0 = \vartheta\left(I + \vartheta k w_0(k)A\right)^{-1} u_0, U^1 = \left(I + \vartheta k w_0(k)A\right)^{-1}\left[\vartheta^{-1}I - \vartheta k w_1(k)A\right]U^0, \tag{7}$$

$$\frac{U^n - U^{n-1}}{k} + \vartheta\sum_{j=0}^{n} w_j(k)AU^{n-j} + (1-\vartheta)\sum_{j=0}^{n-1} w_j(k)AU^{n-1-j} = 0, \quad n \geq 2. \tag{8}$$

1.3. Main results and novelties. We list main theorems and novelties, and carry out proofs in subsequent sections. Throughout this work, C indicates a positive constant independent of the time step size that may assume different values at different occurrences. The norms are defined as $\|u\|_q = \|A^{q/2}u\|$ for $q = 0, 1, 2$.

In addition, we denote the following time-discrete long-time l^∞ and l^p norms for the sequence $U = \{U^n\}_{n\geq 0}^{\infty}$

$$\|U\|_{l^\infty(\mathbf{H})} = \max_{n\geq 0} \|U^n\|, \quad \|U\|_{l^p(\mathbf{H})} = \Big(k\sum_{n=0}^{\infty} \|U^n\|^p\Big)^{1/p}, \quad 1 \leq p < \infty.$$

For the numerical scheme (7)-(8), we first establish the following long-time l^∞ stability theorem for numerical solutions in Section 3.

Theorem 1. *Suppose that* (2) *holds and* U^n *is defined as in* (7)-(8) *with* $\vartheta \in [1/2, 1]$. *Then* $\|U\|_{l^\infty(\mathbf{H})} \leq C\|u_0\|$.

To prove the long-time l^1 properties, we derive the following discrete Wiener-Levy theorem in Section 4 for $\{U^j(\gamma)\}_{j=1}^{\infty}$ defined in (23).

Theorem 2. *Suppose* (2) *holds. Then for* $\gamma \geq \gamma_1$, $\vartheta \in (1/2, 1]$ *and* k *sufficiently small, it holds* $k\sum_{j=1}^{\infty} \left|U^j(\gamma)\right| \leq C\gamma$.

By this theorem, we prove the uniform l^1 stability in Section 5.

Theorem 3. *Assume that* (2) *holds and* U^n *is defined as in* (7)-(8) *with* $\vartheta \in (1/2, 1]$. *Then we have* $\|U\|_{l^1(\mathbf{H})} \leq C\|u_0\|_2$.

Finally, we relax the regularity of u_0 from $\|u_0\|_2 \leq C$ in Theorem 3 and also in the literature to $\|u_0\| \leq C$ in the following theorem, proved in Section 5.

Theorem 4. *If* (2) *holds and* U^n *is defined as in* (7)-(8) *with* $\vartheta \in (1/2, 1]$, *then it holds that* $\|U\|_{l^1(\mathbf{H})} \leq C\|u_0\|$.

Compared with existing works, the contributions and novelties of this work are enumerated as follows:

- Due to the application of the ϑ-type convolution quadrature discretization, most complex integrals contain additional singular terms in denominators in comparison with those for the special cases $\vartheta = 1$ (i.e. the backward Euler method) and $\vartheta = 1/2$ (i.e. the Crank-Nicolson method), which complicates the analysis and requires more subtle treatments (e.g., to prove (35)-(37) we need to estimate some terms in details involving (40)-(41) which have complex denominators.
- A long-time l^∞ stability is proved, which is rarely considered in the literature and in turn leads to the short-time l^1 stability.
- The proof of Theorem 3 requires evaluating the weighted $U^j(\gamma)$, which is in general difficult due to the complexity of the expressions. Thus, we adopt a novel decomposition (71) and an indirect strategy. Specifically, we firstly estimate $U_4^j(\gamma)$ and $U_5^j(\gamma)$ in (71), and then utilize these and the dependence of each term on γ to bound other parts by a linear algebra method based on the discrete Wiener-Lévy theorem proved in Theorem 2.
- We employ a different relation (13) instead of the commonly-used (12) and modify the analysis procedure to relax the regularity of the initial data.

2. Preliminaries

We rewrite the Laplace transform of the kernel $\chi(t)$ as Stieltjes integral [12,38], i.e.,

$$\widehat{\chi}(s) = \int_0^\infty (s+y)^{-1} d\mu(y), \quad s \in \mathbb{C}' = \mathbb{C} \setminus (-\infty, 0], \tag{9}$$

where $\mu(y)$ is a non-decreasing function on $[0,\infty)$ such that

$$0 = \mu(0) \le \mu(0^+), \quad \mu(y) = \mu(y^-), \quad 0 < y < \infty,$$

$$\int_0^\infty \chi_0(y) d\mu(y) < \infty, \quad \mu(0^+) < \mu(\infty) = \chi(0^+), \quad \chi_0(t) = \int_0^1 e^{-xt} dx,$$

and (9) is uniformly convergent regarding s in any compact subset of $\mathbb{C}'$. By defining

$$\Phi(\lambda,\ell) = \int_0^\infty \frac{(y+\lambda)d\mu(y)}{(y+\lambda)^2+\ell^2}, \;\; \Theta(\lambda,\ell) = \int_0^\infty \frac{d\mu(y)}{(y+\lambda)^2+\ell^2}, \;\; \lambda, \ell \in (0,\infty),$$

with $\Phi(\ell) = \Phi(0,\ell)$ and $\Theta(\ell) = \Theta(0,\ell)$ for $\ell \in (0,\infty)$, we use (9) and $\lambda > 0$ to get

$$\int_0^\infty \frac{d\mu(y)}{y+\lambda+\mathrm{i}\ell} = \int_0^\infty e^{-t(\lambda+\mathrm{i}\ell)}\chi(t)dt = \Phi(\lambda,\ell) - \mathrm{i}\ell\Theta(\lambda,\ell). \tag{10}$$

Then, the Fourier transform of $\chi(t)$ yields

$$\widetilde{\chi}(\ell) = \int_0^\infty e^{-\mathrm{i}\ell t}\chi(t)dt = \int_0^\infty \frac{d\mu(y)}{y+\mathrm{i}\ell} = \widehat{\chi}(\mathrm{i}\ell) = \Phi(\ell) - \mathrm{i}\ell\Theta(\ell), \quad \ell \ne 0, \tag{11}$$

where $\Phi(\ell) > 0$ with $\ell > 0$ and $\Theta(\ell) > 0$ is continuously differentiable and strictly decreasing on $(0,+\infty)$. From Carr and Hannsgen [4]

$$\textbf{(i)} \;\; \lim_{\ell\to\infty} \sup \frac{\Theta(\ell)}{\Phi(\ell)} < \infty, \quad \textbf{(ii)} \;\; \Phi(\ell) > 0, \quad \ell > 0, \tag{12}$$

and from Noren [29, Theorem B (ii)]

$$\lim_{\ell\to\infty} \sup \frac{\ell[\Theta(\ell)]^{4/3}}{\Phi(\ell)} < \infty. \tag{13}$$

We further define $\mathcal{A}(t) = \int_0^t \chi(\xi)d\xi$ and $\mathcal{A}_1(t) = \int_0^t \xi\chi(\xi)d\xi$ for $t > 0$. Then the following lemma is given by recalling [4, Lemma 4.1].

Lemma 5. *Assume that* (2) *holds. If* $\widetilde{\chi}(\ell)$ *and* $\Theta(\ell)$ *are continuously differentiable with* $\ell > 0$*, then we get*

$$\textbf{(i)} \;\; \frac{\sqrt{2}}{4}\mathcal{A}(1/\ell) \le |\widetilde{\chi}(\ell)| \le 4\mathcal{A}(1/\ell), \quad \ell > 0; \;\; \textbf{(ii)} \;\; |\widetilde{\chi}'(\ell)| \le 40\mathcal{A}_1(1/\ell), \quad \ell > 0;$$

$$\textbf{(iii)} \;\; \frac{1}{5}\mathcal{A}_1(1/\ell) \le \Theta(\ell) \le 12\mathcal{A}_1(1/\ell), \quad \ell > 0.$$

As in [4,39], we can define a continuous, strictly increasing function $\omega : [\sigma_0,\infty) \to [\rho,\infty)$ (here $\rho > 0$ and $\sigma_0 := \max\{1/\Theta(\rho), 1\}$) with $\omega(\gamma) \to \infty$ ($\gamma \to \infty$) and $\Theta(\omega(\gamma)) = 1/\gamma$. We extend ω to $[\gamma_1,\infty)$, if necessary by defining $\omega(\gamma) = \rho$ ($\gamma_1 \le \gamma \le \sigma_0$). Then, by [4, p. 970], we obtain $\frac{\chi(6/\rho)}{10\omega^2} \le \frac{1}{5}\mathcal{A}_1(1/\omega) \le \gamma^{-1} \le C\mathcal{A}_1(1/\omega)$ for $\gamma \ge \gamma_1$. From Lemma 5 (**iii**) and [4, Eq. (6.8)], we get

$$\Theta(\ell) \le 12\int_0^{1/\ell} \xi\chi(\xi)d\xi \le 12\int_0^{2/\omega} \xi\chi(\xi)d\xi \le C\gamma^{-1}, \quad \ell \ge \frac{\omega}{2}, \tag{14}$$

and we follow [5, p. 462] to obtain

$$c\gamma^{1/2} \le \omega(\gamma) \le C\gamma \left|\log\gamma\right|^{-1/2}, \quad \gamma \ge \gamma_1. \tag{15}$$

Furthermore, the estimate of Shea and Wainger [35, pp. 322–323] gives

$$\int_0^{\rho} \frac{\mathcal{A}_1(1/\ell)}{\mathcal{A}^2(1/\ell)} d\ell < \infty. \tag{16}$$

Next, define $\mathcal{Q}(z) = z + e^{-z} - 1$. By integration by parts twice, we have

$$\widehat{\chi}(\lambda) = \int_0^{\infty} e^{-\lambda t}\chi(t)dt = \frac{1}{\lambda^2}\int_0^{\infty} \mathcal{Q}(\lambda t) d\chi'(t), \quad \lambda > 0, \tag{17}$$

where we utilize the result $\chi(t) + t\chi'(t) \to 0$ $(t \to \infty)$; see [4]. Then we get $|\mathcal{Q}(z)| \leq \frac{1}{2}z^2$ with $0 \leq z \leq 1$ and $|\mathcal{Q}(z)| \leq z + 1$ with $z \geq 0$.

