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AUTO-STABILIZED WEAK GALERKIN FINITE ELEMENT METHODS FOR BIHARMONIC EQUATIONS ON POLYTOPAL MESHES WITHOUT CONVEXITY ASSUMPTIONS

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Abstract. This paper introduces an auto-stabilized weak Galerkin (WG) finite element method for biharmonic equations with built-in stabilizers. Unlike existing stabilizer-free WG methods limited to convex elements in finite element partitions, our approach accommodates both convex and non-convex polytopal meshes, offering enhanced versatility. It employs bubble functions without the restrictive conditions required by existing stabilizer-free WG methods, thereby simplifying implementation and broadening application to various partial differential equations (PDEs). Additionally, our method supports flexible polynomial degrees in discretization and is applicable in any dimension, unlike existing stabilizer-free WG methods that are confined to specific polynomial degree combinations and 2D or 3D settings. We demonstrate optimal order error estimates for WG approximations in both a discrete H^2 norm for $k \geq 2$ and an L^2 norm for $k > 2$, as well as a sub-optimal error estimate in L^2 when $k = 2$, where $k \geq 2$ denotes the degree of polynomials in the approximation.

Key words. Weak Galerkin, finite element methods, auto-stabilized, non-convex, ploytopal meshes, bubble function, weak Laplacian, biharmonic equation.

1. Introduction

In this paper, we propose an auto-stabilized weak Galerkin finite element method with built-in stabilizers suitable for non-convex polytopal meshes, specifically applied to biharmonic equations with Dirichlet and Neumann boundary conditions. Specifically, we seek to determine an unknown function u such that

$$\begin{aligned} \Delta^2 u =& f, && \text{in} \quad \Omega, \\ u =& \xi, && \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \mathbf{n}} =& \nu, && \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded domain with a Lipschitz continuous boundary $\partial\Omega$. Note that the domain Ω considered in this paper can be of any dimension d.

The variational formulation of the model problem (1) can be formulated as follows: Find an unknown function $u \in H^2(\Omega)$ satisfying $u|_{\partial\Omega} = \xi$ and $\frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \nu$, and the following equation

$$(\Delta u, \Delta v) = (f, v), \qquad \forall v \in H_0^2(\Omega), \tag{2}$$

where $H_0^2(\Omega) = \{v \in H^2(\Omega) : v|_{\partial\Omega} = 0, \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = 0\}$.

The weak Galerkin finite element method marks a significant advancement in numerical solutions for PDEs. This innovative approach redefines or approximates differential operators within a framework akin to the distribution theory tailored for

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piecewise polynomials. Unlike conventional techniques, the WG method alleviates the usual regularity constraints on approximating functions by employing carefully crafted stabilizers. Extensive research has demonstrated the WG method's versatility across various model PDEs, bolstered by a substantial list of references [9, 10, 36, 40, 11, 12, 13, 14, 38, 42, 4, 35, 18, 8, 22, 44, 30, 34, 31, 32, 33, 37, 39, 6, 20, 47, 46, 41, 7], underscoring its potential as a powerful tool in scientific computing. What sets WG methods apart from other finite element approaches is their use of weak derivatives and weak continuities to create numerical schemes based on the weak formulations of the underlying PDE problems. This structural versatility makes WG methods exceptionally effective across a wide range of PDEs, ensuring both stability and precision in their approximations.

A significant innovation within the weak Galerkin methodology is the "Primal-Dual Weak Galerkin (PDWG)" approach. This novel method addresses difficulties that traditional numerical strategies often encounter [15, 16, 1, 2, 3, 17, 23, 24, 43, 5, 26, 27, 25, 28, 29]. PDWG interprets numerical solutions as constrained optimization problems, with the constraints mimicking the weak formulation of PDEs through the application of weak derivatives. This innovative formulation leads to the derivation of an Euler-Lagrange equation that integrates both the primary variables and the dual variables (Lagrange multipliers), thereby creating a symmetric numerical scheme.

This paper introduces a straightforward formulation of the weak Galerkin finite element method for biharmonic equations that operates on both convex and non-convex polytopal meshes without the use of stabilizers. The key trade-off for eliminating stabilizers involves using higher-degree polynomials for computing the discrete weak Laplacian operator, which may impact practical applicability. Unlike existing stabilizer-free WG schemes limited to convex elements [45], our method accommodates non-convex polytopal meshes, preserving the size and global sparsity of the stiffness matrix while significantly reducing programming complexity. Theoretical analysis confirms optimal error estimates for WG approximations in both the discrete H^2 norm for $k \geq 2$ and the L^2 norm for $k > 2$, along with a sub-optimal error estimate in L^2 when $k = 2$, where $k \geq 2$ is the polynomial degree in the approximation.

Our method introduces several significant enhancements over the stabilizer-free weak Galerkin finite element method for biharmonic equations presented by [45]. The key contributions are summarized as follows: **1. Theoretical Foundation for Non-Convex Polytopal Meshes:** Our method provides a theoretical foundation for an auto-stabilized WG scheme that handles convex and non-convex elements in finite element partitions through the innovative use of bubble functions, while the existing stabilizer-free WG method [45] is limited to convex meshes. This enhances the practical applicability of our method, making it more versatile for real-world computational scenarios. **2. Superior Flexibility with Bubble Functions:** Unlike the method in [45], which is limited by restrictive conditions imposed in the analysis, our approach employs bubble functions as a critical analysis tool without these constraints. This flexibility allows our method to generalize to various types of PDEs without the complexities imposed by such conditions, thereby simplifying the implementation process. **3. Dimensional Versatility:** Our method is applicable in any dimension d, whereas the method in [45] is confined to 2D or 3D settings. This broader applicability makes our method suitable for higher-dimensional problems. **4. Adaptable Polynomial Degrees:** Our method supports flexible degree of polynomials in the discretization process, unlike the specific polynomial degree combinations in [45]. This adaptability allows for greater precision and control in computational implementations, catering to a wide range of problem-specific requirements. Given these improvements, our method offers enhanced flexibility, broader applicability, and ease of implementation in various computational settings.

Our research introduces a more versatile WG scheme applicable to both convex and non-convex polytopal meshes, as detailed above, making our method a significant advancement over the one in [45]. To provide a comprehensive understanding of our contributions, we include an in-depth analysis of the error estimates in Sections 5-7, even though these sections share some similarities with the work presented in [45]. This analysis is essential for demonstrating the significant improvements and expanded applicability of our method.

While our algorithms share similarities with those introduced by [45], our primary contribution lies in advancing the theoretical framework rather than performing additional empirical validation. The extensive numerical tests detailed in [45] already establish the effectiveness of these methods, rendering further empirical tests unnecessary. This paper, therefore, places a strong emphasis on theoretical analysis. By focusing on theoretical advancements, we provide vital insights that are essential for future development and application of these algorithms.

This paper is organized as follows: In Section 2, we briefly review the definition of the weak Laplacian and its discrete version. In Section 3, we present the simple weak Galerkin scheme without the use of a stabilizer. Section 4 is dedicated to deriving the existence and uniqueness of the solution. In Section 5, we derive the error equation for the proposed weak Galerkin scheme. Section 6 focuses on deriving the error estimate for the numerical approximation in the energy norm. Finally, Section 7 establishes the error estimate for the numerical approximation in the L^2 norm.

The standard notations are adopted throughout this paper. Let D be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d$. We use $(\cdot,\cdot)_{s,D}$, $|\cdot|_{s,D}$ and $\|\cdot\|_{s,D}$ to denote the inner product, semi-norm and norm in the Sobolev space $H^s(D)$ for any integer $s \geq 0$, respectively. For simplicity, the subscript D is dropped from the notations of the inner product and norm when the domain D is chosen as $D = \Omega$. For the case of $s = 0$, the notations $(\cdot,\cdot)_{0,D}$, $|\cdot|_{0,D}$ and $\|\cdot\|_{0,D}$ are simplified as $(\cdot,\cdot)_D$, $|\cdot|_D$ and $\|\cdot\|_D$, respectively.

2. Weak Laplacian Operator and Discrete Weak Laplacian Operator

In this section, we will briefly review the definition of the weak Laplacian operator and its discrete version as introduced in [45].

