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NUMERICAL ANALYSIS OF THE FINITE DIFFERENCE TIME DOMAIN METHODS WITH HIGH ACCURACY IN TIME FOR MAXWELL EQUATIONS

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Abstract. In this paper, we give a rigorous analysis of the finite difference time domain (FDTD) method with high accuracy in time (HAIT) (named HAIT-FDTD(M)) for the three dimensional Maxwell equations, where the time discretization is based on the Taylor expansion of the form: $U^n = C_n^0 + C_n^1\Delta t + \cdots + \frac{1}{M!}C_n^M(\Delta t)^M$ to approximate the fields in time. It is proven that the solutions of the schemes and the vectors representing the coefficients are divergence free. By using the energy method, the numerical energy identities of HAIT-FDTD(M) with $3 \leq M \leq 8$ are derived. It is then proved that these schemes are numerically and monotonically energy conserved as the polynomial degree M becomes large. With the help of the energy identities, stability conditions for the six schemes are derived, and how to select M and Δt in practice is given. By deriving error equations, we prove that the six schemes have convergence of the Mth order in time and the second order in space. Numerical experiments are provided and confirm the analysis on free divergence, approximate energy conservation, stability, and convergence.

Key words. Maxwell equations, finite difference time domain method, stability, energy conservation, convergence, Taylor expansion.

1. Introduction

The finite difference time domain (FDTD) method is one of the methods for numerical solutions of time dependent Maxwell equations, and causes many people's interests and much good research work. For example, the Yee scheme ([31]), proposed by Yee in 1966, is a very popular and efficient method (see Taflove [26]). Monk and S*ü*li [15] proved that the Yee scheme over non-uniform grids is of super convergence of second order in L^2 norm. For the Yee scheme in metamaterials, Li and Shields [12] proved that this scheme is also super convergent in L^2 norm. The stability and second order convergence of the Yee scheme under H^1 norm were proved in [7]. Recently, convergence analysis of the Yee schemes in linear dispersive media was given by Sakkaplankul and Bokil in [20]. The other FDTD methods, including the alternating direction implicit FDTD (ADI-FDTD) methods, the energy-conserved splitting FDTD methods, symplectic FDTD method, locally-one-dimensional (LOD) FDTD method, etc. and their analysis are seen in [33, 17, 6], [1, 2], [8, 10, 24, 25],[32], [9], [23], [3], [11], [4], [14, 29], [21], [30], [19, 27], [18] and the references therein.

Time discretization is important for accuracy, efficiency, stability, and convergence. There are many good time-stepping methods in numerical solutions of Maxwell equations [26, 16, 13]. For example, leap-frog method in [31, 18], Runge-Kutta method in [9, 21], ADI method in [33, 17], splitting methods in [11, 5], energy splitting conserving methods in [1, 2, 14, 30], symplectic method in [10, 25, 24], fourth order method based on the relation between time derivatives and spatial derivatives [29, 32], time-domain moment method based on weighted Laguerre

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polynomials in [3], Newmark time-stepping method in [4], LOD method in [23], Crank-Nicolson method in [28] and explicit-implicit hybrid time-stepping method in [19, 27] and the others in the references.

Different from the above FDTD methods, a new explicit FDTD method with high accuracy in time (HAIT)(called HAIT-FDTD(M)) was proposed in [22] by using Taylor expansion of the form: $U^n = C_n^0 + C_n^1 \Delta t + \cdots + \frac{1}{M!} C_n^M (\Delta t)^M$ to approximate the fields in time, which transforms the Maxwell equations into a system of time-independent differential equations in the coefficients $C_n^k (k = 0, 1, \ldots, M)$, and using the central difference methods to approximate the spatial derivatives of C_n^k. Numerical experiments demonstrate that HAIT-FDTD(M) has the following features: easy implementation, divergence free, numerical energy conservation, and good stability and convergence. However, the rigorous analysis of HAIT-FDTD(M) on stability, error estimate, and convergence by the energy method is not available, since the form of the scheme is very different from traditional ones, which makes the usual analysis methods on stability and convergence (see [15, 1, 7], etc.) do not work on HAIT-FDTD(M). In addition, how to select the polynomial degree M and time step sizes is not clear. Therefore, it is significant to give a rigorous analysis of HAIT-FDTD(M) on these issues.

In this paper, we analyze the HAIT-FDTD(M) schemes for the 3D Maxwell equations with perfectly electric conducting (PEC) boundary conditions. The research methods and results are as follows:

(i) It is proved that the solutions of the HAIT-FDTD(M) schemes and the representing coefficients retain the free divergence property.

(ii) By using the energy method, numerical energy identities of the HAIT-FDTD(M) schemes with $3 \leq M \leq 8$ are derived, and it is then proved that these schemes are approximately energy conserved. With the help of the energy identities, the stability conditions of the six schemes (which are weaker than the CFL (Courant-Friedrichs-Lewy) condition and can be used to select time step sizes and degree M) are derived, and the stability in L^2 norm is then proved.

(iii) It is proved that the HAIT-FDTD(M) schemes with $3 \leq M \leq 8$ are of convergence of M-th order in time and second order in space by using different error analysis from the traditional ones.

(iv) Numerical experiments are carried out and confirm the theoretical analysis of the schemes on free divergence, numerical energy conservation, good stability, and convergence.

2. Maxwell equations and some properties.

2.1. Maxwell equations and properties of the solution. Consider the 3D Maxwell equations in a domain of $\Omega \times (0, T]$:

$$\varepsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \qquad \mu \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}, \tag{1}$$

$$\varepsilon \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \qquad \mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}, \tag{2}$$

$$\varepsilon \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}, \qquad \mu \frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}, \tag{3}$$

where Ω is filled with homogeneous and isotropic medium, so the electric permittivity ε and the magnetic permeability μ are constants, and for $p = (x, y, z) \in \Omega$, $u = x, y, z$

$$E_u = E_u(p, t), \;\; H_u = H_u(p, t), \;\; (E_x, E_y, E_z) =: \mathbf{E}, \;\; (H_x, H_y, H_z) =: \mathbf{H}$$

are respectively the electric and magnetic fields. Suppose that the solution is subject to the PEC boundary conditions:

$$\nu \times \mathbf{E}(p,t) = 0, \quad \nu \cdot \mathbf{H}(p,t) = 0, \quad \forall\,(p,t) \in \partial\Omega \times [0,T], \tag{4}$$

and the initial conditions:

$$\mathbf{E}(p,0) = (E_{x0}, E_{y0}, E_{z0}) =: \mathbf{E}_0, \quad \mathbf{H}(p,0) = (H_{x0}, H_{y0}, H_{z0}) =: \mathbf{H}_0, \tag{5}$$

where ν is the unit normal vector of the boundary of Ω. In this paper, we assume that $\Omega = [0,1]^3$ and that $\mathbf{E}_0$ and $\mathbf{H}_0$ are divergence free. It is easy to see that the Maxwell system has the following energy conservation and free divergence.

Lemma 2.1. The solution of the problem (1)-(5) satisfies that

$$\|\varepsilon^{\frac{1}{2}}\mathbf{E}(\cdot,t)\|^2 + \|\mu^{\frac{1}{2}}\mathbf{H}(\cdot,t)\|^2 \;=\; \|\varepsilon^{\frac{1}{2}}\mathbf{E}_0\|^2 + \|\mu^{\frac{1}{2}}\mathbf{H}_0\|^2, \tag{6}$$

$$\nabla\cdot(\varepsilon\mathbf{E})(p,t) = \nabla\cdot(\mu\mathbf{H})(p,t) = 0, \tag{7}$$

where $p \in \Omega$, $t \geq 0$, and for a vector-valued function $\mathbf{F} = (F_x,\ F_y,\ F_z)$, $\nabla\cdot\mathbf{F} = \partial F_x/\partial x + \partial F_y/\partial y + \partial F_z/\partial z$, and $\|\mathbf{F}\|^2$ is square of the standard L^2 norm.

Lemma 2.2. Let $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{H} = (H_x, H_y, H_z)$ be the solution of (1)-(5) and have the continuous M-th derivatives and third derivatives with respect to time and space respectively, i.e. $\mathbf{E}, \mathbf{H} \in C^M([0,T], C^3(\Omega))$. Then, the following boundary conditions hold.

$$\frac{\partial^m}{\partial t^m}\big(H_x, E_y, E_z,\ \frac{\partial E_x}{\partial x}, \frac{\partial H_y}{\partial x}, \frac{\partial H_z}{\partial x}, \frac{\partial^2 H_x}{\partial x^2}, \frac{\partial^2 E_y}{\partial x^2}, \frac{\partial^2 E_z}{\partial x^2},\ \frac{\partial^3 H_y}{\partial x^3}, \frac{\partial^3 H_z}{\partial x^3}\big)\Big|_{x'} = 0, \tag{8}$$

$$\frac{\partial^m}{\partial t^m}(E_x, H_y, E_z,\ \frac{\partial H_x}{\partial y}, \frac{\partial E_y}{\partial y}, \frac{\partial H_z}{\partial y}, \frac{\partial^2 E_x}{\partial y^2}, \frac{\partial^2 H_y}{\partial y^2}, \frac{\partial^2 E_z}{\partial y^2},\ \frac{\partial^3 H_x}{\partial y^3}, \frac{\partial^3 H_z}{\partial y^3})\Big|_{y'} = 0, \tag{9}$$

$$\frac{\partial^m}{\partial t^m}(E_x, E_y, H_z,\ \frac{\partial H_x}{\partial z}, \frac{\partial H_y}{\partial z}, \frac{\partial E_z}{\partial z}, \frac{\partial^2 E_x}{\partial z^2}, \frac{\partial^2 E_y}{\partial z^2}, \frac{\partial^2 H_z}{\partial z^2},\ \frac{\partial^3 H_x}{\partial z^3}, \frac{\partial^3 H_y}{\partial z^3})\Big|_{z'} = 0, \tag{10}$$

where x', y' and z' are numbers in the set $\{0,1\}$, and $F|_{u'}$ denotes the value of F at the boundary points, and $0 \leq m \leq M$. For example, $F|_{x'} = F(x', y, z, t)$, $x' = 0$ or 1, and $y \in [0,1]$, $z \in [0,1]$, $t \geq 0$.

Lemma 2.2 is proved by using the boundary conditions (4), Maxwell equations (1)-(3) and the free-divergence property (7). Some of the proof is seen in [14].