Using the above analysis, the following lemma holds.

Lemma 6. *If the assumption* (2) *holds and* $\lambda > 0$, *we have*

$$\textbf{(i)}\ e^{-1}\int_0^{1/\lambda} \chi(t)dt \leq \widehat{\chi}(\lambda) \leq 4\int_0^{1/\lambda} \chi(t)dt; \ \textbf{(ii)}\ |\widehat{\chi}'(\lambda)| \leq 30\int_0^{1/\lambda} t\chi(t)dt.$$

Proof. To derive (**i**), we first employ $e^{-\lambda t} \geq e^{-1}$ with $0 \leq t \leq 1/\lambda$ to get

$$\widehat{\chi}(\lambda) = \int_0^{1/\lambda} e^{-\lambda t}\chi(t)dt + \int_{1/\lambda}^{+\infty} e^{-\lambda t}\chi(t)dt \geq \int_0^{1/\lambda} e^{-1}\chi(t)dt.$$

Then from (17), it holds that $|\widehat{\chi}(\lambda)| \leq \frac{1}{2}\int_0^{1/\lambda} t^2 d\chi'(t) + \frac{1}{\lambda^2}\int_{1/\lambda}^{\infty}(\lambda t + 1)d\chi'(t)$ where

$$\frac{1}{2}\int_0^{1/\lambda} t^2 d\chi'(t) = \frac{1}{2}\chi'(1/\lambda)/\lambda^2 - \chi(1/\lambda)/\lambda + \int_0^{1/\lambda} \chi(t)dt, \tag{18}$$

$$\frac{1}{\lambda^2}\int_{1/\lambda}^{\infty}(\lambda t + 1)d\chi'(t) = -2\chi'(1/\lambda)/\lambda^2 + \chi(1/\lambda)/\lambda. \tag{19}$$

We apply $\int_0^{1/\lambda} t^2 d\chi'(t) = \int_0^{1/\lambda} t^2\chi''(t)dt \geq 0$ to add (19) and the quadruple of (18) to get $|\widehat{\chi}(\lambda)| \leq 4\int_0^{1/\lambda}\chi(t)dt - 3\chi(1/\lambda)/\lambda$, which in turn leads to (**i**) by $\chi(t) > 0$.

To derive (**ii**), we denote $\mathcal{J}(z) = -2\mathcal{Q}(z) + z(1 - e^{-z})$ such that $|\mathcal{J}(z)| \leq 3z + 2$ with $z \geq 0$ and $|\mathcal{J}(z)| \leq \frac{1}{4}z^3$ with $0 \leq z \leq 1$. Differentiating (17) leads to $\widehat{\chi}'(\lambda) = \frac{1}{\lambda^3}\int_0^{\infty}\mathcal{J}(\lambda t)d\chi'(t)$ for $\lambda > 0$, which implies $|\widehat{\chi}'(\lambda)| \leq \frac{1}{4}\int_0^{1/\lambda} t^3 d\chi'(t) + \frac{1}{\lambda^3}\int_{1/\lambda}^{+\infty}(2 + 3\lambda t)d\chi'(t)$ for $\lambda > 0$. Then, we apply integration by parts and (2) to get

$$\frac{1}{4}\int_0^{1/\lambda} t^3 d\chi'(t) = \frac{1}{4}\left[\lambda^{-3}\chi'(1/\lambda) - 3\lambda^{-2}\chi(1/\lambda) + 6\int_0^{1/\lambda} t\chi(t)dt\right], \tag{20}$$

$$\frac{1}{\lambda^3}\int_{1/\lambda}^{+\infty}(2 + 3\lambda t)d\chi'(t) = -5\lambda^{-3}\chi'(1/\lambda) + 3\lambda^{-2}\chi(1/\lambda). \tag{21}$$

We apply $\int_0^{1/\lambda} t^3 d\chi'(t) = \int_0^{1/\lambda} t^3\chi''(t)dt \geq 0$ to add (21) and twentyfold of (20) to get $|\widehat{\chi}(\lambda)| \leq 30\int_0^{1/\lambda} t\chi(t)dt - 12\lambda^{-2}\chi(1/\lambda)$, which implies (**ii**) by $\chi(t) > 0$. □

3. Proof of Theorem 1

We first give the following lemma.

Lemma 7. *If* (2) *is satisfied,* $\chi(t)$ *is a positive-type kernel and* $\Re(\widehat{\chi}(\lambda + \mathrm{i}\ell)) \geq 0$ *for* $\lambda \geq 0$.

Proof. The proof is completed by [32, Lemma 1] and [28, Theorem 2]. □

Based on the above lemma, we obtain the following result by [22, pp. 26-27]

$$\sum_{n=0}^{N}\Big(\sum_{j=0}^{n} w_j(k)W_{n-j}\Big)W_n \geq 0. \tag{22}$$

To prove this theorem, we consider the real sequence $\{U^n(\gamma)\}$ for $\gamma > 0$ such that

$$k\sum_{n=0}^{\infty} U^n(\gamma)\xi^n = \widehat{u}\left(\frac{\delta(\xi)}{k};\gamma\right), \quad \xi \in \mathbb{C}, \quad |\xi| \leq 1, \tag{23}$$

where $\widehat{u}(s;\gamma) = \gamma^{-1}[\gamma^{-1}s + \widehat{\chi}(s)]^{-1}$ is the Laplace transform of the solution of the scalar problem $\frac{du(t;\gamma)}{dt} + \gamma\int_0^t \chi(t-r)u(r;\gamma)dr = 0$ with $u(0,\gamma) = 1$ for $\gamma > 0$ and $0 \leq t < \infty$. We denote

$$\mathcal{D}(s;\gamma) = \widehat{\chi}(s) + \gamma^{-1}s = \widehat{\chi}(\lambda + \mathrm{i}\ell) + \gamma^{-1}(\lambda + \mathrm{i}\ell), \quad \mathcal{D}(s) = \mathcal{D}(s;\infty), \tag{24}$$

such that $\widehat{u}(s;\gamma) = \gamma^{-1}[\mathcal{D}(s;\gamma)]^{-1}$. Thus the solution of (3) can be expressed as $U^n = \sum\limits_{m=1}^{\infty} U^n(\gamma_m)(u_0,\varphi_m)\varphi_m$. Then we can rewrite the solution of (23) as

$$U^0(\gamma) = \vartheta\left(1 + \vartheta k w_0(k)\gamma\right)^{-1}, \tag{25}$$

$$U^1(\gamma) = \left(1 + \vartheta k w_0(k)\gamma\right)^{-1}\left[\vartheta^{-1} - \vartheta k w_1(k)\gamma\right]U^0(\gamma), \tag{26}$$

$$\begin{aligned}\frac{U^n(\gamma) - U^{n-1}(\gamma)}{k} &+ \vartheta\sum_{j=0}^{n} w_j(k)\gamma U^{n-j}(\gamma)\\ &+ (1-\vartheta)\sum_{j=0}^{n-1} w_j(k)\gamma U^{n-1-j}(\gamma) = 0, \quad n \geq 2.\end{aligned} \tag{27}$$

Define $U^{\vartheta,n}(\gamma) = \vartheta U^n(\gamma) + (1-\vartheta)U^{n-1}(\gamma)$ for $n \geq 1$ and $U^{\vartheta,0}(\gamma) = \vartheta U^0(\gamma)$ for $1/2 \leq \vartheta \leq 1$. Then for $n \geq 2$, we use (27) and

$$\begin{aligned}U^{\vartheta,n}(\gamma) &= \frac{U^n(\gamma) + U^{n-1}(\gamma)}{2} + \frac{2\vartheta - 1}{2}[U^n(\gamma) - U^{n-1}(\gamma)],\\ \frac{U^n(\gamma) - U^{n-1}(\gamma)}{k} \times U^{\vartheta,n}(\gamma) &= \frac{(U^n(\gamma))^2 - (U^{n-1}(\gamma))^2}{2k}\\ &+ \frac{2\vartheta - 1}{2k}[U^n(\gamma) - U^{n-1}(\gamma)]^2 \geq \frac{(U^n(\gamma))^2 - (U^{n-1}(\gamma))^2}{2k}\end{aligned}$$

to obtain that

$$\frac{(U^n(\gamma))^2 - (U^{n-1}(\gamma))^2}{2k} + \gamma\sum_{j=0}^{n} w_j(k)U^{\vartheta,n-j}(\gamma)U^{\vartheta,n}(\gamma) \leq 0,$$

in which, summing over $n = 2, 3, \cdots, N$, we get

$$\begin{aligned}&\frac{(U^N(\gamma))^2 - (U^1(\gamma))^2}{2k} + \gamma\sum_{n=0}^{N}\sum_{j=0}^{n} w_j(k)U^{\vartheta,n-j}(\gamma)U^{\vartheta,n}(\gamma)\\ &\leq \gamma w_0(k)\left[\left(U^{\vartheta,0}(\gamma)\right)^2 + \left(U^{\vartheta,1}(\gamma)\right)^2\right] + \gamma|w_1(k)|\left|U^{\vartheta,0}(\gamma)U^{\vartheta,1}(\gamma)\right|.\end{aligned}$$