Let $\mathcal{T}_h$ be a finite element partition of the domain $\Omega \subset \mathbb{R}^d$ into polytopes. Assume that $\mathcal{T}_h$ is shape regular [39]. Denote by $\mathcal{E}_h$ the set of all edges/faces in $\mathcal{T}_h$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior edges/faces. Denote by h_T the diameter of $T \in \mathcal{T}_h$ and $h = \max_{T\in\mathcal{T}_h} h_T$ the meshsize of the finite element partition $\mathcal{T}_h$. Denote by $\mathbf{n}_e$ an unit and normal direction to e for $e \in \mathcal{E}_h$.

Let T be a polytopal element with boundary ∂T. A weak function on T refers to $v = \{v_0, v_b, v_n\mathbf{n}_e\}$ such that $v_0 \in L^2(T)$, $v_b \in L^2(\partial T)$ and $v_n \in L^2(\partial T)$. The first component v_0 and the second component v_b represent the value of v in the interior of T and on the boundary of T, respectively. The third component v_n intends to represent the value of $\nabla v_0 \cdot \mathbf{n}_e$ on the boundary of T. In general, v_b and v_n are assumed to be independent of the traces of v_0 and $\nabla v_0 \cdot \mathbf{n}_e$ respectively.

Denote by $W(T)$ the space of all weak functions on T; i.e.,

$$W(T) = \{v = \{v_0, v_b, v_n\mathbf{n}_e\} : v_0 \in L^2(T), v_b \in L^2(\partial T), v_n \in L^2(\partial T)\}. \tag{3}$$

The weak Laplacian operator, denoted by Δ_w, is a linear operator from $W(T)$ to the dual space of $H^2(T)$ such that for any $v \in W(T)$, $\Delta_w v$ is a bounded linear functional on $H^2(T)$ defined by

$$(\Delta_w v, \varphi)_T = (v_0, \Delta\varphi)_T - \langle v_b, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e \cdot \mathbf{n}, \varphi\rangle_{\partial T}, \quad \forall \varphi \in H^2(T), \tag{4}$$

where $\mathbf{n}$ is an unit outward normal direction to ∂T.

For any non-negative integer $r \geq 0$, let $P_r(T)$ be the space of polynomials on T with total degree r and less. A discrete weak Laplacian operator on T, denoted by $\Delta_{w,r,T}$, is a linear operator from $W(T)$ to $P_r(T)$ such that for any $v \in W(T)$, $\Delta_{w,r,T}v$ is the unique polynomial in $P_r(T)$ satisfying

$$(\Delta_{w,r,T}v, \varphi)_T = (v_0, \Delta\varphi)_T - \langle v_b, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e \cdot \mathbf{n}, \varphi\rangle_{\partial T}, \quad \forall \varphi \in P_r(T). \tag{5}$$

For a smooth $v_0 \in H^2(T)$, applying the usual integration by parts to the first term on the right-hand side of (5) gives

$$(\Delta_{w,r,T}v, \varphi)_T = (\Delta v_0, \varphi)_T - \langle v_b - v_0, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e \cdot \mathbf{n} - \nabla v_0 \cdot \mathbf{n}, \varphi\rangle_{\partial T}, \tag{6}$$

for any $\varphi \in P_r(T)$.

3. Auto-Stabilized Weak Galerkin Algorithms

Let $k \geq 2$, $p \geq 1$ and $q \geq 1$ be integers. Assume that $k \geq p \geq q$. For any element $T \in \mathcal{T}_h$, define a local weak finite element space; i.e.,

$$V(k, p, q, T) = \{\{v_0, v_b, v_n\mathbf{n}_e\} : v_0 \in P_k(T), v_b \in P_p(e), v_n \in P_q(e), e \subset \partial T\}.$$

By patching $V(k, p, q, T)$ over all the elements $T \in \mathcal{T}_h$ through a common value v_b on the interior interface $\mathcal{E}_h^0$, we obtain a global weak finite element space; i.e.,

$$V_h = \big\{\{v_0, v_b, v_n\mathbf{n}_e\} : \ \{v_0, v_b, v_n\mathbf{n}_e\}|_T \in V(k, p, q, T), \forall T \in \mathcal{T}_h\big\}.$$

Denote by V_h^0 the subspace of V_h with vanishing boundary values on $\partial\Omega$; i.e.,

$$V_h^0 = \{\{v_0, v_b, v_n\mathbf{n}_e\} \in V_h : v_b|_e = 0, v_n\mathbf{n}_e \cdot \mathbf{n}|_e = 0, e \subset \partial\Omega\}.$$

For simplicity of notation and without confusion, for any $v \in V_h$, denote by $\Delta_w v$ the discrete weak Laplacian operator $\Delta_{w,r,T}v$ computed by (5) on each element T; i.e.,

$$(\Delta_w v)|_T = \Delta_{w,r,T}(v|_T), \qquad \forall T \in \mathcal{T}_h.$$

On each element $T \in \mathcal{T}_h$, let Q_0 be the L^2 projection onto $P_k(T)$. On each edge/face $e \subset \partial T$, let Q_b and Q_n be the L^2 projection operators onto $P_p(e)$ and $P_q(e)$, respectively. For any $w \in H^2(\Omega)$, denote by $Q_h w$ the L^2 projection into the weak finite element space V_h such that

$$(Q_h w)|_T := \{Q_0(w|_T), Q_b(w|_{\partial T}), Q_n(\nabla w|_{\partial T} \cdot \mathbf{n}_e)\mathbf{n}_e\}, \qquad \forall T \in \mathcal{T}_h.$$

The straightforward WG numerical scheme, which avoids the use of stabilizers for the biharmonic equation (1), is formulated as follows.

Auto-Stabilized Weak Galerkin Algorithm 3.1. *Find* $u_h = \{u_0, u_b, u_n\mathbf{n}_e\} \in V_h$ *satisfying* $u_b = Q_b\xi$, $u_n\mathbf{n}_e \cdot \mathbf{n} = Q_n\nu$ *on* $\partial\Omega$ *and the following equation*

$$(\Delta_w u_h, \Delta_w v) = (f, v_0), \qquad \forall v = \{v_0, v_b, v_n\mathbf{n}_e\} \in V_h^0, \tag{7}$$

where

$$(\Delta_w u_h, \Delta_w v) = \sum_{T\in\mathcal{T}_h} (\Delta_w u_h, \Delta_w v)_T,$$
$$(f, v_0) = \sum_{T\in\mathcal{T}_h} (f, v_0)_T.$$

4. Solution Existence and Uniqueness

Recall that $\mathcal{T}_h$ is a shape-regular finite element partition of the domain Ω. Thus, for any $T \in \mathcal{T}_h$ and $\phi \in H^1(T)$, the following trace inequality holds true [39]; i.e.,

$$\|\phi\|_{\partial T}^2 \le C(h_T^{-1}\|\phi\|_T^2 + h_T\|\nabla\phi\|_T^2). \tag{8}$$

If ϕ is a polynomial on the element $T \in \mathcal{T}_h$, the following trace inequality holds true [39]; i.e.,

$$\|\phi\|_{\partial T}^2 \le C h_T^{-1}\|\phi\|_T^2. \tag{9}$$

Given a weak function $v = \{v_0, v_b, v_n\mathbf{n}_e\} \in V_h$, we define the energy norm as:

$$|||v||| = (\Delta_w v, \Delta_w v)^{\frac{1}{2}}. \tag{10}$$

Next, we define the discrete H^2 semi-norm as:

$$\|v\|_{2,h} = \Big(\sum_{T\in\mathcal{T}_h} \|\Delta v_0\|_T^2 + h_T^{-3}\|v_0 - v_b\|_{\partial T}^2 + h_T^{-1}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}^2\Big)^{\frac{1}{2}}. \tag{11}$$

Lemma 4.1. *For* $v = \{v_0, v_b, v_n\mathbf{n}_e\} \in V_h$, *there exists a constant* C *such that*

$$\|\Delta v_0\|_T \le C\|\Delta_w v\|_T.$$

Proof. Let $T \in \mathcal{T}_h$ be a polytopal element with N edges/faces denoted by $e_1, \cdots, e_N$. It is important to emphasis that the polytopal element T can be non-convex. On each edge/face e_i, we construct a linear equation $l_i(x)$ such that $l_i(x) = 0$ on edge/face e_i as follows:

$$l_i(x) = \frac{1}{h_T}\overrightarrow{AX}\cdot\mathbf{n}_i,$$

where $A = (A_1, \cdots, A_{d-1})$ is a given point on the edge/face e_i, $X = (x_1, \cdots, x_{d-1})$ is any point on the edge/face e_i, $\mathbf{n}_i$ is the normal direction to the edge/face e_i, and h_T is the size of the element T.