2.2. Notations of grids, norms, inner products and operators. We use the Yee's staggered gridding technique (see [31]) to partition Ω as eight grids of points, $\Omega^h_{e_u}$ and $\Omega^h_{h_u}(u = x, y, z, 0)$:

$$\Omega^h_{e_x} = \big\{(x_{i+\frac{1}{2}}, y_j, z_k)\big|_{i=0,\ j=1,k=1}^{I-1,J-1,K-1}\big\}, \quad \Omega^h_{h_x} = \big\{(x_i, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}})\big|_{i=1,\ j=0,k=0}^{I-1,J-1,K-1}\big\},$$

$$\Omega^h_{e_0} = \big\{(x_i, y_j, z_k)\big|_{i=1,\ j=1,k=1}^{I-1,J-1,K-1}\big\}, \qquad \Omega^h_{h_0} = \big\{(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}})\big|_{i=0,\ j=0,k=0}^{I-1,J-1,K-1}\big\},$$

and the other grids: $\Omega^h_{e_y} = \{(x_i, y_{j+1/2}, z_k)\}$, $\Omega^h_{e_z} = \{(x_i, y_j, z_{k+1/2})\}$, $\Omega^h_{h_y} = \{(x_{i+1/2}, y_j, z_{k+1/2})\}$ and $\Omega^h_{h_z} = \{(x_{i+1/2}, y_{j+1/2}, z_k)\}$ are similarly defined, where

$$x_i = i\Delta x, \quad x_{i+\frac{1}{2}} = x_i + \frac{1}{2}\Delta x, \quad y_j = j\Delta y, \quad y_{j+\frac{1}{2}} = y_j + \frac{1}{2}\Delta y,$$

$$z_k = k\Delta z, \quad z_{k+\frac{1}{2}} = z_k + \frac{1}{2}\Delta z, \quad x_I = 1, \quad y_J = 1, \quad z_K = 1,$$

where $\Delta u(u = x, y, z)$ are the mesh sizes along the x, y and z directions, I, J and K are positive integers. Time domain $[0,T]$ is divided into: $0 = t^0 < t^1 < \cdots < t^N = T$, where $t^n = n\Delta t$.

For a function $\mathrm{F}(x,y,z,t)$, let $\mathrm{F}^m_{\alpha,\beta,\gamma}$ denote the approximations to the correct value: $\mathrm{F}(x_\alpha, y_\beta, z_\gamma, t^m)$, where $\alpha = i$ or $i+1/2$, $\beta = j$ or $j+1/2$, $\gamma = k$ or $k+1/2$. The (discrete)

L^2 norms of $\mathrm{F}^m_{\alpha,\beta,\gamma}$ over the different grids are defined as follows.

$$\|\mathrm{F}^n\|^2_{e_x} = \sum\nolimits_{E_x} \big(\mathrm{F}^n_{i+\frac{1}{2},j,k}\big)^2 \Delta v \quad =: \sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\sum_{k=1}^{K-1} \big(\mathrm{F}^n_{i+\frac{1}{2},j,k}\big)^2 \Delta v,$$

$$\|\mathrm{F}^n\|^2_{h_x} = \sum\nolimits_{H_x} \big(\mathrm{F}^n_{i,j+\frac{1}{2},k+\frac{1}{2}}\big)^2 \Delta v \;=: \sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\sum_{k=0}^{K-1} \big(\mathrm{F}^n_{i,j+\frac{1}{2},k+\frac{1}{2}}\big)^2 \Delta v,$$

$$\|\mathrm{F}^n\|^2_{e_0} = \sum\nolimits_{E_0} \big(\mathrm{F}^n_{i,j,k}\big)^2 \Delta v \quad =: \sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\sum_{k=1}^{K-1} \big(\mathrm{F}^n_{i,j,k}\big)^2 \Delta v,$$

where $\Delta v = \Delta x \Delta y \Delta z$. The other norms $\|\mathrm{F}^m\|_{e_u}$, $\|\mathrm{F}^m\|_{h_u}$ and $\|F^n\|^2_{h_0}$ and the other summation notations: $\sum_{E_u}$, $\sum_{H_u}$ and $\sum_{H_0}$ over the grids $\Omega^h_{e_u}$, $\Omega^h_{h_u}$ and $\Omega^h_{h_0}$ with $u = y, z$ are similarly defined.

For any vector-valued functions $\mathbf{F}^n = (\mathrm{F}^n_x, \mathrm{F}^n_y, \mathrm{F}^n_z)$ and $\mathbf{G}^n = (\mathrm{G}^n_x, \mathrm{G}^n_y, \mathrm{G}^n_z)$ over $\Omega^h_{e_x}\times\Omega^h_{e_y}\times\Omega^h_{e_z}$ and $\Omega^h_{h_x}\times\Omega^h_{h_y}\times\Omega^h_{h_z}$, define the inner-products $(\cdot,\cdot)_{e_u}$, $(\cdot,\cdot)_{h_u} (u = x, y, z)$, $(\cdot,\cdot)_e$, $(\cdot,\cdot)_h$, and the norms $\|\mathbf{F}^n\|_e$, $\|\mathbf{G}^n\|_h$ as follows.

$$\begin{aligned}
(\mathbf{F}^n, \mathbf{G}^n)_e \;&=\; (\mathrm{F}^n_x, \mathrm{G}^n_x)_{e_x} + (\mathrm{F}^n_y, \mathrm{G}^n_y)_{e_y} + (\mathrm{F}^n_z, \mathrm{G}^n_z)_{e_z} \\
&=\; \Big(\sum\nolimits_{E_x} \mathrm{F}^n_{x_{i+\frac{1}{2},j,k}} \cdot \mathrm{G}^n_{x_{i+\frac{1}{2},j,k}} + \sum\nolimits_{E_y} \mathrm{F}^n_{y_{i,j+\frac{1}{2},k}} \cdot \mathrm{G}^n_{y_{i,j+\frac{1}{2},k}} \\
&+\; \sum\nolimits_{E_z} \mathrm{F}^n_{z_{i,j,k+\frac{1}{2}}} \cdot \mathrm{G}^n_{z_{i,j,k+\frac{1}{2}}}\Big)\Delta v, \qquad \|\mathbf{F}^n\|^2_e = (\mathbf{F}^n, \mathbf{F}^n)_e, \\
(\mathbf{F}^n, \mathbf{G}^n)_h \;&=\; (\mathrm{F}^n_x, \mathrm{G}^n_x)_{h_x} + (\mathrm{F}^n_y, \mathrm{G}^n_y)_{h_y} + (\mathrm{F}^n_z, \mathrm{G}^n_z)_{h_z} \\
&=\; \Big(\sum\nolimits_{H_x} \mathrm{F}^n_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}} \cdot \mathrm{G}^n_{x_{i,j+\frac{1}{2},k+\frac{1}{2}}} + \sum\nolimits_{H_y} \mathrm{F}^n_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}} \cdot \mathrm{G}^n_{y_{i+\frac{1}{2},j,k+\frac{1}{2}}} \\
&+\; \sum\nolimits_{H_z} \mathrm{F}^n_{z_{i+\frac{1}{2},j+\frac{1}{2},k}} \cdot \mathrm{G}^n_{z_{i+\frac{1}{2},j+\frac{1}{2},k}}\Big)\Delta v, \quad \|\mathbf{G}^n\|^2_h = (\mathbf{G}^n, \mathbf{G}^n)_h.
\end{aligned}$$

For a grid function $\mathrm{F}^n_{\alpha,\beta,\gamma}$ with $\alpha = i$ or $i + 1/2$, $\beta = j$ or $j + 1/2$, $\gamma = k$ or $k + 1/2$, denote the central difference operators by δ_x, δ_y and δ_z :

$$\delta_x \mathrm{F}^n_{\alpha,\beta,\gamma} = \frac{\mathrm{F}^n_{\alpha+\frac{1}{2},\beta,\gamma} - \mathrm{F}^n_{\alpha-\frac{1}{2},\beta,\gamma}}{\Delta x}, \qquad \delta_y \mathrm{F}^n_{\alpha,\beta,\gamma} = \frac{\mathrm{F}^n_{\alpha,\beta+\frac{1}{2},\gamma} - \mathrm{F}^n_{\alpha,\beta-\frac{1}{2},\gamma}}{\Delta y},$$

$\delta_z \mathrm{F}^n_{\alpha,\beta,\gamma}$ is similarly defined by adjusting γ. Let $\nabla^h = (\delta_x, \delta_y, \delta_z)$ be the vector of operators. Define the discrete divergence and curl of $\mathbf{F}^n = (\mathrm{F}^n_x, \mathrm{F}^n_y, \mathrm{F}^n_z)$ by

$$\begin{aligned}
&\nabla^h \cdot \mathbf{F}^n = \delta_x \mathrm{F}^n_x + \delta_y \mathrm{F}^n_y + \delta_z \mathrm{F}^n_z, \\
&\nabla^h \times \mathbf{F}^n = (\delta_y \mathrm{F}^n_z - \delta_z \mathrm{F}^n_y, \delta_z \mathrm{F}^n_x - \delta_x \mathrm{F}^n_z, \delta_x \mathrm{F}^n_y - \delta_y \mathrm{F}^n_x).
\end{aligned}$$

For simplicity in notation, let $\bar{i} = i + 1/2$, $\bar{j} = j + 1/2$, $\bar{k} = k + 1/2$, and

$$O_{cfl} = \frac{(\Delta t)^2}{\mu\varepsilon}\Big(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}\Big), \quad \Delta_{xyz} = (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4.$$

3. HAIT-FDTD(M) and analysis on divergence and stability

In this section we analyze the HAIT-FDTD(M) schemes on divergence of the solution fields and stability by the energy method.

3.1. The HAIT-FDTD(M) **scheme and free divergence property.** Let $\mathbf{E}^n = (E^n_{x_{\bar{i},j,k}}$, $E^n_{y_{i,\bar{j},k}}$, $E^n_{z_{i,j,\bar{k}}})$ and $\mathbf{H}^n = (H^n_{x_{i,\bar{j},\bar{k}}}$, $H^n_{y_{\bar{i},j,\bar{k}}}$, $H^n_{z_{\bar{i},\bar{j},k}})$ be the approximations to the exact values $\mathbf{E}(t^n) = \big(E_x(x_{\bar{i}}, y_j, z_k, t^n), E_y(x_i, y_{\bar{j}}, z_k, t^n),\ E_z(x_i, y_j, z_{\bar{k}}, t^n)\big)$ and $\mathbf{H}(t^n) = \big(H_x(x_i, y_{\bar{j}}, z_{\bar{k}}, t^n),\ H_y(x_{\bar{i}}, y_j, z_{\bar{k}}, t^n),\ H_z(x_{\bar{i}}, y_{\bar{j}}, z_k, t^n)\big)$, respectively. Then, the FDTD method or scheme with high accuracy in time (HAIT) (called HAIT-FDTD(M), see [22]), is: Given $\mathbf{E}^n$ and $\mathbf{H}^n$, find $\mathbf{E}^{n+1}$ and $\mathbf{H}^{n+1}$ $(n = 0, 1, \dots, N-1)$ such that for $u = x, y, z$,

$$E^{n+1}_{u_{\alpha,\beta,\gamma}} = \sum_{m=0}^{M} \frac{(\Delta t)^m}{m!} C^m_{u_{\alpha,\beta,\gamma,n}}, \ (\alpha,\beta,\gamma) = (\bar{i}, j, k), (i, \bar{j}, k), \text{ or } (i, j, \bar{k}), \tag{11}$$

$$H^{n+1}_{u_{\alpha,\beta,\gamma}} = \sum_{m=0}^{M} \frac{(\Delta t)^m}{m!} D^m_{u_{\alpha,\beta,\gamma,n}}, \ (\alpha,\beta,\gamma) = (i, \bar{j}, \bar{k}), (\bar{i}, j, \bar{k}), \text{ or } (\bar{i}, \bar{j}, k), \tag{12}$$