Then utilizing Lemma 7 and (22), we have

$$\begin{aligned}(U^N(\gamma))^2 \leq (U^1(\gamma))^2 &+ 2k\gamma w_0(k)\left[\left(U^{\vartheta,0}(\gamma)\right)^2 + \left(U^{\vartheta,1}(\gamma)\right)^2\right]\\ &+ 2k\gamma\left|w_1(k)\right|\left|U^{\vartheta,0}(\gamma)\right|\left|U^{\vartheta,1}(\gamma)\right|.\end{aligned}$$

By choosing a suitable κ such that $|U^\kappa(\gamma)| = \max_{0\le n\le N} |U^n(\gamma)|$, we have

$$|U^\kappa(\gamma)|^2 \le \left|U^1(\gamma)\right| |U^\kappa(\gamma)| + 2k\gamma w_0(k)\left(\left|U^{\vartheta,0}(\gamma)\right| + \left|U^{\vartheta,1}(\gamma)\right|\right)|U^\kappa(\gamma)| + 2k\gamma\,|w_1(k)|\left|U^{\vartheta,0}(\gamma)\right||U^\kappa(\gamma)|,$$

which leads to

$$\left|U^N(\gamma)\right| \le [1+2k\vartheta\gamma w_0(k)]\left|U^1(\gamma)\right| + 2k\gamma\,[w_0(k)+\vartheta|w_1(k)|]\left|U^0(\gamma)\right|. \tag{28}$$

We apply (5) to find

$$w_0(k) = \widehat{\chi}\left(\frac{\delta(0)}{k}\right) = \widehat{\chi}\left(\frac{\vartheta^{-1}}{k}\right) > 0,\ \ w_1(k) = -\widehat{\chi}'\left(\frac{\vartheta^{-1}}{k}\right)\frac{\vartheta^{-2}}{k},$$

and from Lemma 6

$$|w_1(k)| \le \left|\widehat{\chi}'\left(\frac{\vartheta^{-1}}{k}\right)\right|\frac{\vartheta^{-2}}{k} \le 30\vartheta^{-1}e^1 w_0(k). \tag{29}$$

Then (25), (26) and (29) lead to

$$\left|U^0(\gamma)\right| \le \min\left\{\vartheta, \frac{1}{kw_0(k)\gamma}\right\},\quad \left|U^1(\gamma)\right| \le \min\left\{30e^1+1, \frac{30\vartheta^{-1}e^1}{kw_0(k)\gamma}+1\right\}. \tag{30}$$

Employing (28), (29) and (30), we obtain $|U^n(\gamma)| \le C$ for $n \ge 0$ and $\gamma \ge \gamma_1$, which in turn leads to $\|U^n\| \le \sup_{\gamma\in[\gamma_1,\infty)} |U^n(\gamma)|\|u_0\| \le C\|u_0\|$ and thus proves Theorem 1.

4. Proof of Theorem 2

To derive Theorem 2, we first introduce Hardy's inequality (see [8, pp. 48-49]) as follows

Lemma 8. *Denote* $\|g\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |g(e^{-\mathrm{i}\vartheta})|^p d\vartheta\right\}^{1/p}$. *If* $g(\xi) = \sum_{n=0}^\infty g_n\xi^n$ *satisfies* $\|g\|_1 < \infty$, *then* $\sum_{n=0}^\infty \frac{|g_n|}{n+1} \le \pi\|g\|_1$.

By (3) and (25)-(27), $\mathcal{U}(\xi;\gamma)$ defined as

$$\mathcal{U}(\xi;\gamma) = \sum_{j=0}^\infty U^j(\gamma)\,\xi^j, \tag{31}$$

satisfies the following relation

$$\mathcal{U}(\xi;\gamma) = \frac{1}{k}\widehat{u}\left(\frac{\delta(\xi)}{k};\gamma\right) = \frac{1}{k}\frac{1}{\gamma\mathcal{D}\left(\delta(\xi)/k;\gamma\right)}, \tag{32}$$

and we further yield

$$\mathcal{U}'(\xi;\gamma) = \sum_{j=1}^\infty jU^j(\gamma)\,\xi^{j-1} = \sum_{n=0}^\infty (n+1)U^{n+1}(\gamma)\,\xi^n. \tag{33}$$

We apply (33) and Lemma 8 to get

$$\sum_{j=1}^\infty \left|U^j(\gamma)\right| = \sum_{n=0}^\infty \frac{\left|(n+1)U^{n+1}(\gamma)\right|}{n+1} \le \frac{1}{2}\int_0^{2\pi}\left|\mathcal{U}'_\xi(e^{-\mathrm{i}v};\gamma)\right|dv,$$

which leads to

(34)

$$\sum_{n=1}^\infty |U^n(\gamma)| \le \frac{1}{2}\int_{-\pi}^{\pi}\left|\mathcal{U}'_\xi(e^{-\mathrm{i}v};\gamma)\right|dv \le k\left[\int_0^\rho + \int_\rho^{\frac{\varepsilon}{k}} + \int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}\right]\left|\mathcal{U}'_\xi(e^{-\mathrm{i}kv};\gamma)\right|dv.$$

To prove Theorem 2, we need to show that all three parts on the right-hand side of the above equation multiplied by k are bounded by $C\gamma$. However, as $\gamma \geq \gamma_1 > 0$, we instead analyze the following sufficient and stronger conditions for future use

$$k^2 \int_0^\rho \left|\mathcal{U}'_\xi(e^{-\mathrm{i}kv};\gamma)\right| dv \leq C\gamma^{-1}, \tag{35}$$

$$k^2 \int_\rho^{\frac{\varepsilon}{k}} \left|\mathcal{U}'_\xi(e^{-\mathrm{i}kv};\gamma)\right| dv \leq C\gamma, \tag{36}$$

$$k^2 \int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}} \left|\mathcal{U}'_\xi(e^{-\mathrm{i}kv};\gamma)\right| dv \leq Ck. \tag{37}$$

The method we adopt lies in using (32) to differentiate $\mathcal{U}(\xi;\gamma)$ in the integrals as

$$\mathcal{U}'_\xi(\xi;\gamma) = \frac{1}{k}\frac{\mathcal{D}_s\left(\delta(\xi)/k;\gamma\right)}{\gamma\mathcal{D}^2\left(\delta(\xi)/k;\gamma\right)}\frac{1}{k\left[\vartheta+(1-\vartheta)\xi\right]^2}, \tag{38}$$

where $\xi = e^{-\mathrm{i}kv}$ and for $\vartheta \in (1/2, 1]$

$$s = s(k,v) = \frac{\delta(e^{-\mathrm{i}kv})}{k} = \lambda(k,v) + \mathrm{i}\ell(k,v), \tag{39}$$

$$\lambda = \lambda(k,v) = \frac{1}{k}\frac{(2\vartheta-1)[1-\cos(kv)]}{\vartheta^2+(1-\vartheta)^2+2\vartheta(1-\vartheta)\cos(kv)}, \tag{40}$$

$$\ell = \ell(k,v) = \frac{1}{k}\frac{\sin(kv)}{\vartheta^2+(1-\vartheta)^2+2\vartheta(1-\vartheta)\cos(kv)}. \tag{41}$$

Furthermore, the following estimates will be applied.

Lemma 9. *[39] Assume that* (2) *holds,* $0 \leq \lambda \leq 2\varepsilon\ell < \ell$. *Then, we have*

$$|\Theta(\lambda,\ell) - \Theta(\ell)| \leq \frac{9\lambda}{\ell}\Theta(\ell),\ \ \Phi(\lambda,\ell) - \Phi(\ell) \geq -\frac{4\lambda}{\ell}\Phi(\ell) \geq -8\varepsilon\Phi(\ell), \tag{42}$$

$$|\mathcal{D}(\lambda+\mathrm{i}\ell;\gamma)| \geq \frac{1}{10}\widehat{\chi}(\ell) \geq \frac{\sqrt{2}}{20}|\widetilde{\chi}(\ell)|, \quad \ell \leq v \leq \varepsilon, \tag{43}$$

$$|\mathcal{D}(\lambda+\mathrm{i}\ell;\gamma)| \geq C\big[\mathcal{A}(1/\ell)+\ell\Theta(\ell)\big], \quad \left[c_0\rho, \frac{\omega}{2}\right] \cap \left[c_0\rho, \frac{c_1\varepsilon}{k}\right], \tag{44}$$

$$|\mathcal{D}(\lambda+\mathrm{i}\ell;\gamma)| \geq \Phi(\lambda,\ell) \geq \frac{1}{3}\Phi(\ell), \quad \varepsilon_0 < \frac{1}{12}, \tag{45}$$

$$|\mathcal{D}(\lambda+\mathrm{i}\ell;\gamma)| \geq C\gamma^{-1}|\ell-\omega|, \quad \omega < \frac{\omega}{1-\widetilde{\varepsilon}_0} \leq \ell, \quad \widetilde{\varepsilon}_0 \leq \frac{1}{2}, \tag{46}$$

$$|\widehat{\chi}(\lambda+\mathrm{i}\ell)| \geq \frac{1}{2}\left|\widehat{\chi}(\mathrm{i}\ell)\right|, \quad \lambda > 0. \tag{47}$$