The bubble function of the element T can be defined as

$$\Phi_B = l_1^2(x)l_2^2(x)\cdots l_N^2(x) \in P_{2N}(T).$$

It is straightforward to verify that $\Phi_B = 0$ on the boundary ∂T. The function Φ_B can be scaled such that $\Phi_B(M) = 1$ where M represents the barycenter of the element T. Additionally, there exists a sub-domain $\hat{T} \subset T$ such that $\Phi_B \ge \rho_0$ for some constant $\rho_0 > 0$.

For $v=\{v_0,v_b,v_n\mathbf{n}_e\}\in V_h$, letting $r=2N+k-2$ and $\varphi=\Phi_B\Delta v_0\in P_r(T)$ in (6) yields

$$
\begin{aligned}
&(\Delta_w v,\Phi_B\Delta v_0)_T\\
=&(\Delta v_0,\Phi_B\Delta v_0)_T-\langle v_b-v_0,\nabla(\Phi_B\Delta v_0)\cdot\mathbf{n}\rangle_{\partial T}\\
&+\langle v_n\mathbf{n}_e\cdot\mathbf{n}-\nabla v_0\cdot\mathbf{n},\Phi_B\Delta v_0\rangle_{\partial T}\\
=&(\Delta v_0,\Phi_B\Delta v_0)_T,
\end{aligned}
\tag{12}
$$

where we used $\Phi_B=0$ on ∂T.

According to the domain inverse inequality [39], there exists a constant C such that

$$
(\Delta v_0,\Phi_B\Delta v_0)_T\geq C(\Delta v_0,\Delta v_0)_T. \tag{13}
$$

By applying the Cauchy-Schwarz inequality along with (12)-(13), we have

$$
(\Delta v_0,\Delta v_0)_T\leq C(\Delta_w v,\Phi_B\Delta v_0)_T\leq C\|\Delta_w v\|_T\|\Phi_B\Delta v_0\|_T\leq C\|\Delta_w v\|_T\|\Delta v_0\|_T,
$$

which gives

$$
\|\Delta v_0\|_T\leq C\|\Delta_w v\|_T.
$$

This completes the proof of the lemma. □

Remark 4.1. *If the polytopal element T is convex, the bubble function of the element T in Lemma 4.1 can be simplified to*

$$
\Phi_B=l_1(x)l_2(x)\cdots l_N(x).
$$

It can be verified that this simplified bubble function Φ_B satisfies (1) $\Phi_B=0$ on the boundary ∂T, (2) there exists a sub-domain $\hat{T}\subset T$ such that $\Phi_B\geq\rho_0$ for some constant $\rho_0>0$. Lemma 4.1 can be proved in the same manner using this simplified construction. In this case, we take $r=N+k-2$.

By constructing an edge/face-based bubble function

$$
\varphi_{e_k}=\Pi_{i=1,\cdots,N,i\neq k}l_i^2(x),
$$

it can be easily verified that (1) $\varphi_{e_k}=0$ on each edge/face e_i for $i\neq k$, (2) there exists a sub-domain $\widehat{e_k}\subset e_k$ such that $\varphi_{e_k}\geq\rho_1$ for some constant $\rho_1>0$. Let $\varphi=(v_b-v_0)l_k\varphi_{e_k}$. It is straightforward to check that $\varphi=0$ on each edge/face e_i for $i=1,\cdots,N$, $\nabla\varphi=0$ on each edge/face e_i for $i\neq k$ and $\nabla\varphi=(v_0-v_b)(\nabla l_k)\varphi_{e_k}=\mathcal{O}(\frac{(v_0-v_b)\varphi_{e_k}}{h_T}\mathbf{C})$ on edge/face e_k for some vector constant $\mathbf{C}$.

Lemma 4.2. *For $\{v_0,v_b,v_n\mathbf{n}_e\}\in V_h$, let $\varphi=(v_b-v_0)l_k\varphi_{e_k}$. The following inequality holds:*

$$
\|\varphi\|_T^2\leq Ch_T\int_{e_k}(v_b-v_0)^2ds. \tag{14}
$$

Proof. We first extend v_b, initially defined on the $(d-1)$-dimensional edge/face e_k, to the entire d-dimensional polytopal element T using the following formula:

$$
v_b(X)=v_b(Proj_{e_k}(X)),
$$

where $X=(x_1,\cdots,x_d)$ is any point in the element T, $Proj_{e_k}(X)$ denotes the orthogonal projection of the point X onto the hyperplane $H\subset\mathbb{R}^d$ containing the edge/face e_k. When the projection $Proj_{e_k}(X)$ is not on the edge/face e_k, $v_b(Proj_{e_k}(X))$ is defined to be the extension of v_b from e_k to the hyperplane H.

We claim that v_b remains a polynomial defined on the element T after the extension.

Let the hyperplane H containing the edge/face e_k be defined by $d-1$ linearly independent vectors $\boldsymbol{\eta}_1, \cdots, \boldsymbol{\eta}_{d-1}$ originating from a point A on the edge/face e_k. Any point P on the edge/face e_k can be parametrized as

$$P(t_1, \cdots, t_{d-1}) = A + t_1\boldsymbol{\eta}_1 + \cdots + t_{d-1}\boldsymbol{\eta}_{d-1},$$

where $t_1, \cdots, t_{d-1}$ are parameters.

Note that $v_b(P(t_1, \cdots, t_{d-1}))$ is a polynomial of degree p defined on the edge/face e_k. It can be expressed as:

$$v_b(P(t_1, \cdots, t_{d-1})) = \sum_{|\alpha|\leq p} c_\alpha \mathbf{t}^\alpha,$$

where $\mathbf{t}^\alpha = t_1^{\alpha_1} \cdots t_{d-1}^{\alpha_{d-1}}$ and $\alpha = (\alpha_1, \cdots, \alpha_{d-1})$ is a multi-index.

For any point $X = (x_1, \cdots, x_d)$ in the element T, the projection of the point X onto the hyperplane $H \subset \mathbb{R}^d$ containing the edge/face e_k is the point on the hyperplane H that minimizes the distance to X. Mathematically, this projection $Proj_{e_k}(X)$ is an affine transformation which can be expressed as

$$Proj_{e_k}(X) = A + \sum_{i=1}^{d-1} t_i(X)\boldsymbol{\eta}_i,$$

where $t_i(X)$ are the projection coefficients, and A is the origin point on e_k. The coefficients $t_i(X)$ are determined by solving the orthogonality condition:

$$(X - Proj_{e_k}(X)) \cdot \boldsymbol{\eta}_j = 0, \qquad \forall j = 1, \cdots, d-1.$$

This results in a system of linear equations in $t_1(X)$, $\cdots$, $t_{d-1}(X)$, which can be solved to yield:

$$t_i(X) = \text{linear function of } X.$$

Hence, the projection $Proj_{e_k}(X)$ is an affine linear function of X.

We extend the polynomial v_b from the edge/face e_k to the entire element T by defining

$$v_b(X) = v_b(Proj_{e_k}(X)) = \sum_{|\alpha|\leq p} c_\alpha \mathbf{t}(X)^\alpha,$$

where $\mathbf{t}(X)^\alpha = t_1(X)^{\alpha_1} \cdots t_{d-1}(X)^{\alpha_{d-1}}$. Since $t_i(X)$ are linear functions of X, each term $\mathbf{t}(X)^\alpha$ is a polynomial in $X = (x_1, \cdots, x_d)$. Thus, $v_b(X)$ is a polynomial in the d-dimensional coordinates $X = (x_1, \cdots, x_d)$.