where C^m_u and D^m_u $(u = x, y, z)$ are determined by the following formulas:

$$\varepsilon\, C^{m+1}_{x_{i+\frac{1}{2},j,k,n}} = \delta_y D^m_{z_{i+\frac{1}{2},j,k,n}} - \delta_z D^m_{y_{i+\frac{1}{2},j,k,n}}, \tag{13}$$

$$\varepsilon\, C^{m+1}_{y_{i,j+\frac{1}{2},k,n}} = \delta_z D^m_{x_{i,j+\frac{1}{2},k,n}} - \delta_x D^m_{z_{i,j+\frac{1}{2},k,n}}, \tag{14}$$

$$\varepsilon\, C^{m+1}_{z_{i,j,k+\frac{1}{2},n}} = \delta_x D^m_{y_{i,j,k+\frac{1}{2},n}} - \delta_y D^m_{x_{i,j,k+\frac{1}{2},n}}, \tag{15}$$

$$\mu\, D^{m+1}_{x_{i,j+\frac{1}{2},k+\frac{1}{2},n}} = \delta_z C^m_{y_{i,j+\frac{1}{2},k+\frac{1}{2},n}} - \delta_y C^m_{z_{i,j+\frac{1}{2},k+\frac{1}{2},n}}, \tag{16}$$

$$\mu\, D^{m+1}_{y_{i+\frac{1}{2},j,k+\frac{1}{2},n}} = \delta_x C^m_{z_{i+\frac{1}{2},j,k+\frac{1}{2},n}} - \delta_z C^m_{x_{i+\frac{1}{2},j,k+\frac{1}{2},n}}, \tag{17}$$

$$\mu\, D^{m+1}_{z_{i+\frac{1}{2},j+\frac{1}{2},k,n}} = \delta_y C^m_{x_{i+\frac{1}{2},j+\frac{1}{2},k,n}} - \delta_x C^m_{y_{i+\frac{1}{2},j+\frac{1}{2},k,n}}, \tag{18}$$

where $m = 0, 1, \dots, M-1$. The starting values for (13)-(18) are:

$$C^0_{x_{\bar{i},j,k,n}} = E^n_{x_{\bar{i},j,k}}, \quad C^0_{y_{i,\bar{j},k,n}} = E^n_{y_{i,\bar{j},k}}, \quad C^0_{z_{i,j,\bar{k},n}} = E^n_{z_{i,j,\bar{k}}}, \tag{19}$$

$$D^0_{x_{i,\bar{j},\bar{k},n}} = H^n_{x_{i,\bar{j},\bar{k}}}, \quad D^0_{y_{\bar{i},j,\bar{k},n}} = H^n_{y_{\bar{i},j,\bar{k}}}, \quad D^0_{z_{\bar{i},\bar{j},k,n}} = H^n_{z_{\bar{i},\bar{j},k}}. \tag{20}$$

The boundary conditions of C^m_u and D^m_u $(u = x, y, z)$ used in (13)-(18) are:

$$C^m_{x_{\bar{i},j',k,n}} = C^m_{x_{\bar{i},j,k',n}} = C^m_{y_{i',\bar{j},k,n}} = C^m_{y_{i,\bar{j},k',n}} = C^m_{z_{i',j,\bar{k},n}} = 0, \tag{21}$$

$$C^m_{z_{i,j',\bar{k},n}} = D^m_{x_{i',\bar{j},\bar{k},n}} = D^m_{y_{\bar{i},j',\bar{k},n}} = D^m_{z_{\bar{i},\bar{j},j',n}} = 0, \ 0 \le m \le M, \tag{22}$$

where and in what follows $i' = 0$ or I, $j' = 0$ or J, $k' = 0$ or K denote the indices of the points on the boundary.

The vector form of HAIT-FDTD(M) is: To find $\mathbf{E}^{n+1}$ and $\mathbf{H}^{n+1}$ such that

$$\begin{aligned} \mathbf{E}^{n+1} &= \mathbf{C}^0_n + \tfrac{\Delta t}{1!}\mathbf{C}^1_n + \tfrac{(\Delta t)^2}{2!}\mathbf{C}^2_n + \cdots + \tfrac{(\Delta t)^M}{M!}\mathbf{C}^M_n, \\ \mathbf{H}^{n+1} &= \mathbf{D}^0_n + \tfrac{\Delta t}{1!}\mathbf{D}^1_n + \tfrac{(\Delta t)^2}{2!}\mathbf{D}^2_n + \cdots + \tfrac{(\Delta t)^M}{M!}\mathbf{D}^M_n, \end{aligned} \tag{23}$$

where $\mathbf{C}^m_n = (C^m_{x_{\bar{i},j,k,n}}, C^m_{y_{i,\bar{j},k,n}}, C^m_{z_{i,j,\bar{k},n}})$ and $\mathbf{D}^m_n = (D^m_{x_{i,\bar{j},\bar{k},n}}, D^m_{y_{\bar{i},j,\bar{k},n}}, D^m_{z_{\bar{i},\bar{j},k,n}})$ are the vectors of coefficients defined by

$$\varepsilon \mathbf{C}^{m+1}_n = \nabla^h \times \mathbf{D}^m_n, \quad \mu \mathbf{D}^{m+1}_n = -\nabla^h \times \mathbf{C}^m_n, \ \ m = 0, 1, \dots, M-1. \tag{24}$$

The boundary conditions and starting values for (24) are:

$$\nu \times \mathbf{C}^m_n = \mathbf{0}, \quad \nu \cdot \mathbf{D}^m_n = 0, \quad \text{and} \quad \mathbf{C}^0_n = \mathbf{E}^n, \quad \mathbf{D}^0_n = \mathbf{H}^n. \tag{25}$$

By the discrete divergence of the fields in (11)-(18), it is easy to prove that

Theorem 3.1 (free divergence). *Let* $\mathbf{E}^n$ *and* $\mathbf{H}^n$ *be the solutions of* HAIT-FDTD(M) *(*$M \ge 3$*), and* $\mathbf{C}^m_n$ *and* $\mathbf{D}^m_n$ *be the vectors of coefficients. Then,*

$$\nabla^h \cdot \varepsilon \mathbf{E}^{n+1} = \nabla^h \cdot \varepsilon \mathbf{E}^n = \nabla^h \cdot \varepsilon \mathbf{C}^m_n = 0, \quad \forall\, 0 \le m \le M, \tag{26}$$

$$\nabla^h \cdot \mu \mathbf{H}^{n+1} = \nabla^h \cdot \mu \mathbf{H}^n = \nabla^h \cdot \mu \mathbf{D}^m_n = 0, \ \forall\, 0 \le n \le N-1. \tag{27}$$

3.2. The L^2 norms and the inner-product of vector coefficients.

Lemma 3.2. *For $m = 0, 1, \ldots, M (M \geq 1)$, let $\mathbf{C}_n^m = (C^m_{x_{\bar{i},j,k,n}}, C^m_{y_{i,\bar{j},k,n}}, C^m_{z_{i,j,\bar{k},n}})$ and $\mathbf{D}_n^m = (D^m_{x_{i,\bar{j},\bar{k},n}}, D^m_{y_{\bar{i},j,\bar{k},n}}, D^m_{z_{\bar{i},\bar{j},k,n}})$ be the vectors of coefficients for $\mathbf{E}^{n+1}$ and $\mathbf{H}^{n+1}$ defined in* (13)-(18). *Then for any $m \geq 0$ and $n \geq 1$, it holds that*

$$\begin{aligned}
\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^{m+1}\|_e^2 &= \|\varepsilon^{-\frac{1}{2}}\delta_x D_x^m\|_{h_0}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_y D_x^m\|_{e_z}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_z D_x^m\|_{e_y}^2 \\
&+ \|\varepsilon^{-\frac{1}{2}}\delta_x D_y^m\|_{e_z}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_y D_y^m\|_{h_0}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_z D_y^m\|_{e_x}^2 \\
&+ \|\varepsilon^{-\frac{1}{2}}\delta_x D_z^m\|_{e_y}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_y D_z^m\|_{e_x}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_z D_z^m\|_{h_0}^2 \\
&\leq \frac{4}{\mu\varepsilon}\Big(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}\Big)\|\mu^{\frac{1}{2}}\mathbf{D}_n^m\|_h^2, \qquad (28)
\end{aligned}$$

$$\begin{aligned}
\|\mu^{\frac{1}{2}}\mathbf{D}_n^{m+1}\|_h^2 &= \|\mu^{-\frac{1}{2}}\delta_x C_x^m\|_{e_0}^2 + \|\mu^{-\frac{1}{2}}\delta_y C_x^m\|_{h_z}^2 + \|\mu^{-\frac{1}{2}}\delta_z C_x^m\|_{h_y}^2 \\
&+ \|\mu^{-\frac{1}{2}}\delta_x C_y^m\|_{h_z}^2 + \|\mu^{-\frac{1}{2}}\delta_y C_y^m\|_{e_0}^2 + \|\mu^{-\frac{1}{2}}\delta_z C_y^m\|_{h_x}^2 \\
&+ \|\mu^{-\frac{1}{2}}\delta_x C_z^m\|_{h_y}^2 + \|\mu^{-\frac{1}{2}}\delta_y C_z^m\|_{h_x}^2 + \|\mu^{-\frac{1}{2}}\delta_z C_z^m\|_{e_0}^2 \\
&\leq \frac{4}{\mu\varepsilon}\Big(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}\Big)\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^m\|_e^2. \qquad (29)
\end{aligned}$$

Proof. First we prove (28). By the expressions of $C_u^{m+1}(u = x, y, z)$, we have

$$\begin{aligned}
\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^{m+1}\|_e^2 = &\|\varepsilon^{-\frac{1}{2}}\delta_y D_z^m\|_{e_x}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_z D_y^m\|_{e_x}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_z D_x^m\|_{e_y}^2 \\
&+ \|\varepsilon^{-\frac{1}{2}}\delta_x D_z^m\|_{e_y}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_x D_y^m\|_{e_z}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_y D_x^m\|_{e_z}^2 + \mathrm{T}_d, \qquad (30)
\end{aligned}$$

where the last term T_d is

$$\mathrm{T}_d = -2\Big(\sum\nolimits_{E_x}\varepsilon^{-1}\delta_y D_z^m \delta_z D_y^m + \sum\nolimits_{E_y}\varepsilon^{-1}\delta_z D_x^m \delta_x D_z^m + \sum\nolimits_{E_z}\varepsilon^{-1}\delta_x D_y^m \delta_y D_x^m\Big)\Delta v.$$

By using summation by parts and the boundary conditions: (21)-(22), we get

$$\begin{aligned}
\mathrm{T}_d &= -2\sum\nolimits_{H_0}\varepsilon^{-1}\Big(\delta_z D_z^m \delta_y D_y^m + \delta_x D_x^m \delta_z D_z^m + \delta_x D_x^m \delta_y D_y^m\Big)\Delta v \\
&= \|\varepsilon^{-\frac{1}{2}}\delta_x D_x^m\|_{h_0}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_y D_y^m\|_{h_0}^2 + \|\varepsilon^{-\frac{1}{2}}\delta_z D_z^m\|_{h_0}^2, \qquad (31)
\end{aligned}$$

where the following free-divergence (see Theorem 3.1) is used:

$$\|\nabla^h \cdot \mathbf{D}_n^m\|_{h_0}^2 = \|\delta_x D_x^m + \delta_y D_y^m + \delta_z D_z^m\|_{h_0}^2 = 0.$$