(I) We first derive (35). We use (9) with $\lambda \geq 0$ and $\ell > 0$ to obtain

$$|\widehat{\chi}(\lambda+\mathrm{i}\ell)| \leq \int_0^\infty \frac{d\mu(y)}{\sqrt{y^2+\ell^2}} \leq \int_0^\infty \frac{\sqrt{2}d\mu(y)}{y+\ell} \leq 2\left|\widetilde{\chi}(\ell)\right|, \tag{48}$$

$$|\widehat{\chi}'(\lambda+\mathrm{i}\ell)| \leq \int_0^\infty \frac{d\mu(y)}{(y+\lambda)^2+\ell^2} = \Theta(\lambda,\ell) \leq \Theta(\ell). \tag{49}$$

Selecting k, ρ sufficiently small with $0 \leq v \leq \rho$ such that

$$c_0 v \leq \ell(k,v) \leq \frac{v}{\vartheta^2+(1-\vartheta)^2}, \quad \lambda(k,v) \leq (2\vartheta-1)\rho\ell \leq \ell, \tag{50}$$

where $\frac{1}{2} \le c_0 < 1$. Therefore, using (16), (43) and (49)-(50) and utilizing Lemmas 5-6, we have

$$
\begin{aligned}
k^2 \int_0^{\rho} \left|\mathcal{U}_\xi'(e^{-\mathrm{i}kv};\gamma)\right| dv &= \int_0^{\rho} \left| \frac{(\widehat{\chi}'(s)+\gamma^{-1})\left[\vartheta+(1-\vartheta)e^{-\mathrm{i}kv}\right]^{-2}}{\gamma[\mathcal{D}(s;\gamma)]^2} \right| dv \\
&\le C\gamma^{-2}\rho|\widetilde{\chi}(\rho)|^{-2} + C\gamma^{-1}\int_0^{\rho} \frac{\mathcal{A}_1(1/\ell)}{\mathcal{A}^2(1/\ell)} d\ell \le C\gamma^{-1},
\end{aligned} \tag{51}
$$

where we employ $\left|\vartheta+(1-\vartheta)e^{-\mathrm{i}kv}\right|^{-2} \le 1/(\vartheta^2+(1-\vartheta)^2) \le 2$ for $v \in (0,\rho)$ and $\vartheta \in (1/2, 1]$.

(II) Second, we deduce (37). Note that

$$
\left|\vartheta+(1-\vartheta)e^{-\mathrm{i}kv}\right|^2 = \vartheta^2+(1-\vartheta)^2+2\vartheta(1-\vartheta)\cos(vk), \quad v \in \left[\rho, \frac{\pi}{k}\right]. \tag{52}
$$

Using (40) with $v \in \left(\frac{\pi}{2k}, \frac{\pi}{k}\right]$, we have

$$
\begin{aligned}
\frac{1}{\vartheta^2+(1-\vartheta)^2+2\vartheta(1-\vartheta)\cos(vk)} &= k^2\lambda^2 \frac{\vartheta^2+(1-\vartheta)^2+2\vartheta(1-\vartheta)\cos(vk)}{(2\vartheta-1)^2[1-\cos(kv)]^2} \\
&< \frac{k^2\lambda^2[\vartheta^2+(1-\vartheta)^2]}{(2\vartheta-1)^2[1-\cos(kv)]^2} \le (k\lambda)^2,
\end{aligned} \tag{53}
$$

and with $v \in \left(\rho, \frac{\pi}{2k}\right]$, we obtain $1/(\vartheta^2+(1-\vartheta)^2+2\vartheta(1-\vartheta)\cos(vk)) \le 1/(\vartheta^2+(1-\vartheta)^2) \le 2$.

(II.A) When $v \in \left[\frac{\varepsilon}{k}, \frac{\pi-\varepsilon k}{k}\right]$, we get

$$
\frac{\sin\varepsilon}{k} \le \ell(k,v) \le \frac{1}{(2\vartheta-1)^2 k}, \quad \frac{(2\vartheta-1)(1-\cos\varepsilon)}{k} \le \lambda(k,v) \le \frac{1+\cos(\varepsilon k)}{(2\vartheta-1)k}, \tag{54}
$$

which leads to

$$
\Phi(\lambda,\ell)-\Phi(\lambda,0) = -\int_0^{\infty} \frac{d\mu(y)}{(y+\lambda)\left[\left(\frac{y}{\ell}+\frac{\lambda}{\ell}\right)^2+1\right]} \ge -\frac{1}{1+(2\vartheta-1)^6(1-\cos\varepsilon)^2}\widehat{\chi}(\lambda).
$$

Thus we obtain

$$
\Phi(\lambda,\ell) \ge \left[1-\frac{1}{1+(2\vartheta-1)^6(1-\cos\varepsilon)^2}\right]\widehat{\chi}(\lambda), \quad |\mathcal{D}(s;\gamma)| \ge \frac{\lambda}{\gamma} \ge \frac{C(\varepsilon,\vartheta)}{\gamma k}. \tag{55}
$$

In addition, by (9), we have

$$
|\widehat{\chi}'(\lambda+\mathrm{i}\ell)| \le \int_0^{\infty} \frac{d\mu(y)}{(y+\lambda)^2} = |\widehat{\chi}'(\lambda)| \le \frac{\widehat{\chi}(\lambda)}{\lambda}. \tag{56}
$$

Thus, based on the above analysis and Lemma 6, we have

$$
\begin{aligned}
k^2 \int_{\frac{\varepsilon}{k}}^{\frac{\pi-\varepsilon k}{k}} \left|\mathcal{U}_\xi'(e^{-\mathrm{i}kv};\gamma)\right| dv &= \int_{\frac{\varepsilon}{k}}^{\frac{\pi-\varepsilon k}{k}} \left| \frac{(\widehat{\chi}'(s)+\gamma^{-1})\left[\vartheta+(1-\vartheta)e^{-\mathrm{i}kv}\right]^{-2}}{\gamma[\mathcal{D}(s;\gamma)]^2} \right| dv \\
&\le Ck + C\lambda^{-1} \le Ck.
\end{aligned} \tag{57}
$$

(II.B) When $v \in \left[\frac{\pi-\varepsilon k}{k}, \frac{\pi}{k}\right]$, we have

$$
0 \le \ell(k,v) \le \frac{\sin(\varepsilon k)}{(2\vartheta-1)^2 k} \le c_1\varepsilon, \quad \frac{c_2'}{k} \ge \lambda(k,v) \ge \frac{(2\vartheta-1)[1+\cos(\varepsilon k)]}{[1+2\vartheta(1-\vartheta)(1-\cos(\varepsilon k))]k} \ge \frac{c_2}{k},
$$

$$
\Phi(\lambda,\ell)-\Phi(\lambda,0) \ge -\left(\frac{c_1\varepsilon k}{c_2}\right)^2 \Phi(\lambda,0).
$$

Then by selecting suitable ε such that $\varepsilon k \leq \frac{\sqrt{6}c_2}{3c_1}$, we have

$$\Phi(\lambda,\ell) \geq \left[1-\left(\frac{c_1\varepsilon k}{c_2}\right)^2\right]\widehat{\chi}(\lambda) \geq \frac{1}{3}\widehat{\chi}(\lambda), \quad |\mathcal{D}(s;\gamma)| \geq \frac{\lambda}{\gamma} \geq \frac{c_2}{\gamma k}. \tag{58}$$

Analogous to (57), using (56) we obtain

$$\begin{aligned} k^2\int_{\frac{\pi-\varepsilon k}{k}}^{\frac{\pi}{k}} \left|\mathcal{U}_\xi'(e^{-\mathrm{i}kv};\gamma)\right| dv &\leq \int_{\frac{\pi-\varepsilon k}{k}}^{\frac{\pi}{k}} \frac{|\widehat{\chi}'(s)+\gamma^{-1}|}{\gamma|\mathcal{D}(s;\gamma)|^2}dv \\ &\leq \int_{\frac{\pi-\varepsilon k}{k}}^{\frac{\pi}{k}} \Big[\frac{\widehat{\chi}(\lambda)\lambda^{-1}}{\gamma\frac{1}{3}\widehat{\chi}(\lambda)\frac{\lambda}{\gamma}} + \frac{\gamma^{-1}}{\gamma(\lambda/\gamma)^2}\Big]dv \leq Ck^2. \end{aligned} \tag{59}$$

Employing (57) and (59), we obtain (37).