Secondly, let v_{trace} denote the trace of v_0 on the edge/face e_k. We extend v_{trace} to the entire element T using the following formula:

$$v_{trace}(X) = v_{trace}(Proj_{e_k}(X)),$$

where X is any point in the element T, $Proj_{e_k}(X)$ denotes the projection of the point X onto the hyperplane H containing the edge/face e_k. When the projection $Proj_{e_k}(X)$ is not on the edge/face e_k, $v_{trace}(Proj_{e_k}(X))$ is defined to be the extension of v_{trace} from e_k to the hyperplane H. Similar to the case for v_b, v_{trace} remains a polynomial after this extension.

Let $\varphi = (v_b - v_0) l_k \varphi_{e_k}$. We have

$$\begin{aligned}
\|\varphi\|_T^2 = \int_T \varphi^2 dT \leq & Ch_T^2 \int_T (\nabla\varphi)^2 dT \\
\leq & Ch_T^2 \int_T (\nabla((v_b - v_{trace})(X) l_k \varphi_{e_k}))^2 dT \\
\leq & Ch_T^3 \int_{e_k} ((v_b - v_{trace})(Proj_{e_k}(X))(\nabla l_k)\varphi_{e_k})^2 ds \\
\leq & Ch_T \int_{e_k} (v_b - v_0)^2 ds,
\end{aligned}$$

where we used Poincare inequality since $\varphi = 0$ on each edge/face e_i for $i = 1, \cdots, N$, $\nabla\varphi = 0$ on each edge/face e_i for $i \neq k$, $\nabla\varphi = (v_0 - v_b)(\nabla l_k)\varphi_{e_k} = \mathcal{O}(\frac{(v_0 - v_b)\varphi_{e_k}}{h_T}\mathbf{C})$ on edge/face e_k for some vector constant $\mathbf{C}$, and the properties of the projection.

This completes the proof of the lemma.

□

Lemma 4.3. *For* $\{v_0, v_b, v_n\mathbf{n}_e\} \in V_h$*, let* $\varphi = (v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n}\varphi_{e_k}$*. The following inequality holds:*

$$\|\varphi\|_T^2 \leq Ch_T \int_{e_k} ((v_n\mathbf{n}_e - \nabla v_0)\cdot\mathbf{n})^2 ds. \tag{15}$$

Proof. We first extend v_n, initially defined on the $(d-1)$-dimensional edge/face e_k, to the entire d-dimensional polytopal element T using the following formula:

$$v_n(X) = v_n(Proj_{e_k}(X)),$$

where $X = (x_1, \cdots, x_d)$ is any point in the element T, $Proj_{e_k}(X)$ denotes the orthogonal projection of the point X onto the hyperplane H containing the edge/face e_k. When the projection $Proj_{e_k}(X)$ is not on the edge/face e_k, $v_n(Proj_{e_k}(X))$ is defined to be the extension of v_n from e_k to the hyperplane H.

We claim that v_n remains a polynomial defined on the element T after the extension. This can be proved in the same manner as demonstrated in Lemma 4.2.

Secondly, let v_{trace} denote the trace of v_0 on the edge/face e_k. We extend v_{trace} to the entire element T using the following formula:

$$v_{trace}(X) = v_{trace}(Proj_{e_k}(X)),$$

where X is any point in the element T, $Proj_{e_k}(X)$ denotes the projection of the point X onto the hyperplane H containing the edge/face e_k. When the projection $Proj_{e_k}(X)$ is not on the edge/face e_k, $v_{trace}(Proj_{e_k}(X))$ is defined to be the extension of v_{trace} from e_k to the hyperplane H. v_{trace} remains a polynomial after this extension. This proof can be found in Lemma 4.2.

Let $\varphi = (v_n \mathbf{n}_e - \nabla v_0) \cdot \mathbf{n}\varphi_{e_k}$. We have

$$\begin{aligned}\|\varphi\|_T^2 = \int_T \varphi^2 dT = &\int_T ((v_n \mathbf{n}_e - \nabla v_0)(X) \cdot \mathbf{n}\varphi_{e_k})^2 dT \\ \leq &Ch_T \int_{e_k} ((v_n \mathbf{n}_e - \nabla v_{trace})(Proj_{e_k}(X)) \cdot \mathbf{n}\varphi_{e_k})^2 dT \\ \leq &Ch_T \int_{e_k} ((v_n \mathbf{n}_e - \nabla v_0)) \cdot \mathbf{n})^2 ds,\end{aligned}$$

where we used the facts that (1) $\varphi_{e_k} = 0$ on each edge/face e_i for $i \neq k$, (2) there exists a sub-domain $\widehat{e_k} \subset e_k$ such that $\varphi_{e_k} \geq \rho_1$ for some constant $\rho_1 > 0$, and applied the properties of the projection.

This completes the proof of the lemma. □

Lemma 4.4. *There exists two positive constants C_1 and C_2 such that for any $v = \{v_0, v_b, v_n \mathbf{n}_e\} \in V_h$, we have*

$$C_1 \|v\|_{2,h} \leq |||v||| \leq C_2 \|v\|_{2,h}. \tag{16}$$

Proof. Recall that an edge/face-based bubble function is defined as

$$\varphi_{e_k} = \Pi_{i=1,\cdots,N, i\neq k} l_i^2(x).$$

We first extend v_b from the edge/face e_k to the element T. Next, let v_{trace} denote the trace of v_0 on the edge/face e_k and extend v_{trace} to the element T. For simplicity, we continue to denote these extensions as v_b and v_0. Details of the extensions can be found in Lemma 4.2. By substituting $\varphi = (v_b - v_0) l_k \varphi_{e_k}$ into (6), we obtain

$$\begin{aligned}(\Delta_w v, \varphi)_T = &(\Delta v_0, \varphi)_T - \langle v_b - v_0, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial T} + \langle v_n \mathbf{n}_e \cdot \mathbf{n} - \nabla v_0 \cdot \mathbf{n}, \varphi\rangle_{\partial T} \\ = &(\Delta v_0, \varphi)_T + \int_{e_k} |v_b - v_0|^2 (\nabla l_k)\varphi_{e_k} \cdot \mathbf{n} ds \\ = &(\Delta v_0, \varphi)_T + Ch_T^{-1} \int_{e_k} |v_b - v_0|^2 \varphi_{e_k} ds,\end{aligned} \tag{17}$$

where we used $\varphi = 0$ on each edge/face e_i for $i = 1, \cdots, N$, $\nabla\varphi = 0$ on each edge/face e_i for $i \neq k$ and $\nabla\varphi = (v_0 - v_b)(\nabla l_k)\varphi_{e_k} = \mathcal{O}(\frac{(v_0 - v_b)\varphi_{e_k}}{h_T} \mathbf{C})$ on edge/face e_k for some vector constant $\mathbf{C}$.

Recall that (1) $\varphi_{e_k} = 0$ on each edge/face e_i for $i \neq k$, (2) there exists a sub-domain $\widehat{e_k} \subset e_k$ such that $\varphi_{e_k} \geq \rho_1$ for some constant $\rho_1 > 0$. Using Cauchy-Schwarz inequality, the domain inverse inequality [39], (17) and Lemma 4.2 gives

$$\begin{aligned}\int_{e_k} |v_b - v_0|^2 ds \leq &C \int_{e_k} |v_b - v_0|^2 \varphi_{e_k} ds \\ \leq &Ch(\|\Delta_w v\|_T + \|\Delta v_0\|_T)\|\varphi\|_T \\ \leq &Ch_T^{\frac{3}{2}}(\|\Delta_w v\|_T + \|\Delta v_0\|_T)(\int_{e_k} |v_b - v_0|^2 ds)^{\frac{1}{2}},\end{aligned}$$

which, from Lemma 4.1, gives

$$h_T^{-3}\int_{e_k}|v_b-v_0|^2ds\leq C(\|\Delta_w v\|_T^2+\|\Delta v_0\|_T^2)\leq C\|\Delta_w v\|_T^2. \tag{18}$$