Combining of (30) with (31) gives the equality in (28). Similarly, we obtain the equality in (29). Next we prove the inequality in (28). By using the inequality $2fg \leq f^2 + g^2$ and the boundary conditions: (22), we see that

$$\|\varepsilon^{-\frac{1}{2}}\delta_x D_x^m\|_{h_0}^2 \leq \frac{4}{(\Delta x)^2}\|\varepsilon^{-\frac{1}{2}}D_x^m\|_{h_x}^2, \quad \|\varepsilon^{-\frac{1}{2}}\delta_y D_x^m\|_{e_z}^2 \leq \frac{4}{(\Delta y)^2}\|\varepsilon^{-\frac{1}{2}}D_x^m\|_{h_x}^2, \qquad (32)$$

$$\|\varepsilon^{-\frac{1}{2}}\delta_z D_x^m\|_{e_y}^2 \leq \frac{4}{(\Delta z)^2}\|\varepsilon^{-\frac{1}{2}}D_x^m\|_{h_x}^2, \quad \|\varepsilon^{-\frac{1}{2}}\delta_x D_y^m\|_{e_z}^2 \leq \frac{4}{(\Delta x)^2}\|\varepsilon^{-\frac{1}{2}}D_y^m\|_{h_y}^2, \qquad (33)$$

Similar to (32)-(33), the estimates of $\delta_u D_y^m$ and $\delta_u D_z^m(u = x, y, z)$ are obtained. Substituting (32)-(33), etc., into the equality in (28), we obtain the inequality in (28). Symmetrically, the inequality in (29) is proved. □

Lemma 3.3. *Let $\mathbf{C}_n^m$ and $\mathbf{D}_n^m$ be the vectors of coefficients for $\mathbf{E}^{n+1}$ and $\mathbf{H}^{n+1}$ in the* HAIT-FDTD(M) *schemes with $M \geq 3$. Then, if $M = 3$, it holds that*

$$(\varepsilon\mathbf{C}_n^0, \mathbf{C}_n^1)_e + (\mu\mathbf{D}_n^0, \mathbf{D}_n^1)_h = 0, \quad (\varepsilon\mathbf{C}_n^0, \mathbf{C}_n^3)_e + (\mu\mathbf{D}_n^0, \mathbf{D}_n^3)_h = 0, \qquad (34)$$

$$(\varepsilon\mathbf{C}_n^1, \mathbf{C}_n^2)_e + (\mu\mathbf{D}_n^1, \mathbf{D}_n^2)_h = 0, \quad (\varepsilon\mathbf{C}_n^2, \mathbf{C}_n^3)_e + (\mu\mathbf{D}_n^2, \mathbf{D}_n^3)_h = 0, \qquad (35)$$

$$(\varepsilon\mathbf{C}_n^0, \mathbf{C}_n^2)_e + (\mu\mathbf{D}_n^0, \mathbf{D}_n^2)_h = -\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^1\|_e^2 - \|\mu^{\frac{1}{2}}\mathbf{D}_n^1\|_h^2, \qquad (36)$$

$$(\varepsilon\mathbf{C}_n^1, \mathbf{C}_n^3)_e + (\mu\mathbf{D}_n^1, \mathbf{D}_n^3)_h = -\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^2\|_e^2 - \|\mu^{\frac{1}{2}}\mathbf{D}_n^2\|_h^2. \qquad (37)$$

In general, if $M \geq 4$ and $p, q \in \{0, 1, \ldots, M\}$ with $p < q$, then the inner products of the vectors of coefficients in HAIT-FDTD(M) *with $M \geq 4$ are:*

$$(\mathbf{C}_n^p, \nabla^h \times \mathbf{D}_n^q)_e = (\nabla^h \times \mathbf{C}_n^p, \mathbf{D}_n^q)_h, \tag{38}$$

$$(\varepsilon \mathbf{C}_n^p, \mathbf{C}_n^q)_e + (\mu \mathbf{D}_n^p, \mathbf{D}_n^q)_h = \begin{cases} 0, & \text{if } p+q \text{ is odd}, \\ (-1)^{\frac{q-p}{2}} \left(\|\varepsilon^{\frac{1}{2}} \mathbf{C}_n^{\frac{p+q}{2}}\|_e^2 + \|\mu^{\frac{1}{2}} \mathbf{D}_n^{\frac{p+q}{2}}\|_h \right), & \text{others}. \end{cases} \tag{39}$$

Proof. We only derive (38) and the first equation in (34) since all the other equations in (34)-(37) and (39) can be proved similarly by using (38) and the definitions of $\mathbf{C}_n^p$ and $\mathbf{D}_n^q$ in (24) or (13)-(18). By summation by parts, we have

$$(\mathbf{C}_n^p, \nabla^h \times \mathbf{D}_n^q)_e = \sum\nolimits_{H_x} D_x^q (\delta_y C_z^p - \delta_z C_y^p) \Delta v + \sum\nolimits_{H_y} D_y^q (\delta_z C_x^p - \delta_x C_z^p) \Delta v$$
$$+ \sum\nolimits_{H_z} D_z^q (\delta_x C_y^p - \delta_y C_x^p)) \Delta v = (\nabla^h \times \mathbf{C}_n^p, \mathbf{D}_n^q)_h,$$

where the boundary conditions in (25) are used.

Next we consider the first relation in (34). By using (38), we see that

$$(\varepsilon \mathbf{C}_n^0, \mathbf{C}_n^1)_e + (\mu \mathbf{D}_n^0, \mathbf{D}_n^1)_h = (\mathbf{C}_n^0, \nabla^h \times \mathbf{D}_n^0)_e + (\mathbf{D}_n^0, -\nabla^h \times \mathbf{C}_n^0)_h$$
$$= (\nabla^h \times \mathbf{C}_n^0, \mathbf{D}_n^0)_h - (\mathbf{D}_n^0, \nabla^h \times \mathbf{C}_n^0)_h = 0.$$

Similarly, the other relations in (34)-(39) are derived, where the sign "-" is changed into "+" if the number of summation by parts used is even; otherwise, the sign is negative. This proves Lemma 3.3. □

3.3. Numerical energy identities and stability analysis. By the virtue of Lemmas 3.2 and 3.3, we obtain the following theorem.

Theorem 3.4. *Let $\mathbf{E}^{n+1}$ and $\mathbf{H}^{n+1}$ be the solution of* HAIT-FDTD(M) *($M \geq 3$), and let $\mathbf{C}_n^m$ and $\mathbf{D}_n^m$ be the vectors of coefficients. Then the energy of the fields satisfies that*

$$\|\varepsilon^{\frac{1}{2}} \mathbf{E}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}} \mathbf{H}^{n+1}\|_h^2 = \|\varepsilon^{\frac{1}{2}} \mathbf{E}^n\|_e^2 + \|\mu^{\frac{1}{2}} \mathbf{H}^n\|_h^2 + \mathrm{PT}_M, \tag{40}$$

where $\mathrm{PT}_M (3 \leq M \leq 8)$ *are the perturbation terms, defined by*

$$\begin{aligned}
\mathrm{PT}_3 &= -\frac{(\Delta t)^4}{12}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^2\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^2\|_h^2\right) + \frac{(\Delta t)^6}{36}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^3\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^3\|_h^2\right);\\
\mathrm{PT}_4 &= -\frac{(\Delta t)^6}{72}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^3\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^3\|_h^2\right) + \frac{(\Delta t)^8}{(4!)^2}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^4\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^4\|_h^2\right);\\
\mathrm{PT}_5 &= \frac{(\Delta t)^6}{360}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^3\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^3\|_h^2\right)\\
&- \frac{(\Delta t)^8}{960}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^4\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^4\|_h^2\right) + \frac{(\Delta t)^{10}}{(5!)^2}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^5\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^5\|_h^2\right);\\
\mathrm{PT}_6 &= \frac{(\Delta t)^8}{2880}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^4\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^4\|_h^2\right)\\
&- \frac{(\Delta t)^{10}}{21600}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^5\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^5\|_h^2\right) + \frac{(\Delta t)^{12}}{(6!)^2}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^6\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^6\|_h^2\right);\\
\mathrm{PT}_7 &= \frac{-(\Delta t)^8}{20160}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^4\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^4\|_h^2\right) + \frac{(\Delta t)^{10}}{50400}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^5\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^5\|_h^2\right)\\
&- \frac{(\Delta t)^{12}}{725760}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^6\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^6\|_h^2\right) + \frac{(\Delta t)^{14}}{(7!)^2}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^7\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^7\|_h^2\right);\\
\mathrm{PT}_8 &= \frac{-(\Delta t)^{10}}{201600}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^5\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^5\|_h^2\right) + \frac{(\Delta t)^{12}}{1451520}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^6\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^6\|_h^2\right)\\
&- \frac{(\Delta t)^{14}}{33868800}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^7\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^7\|_h^2\right) + \frac{(\Delta t)^{16}}{(8!)^2}\left(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^8\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^8\|_h^2\right).
\end{aligned}$$

Proof. We only derive (40) with $M = 3$, since the other identities are similarly obtained. By the definitions of $\| * \|_e$, $\| * \|_h$, $\mathbf{E}^{n+1}$ and $\mathbf{H}^{n+1}$, we have

$$\|\varepsilon^{\frac{1}{2}}\mathbf{E}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^{n+1}\|_h^2 = \sum_{l=0}^{3}\Big(\|\varepsilon^{\frac{1}{2}}\frac{(\Delta t)^l}{l!}C_x^l\|_{e_x}^2 + \|\varepsilon^{\frac{1}{2}}\frac{(\Delta t)^l}{l!}C_y^l\|_{e_y}^2\Big)$$
$$+\sum_{l=0}^{3}\Big(\|\varepsilon^{\frac{1}{2}}\frac{(\Delta t)^l}{l!}C_z^l\|_{e_z}^2 + \|\mu^{\frac{1}{2}}\frac{(\Delta t)^l}{l!}D_x^l\|_{h_x}^2 + \|\mu^{\frac{1}{2}}\frac{(\Delta t)^l}{l!}D_y^l\|_{h_y}^2 + \|\mu^{\frac{1}{2}}\frac{(\Delta t)^l}{l!}D_z^l\|_{h_z}^2\Big)$$
$$+2\Big(\frac{\Delta t}{1!}\mathrm{T}_{01} + \frac{(\Delta t)^2}{2!}\mathrm{T}_{02} + \frac{(\Delta t)^3}{3!}\mathrm{T}_{03} + \frac{(\Delta t)^3}{2!}\mathrm{T}_{12} + \frac{(\Delta t)^4}{3!}\mathrm{T}_{13} + \frac{(\Delta t)^5}{2!3!}\mathrm{T}_{23}\Big), \tag{41}$$

where the terms $\mathrm{T}_{lm}(l, m = 0, 1, 2, 3,\ l < m)$ are

$$\begin{aligned}
\mathrm{T}_{01} &= (\varepsilon\mathbf{C}_n^0, \mathbf{C}_n^1)_e + (\mu\mathbf{D}_n^0, \mathbf{D}_n^1)_h, & \mathrm{T}_{02} &= (\varepsilon\mathbf{C}_n^0, \mathbf{C}_n^2)_e + (\mu\mathbf{D}_n^0, \mathbf{D}_n^2)_h,\\
\mathrm{T}_{03} &= (\varepsilon\mathbf{C}_n^0, \mathbf{C}_n^3)_e + (\mu\mathbf{D}_n^0, \mathbf{D}_n^3)_h, & \mathrm{T}_{12} &= (\varepsilon\mathbf{C}_n^1, \mathbf{C}_n^2)_e + (\mu\mathbf{D}_n^1, \mathbf{D}_n^2)_h,\\
\mathrm{T}_{13} &= (\varepsilon\mathbf{C}_n^1, \mathbf{C}_n^3)_e + (\mu\mathbf{D}_n^1, \mathbf{D}_n^3)_h, & \mathrm{T}_{23} &= (\varepsilon\mathbf{C}_n^2, \mathbf{C}_n^3)_e + (\mu\mathbf{D}_n^2, \mathbf{D}_n^3)_h.
\end{aligned}$$