(III) Finally, we turn to prove (36). Notice that if $v \in \left[\rho, \frac{\varepsilon}{k}\right]$, we can select ε and k small enough such that $c_0 v \leq \ell(k,v) \leq c_1 v$ and $\lambda(k,v) \leq 2\varepsilon_0\ell \leq \ell(k,v)$ where $1 < c_1 < \infty$. Then we split the interval $\left[c_0\rho, \frac{c_1\varepsilon}{k}\right]$ into three parts, i.e.,

$$\begin{aligned} \ell \in \left[c_0\rho, \frac{c_1\varepsilon}{k}\right] &= \left(\left[c_0\rho, \frac{\omega}{2}\right] \cap \left[c_0\rho, \frac{c_1\varepsilon}{k}\right]\right) \cup \left(\left[\frac{\omega}{2}, 2\omega\right] \cap \left[c_0\rho, \frac{c_1\varepsilon}{k}\right]\right) \\ &\quad \cup \left([2\omega,\infty] \cap \left[c_0\rho, \frac{c_1\varepsilon}{k}\right]\right) := \Lambda_1 + \Lambda_2 + \Lambda_3. \end{aligned} \tag{60}$$

It then follows from (38) that

$$\begin{aligned} &k^2\int_\rho^{\frac{\varepsilon}{k}} \left|\mathcal{U}_\xi'(e^{-\mathrm{i}kv};\gamma)\right| dv = \int_\rho^{\frac{\varepsilon}{k}} \left|\frac{\left(\widehat{\chi}'(s)+\gamma^{-1}\right)\left[\vartheta + (1-\vartheta)e^{-\mathrm{i}kv}\right]^{-2}}{\gamma[\mathcal{D}(s;\gamma)]^2}\right| dv \\ &\leq C\int_{c_0\rho}^{c_1\frac{\varepsilon}{k}} \left|\frac{\widehat{\chi}'(\lambda+\mathrm{i}\ell)+\gamma^{-1}}{\gamma[\mathcal{D}(\lambda+\mathrm{i}\ell;\gamma)]^2}\right| d\ell = C\left[\int_{\Lambda_1} + \int_{\Lambda_2} + \int_{\Lambda_3}\right]\left|\frac{\widehat{\chi}'(\lambda+\mathrm{i}\ell)+\gamma^{-1}}{\gamma[\mathcal{D}(\lambda+\mathrm{i}\ell;\gamma)]^2}\right| d\ell. \end{aligned}$$

For the first part, we use (44) and (49) to yield

$$\int_{\Lambda_1}\left|\frac{\widehat{\chi}'(\lambda+\mathrm{i}\ell)+\gamma^{-1}}{\gamma[\mathcal{D}(\lambda+\mathrm{i}\ell;\gamma)]^2}\right| d\ell \leq C\int_{\Lambda_1} \frac{[1+\gamma\Theta(\ell)]d\ell}{\gamma^2\left[\mathcal{A}(\ell^{-1})+\ell\Theta(\ell)\right]^2} \leq C\int_\rho^{\omega/2}\frac{d\ell}{\ell^2} \leq C. \tag{61}$$

Then for the second part, we employ (12) and (15) to get

$$\int_{\Lambda_2}\left|\frac{\widehat{\chi}'(\lambda+\mathrm{i}\ell)+\gamma^{-1}}{\gamma[\mathcal{D}(\lambda+\mathrm{i}\ell;\gamma)]^2}\right| d\ell \leq C\omega\frac{\Theta(\omega)^2}{\Phi(\omega)^2} \leq C\omega \leq C\gamma. \tag{62}$$

Finally, for the third part we employ (14), (46) and (15) to obtain

$$\int_{\Lambda_3}\left|\frac{\widehat{\chi}'(\lambda+\mathrm{i}\ell)+\gamma^{-1}}{\gamma[\mathcal{D}(\lambda+\mathrm{i}\ell;\gamma)]^2}\right| d\ell \leq C\int_{\Lambda_3}\frac{[1+\gamma\Theta(\ell)]d\ell}{(\ell-\omega)^2} \leq C\int_{2\omega}^\infty \frac{d\ell}{\ell^2} \leq C\gamma^{-\frac{1}{2}}. \tag{63}$$

Using (61)-(63), we obtain (36) and thus complete the proof.

5. Proof of Theorems 3 and 4

5.1. Proof of Theorem 3. We utilize Theorem 1 to arrive at

$$k\sum_{j=0}^{m+1}\|U^j\| \leq C\|u_0\|k(m+2) = C\|u_0\|(t_m+2k) \leq C\|u_0\| \tag{64}$$

for some t_m such that $1 \leq t_m \leq C$ for some constant C. Thus we remain to derive

$$k\sum_{j=m+2}^{\infty}\sup_{\gamma\in[\gamma_1,\infty)}|\gamma^{-1}U^j(\gamma)| \leq C \tag{65}$$

such that we could combine this with

$$\|U^n\| \leq \sup_{\gamma\in[\gamma_1,\infty)} \left|\gamma^{-1}U^n(\gamma)\right| \|u_0\|_2$$

to conclude

$$k\sum_{j=m+2}^{\infty} \|U^j\| \leq C\|u_0\|_2,$$

which, together with (64), implies the conclusion of Theorem 3.

With suitably small ρ, we use (31) and (23) to get

$$U^j(\gamma) = \frac{1}{2\pi\mathrm{i}}\int_{|\xi|=\rho} \xi^{-(j+1)}\mathcal{U}(\xi;\gamma)d\xi = \frac{1}{2\pi k\mathrm{i}}\int_{|\xi|=\rho} \xi^{-(j+1)}\widehat{u}\left(\frac{\delta(\xi)}{k};\gamma\right)d\xi. \tag{66}$$

By the transform $\xi = e^{-kz}$, the contour becomes $\{z = k^{-1}\log(\rho^{-1}) + \mathrm{i}y : |y| \leq \pi/k\}$. As $\widehat{u}\left(\frac{\delta(e^{-kz})}{k};\gamma\right)$ is analytic in $\Xi^+ = \{z : \Re(z) \geq 0, z \neq 0\}$ and contributions from the line segments on $\Im(z) = \pm\frac{\pi}{k}$ ($\Im(z)$ indicates the imaginary part of z) are canceled via periodicity, the contour could be further simplified as $\{z = \mathrm{i}v : |v| \leq \pi/k\}$ such that

$$U^j(\gamma) = \frac{1}{2\pi}\int_{-\pi/k}^{\pi/k} e^{\mathrm{i}vt_j}\widehat{u}\left(\frac{\delta(e^{-\mathrm{i}vk})}{k};\gamma\right)dv, \tag{67}$$

based on which we employ (24), the integrate by parts and (39)-(41) to get

$$U^j(\gamma) = \frac{\gamma^{-1}}{2t_j\pi\mathrm{i}}\left(\frac{e^{\mathrm{i}vt_j}}{\mathcal{D}(s;\gamma)}\Big|_{-\pi/k}^{\pi/k} - \int_{-\pi/k}^{\pi/k} e^{\mathrm{i}vt_j}\frac{-\mathcal{D}_s(s;\gamma)}{\mathcal{D}^2(s;\gamma)}\frac{\partial s}{\partial v}dv\right). \tag{68}$$

By symmetry and change of variable for the integral on $(-\pi/k, 0]$ we obtain

$$U^j(\gamma) = \Re\left\{\frac{\gamma^{-1}}{t_j\pi}\int_0^{\pi/k} \frac{e^{\mathrm{i}vt_{j-1}}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}]^2}\frac{\mathcal{D}_s(s(k,v);\gamma)}{\mathcal{D}^2(s(k,v);\gamma)}dv\right\}. \tag{69}$$

Similar to [4, (4.34)], we obtain

$$\frac{\mathcal{D}_s(s;\gamma)}{\mathcal{D}^2(s;\gamma)} = \frac{\mathcal{D}'(s) + \gamma^{-1}}{[\mathcal{D}(s)]^2}\left(1 - \frac{2s\gamma^{-1}}{\mathcal{D}(s)}\right) + \frac{\gamma^{-2}s^2\mathcal{D}_s(s;\gamma)}{[\mathcal{D}(s)]^2\mathcal{D}(s;\gamma)}\left[\frac{2}{\mathcal{D}(s)} + \frac{1}{\mathcal{D}(s;\gamma)}\right]. \tag{70}$$

Then for $\forall\lambda \in [\lambda_1, \infty)$, (69) and (70) give

$$U^j(\gamma) = \Re\left\{\gamma^{-1}U_1^j + \gamma^{-2}U_2^j + \gamma^{-3}U_3^j + U_4^j(\gamma) + U_5^j(\gamma)\right\}, \tag{71}$$

where $\Re(z)$ represents the real part of z and

$$U_1^j = \frac{1}{t_j\pi}\int_0^{\rho} \frac{e^{\mathrm{i}vt_{j-1}}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}]^2}\frac{\mathcal{D}'(s)}{[\mathcal{D}(s)]^2}dv, \tag{72}$$

$$U_2^j = \frac{1}{t_j\pi}\int_0^{\rho} \frac{e^{\mathrm{i}vt_{j-1}}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}]^2}\left[\frac{1}{[\mathcal{D}(s)]^2} - \frac{2s\mathcal{D}'(s)}{[\mathcal{D}(s)]^3}\right]dv, \tag{73}$$

$$U_3^j = \frac{1}{t_j\pi}\int_0^{\rho} \frac{e^{\mathrm{i}vt_{j-1}}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}]^2}\frac{-2s}{[\mathcal{D}(s)]^3}dv, \tag{74}$$

$$U_4^j(\gamma) = \frac{\gamma^{-3}}{t_j\pi}\int_0^{\rho} \frac{e^{\mathrm{i}vt_{j-1}}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}]^2}\frac{s^2\mathcal{D}_s(s;\gamma)}{\mathcal{D}^2(s)\mathcal{D}(s;\gamma)}\left[\frac{2}{\mathcal{D}(s)} + \frac{1}{\mathcal{D}(s;\gamma)}\right]dv, \tag{75}$$

$$U_5^j(\gamma) = \frac{\gamma^{-1}}{t_j\pi}\int_{\rho}^{\pi/k} \frac{e^{\mathrm{i}vt_{j-1}}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}]^2}\frac{\mathcal{D}_s(s;\gamma)}{\mathcal{D}^2(s;\gamma)}dv. \tag{76}$$