Next, we extend v_n from the edge/face e_k to the element T. For simplicity, we continue to denote this extension as v_n. Details of this extension can be found in Lemma 4.3. Letting $\varphi=(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}\varphi_{e_k}$ in (6) gives

$$\begin{aligned}&(\Delta_w v,\varphi)_T\\ =&(\Delta v_0,\varphi)_T-\langle v_b-v_0,\nabla\varphi\cdot\mathbf{n}\rangle_{\partial T}+\langle v_n\mathbf{n}_e\cdot\mathbf{n}-\nabla v_0\cdot\mathbf{n},\varphi\rangle_{\partial T}\\ =&(\Delta v_0,\varphi)_T-\langle v_b-v_0,\nabla\varphi\cdot\mathbf{n}\rangle_{\partial T}+\int_{e_k}|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}|^2\varphi_{e_k}ds,\end{aligned}$$

where we used $\varphi_{e_k}=0$ on edge/face e_i for $i\neq k$, and the fact that there exists a sub-domain $\widehat{e_k}\subset e_k$ such that $\varphi_{e_k}\geq\rho_1$ for some constant $\rho_1>0$. This, together with Cauchy-Schwarz inequality, the domain inverse inequality [39], the inverse inequality, the trace inequality (9), (18) and Lemma 4.3, gives

$$\begin{aligned}&\int_{e_k}|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}|^2ds\\ \leq&C\int_{e_k}|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}|^2\varphi_{e_k}ds\\ \leq&C(\|\Delta_w v\|_T+\|\Delta v_0\|_T)\|\varphi\|_T+C\|v_0-v_b\|_{\partial T}\|\nabla\varphi\cdot\mathbf{n}\|_{\partial T}\\ \leq&Ch_T^{\frac12}(\|\Delta_w v\|_T+\|\Delta v_0\|_T)(\int_{e_k}|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}|^2ds)^{\frac12}\\ &+Ch_T^{\frac32}\|\Delta_w v\|_T h_T^{-\frac12}h_T^{-1}h_T^{\frac12}(\int_{e_k}|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}|^2ds)^{\frac12}.\end{aligned}$$

This, together with Lemma 4.1, gives

$$h_T^{-1}\int_{e_k}|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}|^2ds\leq C(\|\Delta_w v\|_T^2+\|\Delta v_0\|_T^2)\leq C\|\Delta_w v\|_T^2. \tag{19}$$

Using Lemma 4.1, (18), (19), (10) and (11), gives

$$C_1\|v\|_{2,h}\leq{|\!|\!|}v{|\!|\!|}.$$

Next, applying Cauchy-Schwarz inequality, the inverse inequality, and the trace inequality (9) to (6), gives

$$\begin{aligned}\Big|(\Delta_w v,\varphi)_T\Big|\leq&\|\Delta v_0\|_T\|\varphi\|_T+\|v_b-v_0\|_{\partial T}\|\nabla\varphi\cdot\mathbf{n}\|_{\partial T}+\|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}\|_{\partial T}\|\varphi\|_{\partial T}\\ \leq&\|\Delta v_0\|_T\|\varphi\|_T+h_T^{-\frac32}\|v_b-v_0\|_{\partial T}\|\varphi\|_T+h_T^{-\frac12}\|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}\|_{\partial T}\|\varphi\|_T.\end{aligned}$$

This yields

$$\|\Delta_w v\|_T^2\leq C(\|\Delta v_0\|_T^2+h_T^{-3}\|v_b-v_0\|_{\partial T}^2+h_T^{-1}\|(v_n\mathbf{n}_e-\nabla v_0)\cdot\mathbf{n}\|_{\partial T}^2),$$

and further gives

$${|\!|\!|}v{|\!|\!|}\leq C_2\|v\|_{2,h}.$$

This completes the proof of the lemma. □

Remark 4.2. *Consider any d-dimensional polytopal element T. There exists a hyperplane $H \subset R^d$ such that a finite number l of distinct $(d-1)$-dimensional edges/faces containing e_i are completely contained within H. In such cases, Lemmas 4.2, 4.3, and 4.4 can be proved with additional techniques. For more details, see* [41]. *The techniques in* [41] *can be readily generalized to Lemmas 4.2-4.4.*

Theorem 4.5. *The WG Algorithm 3.1 has a unique solution.*

Proof. Assume $u_h^{(1)} \in V_h$ and $u_h^{(2)} \in V_h$ are two distinct solutions of the WG Algorithm 3.1. Define $\eta_h = u_h^{(1)} - u_h^{(2)}$. Then, $\eta_h \in V_h^0$ and satisfies

$$(\Delta_w \eta_h, \Delta_w v) = 0, \qquad \forall v \in V_h^0.$$

Letting $v = \eta_h$ in the above equation gives $|||\eta_h||| = 0$. From (16) we have $\|\eta_h\|_{2,h} = 0$, which implies $\Delta\eta_0 = 0$ on each T, $\eta_0 = \eta_b$ and $\nabla\eta_0 \cdot \mathbf{n} = \eta_n \mathbf{n}_e \cdot \mathbf{n}$ on each ∂T. Thus η_0 is a smooth harmonic function in Ω. Using the facts $\eta_0 = \eta_b$ and $\nabla\eta_0 \cdot \mathbf{n} = \eta_n \mathbf{n}_e \cdot \mathbf{n}$ on each ∂T and the boundary conditions of $\eta_b = 0$ and $\eta_n \mathbf{n}_e \cdot \mathbf{n} = 0$ on $\partial\Omega$ implies $\eta_0 = 0$ and $\nabla\eta_0 \cdot \mathbf{n} = 0$ on $\partial\Omega$. Therefore, we obtain $\eta_0 \equiv 0$ in Ω and further $\eta_b \equiv 0$ and $\eta_n \equiv 0$ in Ω. This gives $\eta_h \equiv 0$ in Ω. Therefore, we have $u_h^{(1)} \equiv u_h^{(2)}$.

This completes the proof of this theorem. □

5. Error Equations

Let Q_r denote the L^2 projection operator onto the finite element space consisting of piecewise polynomials of degree at most r.

Lemma 5.1. *The following property holds true, namely:*

$$\Delta_w u = Q_r(\Delta u), \qquad \forall u \in H^2(T). \tag{20}$$

Proof. For any $u \in H^2(T)$, using (6) gives

$$\begin{aligned}
&(\Delta_w u, \varphi)_T \\
=&(\Delta u, \varphi)_T - \langle u|_{\partial T} - u|_T, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial T} + \langle (\nabla u \cdot \mathbf{n}_e)|_{\partial T} \mathbf{n}_e \cdot \mathbf{n} - \nabla(u|_T) \cdot \mathbf{n}, \varphi\rangle_{\partial T} \\
=&(\Delta u, \varphi)_T = (Q_r(\Delta u), \varphi)_T,
\end{aligned}$$

for any $\varphi \in P_r(T)$. This completes the proof of this lemma. □

Let u be the exact solution of the biharmonic equation (1), and let $u_h \in V_h$ be its numerical approximation obtained from the Weak Galerkin scheme 3.1. We define the error function, denoted by e_h, as follows

$$e_h = u - u_h. \tag{21}$$

Lemma 5.2. *The error function e_h, defined in (21), satisfies the following error equation, namely:*

$$(\Delta_w e_h, \Delta_w v) = \ell(u, v), \qquad \forall v \in V_h^0, \tag{22}$$

where

$$\ell(u,v) = \sum_{T \in \mathcal{T}_h} -\langle v_b - v_0, \nabla((Q_r - I)\Delta u) \cdot \mathbf{n}\rangle_{\partial T} + \langle v_n \mathbf{n}_e \cdot \mathbf{n} - \nabla v_0 \cdot \mathbf{n}, (Q_r - I)\Delta u\rangle_{\partial T}.$$

Proof. By utilizing (20), the ususal integration by parts, and setting $\varphi = Q_r\Delta u$ in (6), we obtain the following:

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}(\Delta_w u, \Delta_w v)_T \\
=&\sum_{T\in\mathcal{T}_h}(Q_r\Delta u, \Delta_w v)_T \\
=&\sum_{T\in\mathcal{T}_h}(\Delta v_0, Q_r\Delta u)_T - \langle v_b - v_0, \nabla(Q_r\Delta u)\cdot\mathbf{n}\rangle_{\partial T} \\
&+ \langle v_n\mathbf{n}_e\cdot\mathbf{n} - \nabla v_0\cdot\mathbf{n}, Q_r\Delta u\rangle_{\partial T} \\
=&\sum_{T\in\mathcal{T}_h}(\Delta v_0, \Delta u)_T - \langle v_b - v_0, \nabla(Q_r\Delta u)\cdot\mathbf{n}\rangle_{\partial T} \\
&+ \langle v_n\mathbf{n}_e\cdot\mathbf{n} - \nabla v_0\cdot\mathbf{n}, Q_r\Delta u\rangle_{\partial T} \\
=&\sum_{T\in\mathcal{T}_h}(\Delta^2 u, v_0)_T - \langle\nabla(\Delta u)\cdot\mathbf{n}, v_0\rangle_{\partial T} + \langle\Delta u, \nabla v_0\cdot\mathbf{n}\rangle_{\partial T} \\
&- \langle v_b - v_0, \nabla(Q_r\Delta u)\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e\cdot\mathbf{n} - \nabla v_0\cdot\mathbf{n}, Q_r\Delta u\rangle_{\partial T} \\
=&(f, v_0) + \sum_{T\in\mathcal{T}_h} -\langle v_b - v_0, \nabla((Q_r - I)\Delta u)\cdot\mathbf{n}\rangle_{\partial T} \\
&+ \langle v_n\mathbf{n}_e\cdot\mathbf{n} - \nabla v_0\cdot\mathbf{n}, (Q_r - I)\Delta u\rangle_{\partial T},
\end{aligned}
\tag{23}
$$

where we used (1), $\Delta v_0 \in P_{k-2}(T)$, $r = 2N+k-2 \geq k-2$, $\sum_{T\in\mathcal{T}_h}\langle\Delta u, v_n\mathbf{n}_e\cdot\mathbf{n}\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\langle\Delta u, v_n\mathbf{n}_e\cdot\mathbf{n}\rangle_{\partial\Omega} = 0$ since $v_n\mathbf{n}_e\cdot\mathbf{n} = 0$ on $\partial\Omega$, and $\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta u)\cdot\mathbf{n}, v_b\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta u)\cdot\mathbf{n}, v_b\rangle_{\partial\Omega} = 0$ since $v_b = 0$ on $\partial\Omega$.

Subtracting (7) from (23) gives

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}(\Delta_w e_h, \Delta_w v)_T \\
=&\sum_{T\in\mathcal{T}_h} -\langle v_b - v_0, \nabla((Q_r - I)\Delta u)\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e\cdot\mathbf{n} - \nabla v_0\cdot\mathbf{n}, (Q_r - I)\Delta u\rangle_{\partial T}.
\end{aligned}
$$

This completes the proof of the lemma. □

6. Error Estimates

Lemma 6.1. [45] *Assume that w is sufficiently regular such that $w \in H^{\max\{k+1,4\}}(\Omega)$. There exists a constant C such that the following estimates hold true, namely:*

$$
\Big(\sum_{T\in\mathcal{T}_h} h_T\|\Delta w - Q_r\Delta w\|_{\partial T}^2\Big)^{\frac{1}{2}} \leq Ch^{k-1}\|w\|_{k+1}, \tag{24}
$$

$$
\Big(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w - Q_r\Delta w)\|_{\partial T}^2\Big)^{\frac{1}{2}} \leq Ch^{k-1}(\|w\|_{k+1} + h\delta_{r,0}\|w\|_4), \tag{25}
$$

where $\delta_{r,0}$ is the Kronecker delta such that $\delta_{r,0} = 1$ for $r = 0$ and otherwise $\delta_{r,0} = 0$.

Lemma 6.2. *Assume the exact solution u of the biharmonic equation (1) is sufficiently regular, so that $u \in H^{k+1}(\Omega)$. Then, there exists a constant C, such that the following estimate holds true; i.e.,*

$$
|\!|\!|u - Q_h u|\!|\!| \leq Ch^{k-1}\|u\|_{k+1}. \tag{26}
$$

Proof. Using (6), the trace inequalities (8)-(9), and the inverse inequality, we have, for any $\varphi \in P_r(T)$,

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}(\Delta_w(u-Q_hu),\varphi)_T\\
=&\sum_{T\in\mathcal{T}_h}(\Delta(u-Q_0u),\varphi)_T-\langle Q_0u-Q_bu,\nabla\varphi\cdot\mathbf{n}\rangle_{\partial T}\\
&+\langle\nabla(Q_0u)\cdot\mathbf{n}-Q_n(\nabla u\cdot\mathbf{n}_e)\mathbf{n}_e\cdot\mathbf{n},\varphi\rangle_{\partial T}\\
\le&\Big(\sum_{T\in\mathcal{T}_h}\|\Delta(u-Q_0u)\|_T^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}\|\varphi\|_T^2\Big)^{\frac12}\\
&+\Big(\sum_{T\in\mathcal{T}_h}\|Q_0u-Q_bu\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla\varphi\cdot\mathbf{n}\|_{\partial T}^2\Big)^{\frac12}\\
&+\Big(\sum_{T\in\mathcal{T}_h}\|\nabla(Q_0u)\cdot\mathbf{n}-Q_n(\nabla u\cdot\mathbf{n}_e)\mathbf{n}_e\cdot\mathbf{n}\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}\|\varphi\|_{\partial T}^2\Big)^{\frac12}\\
\le&\Big(\sum_{T\in\mathcal{T}_h}\|\Delta(u-Q_0u)\|_T^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}\|\varphi\|_T^2\Big)^{\frac12}\\
&+\Big(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|Q_0u-u\|_T^2+h_T\|Q_0u-u\|_{1,T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}h_T^{-3}\|\varphi\|_T^2\Big)^{\frac12}\\
&+\Big(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\nabla Q_0u-\nabla u\|_T^2+h_T\|\nabla Q_0u-\nabla u\|_{1,T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\varphi\|_T^2\Big)^{\frac12}\\
\le&Ch^{k-1}\|u\|_{k+1}\Big(\sum_{T\in\mathcal{T}_h}\|\varphi\|_T^2\Big)^{\frac12}.
\end{aligned}
$$

Letting $\varphi=\Delta_w(u-Q_hu)$ gives

$$\sum_{T\in\mathcal{T}_h}(\Delta_w(u-Q_hu),\Delta_w(u-Q_hu))_T\le Ch^{k-1}\|u\|_{k+1}\interleave u-Q_hu\interleave.$$

This completes the proof of the lemma. □

Theorem 6.3. *Assume that the exact solution u of the biharmonic equation (1) is sufficiently regular, so that $u\in H^{\max\{k+1,4\}}(\Omega)$. There exists a constant C, such that the following error estimate holds true, namely:*

$$\interleave u-u_h\interleave\le Ch^{k-1}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4). \tag{27}$$

Proof. For the first term on the right-hand side of the error equation (22), using Cauchy-Schwarz inequality, the estimate (25), and (16) implies

$$
\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_h}-\langle v_b-v_0,\nabla((Q_r-I)\Delta u)\cdot\mathbf{n}\rangle_{\partial T}\Big|\\
\le&C(\sum_{T\in\mathcal{T}_h}h_T^{-3}\|v_b-v_0\|_{\partial T}^2)^{\frac12}\cdot(\sum_{T\in\mathcal{T}_h}h_T^3\|\nabla((Q_r-I)\Delta u)\cdot\mathbf{n}\|_{\partial T}^2)^{\frac12}\\
\le&Ch^{k-1}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4)\interleave v\interleave.
\end{aligned}\tag{28}
$$