By using Lemma 3.3 and simplifying the expressions in (41), we get (40) with $M = 3$. This completes the proof of Theorem 3.4. □

Remark 3.5. *From Theorem 3.4 we see that* HAIT-FDTD(M) *is numerically energy conserved when M is so large that the perturbation terms are less than the machine error.*

Theorem 3.6. *The solutions of* HAIT-FDTD(M) *with* $3 \leq M \leq 8$ *are bounded by*

$$\|\varepsilon^{\frac{1}{2}}\mathbf{E}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^{n+1}\|_h^2 \leq O_M\big(\|\varepsilon^{\frac{1}{2}}\mathbf{E}^0\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^0\|_h^2\big), \tag{42}$$

where $O_3 = O_4 = O_7 = O_8 = 1$ and $O_5 = O_6 = exp(rT)$ (r is a small positive number), if the step sizes Δt and $\Delta u(u = x, y, z)$ satisfy that

$$O_{cfl} =: (c\Delta t)^2\Big(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}\Big) \leq r_M, \tag{43}$$

where r_M with $3 \leq M \leq 8$ are constants, defined by

$$r_3 = \frac{3}{4},\quad r_4 = 2,\quad r_5 = \min\Big\{\frac{15}{4}, \sqrt[3]{\frac{45}{8}\ r\Delta t}\Big\}$$
$$r_6 = \min\Big\{6, \sqrt[4]{\frac{45}{4}\ r\Delta t}\Big\},\quad r_7 = \frac{5}{8},\quad r_8 = \frac{9}{5}.$$

Proof. First we prove (42) with $M = 3$. By using Lemma 3.2, we obtain

$$(\Delta t)^2\big(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^{m+1}\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^{m+1}\|_h^2\big) \leq 4O_{cfl}\big(\|\mu^{\frac{1}{2}}\mathbf{D}_n^m\|_h^2 + \|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^m\|_e^2\big). \tag{44}$$

Combining (44) with $m = 2$ and the identity (40) with $M = 3$, we have

$$\begin{aligned}
\|\varepsilon^{\frac{1}{2}}\mathbf{E}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^{n+1}\|_h^2 \ &\leq\ \|\varepsilon^{\frac{1}{2}}\mathbf{E}^n\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^n\|_h^2\\
&- \frac{(\Delta t)^4}{12}\Big(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^2\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^2\|_h^2\Big)\Big(1 - \frac{4}{3}O_{cfl}\Big).
\end{aligned} \tag{45}$$

From (45) we see that if (43) holds, then the inequality (42) is obtained by using (45) repeatedly. Similarly, (42) and (43) with $M = 4, 7, 8$ are proved.

Next, we prove (42) and (43) with $M = 5, 6$. By (40) with $M = 5$, we see that

$$\begin{aligned}
\|\varepsilon^{\frac{1}{2}}\mathbf{E}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^{n+1}\|_h^2 \ &\leq\ \|\varepsilon^{\frac{1}{2}}\mathbf{E}^n\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^n\|_h^2 + \frac{(\Delta t)^6}{360}\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^3\|_e^2\\
&+ \frac{(\Delta t)^6}{360}\|\mu^{\frac{1}{2}}\mathbf{D}_n^3\|_h^2 - \frac{(\Delta t)^8}{960}(1 - \frac{4}{15}O_{cfl})(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^4\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^4\|_h^2).
\end{aligned} \tag{46}$$

Then, if $O_{cfl} \leq 15/4$, applying (44) with $m = 2, 1, 0$ into (46) repeatedly leads

$$\|\varepsilon^{\frac{1}{2}}\mathbf{E}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^{n+1}\|_h^2 \leq \Big(1 + \frac{8(O_{cfl})^3}{45}\Big)(\|\sqrt{\varepsilon}\mathbf{E}^n\|_e^2 + \|\sqrt{\mu}\mathbf{H}^n\|_h^2). \tag{47}$$

Thus, if $8(O_{cfl})^3/45 \leq r\Delta t$, or $O_{cfl} \leq \sqrt[3]{5.625 r\Delta t}$ (where r is a positive number such that $exp(rT)$ is not large), then (47) becomes

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\mathbf{E}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^{n+1}\|_h &\leq \big(1 + r\Delta t\big)(\|\varepsilon^{\frac{1}{2}}\mathbf{E}^n\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^n\|_h^2)\\ &\leq e^{rT}(\|\varepsilon^{\frac{1}{2}}\mathbf{E}^0\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^0\|_h^2),\end{aligned} \tag{48}$$

which is (42) for $M = 5$. Similarly, (42) and (43) with $M = 6$ are derived. This completes the proof. □

Remark 3.7. (1) *The stability conditions shown in Theorem 3.6 are sufficient but not necessary, and the stability conditions of the four schemes with* $M = 3, 4, 7$ *and* 8 *are weaker than the CFL stability condition:* $O_{cfl} \leq 1/3$.

(2) *The* HAIT-FDTD(M) *schemes with* $M = 3, 4, 7, 8$ *are energy decreased, while* HAIT-FDTD(M=5, 6) are energy-increased. For example, by applying Lemma 3.2 to the negative term of PT_M in (40) with $M = 5$, we see that

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\mathbf{E}^{n+1}\|_e^2 &+ \|\mu^{\frac{1}{2}}\mathbf{H}^{n+1}\|_h^2 \geq \|\varepsilon^{\frac{1}{2}}\mathbf{E}^n\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{H}^n\|_h^2\\ &+ \frac{(\Delta t)^6}{360}\Big(1 - \frac{3}{2}O_{cfl}\Big)\big(\|\varepsilon^{\frac{1}{2}}\mathbf{C}_n^3\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{D}_n^3\|_h^2\big).\end{aligned}$$

Therefore, if $O_{cfl} \leq r_5^* = \min\{2/3, \sqrt[3]{45\, r\Delta t/8}\}$, HAIT-FDTD(M=5) is then energy-increased and stable. Similarly, it is proved that if $O_{cfl} \leq r_6^* = \min\{45/24, \sqrt[4]{45\, r\Delta t/4}\}$, HAIT-FDTD(M=6) is also energy-increased and stable.

4. Error estimate and convergence analysis.

In this section we investigate the error of HAIT-FDTD(M) and estimate the magnitude of the error of the solutions to the schemes.

4.1. Derivation of error equations and truncation errors.

For the solution of the problem (1) − (5), $E_u(p,t)$ and $H_u(p,t)$ $(u = x, y, z)$, the m-th$(0 \leq m \leq M)$ derivatives of $E_u(p,t)$ and $H_u(p,t)$ with respect to time at (p, t^n) are denoted by $\tilde{C}_u^m(p,t^n)$ and $\tilde{D}_u^m(p,t^n)$, respectively. For $u = x, y, z$, by using the Taylor Theorem, we see that

$$\begin{aligned}E_u(p,t^{n+1}) &= \tilde{C}_u^0(p,t^n) + \sum_{m=1}^{M} \tfrac{(\Delta t)^m}{m!}\tilde{C}_u^m(p,t^n) + (\Delta t)^{M+1}S_{e_u}^n,\\ H_u(p,t^{n+1}) &= \tilde{D}_u^0(p,t^n) + \sum_{m=1}^{M} \tfrac{(\Delta t)^m}{m!}\tilde{D}_u^m(p,t^n) + (\Delta t)^{M+1}S_{h_u}^n,\end{aligned} \tag{49}$$

where $\mathbf{S}_e^n =: (S_{e_x}^n, S_{e_y}^n, S_{e_z}^n)$ and $\mathbf{S}_h^n =: (S_{h_x}^n, S_{h_y}^n, S_{h_z}^n)$ are the remainders, defined by

$$S_{e_u}^n = \frac{1}{(M+1)!}\frac{\partial^{M+1}E_u}{\partial t^{M+1}}(p, t_{1u}^n), \quad S_{h_u}^n = \frac{1}{(M+1)!}\frac{\partial^{M+1}H_u}{\partial t^{M+1}}(p, t_{2u}^n), \tag{50}$$

in which t_{1u}^n and t_{2u}^n $(u = x, y, z)$ are the numbers between t^n and t^{n+1}.