Then we intend to bound U_q^j $(1 \le q \le 5)$ in order to show (65). In Sections 5.2–5.3, we shall prove by technical analysis that

$$\left|U_4^j(\gamma)\right| \le C t_{j-1}^{-2}, \quad \gamma \in [\gamma_1, \infty), \tag{77}$$

$$\left|U_5^j(\gamma)\right| \le C t_{j-1}^{-2}\gamma, \quad \gamma \in [\gamma_1, \infty). \tag{78}$$

To bound U_q^j $(q = 1, 2, 3)$, a novel approach is adopted. Note that U_q^j $(q = 1, 2, 3)$ are not related to γ, and (71) holds for any $\gamma \in [\gamma_1, \infty)$. Thus we choose three different γ with $\bar\gamma_1 < \bar\gamma_2 < \bar\gamma_3 \le C$ in (71), leading to the following system

$$\begin{bmatrix} \bar\gamma_1^{-1} & \bar\gamma_1^{-2} & \bar\gamma_1^{-3} \\ \bar\gamma_2^{-1} & \bar\gamma_2^{-2} & \bar\gamma_2^{-3} \\ \bar\gamma_3^{-1} & \bar\gamma_3^{-2} & \bar\gamma_3^{-3} \end{bmatrix} \begin{bmatrix} \Re(U_1^j) \\ \Re(U_2^j) \\ \Re(U_3^j) \end{bmatrix} = \begin{bmatrix} U^j(\bar\gamma_1) - \Re\big(U_4^j(\bar\gamma_1) + U_5^j(\bar\gamma_1)\big) \\ U^j(\bar\gamma_2) - \Re\big(U_4^j(\bar\gamma_2) + U_5^j(\bar\gamma_2)\big) \\ U^j(\bar\gamma_3) - \Re\big(U_4^j(\bar\gamma_3) + U_5^j(\bar\gamma_3)\big) \end{bmatrix}.$$

By the invertability of Vandermonde matrix, we have

$$\sum_{i=1}^{3} \left|\Re(U_i^j)\right| \le C \max_{1\le q\le 3} \left[|U^j(\bar\gamma_q)| + \sum_{i=4}^{5} \left|\Re\big(U_i^j(\bar\gamma_q)\big)\right| \right], \tag{79}$$

which, together with Theorem 2, (77) and (78), implies (65) as follows

$$\begin{aligned} k \sum_{j=m+2}^{\infty} \sup_{\gamma\in[\gamma_1,\infty)} |\gamma^{-1}U^j(\gamma)| &\le Ck \sum_{j=m+2}^{\infty} \left[\sum_{i=1}^{3} \left|\Re(U_i^j)\right| + \sum_{i=4}^{5} \sup_{\gamma\in[\gamma_1,\infty)} |\gamma^{-1}U_i^j(\gamma)| \right] \\ &\le Ck \sum_{j=m+2}^{\infty} \left[\max_{1\le q\le 3} \left|U^j(\bar\gamma_q)\right| + t_{j-1}^{-2} + \sum_{i=4}^{5} \sup_{\gamma\in[\gamma_1,\infty)} |\gamma^{-1}U_i^j(\gamma)| \right] \\ &\le C\bar\gamma_3 + Ck \sum_{j=m+2}^{\infty} t_{j-1}^{-2} \le C, \end{aligned}$$

where we use $k \sum_{j=m+2}^{\infty} t_{j-1}^{-2} \le \int_{t_m}^{+\infty} \zeta^{-2} d\zeta = \frac{1}{t_m} \le 1$. Thus, we remain to prove (77) and (78) to complete the proof of Theorem 3.

5.2. Proof of (77)**.** By integration by parts for (75) we have $U_4^j(\gamma) := U_{4,A}^j(\gamma) + U_{4,B}^j(\gamma) + U_{4,C}^j(\gamma)$ where

$$U_{4,A}^j(\gamma) = \frac{\gamma^{-3}}{\mathrm{i}\pi t_j t_{j-1}} \frac{e^{\mathrm{i}vt_{j-1}}}{\left[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}\right]^2} \widehat{\mathcal{K}}(s(k,v);\gamma)\Big|_{v=0}^{v=\rho}, \tag{80}$$

$$U_{4,B}^j(\gamma) = -\frac{\gamma^{-3}}{\pi t_j t_{j-1}} \int_0^\rho \frac{2(1-\vartheta)ke^{\mathrm{i}vt_{j-2}}}{\left[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}\right]^3} \widehat{\mathcal{K}}(s;\gamma)dv, \tag{81}$$

$$U_{4,C}^j(\gamma) = -\frac{\gamma^{-3}}{\pi t_j t_{j-1}} \int_0^\rho \frac{e^{\mathrm{i}vt_{j-2}}}{\left[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}\right]^4} \widehat{\mathcal{K}}'(s;\gamma)dv \text{ with} \tag{82}$$

$$\widehat{\mathcal{K}}(s;\gamma) = \frac{s^2\mathcal{D}_s(s;\gamma)}{\mathcal{D}^2(s)\mathcal{D}(s;\gamma)} \left[\frac{2}{\mathcal{D}(s)} + \frac{1}{\mathcal{D}(s;\gamma)} \right], \tag{83}$$

$$\begin{aligned} \widehat{\mathcal{K}}'(s;\gamma) = &\frac{2s\mathcal{D}_s(s;\gamma) + s^2\mathcal{D}''(s)}{\mathcal{D}^2(s)\mathcal{D}(s;\gamma)} \left[\frac{2}{\mathcal{D}(s)} + \frac{1}{\mathcal{D}(s;\gamma)} \right] \\ &- \frac{2s^2\mathcal{D}_s(s;\gamma)}{\mathcal{D}^2(s)\mathcal{D}(s;\gamma)} \left[\frac{3\mathcal{D}'(s)}{\mathcal{D}^2(s)} + \frac{\mathcal{D}'(s) + \mathcal{D}_s(s;\gamma)}{\mathcal{D}(s)\mathcal{D}(s;\gamma)} + \frac{\mathcal{D}_s(s;\gamma)}{\mathcal{D}^2(s;\gamma)} \right]. \end{aligned} \tag{84}$$

Using the symmetry and periodicity, we have $U^j_{4,A}(\gamma)|_{v=0} = 0$ such that

$$|U^j_{4,A}(\gamma)| = \frac{\gamma^{-3}}{\pi t_j t_{j-1}} \left| \frac{e^{\mathrm{i}\rho t_{j-1}}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}\rho k}]^2} \widehat{\mathcal{K}}(s(k,\rho);\gamma) \right| \le C\gamma^{-3} t_{j-1}^{-2}. \tag{85}$$

Then for $v \in [0,\rho]$, we get

$$|\widehat{\chi}(\lambda + \mathrm{i}\ell)| \ge \frac{\sqrt{2}}{2}\widehat{\chi}(\ell), \quad |\widetilde{\chi}(\ell)| = \left| \int_0^\infty \frac{d\mu(y)}{y + \mathrm{i}\ell} \right| \le \sqrt{2}\widehat{\chi}(\ell). \tag{86}$$

Besides, from [38, Corollary 2b.2, p. 328], we have $\widehat{\chi}^{(m)}(s) = (-1)^m m! \int_0^\infty \frac{d\mu(y)}{(s+y)^{m+1}}$ for $m = 0, 1, 2, \cdots$. Therefore, we obtain the following estimates

$$|\widehat{\chi}''(s)| \le 2\int_0^\infty \frac{d\mu(y)}{|s+y|^3} = 2\int_0^\infty \frac{d\mu(y)}{[(y+\lambda)^2 + \ell^2]^{3/2}} \le \frac{2\Theta(\ell)}{\ell}, \tag{87}$$

$$|\widehat{\chi}''(s)| \le 2\int_0^\infty \frac{d\mu(y)}{(\lambda+y)^3} \le \frac{2}{\lambda}|\widehat{\chi}'(\lambda)| \le \frac{2}{\lambda^2}\widehat{\chi}(\lambda). \tag{88}$$

Then we use (86) and (49) to obtain

$$\left|\widehat{\mathcal{K}}(s;\gamma)\right| \le \frac{32\ell^2\left(\gamma^{-1} + |\widehat{\chi}'(\lambda + \mathrm{i}\ell)|\right)}{|\widetilde{\chi}(\ell)|^4} \le \frac{32\ell^2\left[\gamma^{-1} + \Theta(\ell)\right]}{|\widetilde{\chi}(\ell)|^4}, \tag{89}$$

and it follows from (48)-(49) and (86)-(87) that

$$\left|\widehat{\mathcal{K}}'(s;\gamma)\right| \le \frac{C\ell\left[\gamma^{-1} + \Theta(\ell)\right]}{|\widetilde{\chi}(\ell)|^4}\left[1 + \ell + \frac{\ell\left[\gamma^{-1} + \Theta(\ell)\right]}{|\widetilde{\chi}(\ell)|}\right]. \tag{90}$$

From (81)-(84), we have

$$\begin{aligned} \left|U^j_{4,B}(\gamma) + U^j_{4,C}(\gamma)\right| &\le C\gamma^{-3}t_{j-1}^{-2}\int_0^\rho \left(\left|\widehat{\mathcal{K}}(s(k,v);\gamma)\right| + \left|\widehat{\mathcal{K}}'(s(k,v);\gamma)\right|\right) dv \\ &\le C\gamma^{-3}t_{j-1}^{-2}, \end{aligned}$$

which, together with (85), proves (77).