For the second term on the right-hand side of the error equation (22), using Cauchy-Schwarz inequality, the trace inequality (8) and (16) gives

$$
\begin{aligned}
&\Big|\sum_{T\in\mathcal{T}_h}\langle v_n\mathbf{n}_e\cdot\mathbf{n}-\nabla v_0\cdot\mathbf{n},(Q_r-I)\Delta u\rangle_{\partial T}\Big|\\
\le& C\Big(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|v_n\mathbf{n}_e\cdot\mathbf{n}-\nabla v_0\cdot\mathbf{n}\|_{\partial T}^2\Big)^{\frac12}\cdot\Big(\sum_{T\in\mathcal{T}_h}h_T\|(Q_r-I)\Delta u\|_{\partial T}^2\Big)^{\frac12}\\
\le& C\|v\|_{1,h}\Big(\sum_{T\in\mathcal{T}_h}\|(Q_r-I)\Delta u\|_T^2+h_T^2\|(Q_r-I)\Delta u\|_{1,T}^2\Big)^{\frac12}\\
\le& Ch^{k-1}\|u\|_{k+1}|||v|||.
\end{aligned}
\tag{29}
$$

Substituting (28)-(29) into (22) gives

$$
(\Delta_w e_h,\Delta_w v)\le Ch^{k-1}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4)|||v|||.
\tag{30}
$$

By setting $v=Q_hu-u_h$ in (30), and applying the Cauchy-Schwarz inequality along with (26), we obtain

$$
\begin{aligned}
&|||u-u_h|||^2\\
=&\sum_{T\in\mathcal{T}_h}(\Delta_w(u-u_h),\Delta_w(u-Q_hu))_T+(\Delta_w(u-u_h),\Delta_w(Q_hu-u_h))_T\\
\le&\Big(\sum_{T\in\mathcal{T}_h}\|\Delta_w(u-u_h)\|_T^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}\|\Delta_w(u-Q_hu)\|_T^2\Big)^{\frac12}\\
&+Ch^{k-1}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4)|||Q_hu-u_h|||\\
\le&|||u-u_h|||\,|||u-Q_hu|||+Ch^{k-1}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4)(|||Q_hu-u|||+|||u-u_h|||)\\
\le&C|||u-u_h|||h^{k-1}\|u\|_{k+1}+Ch^{k-1}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4)h^{k-1}\|u\|_{k+1}\\
&+Ch^{k-1}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4)|||u-u_h|||.
\end{aligned}
$$

This gives

$$
\begin{aligned}
|||u-u_h|||\le&Ch^{k-1}\|u\|_{k+1}+Ch^{k-1}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4)\\
\le&Ch^{k-1}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4).
\end{aligned}
$$

This completes the proof of the theorem. □

7. Error Estimates in L^2 Norm

The standard duality argument is utilized to derive the L^2 error estimate. Recall that $e_h=u-u_h=\{e_0,e_b,\mathbf{e}_g\}$. Let us denote $\zeta_h=Q_hu-u_h=\{\zeta_0,\zeta_b,\boldsymbol{\zeta}_g\}\in V_h^0$. The dual problem for the biharmonic equation (1) seeks $w\in H_0^2(\Omega)$ satisfying

$$
\begin{aligned}
\Delta^2 w&=\zeta_0, &&\text{in }\Omega,\\
w&=0, &&\text{0n }\partial\Omega,\\
\frac{\partial w}{\partial\mathbf{n}}&=0, &&\text{0n }\partial\Omega.
\end{aligned}
\tag{31}
$$

Assume that the H^4-regularity property holds true; that is,

$$
\|w\|_4\le C\|\zeta_0\|.
\tag{32}
$$

Theorem 7.1. *Assume that the exact solution u of the biharmonic equation (1) satisfies $u \in H^{\max\{k+1,4\}}(\Omega)$ and the H^4-regularity assumption (32) for the dual problem (31) holds true. Let $u_h \in V_h$ be the numerical solution of the weak Galerkin scheme 3.1. Then, there exists a constant C such that*

$$\|e_0\| \leq Ch^{k+1-\delta_{r,0}}(\|u\|_{k+1} + h\delta_{r,0}\|u\|_4).$$

Proof. Testing (31) by ζ_0, using the usual integration by parts, we obtain

$$\begin{aligned}
&\|\zeta_0\|^2\\
=&(\Delta^2 w, \zeta_0)\\
=&\sum_{T\in\mathcal{T}_h}(\Delta w, \Delta\zeta_0)_T - \langle \Delta w, \nabla\zeta_0\cdot\mathbf{n}\rangle_{\partial T} + \langle \nabla(\Delta w)\cdot\mathbf{n}, \zeta_0\rangle_{\partial T} \qquad (33)\\
=&\sum_{T\in\mathcal{T}_h}(\Delta w, \Delta\zeta_0)_T - \langle \Delta w, \nabla\zeta_0\cdot\mathbf{n} - \zeta_n\mathbf{n}_e\cdot\mathbf{n}\rangle_{\partial T} + \langle \nabla(\Delta w)\cdot\mathbf{n}, \zeta_0 - \zeta_b\rangle_{\partial T},
\end{aligned}$$

where we used $\sum_{T\in\mathcal{T}_h}\langle \Delta w, \zeta_n\mathbf{n}_e\cdot\mathbf{n}\rangle_{\partial T} = \langle \Delta w, \zeta_n\mathbf{n}_e\cdot\mathbf{n}\rangle_{\partial\Omega} = 0$ due to $\zeta_n\mathbf{n}_e\cdot\mathbf{n} = 0$ on $\partial\Omega$, and $\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta w)\cdot\mathbf{n}, \zeta_b\rangle_{\partial T} = \langle\nabla(\Delta w)\cdot\mathbf{n}, \zeta_b\rangle_{\partial\Omega} = 0$ due to $\zeta_b = 0$ on $\partial\Omega$.

Letting $u = w$ and $v = \zeta_h$ in (23) gives

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}(\Delta_w w, \Delta_w\zeta_h)_T\\
=&\sum_{T\in\mathcal{T}_h}(\Delta\zeta_0, \Delta w)_T - \langle\zeta_b - \zeta_0, \nabla(Q_r\Delta w)\cdot\mathbf{n}\rangle_{\partial T} + \langle\zeta_n\mathbf{n}_e\cdot\mathbf{n} - \nabla\zeta_0\cdot\mathbf{n}, Q_r\Delta w\rangle_{\partial T},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}(\Delta\zeta_0, \Delta w)_T\\
=&\sum_{T\in\mathcal{T}_h}(\Delta_w w, \Delta_w\zeta_h)_T + \langle\zeta_b - \zeta_0, \nabla(Q_r\Delta w)\cdot\mathbf{n}\rangle_{\partial T} - \langle\zeta_n\mathbf{n}_e\cdot\mathbf{n} - \nabla\zeta_0\cdot\mathbf{n}, Q_r\Delta w\rangle_{\partial T}.
\end{aligned}$$

Substituting the above equation into (33) and using (22) gives

$$\begin{aligned}
\|\zeta_0\|^2 =& \sum_{T\in\mathcal{T}_h}(\Delta_w w, \Delta_w\zeta_h)_T + \langle\zeta_b - \zeta_0, \nabla((Q_r - I)\Delta w)\cdot\mathbf{n}\rangle_{\partial T}\\
&- \langle\zeta_n\mathbf{n}_e\cdot\mathbf{n} - \nabla\zeta_0\cdot\mathbf{n}, (Q_r - I)\Delta w\rangle_{\partial T}\\
=& \sum_{T\in\mathcal{T}_h}(\Delta_w w, \Delta_w e_h)_T + (\Delta_w w, \Delta_w(Q_h u - u))_T - \ell(w, \zeta_h)\\
=& \sum_{T\in\mathcal{T}_h}(\Delta_w Q_h w, \Delta_w e_h)_T + (\Delta_w(w - Q_h w), \Delta_w e_h)_T \qquad (34)\\
&+ (\Delta_w w, \Delta_w(Q_h u - u))_T - \ell(w, \zeta_h)\\
=&\ell(u, Q_h w) + \sum_{T\in\mathcal{T}_h}(\Delta_w(w - Q_h w), \Delta_w e_h)_T\\
&+ (\Delta_w w, \Delta_w(Q_h u - u))_T - \ell(w, \zeta_h)\\
=&J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We will estimate the four terms $J_i (i = 1, \cdots, 4)$ on the last line of (34) individually.