Differentiating both sides of (1)-(3) m-times with respect to time, we obtain the relations between $\{\tilde{C}_u^{m+1}, \tilde{D}_u^{m+1}\}$ and $\{\tilde{C}_u^m, \tilde{D}_u^m\}(u = x, y, z,\ 0 \leq m \leq M-1)$:

$$\varepsilon\, \tilde{C}_x^{m+1} = \frac{\partial \tilde{D}_z^m}{\partial y} - \frac{\partial \tilde{D}_y^m}{\partial z}\Big|_{(p,t^n)}, \quad \varepsilon\, \tilde{C}_y^{m+1} = \frac{\partial \tilde{D}_x^m}{\partial z} - \frac{\partial \tilde{D}_z^m}{\partial x}\Big|_{(p,t^n)}, \tag{51}$$

$$\varepsilon\, \tilde{C}_z^{m+1} = \frac{\partial \tilde{D}_y^m}{\partial x} - \frac{\partial \tilde{D}_x^m}{\partial y}\Big|_{(p,t^n)}, \quad \mu\, \tilde{D}_x^{m+1} = \frac{\partial \tilde{C}_y^m}{\partial z} - \frac{\partial \tilde{C}_z^m}{\partial y}\Big|_{(p,t^n)}, \tag{52}$$

$$\mu\, \tilde{D}_y^{m+1} = \frac{\partial \tilde{C}_z^m}{\partial x} - \frac{\partial \tilde{C}_x^m}{\partial z}\Big|_{(p,t^n)}, \quad \mu\, \tilde{D}_z^{m+1} = \frac{\partial \tilde{C}_x^m}{\partial y} - \frac{\partial \tilde{C}_y^m}{\partial x}\Big|_{(p,t^n)}. \tag{53}$$

Let $\boldsymbol{\mathcal{E}}^n = (\mathcal{E}_x^n, \mathcal{E}_y^n, \mathcal{E}_z^n)$ and $\boldsymbol{\mathcal{H}}^n = (\mathcal{H}_x^n, \mathcal{H}_y^n, \mathcal{H}_z^n)$ be the error fields, let $\boldsymbol{\mathcal{C}}_n^m = (\mathcal{C}_x^m, \mathcal{C}_y^m, \mathcal{C}_z^m)$ and $\boldsymbol{\mathcal{D}}_n^m = (\mathcal{D}_x^m, \mathcal{D}_y^m, \mathcal{D}_z^m)$ be the coefficient vectors for the error fields. For example,

$$\mathcal{E}^n_{x_{i+\frac{1}{2},j,k}} = E_x(x_{i+\frac{1}{2}}, y_j, z_k, t^n) - E^n_{x_{i+\frac{1}{2},j,k}},$$
$$\mathcal{C}^m_{x_{i+\frac{1}{2},j,k,n}} = \tilde{C}^m_x(x_{i+\frac{1}{2}}, y_j, z_k, t^n) - C^m_{x_{i+\frac{1}{2},j,k,n}}.$$

Subtracting the equations in (11)-(12) from the equations in (49) over $\Omega^h_{e_u}$ and $\Omega^h_{h_u}(u = x, y, z)$, respectively, we obtain

$$\mathcal{E}^{n+1}_{u_{\alpha,\beta,\gamma}} = \sum_{m=0}^{M} \frac{(\Delta t)^m}{m!} \mathcal{C}^m_{u_{\alpha,\beta,\gamma,n}} + (\Delta t)^{M+1} S^n_{e_u}|_{\alpha,\beta,\gamma}, \quad u = x, y, z, \tag{54}$$

where $(\alpha,\beta,\gamma) = (\bar{i}, j, k)$, $(i, \bar{j}, k)$ and $(i, j, \bar{k})$ respectively, and

$$\mathcal{H}^{n+1}_{u_{\alpha,\beta,\gamma}} = \sum_{m=0}^{M} \frac{(\Delta t)^m}{m!} \mathcal{D}^m_{u_{\alpha,\beta,\gamma,n}} + (\Delta t)^{M+1} S^n_{h_u}|_{\alpha,\beta,\gamma}, \quad u = x, y, z, \tag{55}$$

where $(\alpha,\beta,\gamma)=(i,\bar{j},\bar{k})$, $(\bar{i}, j, \bar{k})$ and $(\bar{i}, \bar{j}, k)$ respectively.

Similarly, subtracting the equations in (13)-(18) from the discretized equations in (51)-(53) over $\Omega^h_{e_u}$ and $\Omega^h_{h_u}(u = x, y, z)$, respectively, we obtain

$$\varepsilon\, \mathcal{C}^{m+1}_{x_{i+\frac{1}{2},j,k,n}} = \delta_y \mathcal{D}^m_{z_{i+\frac{1}{2},j,k,n}} - \delta_z \mathcal{D}^m_{y_{i+\frac{1}{2},j,k,n}} + R^m_{c_{x_{i+\frac{1}{2},j,k,n}}}, \tag{56}$$
$$\varepsilon\, \mathcal{C}^{m+1}_{y_{i,j+\frac{1}{2},k,n}} = \delta_z \mathcal{D}^m_{x_{i,j+\frac{1}{2},k,n}} - \delta_x \mathcal{D}^m_{z_{i,j+\frac{1}{2},k,n}} + R^m_{c_{y_{i,j+\frac{1}{2},k,n}}}, \tag{57}$$
$$\varepsilon\, \mathcal{C}^{m+1}_{z_{i,j,k+\frac{1}{2},n}} = \delta_x \mathcal{D}^m_{y_{i,j,k+\frac{1}{2},n}} - \delta_y \mathcal{D}^m_{x_{i,j,k+\frac{1}{2},n}} + R^m_{c_{z_{i,j,k+\frac{1}{2},n}}}, \tag{58}$$
$$\mu\, \mathcal{D}^{m+1}_{x_{i,j+\frac{1}{2},k+\frac{1}{2},n}} = \delta_z \mathcal{C}^m_{y_{i,j+\frac{1}{2},k+\frac{1}{2},n}} - \delta_y \mathcal{C}^m_{z_{i,j+\frac{1}{2},k+\frac{1}{2},n}} + R^m_{d_{x_{i,j+\frac{1}{2},k+\frac{1}{2},n}}}, \tag{59}$$
$$\mu\, \mathcal{D}^{m+1}_{y_{i+\frac{1}{2},j,k+\frac{1}{2},n}} = \delta_x \mathcal{C}^m_{z_{i+\frac{1}{2},j,k+\frac{1}{2},n}} - \delta_z \mathcal{C}^m_{x_{i+\frac{1}{2},j,k+\frac{1}{2},n}} + R^m_{d_{y_{i+\frac{1}{2},j,k+\frac{1}{2},n}}}, \tag{60}$$
$$\mu\, \mathcal{D}^{m+1}_{z_{i+\frac{1}{2},j+\frac{1}{2},k,n}} = \delta_y \mathcal{C}^m_{x_{i+\frac{1}{2},j+\frac{1}{2},k,n}} - \delta_x \mathcal{C}^m_{y_{i+\frac{1}{2},j+\frac{1}{2},k,n}} + R^m_{d_{z_{i+\frac{1}{2},j+\frac{1}{2},k,n}}}, \tag{61}$$

where $0 \le m \le M-1$, $R^m_{c_u}$ and $R^m_{d_u}$ $(u = x, y, z)$ are the remainders:

$$\begin{aligned}
R^m_{c_{x_{i+\frac{1}{2},j,k,n}}} &= \tfrac{(\Delta y)^2}{24} \frac{\partial^3 \tilde{D}^m_z}{\partial y^3}(p_{i+\frac{1}{2},j_1,k}, t^n) - \tfrac{(\Delta z)^2}{24} \frac{\partial^3 \tilde{D}^m_y}{\partial z^3}(p_{i+\frac{1}{2},j,k_1}, t^n),\\
R^m_{c_{y_{i,j+\frac{1}{2},k,n}}} &= \tfrac{(\Delta z)^2}{24} \frac{\partial^3 \tilde{D}^m_x}{\partial z^3}(p_{i,j+\frac{1}{2},k_2}, t^n) - \tfrac{(\Delta x)^2}{24} \frac{\partial^3 \tilde{D}^m_z}{\partial x^3}(p_{i_1,j+\frac{1}{2},k}, t^n),\\
R^m_{c_{z_{i,j,k+\frac{1}{2},n}}} &= \tfrac{(\Delta x)^2}{24} \frac{\partial^3 \tilde{D}^m_y}{\partial x^3}(p_{i_2,j,k+\frac{1}{2}}, t^n) - \tfrac{(\Delta y)^2}{24} \frac{\partial^3 \tilde{D}^m_x}{\partial y^3}(p_{i,j_2,k+\frac{1}{2}}, t^n),
\end{aligned} \tag{62}$$

$$\begin{aligned}
R^m_{d_{x_{i,j+\frac{1}{2},k+\frac{1}{2},n}}} &= \tfrac{(\Delta z)^2}{24} \frac{\partial^3 \tilde{C}^m_y}{\partial z^3}(p_{i,j+\frac{1}{2},k_3}, t^n) - \tfrac{(\Delta y)^2}{24} \frac{\partial^3 \tilde{C}^m_z}{\partial y^3}(p_{i,j_3,k+\frac{1}{2}}, t^n),\\
R^m_{d_{y_{i+\frac{1}{2},j,k+\frac{1}{2},n}}} &= \tfrac{(\Delta x)^2}{24} \frac{\partial^3 \tilde{C}^m_z}{\partial x^3}(p_{i_3,j,k+\frac{1}{2}}, t^n) - \tfrac{(\Delta z)^2}{24} \frac{\partial^3 \tilde{C}^m_x}{\partial z^3}(p_{i+\frac{1}{2},j,k_4}, t^n),\\
R^m_{d_{z_{i+\frac{1}{2},j+\frac{1}{2},k,n}}} &= \tfrac{(\Delta y)^2}{24} \frac{\partial^3 \tilde{C}^m_x}{\partial y^3}(p_{i+\frac{1}{2},j_4,k}, t^n) - \tfrac{(\Delta x)^2}{24} \frac{\partial^3 \tilde{C}^m_y}{\partial x^3}(p_{i_4,j+\frac{1}{2},k}, t^n),
\end{aligned} \tag{63}$$

where $p_{\alpha,\beta,\gamma} = (x_\alpha, y_\beta, z_\gamma)$, and for $1 \le l \le 4$, $x_{i_l} \in (x_{i-1/2}, x_{i+1/2})$, $y_{j_l} \in (y_{j-1/2}, y_{j+1/2})$ and $z_{k_l} \in (z_{k-1/2}, z_{k+1/2})$ are all constants.

The starting values for $\mathcal{C}^m_u$ and $\mathcal{D}^m_u(u = x, y, z)$ in (56)-(61) are:

$$\mathcal{C}^0_{x_{\bar{i},j,k,n}} = \mathcal{E}^n_{x_{\bar{i},j,k}},\ \ \mathcal{C}^0_{y_{i,\bar{j},k,n}} = \mathcal{E}^n_{y_{i,\bar{j},k}},\ \ \mathcal{C}^0_{z_{i,j,\bar{k},n}} = \mathcal{E}^n_{z_{i,j,\bar{k}}}, \tag{64}$$
$$\mathcal{D}^0_{x_{i,\bar{j},\bar{k},n}} = \mathcal{H}^n_{x_{i,\bar{j},\bar{k}}},\ \ \mathcal{D}^0_{y_{\bar{i},j,\bar{k},n}} = \mathcal{H}^n_{y_{\bar{i},j,\bar{k}}},\ \ \mathcal{D}^0_{z_{\bar{i},\bar{j},k,n}} = \mathcal{H}^n_{z_{\bar{i},\bar{j},k}}. \tag{65}$$

And the boundary conditions for $\mathcal{C}^m_u$ and $\mathcal{D}^m_u(u = x, y, z,\ 0 \le m \le M)$ are:

$$\mathcal{C}^m_{x_{\bar{i},j',k,n}} = C^m_{x_{\bar{i},j,k',n}} = \mathcal{C}^m_{y_{i',\bar{j},k,n}} = \mathcal{C}^m_{y_{i,\bar{j},k',n}} = \mathcal{C}^m_{z_{i',j,\bar{k},n}} = 0, \tag{66}$$
$$\mathcal{C}^m_{z_{i,j',\bar{k},n}} = \mathcal{D}^m_{x_{i',\bar{j},\bar{k},n}} = \mathcal{D}^m_{y_{\bar{i},j',\bar{k},n}} = \mathcal{D}^m_{z_{\bar{i},\bar{j},k',n}} = 0, \tag{67}$$

where $i' = 0$ or I, $j' = 0$ or J, $k' = 0$ or K, denoting the indices on the boundary.