5.3. Proof of (78)**.** For $U^j_5(\gamma)$, we apply integration by parts to obtain

$$\begin{aligned} U^j_{5,A}(\gamma) &= \frac{\gamma^{-1}}{t_j\pi\mathrm{i}t_{j-1}} \frac{e^{\mathrm{i}vt_{j-1}}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}]^2} \frac{\mathcal{D}_s(s;\gamma)}{\mathcal{D}^2(s;\gamma)}\Big|_{v=\rho}^{v=\pi/k} \\ &- \frac{\gamma^{-1}}{t_j\pi\mathrm{i}t_{j-1}} \int_\rho^{\pi/k} \frac{2\mathrm{i}k(1-\vartheta)e^{-\mathrm{i}vk}e^{\mathrm{i}vt_{j-1}}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}]^3} \frac{\mathcal{D}_s(s;\gamma)}{\mathcal{D}^2(s;\gamma)} dv \\ &- \frac{\gamma^{-1}}{t_j\pi\mathrm{i}t_{j-1}} \int_\rho^{\pi/k} \frac{e^{\mathrm{i}vt_{j-1}}\frac{\partial s}{\partial v}}{[\vartheta + (1-\vartheta)e^{-\mathrm{i}vk}]^2} \left[\frac{\mathcal{D}_{ss}(s;\gamma)}{\mathcal{D}^2(s;\gamma)} - \frac{2\mathcal{D}_s^2(s;\gamma)}{\mathcal{D}^3(s;\gamma)}\right] dv \\ &:= U^j_{5,A}(\gamma) - U^j_{5,B}(\gamma) - U^j_{5,C}(\gamma), \end{aligned} \tag{91}$$

in which we have

$$\begin{aligned} |U^j_{5,B}(\gamma) + U^j_{5,C}(\gamma)| &\le \frac{C\gamma^{-1}}{t_{j-1}^2}\int_\rho^{\pi/k}\left[\frac{|\mathcal{D}_s(s;\gamma)| + |\mathcal{D}_{ss}(s;\gamma)|}{|\mathcal{D}(s;\gamma)|^2} + \frac{|\mathcal{D}_s(s;\gamma)|^2}{|\mathcal{D}(s;\gamma)|^3}\right] dv \\ &\le \frac{C\gamma^{-1}}{t_{j-1}^2}\int_\rho^{\pi/k}\left[\frac{|\widehat{\chi}'(s)| + \gamma^{-1} + |\widehat{\chi}''(s)|}{|\mathcal{D}(s;\gamma)|^2} + \frac{|\widehat{\chi}'(s)|^2 + \gamma^{-2}}{|\mathcal{D}(s;\gamma)|^3}\right] dv \\ &=: U^j_{5,D}(\gamma), \quad t_{j-1} \ge 1, \end{aligned}$$

in which we used (24). We will estimate $U^j_{5,A}(\gamma)$ and $U^j_{5,D}(\gamma)$ separately as follows.

(I) Estimation of $U^j_{5,A}(\gamma)$. First, we take $v=\frac{\pi}{k}$ and use (39)-(41) to get

$$\begin{aligned}
&\lambda=\frac{2}{(2\vartheta-1)k},\quad \ell=0,\quad s=\frac{2}{(2\vartheta-1)k},\\
&D\left(\frac{2}{(2\vartheta-1)k};\gamma\right)=\widehat{\chi}\left(\frac{2}{(2\vartheta-1)k}\right)+\frac{2\gamma^{-1}}{(2\vartheta-1)k},\\
&D_s\left(\frac{2}{(2\vartheta-1)k};\gamma\right)=\widehat{\chi}'\left(\frac{2}{(2\vartheta-1)k}\right)+\gamma^{-1},
\end{aligned}$$

which combines (56) to obtain

$$\frac{\left|D_s(\frac{2}{(2\vartheta-1)k};\gamma)\right|}{\left|D(\frac{2}{(2\vartheta-1)k};\gamma)\right|^2}\le\frac{\widehat{\chi}'\left(\frac{2}{(2\vartheta-1)k}\right)}{2\widehat{\chi}\left(\frac{2}{(2\vartheta-1)k}\right)\frac{2\gamma^{-1}}{(2\vartheta-1)k}}+\frac{\gamma^{-1}}{\left(\frac{2\gamma^{-1}}{(2\vartheta-1)k}\right)^2}\le\frac{(2\vartheta-1)^2k^2\gamma}{2}.$$

Thus, we follow (91) to yield

$$|U^j_{5,A}(\gamma)|\le Ck^2t^2_{j-1},\quad t_{j-1}\ge 1. \tag{92}$$

(II) Estimation of $U^j_{5,D}(\gamma)$. We first split $U^j_{5,D}(\gamma)$ as

(93)
$$\begin{aligned}
&|U^j_{5,D}(\gamma)|\\
\le&\frac{C\gamma^{-1}}{t^2_{j-1}}\left(\int_\rho^{\frac{\varepsilon}{k}}+\int_{\frac{\varepsilon}{k}}^{\frac{\pi-\varepsilon k}{k}}+\int_{\frac{\pi-\varepsilon k}{k}}^{\frac{\pi}{k}}\right)\left[\frac{|\widehat{\chi}'(s)|+\gamma^{-1}+|\widehat{\chi}''(s)|}{|\mathcal{D}(s;\gamma)|^2}+\frac{|\widehat{\chi}'(s)|^2+\gamma^{-2}}{|\mathcal{D}(s;\gamma)|^3}\right]dv.
\end{aligned}$$

Then we first discuss $v\in[\frac{\varepsilon}{k},\frac{\pi-\varepsilon k}{k}]$. We use (54)-(56) and (88) to yield

$$\begin{aligned}
&\gamma^{-1}\int_{\frac{\varepsilon}{k}}^{\frac{\pi-\varepsilon k}{k}}\left[\frac{|\widehat{\chi}'(s)|+\gamma^{-1}+|\widehat{\chi}''(s)|}{|\mathcal{D}(s;\gamma)|^2}+\frac{|\widehat{\chi}'(s)|^2+\gamma^{-2}}{|\mathcal{D}(s;\gamma)|^3}\right]dv\\
&\quad\le C\int_{\frac{\varepsilon}{k}}^{\frac{\pi-\varepsilon k}{k}}\left(\frac{\widehat{\chi}(\lambda)\gamma^{-1}}{\lambda\widehat{\chi}(\lambda)\frac{\lambda}{\gamma}}+\frac{\gamma^{-2}}{(\frac{1}{k\gamma})^2}+\frac{\widehat{\chi}(\lambda)\gamma^{-1}}{\lambda^2\widehat{\chi}(\lambda)\frac{\lambda}{\gamma}}+\frac{\widehat{\chi}^2(\lambda)\gamma^{-1}}{\lambda^2\widehat{\chi}^2(\lambda)\frac{\lambda}{\gamma}}+\frac{\gamma^{-3}}{(\frac{\lambda}{\gamma})^3}\right)dv\\
&\quad\le C\int_{\frac{\varepsilon}{k}}^{\frac{\pi-\varepsilon k}{k}}(\lambda^{-2}+k^2+\lambda^{-3})dv\le Ck.
\end{aligned}$$

Similar to this, for $v\in[\frac{\pi-\varepsilon k}{k},\frac{\pi}{k}]$ we employ (52)-(53), (56) and (88) to get

$$\gamma^{-1}\int_{\frac{\pi-\varepsilon k}{k}}^{\frac{\pi}{k}}\left[\frac{|\widehat{\chi}'(s)|+\gamma^{-1}+|\widehat{\chi}''(s)|}{|\mathcal{D}(s;\gamma)|^2}+\frac{|\widehat{\chi}'(s)|^2+\gamma^{-2}}{|\mathcal{D}(s;\gamma)|^3}\right]dv\le Ck.$$

Finally we consider $v\in[\rho,\frac{\varepsilon}{k}]$. From (60), we employ (87), (49), (44)-(46), (14), (12), and

$$dv=\frac{\left[\vartheta^2+(1-\vartheta)^2+2\vartheta(1-\vartheta)\cos(kv)\right]^2}{[\vartheta^2+(1-\vartheta)^2]\cos(kv)+2\vartheta(1-\vartheta)}d\ell$$

to obtain that

$$\gamma^{-1}\int_{\rho}^{\frac{\varepsilon}{k}}\left[\frac{|\widehat{\chi}'(s)|+\gamma^{-1}+|\widehat{\chi}''(s)|}{|\mathcal{D}(s;\gamma)|^2}+\frac{|\widehat{\chi}'(s)|^2+\gamma^{-2}}{|\mathcal{D}(s;\gamma)|^3}\right]dv$$
$$\le C\gamma^{-1}\left[\int_{c_0\rho}^{\frac{\omega}{2}}\frac{\gamma^{-1}+\Theta(\ell)}{\ell^3\Theta^2(\ell)}d\ell+\int_{\frac{\omega}{2}}^{2\omega}\left(\frac{\gamma^{-1}}{\ell\Phi^2(\ell)}+\frac{\gamma^{-2}}{\Phi^3(\ell)}\right)d\ell\right.$$
$$\left.+\int_{2\omega}^{\infty}\frac{\gamma d\ell}{\ell|\ell-\omega|^2}+\int_{2\omega}^{\infty}\frac{\gamma d\ell}{|\ell-\omega|^3}\right]\le C(1+\gamma), \tag{94}$$

in which we used (14)-(15) and (12) so that

$$\gamma^{-1}\int_{\frac{\omega}{2}}^{2\omega}\left(\frac{\gamma^{-1}}{\ell\Phi^2(\ell)}+\frac{\gamma^{-2}}{\Phi^3(\ell)}\right)d\ell\le C\int_{\frac{\omega}{2}}^{2\omega}\left(\frac{\Theta^2(\ell)}{\ell\Phi^2(\ell)}+\frac{\Theta^3(\ell)}{\Phi^3(\ell)}\right)d\ell\le C\omega\le C\gamma.$$

Based on the above analysis, we have

$$|U^j_{5,D}(\gamma)|\le C(k+1+\gamma)t^2_{j-1},\quad t_{j-1}\ge 1. \tag{95}$$

We combine (92) and (95) to complete the proof of (78).