For J_1, using Cauchy-Schwarz inequality, the trace inequality (8), the estimates (24)-(25), gives

(35)
$$
\begin{aligned}
J_1 =&\ell(u, Q_h w)\\
\le&\Big|\sum_{T\in\mathcal{T}_h} -\langle Q_b w - Q_0 w, \nabla((Q_r - I)\Delta u)\cdot \mathbf{n}\rangle_{\partial T}\\
&+ \langle Q_n(\nabla w\cdot \mathbf{n}_e)\mathbf{n}_e\cdot \mathbf{n} - \nabla Q_0 w\cdot \mathbf{n}, (Q_r - I)\Delta u\rangle_{\partial T}\Big|\\
\le&\Big(\sum_{T\in\mathcal{T}_h}\|Q_b w - Q_0 w\|_{\partial T}^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla((Q_r - I)\Delta u)\cdot \mathbf{n}\|_{\partial T}^2\Big)^{\frac{1}{2}}\\
&+\Big(\sum_{T\in\mathcal{T}_h}\|Q_n(\nabla w\cdot \mathbf{n}_e)\mathbf{n}_e\cdot \mathbf{n} - \nabla Q_0 w\cdot \mathbf{n}\|_{\partial T}^2\Big)^{\frac{1}{2}}\\
&\cdot\Big(\sum_{T\in\mathcal{T}_h}\|(Q_r - I)\Delta u\|_{\partial T}^2\Big)^{\frac{1}{2}}\\
\le&\Big(\sum_{T\in\mathcal{T}_h} h_T^{-4}\|Q_b w - Q_0 w\|_T^2 + h_T^{-2}\|Q_b w - Q_0 w\|_{1,T}^2\Big)^{\frac{1}{2}}\\
&\Big(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla((Q_r - I)\Delta u)\cdot \mathbf{n}\|_{\partial T}^2\Big)^{\frac{1}{2}}\\
&+\Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|Q_n(\nabla w\cdot \mathbf{n}_e)\mathbf{n}_e\cdot \mathbf{n} - \nabla Q_0 w\cdot \mathbf{n}\|_T^2\\
&+\|Q_n(\nabla w\cdot \mathbf{n}_e)\mathbf{n}_e\cdot \mathbf{n} - \nabla Q_0 w\cdot \mathbf{n}\|_{1,T}^2\Big)^{\frac{1}{2}}\cdot\Big(\sum_{T\in\mathcal{T}_h} h_T\|(Q_r - I)\Delta u\|_{\partial T}^2\Big)^{\frac{1}{2}}\\
\le&\Big(\sum_{T\in\mathcal{T}_h} h_T^{-4}\|w - Q_0 w\|_T^2 + h_T^{-2}\|w - Q_0 w\|_{1,T}^2\Big)^{\frac{1}{2}}\\
&\cdot Ch^{k-1}(\|u\|_{k+1} + h\delta_{r,0}\|u\|_4)\\
&+\Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|\nabla w - \nabla Q_0 w\|_T^2 + \|\nabla w - \nabla Q_0 w\|_{1,T}^2\Big)^{\frac{1}{2}} Ch^{k-1}\|u\|_{k+1}\\
\le&Ch^{k+1}(\|u\|_{k+1} + h\delta_{r,0}\|u\|_4)\|w\|_4.
\end{aligned}
$$

For J_2, using Cauchy-Schwarz inequality, (26) with $k = 3$ and (27) gives

$$
\begin{aligned}
J_2 \le {|\!|\!|} w - Q_h w{|\!|\!|}\,{|\!|\!|} e_h{|\!|\!|} \le& Ch^{k-1}(\|u\|_{k+1} + h\delta_{r,0}\|u\|_4)h^2\|w\|_4\\
\le& Ch^{k+1}(\|u\|_{k+1} + h\delta_{r,0}\|u\|_4)\|w\|_4.
\end{aligned}
\tag{36}
$$

For J_3, denote by Q^1 a L^2 projection onto $P_1(T)$. Using (5) gives

$$
\begin{aligned}
&(\Delta_w(Q_h u - u), Q^1\Delta_w w)_T\\
=&(Q_0 u - u, \Delta(Q^1\Delta_w w))_T - \langle Q_b u - u, \nabla(Q^1\Delta_w w)\cdot \mathbf{n}\rangle_{\partial T}\\
&+ \langle (Q_n(\nabla u\cdot \mathbf{n}_e)\mathbf{n}_e - \nabla u\cdot \mathbf{n}_e\mathbf{n}_e)\cdot \mathbf{n}, Q^1\Delta_w w\rangle_{\partial T} = 0,
\end{aligned}
\tag{37}
$$

where we used $\Delta(Q^1\partial_{ij,w}^2 w) = 0$, $\nabla(Q^1\partial_{ij,w}^2 w) = C$, the property of the projection opertors Q_b and Q_n, as well as $p \ge 1, q \ge 1$.

Using (37), Cauchy-Schwarz inequality, (20) and (26), gives

$$
\begin{aligned}
J_3 \leq & |\sum_{T\in\mathcal{T}_h} (\Delta_w w, \Delta_w(Q_h u - u))_T| \\
= & |\sum_{T\in\mathcal{T}_h} (\Delta_w w - Q^1\Delta_w w, \Delta_w(Q_h u - u))_T| \\
= & |\sum_{T\in\mathcal{T}_h} (Q_r\Delta w - Q^1 Q_r\Delta w, \Delta_w(Q_h u - u))_T| \\
\leq & \Big(\sum_{T\in\mathcal{T}_h} \|Q_r\Delta w - Q^1 Q^r\Delta w\|_T^2\Big)^{\frac{1}{2}} |||Q_h u - u||| \\
\leq & Ch^{k+1}\|u\|_{k+1}\|w\|_4.
\end{aligned} \tag{38}
$$

For J_4, using Cauchy-Schwarz inequality, the trace inequality (8), Lemma 4.4, the estimates (24)-(25), (26), (27) gives

$$
\begin{aligned}
J_4 = & \ell(w, \zeta_h) \\
\leq & \Big|\sum_{T\in\mathcal{T}_h} -\langle \zeta_b - \zeta_0, \nabla((Q_r - I)\Delta w)\cdot\mathbf{n}\rangle_{\partial T} \\
& + \langle \zeta_n\mathbf{n}_e\cdot\mathbf{n} - \nabla\zeta_0\cdot\mathbf{n}, (Q_r - I)\Delta w\rangle_{\partial T}\Big| \\
\leq & \Big(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|\zeta_b - \zeta_0\|_{\partial T}^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla((Q_r - I)\Delta w)\cdot\mathbf{n}\|_{\partial T}^2\Big)^{\frac{1}{2}} \\
& + \Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\zeta_n\mathbf{n}_e\cdot\mathbf{n} - \nabla\zeta_0\cdot\mathbf{n}\|_{\partial T}^2\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h} h_T\|(Q_r - I)\Delta w\|_{\partial T}^2\Big)^{\frac{1}{2}} \\
\leq & Ch^{2-\delta_{r,0}}\|w\|_4 |||\zeta_h||| \\
\leq & Ch^{2-\delta_{r,0}}\|w\|_4(|||u - u_h||| + |||u - Q_h u|||) \\
\leq & Ch^{k+1-\delta_{r,0}}\|w\|_4(\|u\|_{k+1} + h\delta_{r,0}\|u\|_4).
\end{aligned} \tag{39}
$$

Using (32) and substituting (35)-(36) and (38)-(39) into (34) gives

$$
\|\zeta_0\|^2 \leq Ch^{k+1-\delta_{r,0}}\|w\|_4(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4) \leq Ch^{k+1-\delta_{r,0}}(\|u\|_{k+1}+h\delta_{r,0}\|u\|_4)\|\zeta_0\|.
$$

This gives

$$
\|\zeta_0\| \leq Ch^{k+1-\delta_{r,0}}(\|u\|_{k+1} + h\delta_{r,0}\|u\|_4),
$$

which, using the triangle inequality, gives

$$
\|e_0\| \leq \|\zeta_0\| + \|u - Q_0 u\| \leq Ch^{k+1-\delta_{r,0}}(\|u\|_{k+1} + h\delta_{r,0}\|u\|_4).
$$

This completes the proof of the theorem. □

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