Let $\boldsymbol{\mathcal{C}}_n^m = (\mathcal{C}_x^m, \mathcal{C}_y^m, \mathcal{C}_z^m)$, $\boldsymbol{\mathcal{D}}_n^m = (\mathcal{D}_x^m, \mathcal{D}_y^m, \mathcal{D}_z^m)$, $\mathbf{R}_c^m = (R_{c_x}^m, R_{c_y}^m, R_{c_z}^m)$ and $\mathbf{R}_d^m = (R_{d_x}^m, R_{d_y}^m, R_{d_z}^m)$. Then, the vector form of (54)-(61) is:

$$\boldsymbol{\mathcal{E}}^{n+1} = \sum_{m=0}^{M} \frac{(\Delta t)^m}{m!} \boldsymbol{\mathcal{C}}_n^m + (\Delta t)^{M+1} S_e^n, \ \boldsymbol{\mathcal{H}}^{n+1} = \sum_{m=0}^{M} \frac{(\Delta t)^m}{m!} \boldsymbol{\mathcal{D}}_n^m + (\Delta t)^{M+1} S_h^n,$$

$$\varepsilon \boldsymbol{\mathcal{C}}_n^{m+1} = \nabla^h \times \boldsymbol{\mathcal{D}}_n^m + \mathbf{R}_c^m, \quad \mu \boldsymbol{\mathcal{D}}_n^{m+1} = -\nabla^h \times \boldsymbol{\mathcal{C}}_n^m + \mathbf{R}_d^m. \tag{68}$$

4.2. Estimate of the remainders and the coefficients of error fields. From the remainders (62), (63) and (50) and Lemma 2.2, it can be proved that

Lemma 4.1. *For $0 \le n \le N$ and $0 \le m \le M$, let $\mathbf{S}_e^n$, $\mathbf{S}_h^n$, $\mathbf{R}_c^m$ and $\mathbf{R}_d^m$ be the remainders defined in (50), (62) and (63) respectively. Then,*

$$\nu \times \mathbf{S}_e^m = 0, \ \ \nu \times \mathbf{R}_c^m = 0, \ \ \nu \cdot \mathbf{S}_h^m = 0, \ \ \nu \cdot \mathbf{R}_d^m = 0, \tag{69}$$

$$\|\varepsilon^{-\frac{1}{2}} \mathbf{R}_c^m\|_e^2 + \|\mu^{-\frac{1}{2}} \mathbf{R}_d^m\|_h^2 \le O_{1eh}\Big((\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\Big), \tag{70}$$

$$\|\varepsilon^{-\frac{1}{2}} \nabla^h \times \mathbf{R}_c^m\|_h^2 + \|\mu^{-\frac{1}{2}} \nabla^h \times \mathbf{R}_d^m\|_h^2 \le O_{2eh}\Big((\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\Big), \tag{71}$$

$$\|\varepsilon^{\frac{1}{2}} \mathbf{S}_e^n\|_e^2 + \|\mu^{\frac{1}{2}} \mathbf{S}_h^n\|_h^2 \le O_{3eh}, \ \|\mu^{-\frac{1}{2}} \nabla^h \times \mathbf{S}_e^n\|_h^2 + \|\varepsilon^{-\frac{1}{2}} \nabla^h \times \mathbf{S}_h^n\|_e^2 \le O_{4eh}, \tag{72}$$

where O_{1eh}, O_{2eh}, O_{3eh} and O_{4eh} are constants, dependent of the norms of the derivatives of the exact solution (E_x, E_y, E_z) and (H_x, H_y, H_z).

Lemma 4.2. *Let $\boldsymbol{\mathcal{C}}_n^m$ and $\boldsymbol{\mathcal{D}}_n^m$ with $0 \le m \le M$ be the vectors of coefficients defined in* (56)-(67). *Then, for any $m = 0, 1, \ldots, M-1$, and $n \ge 1$, it holds that*

$$\|\varepsilon^{\frac{1}{2}} \boldsymbol{\mathcal{C}}_n^{m+1}\|_e^2 + \|\mu^{\frac{1}{2}} \boldsymbol{\mathcal{D}}_n^{m+1}\|_h^2 \le (O_{1eh} + O_{2eh})\Big((\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\Big)$$
$$+ \frac{4}{\mu\varepsilon}\Big(\frac{1}{4} + \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}\Big)\big(\|\varepsilon^{\frac{1}{2}} \boldsymbol{\mathcal{C}}_n^m\|_e^2 + \|\mu^{\frac{1}{2}} \boldsymbol{\mathcal{D}}_n^m\|_h^2\big). \tag{73}$$

By the method of proving Lemma 3.2, (73) is derived.

Lemma 4.3. *Let $\boldsymbol{\mathcal{C}}_n^m$ and $\boldsymbol{\mathcal{D}}_n^m$ be the vectors of coefficients of the error fields $\boldsymbol{\mathcal{E}}^{n+1}$ and $\boldsymbol{\mathcal{H}}^{n+1}$ for* HAIT-FDTD(M) *with $M = 3$. Then,*

$$(\varepsilon \boldsymbol{\mathcal{C}}_n^0, \boldsymbol{\mathcal{C}}_n^1)_e + (\mu \boldsymbol{\mathcal{D}}_n^0, \boldsymbol{\mathcal{D}}_n^1)_h = (\boldsymbol{\mathcal{C}}_n^0, \mathbf{R}_c^0)_e + (\boldsymbol{\mathcal{D}}_n^0, \mathbf{R}_d^0)_h; \tag{74}$$

$$(\varepsilon \boldsymbol{\mathcal{C}}_n^0, \boldsymbol{\mathcal{C}}_n^2)_e + (\mu \boldsymbol{\mathcal{D}}_n^0, \boldsymbol{\mathcal{D}}_n^2)_h = (\boldsymbol{\mathcal{C}}_n^0, \mathbf{R}_c^1)_e + (\boldsymbol{\mathcal{D}}_n^0, \mathbf{R}_d^1)_h + (\boldsymbol{\mathcal{C}}_n^1, \mathbf{R}_c^0)_e$$
$$+ (\boldsymbol{\mathcal{D}}_n^1, \mathbf{R}_d^0)_h - (\varepsilon \boldsymbol{\mathcal{C}}_n^1, \boldsymbol{\mathcal{C}}_n^1)_e - (\mu \boldsymbol{\mathcal{D}}_n^1, \boldsymbol{\mathcal{D}}_n^1)_h; \tag{75}$$

$$(\varepsilon \boldsymbol{\mathcal{C}}_n^0, \boldsymbol{\mathcal{C}}_n^3)_e + (\mu \boldsymbol{\mathcal{D}}_n^0, \boldsymbol{\mathcal{D}}_n^3)_h = (\boldsymbol{\mathcal{C}}_n^0, \mathbf{R}_c^2)_e + (\boldsymbol{\mathcal{D}}_n^0, \mathbf{R}_d^2)_h - (\boldsymbol{\mathcal{C}}_n^1, \mathbf{R}_c^1)_e$$
$$- (\boldsymbol{\mathcal{D}}_n^1, \mathbf{R}_d^1)_h + (\boldsymbol{\mathcal{C}}_n^2, \mathbf{R}_c^0)_e + (\boldsymbol{\mathcal{D}}_n^2, \mathbf{R}_d^0)_h; \tag{76}$$

$$(\varepsilon \boldsymbol{\mathcal{C}}_n^1, \boldsymbol{\mathcal{C}}_n^2)_e + (\mu \boldsymbol{\mathcal{D}}_n^1, \boldsymbol{\mathcal{D}}_n^2)_h = (\boldsymbol{\mathcal{C}}_n^1, \mathbf{R}_c^1)_e + (\boldsymbol{\mathcal{D}}_n^1, \mathbf{R}_d^1)_h; \tag{77}$$

$$(\varepsilon \boldsymbol{\mathcal{C}}_n^1, \boldsymbol{\mathcal{C}}_n^3)_e + (\mu \boldsymbol{\mathcal{D}}_n^1, \boldsymbol{\mathcal{D}}_n^3)_h = (\boldsymbol{\mathcal{C}}_n^1, \mathbf{R}_c^2)_e + (\boldsymbol{\mathcal{D}}_n^1, \mathbf{R}_d^2)_h + (\boldsymbol{\mathcal{C}}_n^2, \mathbf{R}_c^1)_e$$
$$+ (\boldsymbol{\mathcal{D}}_n^2, \mathbf{R}_d^1)_h - (\varepsilon \boldsymbol{\mathcal{C}}_n^2, \boldsymbol{\mathcal{C}}_n^2)_e - (\mu \boldsymbol{\mathcal{D}}_n^2, \boldsymbol{\mathcal{D}}_n^2)_h; \tag{78}$$

$$(\varepsilon \boldsymbol{\mathcal{C}}_n^2, \boldsymbol{\mathcal{C}}_n^3)_e + (\mu \boldsymbol{\mathcal{D}}_n^2, \boldsymbol{\mathcal{D}}_n^3)_h = (\boldsymbol{\mathcal{C}}_n^2, \mathbf{R}_c^2)_e + (\boldsymbol{\mathcal{D}}_n^2, \mathbf{R}_d^2)_h. \tag{79}$$

The proof of Lemma 4.3 is similar to that of Lemma 3.3, here omitted for shortness. The inner products of the vectors of coefficients for the other HAIT-FDTD(M) schemes with $M \ge 4$ are similarly derived.

4.3. Convergence analysis of HAIT-FDTD(M)

Theorem 4.4. *Let $\boldsymbol{\mathcal{E}}^{n+1}$ and $\boldsymbol{\mathcal{H}}^{n+1}$ be the error fields for* HAIT-FDTD(M) *with $3 \le M \le 8$. Then, it holds that*

$$\|\varepsilon^{\frac{1}{2}} \boldsymbol{\mathcal{E}}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}} \boldsymbol{\mathcal{H}}^{n+1}\|_h^2 \le O_{eh}\Big((\Delta t)^{2M} + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\Big), \tag{80}$$

where O_{eh} is a constant dependent of $O_{leh}(l = 1, 2, 3, 4)$, if the following convergence conditions for HAIT-FDTD(M)*(*$3 \leq M \leq 8$*), denoted by* $\mathrm{CC}_{M=*}$*, are respectively satisfied.*

$$\mathrm{CC}_{\mathrm{M}=3}: \ O_{cfl} \leq \frac{3}{4} - \frac{(\Delta t)^2}{4\mu\varepsilon}, \quad \mathrm{CC}_{\mathrm{M}=4}: \ O_{cfl} \leq 2 - \frac{(\Delta t)^2}{4\mu\varepsilon},$$
$$\mathrm{CC}_{\mathrm{M}=5}: \ O_{cfl} \leq \min\Big\{\frac{15}{4} - \frac{(\Delta t)^2}{4\mu\varepsilon}, \sqrt[3]{\frac{45r\Delta t}{8}} - \frac{(\Delta t)^2}{4\mu\varepsilon}\Big\},$$
$$\mathrm{CC}_{\mathrm{M}=6}: \ O_{cfl} \leq \min\Big\{6 - \frac{(\Delta t)^2}{4\mu\varepsilon}, \sqrt[4]{\frac{45}{4}r\Delta t} - \frac{(\Delta t)^2}{4\mu\varepsilon}\Big\},$$
$$\mathrm{CC}_{\mathrm{M}=7}: \ O_{cfl} \leq \frac{5}{8} - \frac{(\Delta t)^2}{4\mu\varepsilon}, \quad \mathrm{CC}_{\mathrm{M}=8}: \ O_{cfl} \leq \frac{9}{5} - \frac{(\Delta t)^2}{4\mu\varepsilon},$$

where $r > 0$ is small number.