5.4. Proof of Theorem 4. The key idea of relaxing the regularity of the initial data lies in using (13) instead of (12) in the estimate of (94) to reduce the power of γ. Specifically, we establish the following bound by $1/\Theta(\ell)\le C\gamma$ for $\omega/2\le\ell\le 2\omega$ (cf. [5, (5.5)]), (15), and (13)

$$\gamma^{-1}\int_{\frac{\omega}{2}}^{2\omega}\left(\frac{\gamma^{-1}}{\ell\Phi^2(\ell)}+\frac{\gamma^{-2}}{\Phi^3(\ell)}\right)d\ell\le C\int_{\frac{\omega}{2}}^{2\omega}\left[\frac{\Theta^2(\ell)}{\ell\Phi^2(\ell)}+\frac{\Theta^3(\ell)}{\Phi^3(\ell)}\right]d\ell$$
$$=C\int_{\frac{\omega}{2}}^{2\omega}\left[\frac{\ell^{-3}}{[\Theta(\ell)]^{2/3}}\left(\frac{\ell[\Theta(\ell)]^{4/3}}{\Phi(\ell)}\right)^2+\frac{\ell^{-3}}{\Theta(\ell)}\left(\frac{\ell[\Theta(\ell)]^{4/3}}{\Phi(\ell)}\right)^3\right]d\ell \tag{96}$$
$$\le C\int_{\frac{\omega}{2}}^{2\omega}\left[\frac{\ell^{-3}}{[\Theta(\ell)]^{2/3}}+\frac{\ell^{-3}}{\Theta(\ell)}\right]d\ell\le C(\gamma^{2/3}+\gamma)\int_{\frac{\omega}{2}}^{2\omega}\ell^{-3}d\ell\le C\gamma\omega^{-2}\le C,$$

which immediately leads to the re-estimate of (94)

$$\gamma^{-1}\int_{\rho}^{\frac{\varepsilon}{k}}\left[\frac{|\widehat{\chi}'(s)|+\gamma^{-1}+|\widehat{\chi}''(s)|}{|\mathcal{D}(s;\gamma)|^2}+\frac{|\widehat{\chi}'(s)|^2+\gamma^{-2}}{|\mathcal{D}(s;\gamma)|^3}\right]dv\le C.$$

Therefore, we obtain for this case

$$\left|U^j_5(\gamma)\right|\le Ct^{-2}_{j-1},\quad t_{j-1}\ge 1,\quad \gamma\in[\gamma_1,\infty).$$

The proof of Theorem 4 is thus completed by incorporating this equation, (77), (79), (64) and (71).

6. Numerical experiments

We select $\Omega=(0,1)$, $Au=-\partial^2u/\partial x^2$, the Abel kernel $\chi(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $0<\alpha<1$ and zero Dirichlet boundary conditions. Both the smooth and non-smooth initial data will be applied. The centering finite difference with mesh size $h=1/M$ is applied for spatial discretization, and we take $k=T/N$ for relatively large T. We denote the discrete spatial $\mathcal{L}^2$ norm as $\|U^n\|^2_{\mathcal{L}^2}=h\sum_{j=1}^{M-1}|U^n_j|^2$, the discrete time-space $\mathcal{L}^1(\mathcal{L}^2)$ norm as $U^*=k\sum_{n=1}^{N}\|U^n\|_{\mathcal{L}^2}$, and the error as $E(h,k)=\left\|U^N_{h,k}-U^N_{h,\frac{k}{2}}\right\|_{\mathcal{L}^2}$.

We first evaluate the accuracy of the proposed ϑ-type convolution quadrature method with different ϑ. We take $h=1/100$, $T=800$ and the smooth initial value

$u_0 = \sin(\pi x)$, and present errors and CPU times in Table 1. We observe that the errors become smaller as ϑ gradually approaches 0.5, especially for smaller N.

TABLE 1. Errors and CPU times (seconds) when $h = 1/100$ and $T = 800$ with $u_0 = \sin(\pi x)$.

α	N	$\vartheta = 0.5$		$\vartheta = 0.75$		$\vartheta = 1$	
		$E(h,k)$	CPU(s)	$E(h,k)$	CPU(s)	$E(h,k)$	CPU(s)
0.5	6	8.3622e-4	0.001	2.2993e-2	0.001	2.1495e-1	0.001
	12	7.7253e-3	0.002	1.5950e-2	0.002	3.3644e-2	0.002
	24	8.7164e-3	0.003	1.1945e-2	0.004	1.6716e-2	0.003
	48	6.7309e-3	0.006	8.1305e-2	0.007	9.8504e-3	0.007
	96	4.3490e-3	0.007	5.0356e-3	0.010	5.7938e-3	0.012

We then test the long-time stability of numerical solutions to the proposed method. We take $h = 1/128$, $T = 1000$ and the smooth initial value $u_0 = \sin(\pi x)$, and present U^* and CPU times in Table 2. We observe that as N increases, U^* is gradually decreasing, which illustrates the long-time stability of the proposed scheme. Similar phenomena are also shown in Table 3 where we select $h = 1/128$, $T = 1000$ and the non-smooth initial value $u_0 = x^{-\frac{1}{4}}$. These results demonstrate the validity of our analysis in long-time simulations.

TABLE 2. $\mathcal{L}^1(\mathcal{L}^2)$ norms of U and CPU times (seconds) when $h = 1/128$ and $T = 1000$ with $u_0 = \sin(\pi x)$.

α	N	$\vartheta = 0.5$		$\vartheta = 0.75$		$\vartheta = 1$	
		U^*	CPU(s)	U^*	CPU(s)	U^*	CPU(s)
0.9	128	2.3793e+2	0.060	3.0388e+2	0.067	3.8210e+2	0.071
	256	1.2514e+2	0.154	1.5245e+2	0.173	1.8225e+2	0.167
	512	6.4318e+1	0.368	7.6608e+1	0.411	8.9448e+1	0.389
	1024	3.2639e+1	1.076	3.8457e+1	1.171	4.4402e+1	1.172
	2048	1.6450e+1	7.785	1.9281e+1	8.026	2.2142e+1	8.199
0.1	128	2.0802e-1	0.068	2.0946e-1	0.053	2.1096e-1	0.056
	256	1.0520e-1	0.158	1.0613e-1	0.175	1.0708e-1	0.172
	512	5.3164e-2	0.440	5.3737e-2	0.403	5.4320e-2	0.402
	1024	2.6849e-2	1.169	2.7188e-2	1.162	2.7532e-2	1.177
	2048	1.3549e-2	7.986	1.3744e-2	8.205	1.3941e-2	8.314

7. Concluding remarks

In this work, we derive a unified analysis framework for the long-time uniform stability of time-discrete solutions for Volterra integrodifferential equations with the completely monotonic kernel via the ϑ-type convolution quadrature. An interesting topic is to extend the work to the non-homogeneous case, where one may require certain long-time decay property of the forcing term f to perform the long-time stability analysis. For error estimates, one may consider the truncation errors as the forcing term and then invoke the stability result to get the convergence analysis result. Nevertheless, a substantial modification is at least needed since it is still not clear whether the estimates of the truncation errors satisfy the desired long-time decay properties. We will investigate these issues by combining the frequency domain analysis method in [4,29] and error analysis method in [40,42] in the future.

Another challenging topic is to include the case $\vartheta = 1/2$ in the l^1 stability analysis in Theorems 3 and 4, since (40)–(41) may cause blow-up when $\vartheta = 1/2$

TABLE 3. $\mathcal{L}^1(\mathcal{L}^2)$ norms of U and CPU times (seconds) when $h = 1/128$ and $T = 1000$ with $u_0 = x^{-\frac{1}{4}}$.

α	N	$\vartheta = 0.5$		$\vartheta = 0.75$		$\vartheta = 1$	
		U^*	CPU(s)	U^*	CPU(s)	U^*	CPU(s)
0.8	128	1.7123e+3	0.057	2.0803e+3	0.058	2.4864e+3	0.063
	256	9.0028e+2	0.158	1.0685e+3	0.153	1.2448e+3	0.173
	512	4.6325e+2	0.423	5.4410e+2	0.405	6.2674e+2	0.391
	1024	2.3544e+2	1.211	2.7528e+2	1.172	3.1552e+2	1.184
	2048	1.1882e+2	8.138	1.3866e+2	8.308	1.5859e+2	8.146
0.2	128	1.0126e+1	0.068	1.0355e+1	0.068	1.0587e+1	0.061
	256	5.1953e+0	0.163	5.3381e+0	0.178	5.4820e+0	0.164
	512	2.6558e+0	0.428	2.7406e+0	0.452	2.8257e+0	0.486
	1024	1.3533e+0	1.161	1.4018e+0	1.192	1.4504e+0	1.244
	2048	6.8773e-1	8.389	7.1471e-1	8.326	7.4172e-1	8.464

in frequency domain analysis, which may require more sophisticated techniques or methods that remain to be further explored.

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