Proof. First we consider the case: $M = 3$. By the expressions of $\mathcal{E}_u^{n+1}$ and $\mathcal{H}_u^{n+1}(u = x, y, z)$ in (54)-(55), we see that

$$\begin{aligned}
\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{E}}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{H}}^{n+1}\|_h^2 &= \sum_{m=0}^{3}\Big(\|\varepsilon^{\frac{1}{2}}\frac{(\Delta t)^m}{m!}\boldsymbol{\mathcal{C}}_n^m\|_e^2 + \|\mu^{\frac{1}{2}}\frac{(\Delta t)^m}{m!}\boldsymbol{\mathcal{D}}_n^m\|_h^2\Big) \\
&+ 2\Big(\frac{\Delta t}{1!}\Gamma_{01} + \frac{(\Delta t)^2}{2!}\Gamma_{02} + \frac{(\Delta t)^3}{3!}\Gamma_{03} + \frac{(\Delta t)^3}{2!}\Gamma_{12} + \frac{(\Delta t)^4}{3!}\Gamma_{13} + \frac{(\Delta t)^5}{2!3!}\Gamma_{23}\Big) \\
&+ 2(\Delta t)^4\Big(\big(\varepsilon\sum_{m=0}^{3}\frac{(\Delta t)^m}{m!}\boldsymbol{\mathcal{C}}_n^m, \mathbf{S}_e^n\big)_e + \big(\mu\sum_{m=0}^{3}\frac{(\Delta t)^m}{m!}\boldsymbol{\mathcal{D}}_n^m, \mathbf{S}_h^n\big)_h\Big) \\
&+ (\Delta t)^8\Big(\|\varepsilon^{\frac{1}{2}}\mathbf{S}_e^n\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{S}_h^n\|_h^2\Big),
\end{aligned} \tag{81}$$

where the terms $\Gamma_{pq}(p < q, \ p, q \in \{0, 1, 2, 3\})$ are

$$\begin{aligned}
\Gamma_{01} &= (\varepsilon\boldsymbol{\mathcal{C}}_n^0, \boldsymbol{\mathcal{C}}_n^1)_e + (\mu\boldsymbol{\mathcal{D}}_n^0, \boldsymbol{\mathcal{D}}_n^1)_h, \quad \Gamma_{02} = (\varepsilon\boldsymbol{\mathcal{C}}_n^0, \boldsymbol{\mathcal{C}}_n^2)_e + (\mu\boldsymbol{\mathcal{D}}_n^0, \boldsymbol{\mathcal{D}}_n^2)_h, \\
\Gamma_{03} &= (\varepsilon\boldsymbol{\mathcal{C}}_n^0, \boldsymbol{\mathcal{C}}_n^3)_e + (\mu\boldsymbol{\mathcal{D}}_n^0, \boldsymbol{\mathcal{D}}_n^3)_h, \quad \Gamma_{12} = (\varepsilon\boldsymbol{\mathcal{C}}_n^1, \boldsymbol{\mathcal{C}}_n^2)_e + (\mu\boldsymbol{\mathcal{D}}_n^1, \boldsymbol{\mathcal{D}}_n^2)_h, \\
\Gamma_{13} &= (\varepsilon\boldsymbol{\mathcal{C}}_n^1, \boldsymbol{\mathcal{C}}_n^3)_e + (\mu\boldsymbol{\mathcal{D}}_n^1, \boldsymbol{\mathcal{D}}_n^3)_h, \quad \Gamma_{23} = (\varepsilon\boldsymbol{\mathcal{C}}_n^2, \boldsymbol{\mathcal{C}}_n^3)_e + (\mu\boldsymbol{\mathcal{D}}_n^2, \boldsymbol{\mathcal{D}}_n^3)_h.
\end{aligned}$$

By using Lemma 4.3 and simplifying the right hand side of (81), we have

$$\begin{aligned}
\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{E}}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{H}}^{n+1}\|_h^2 &= \|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^0\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^0\|_h^2 \\
&- \frac{(\Delta t)^4}{12}\big(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^2\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^2\|_h^2\big) + \frac{(\Delta t)^6}{36}\Big(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^3\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^3\|_h^2\Big) \\
&+ \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + (\Delta t)^8\Big(\|\varepsilon^{\frac{1}{2}}\mathbf{S}_e^n\|_e^2 + \|\mu^{\frac{1}{2}}\mathbf{S}_h^n\|_h^2\Big),
\end{aligned} \tag{82}$$

where the terms $\Psi_k(k = 1, 2, 3, 4)$ are:

$$\begin{aligned}
\Psi_1 &= 2\Delta t\big((\boldsymbol{\mathcal{C}}_n^0, \Psi_{1a})_e + (\boldsymbol{\mathcal{D}}_n^0, \Psi_{1b})_h\big), \\
\Psi_2 &= (\Delta t)^2\big((\boldsymbol{\mathcal{C}}_n^1, \Psi_{2a})_e + (\boldsymbol{\mathcal{D}}_n^1, \Psi_{2b})_h\big), \\
\Psi_3 &= \frac{(\Delta t)^3}{3}\big((\boldsymbol{\mathcal{C}}_n^2, \Psi_{3a})_e + (\boldsymbol{\mathcal{D}}_n^2, \Psi_{3b})_h\big), \\
\Psi_4 &= \frac{(\Delta t)^7}{3}\big((\varepsilon\boldsymbol{\mathcal{C}}_n^3, \mathbf{S}_e^n)_e + (\mu\boldsymbol{\mathcal{D}}_n^3, \mathbf{S}_h^n)_h\big), \\
\Psi_{1a} &= \mathbf{R}_c^0 + \frac{\Delta t}{2}\mathbf{R}_c^1 + \frac{(\Delta t)^2}{6}\mathbf{R}_c^2 + (\Delta t)^3\varepsilon\mathbf{S}_e^n, \\
\Psi_{1b} &= \mathbf{R}_d^0 + \frac{\Delta t}{2}\mathbf{R}_d^1 + \frac{(\Delta t)^2}{6}\mathbf{R}_d^2 + (\Delta t)^3\mu\mathbf{S}_h^n,
\end{aligned}$$

$$\Psi_{2a} = \mathbf{R}_c^0 + \frac{2\Delta t}{3}\mathbf{R}_c^1 + \frac{(\Delta t)^2}{3}\mathbf{R}_c^2 + 2(\Delta t)^3\varepsilon\mathbf{S}_e^n,$$
$$\Psi_{2b} = \mathbf{R}_d^0 + \frac{2\Delta t}{3}\mathbf{R}_d^1 + \frac{(\Delta t)^2}{3}\mathbf{R}_d^2 + 2(\Delta t)^3\mu\mathbf{S}_h^n,$$
$$\Psi_{3a} = \mathbf{R}_c^0 + \Delta t\mathbf{R}_c^1 + \frac{(\Delta t)^2}{2}\mathbf{R}_c^2 + 3(\Delta t)^3\varepsilon\mathbf{S}_e^n,$$
$$\Psi_{3b} = \mathbf{R}_d^0 + \Delta t\mathbf{R}_d^1 + \frac{(\Delta t)^2}{2}\mathbf{R}_d^2 + 3(\Delta t)^3\mu\mathbf{S}_h^n.$$

By Lemmas 4.1 and 4.2, and summation by parts, we have

$$\Psi_1 \le \Delta t\big(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^0\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^0\|_h^2 + O_{eh}(\Delta_{xyz} + (\Delta t)^6)\big), \tag{83}$$

$$\Psi_2 \le \frac{(\Delta t)^2}{2}\Big(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^0\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^0\|_h^2 + O_{eh}(\Delta_{xyz} + (\Delta t)^6)\Big), \tag{84}$$

$$\begin{aligned}\Psi_3 \le\ & \Delta t O_{eh}\big(\Delta_{xyz} + (\Delta t)^6\big)\\ & + \Big(\frac{(\Delta t)^5}{4\mu\varepsilon} + (\Delta t)^3 O_{cfl}\Big)\big(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^0\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^0\|_h^2\big),\end{aligned} \tag{85}$$

$$\begin{aligned}\Psi_4 \le\ & \frac{(\Delta t)^3}{6}\Big(\frac{(\Delta t)^2}{\mu\varepsilon} + 4O_{cfl}\Big)^2\big(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^0\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^0\|_h^2\big)\\ & + \Delta t O_{eh}\big(\Delta_{xyz} + (\Delta t)^6\big),\end{aligned} \tag{86}$$

where O_{eh} be a generic constant which is dependent of O_{leh} with $l = 1, \dots, 4$ and different from each other in different places. Next, we consider the terms with $(\Delta t)^4$ and $(\Delta t)^6$ in (82). By Lemma 4.2, we have

$$\begin{aligned}& -\frac{\Delta t^4}{12}\big(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^2\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^2\|_h^2\big) + \frac{\Delta t^6}{36}\Big(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^3\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^3\|_h^2\Big) \le (\Delta t)^6 O_{eh}\Delta_{xyz}\\ & \quad -\frac{\Delta t^4}{12}\Big(1 - \frac{4}{3}\big(\frac{(\Delta t)^2}{4\mu\varepsilon} + O_{cfl}\big)\Big)\big(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{C}}_n^2\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{D}}_n^2\|_h^2\big).\end{aligned} \tag{87}$$

If $O_{cfl} \le 3/4 - (\Delta t)^2/(4\mu\varepsilon)$, substituting (83), (84), (85), (86) and (87) into (82), and using (72), we have

$$\begin{aligned}& \|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{E}}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{H}}^{n+1}\|_h^2 \le (1 + \beta_1\Delta t)(\|\varepsilon^{\frac{1}{2}}\boldsymbol{\mathcal{E}}^n\|_e^2 + \|\mu^{\frac{1}{2}}\boldsymbol{\mathcal{H}}^n\|_h^2)\\ & \quad + O_{eh}\Delta t\big((\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4 + (\Delta t)^6\big),\end{aligned} \tag{88}$$

where β_1 is a constant defined by

$$\beta_1 = 1 + \frac{\Delta t}{2} + \frac{(\Delta t)^2}{6}\Big(\frac{(\Delta t)^2}{\mu\varepsilon} + 4O_{cfl}\Big)^2 + (\Delta t)^2 O_{cfl} + \frac{(\Delta t)^4}{4\mu\varepsilon}.$$

By using (88) repeatedly, we obtain

$$\begin{aligned}\|\varepsilon^{\frac{1}{2}}\mathcal{E}^{n+1}\|_e^2 + \|\mu^{\frac{1}{2}}\mathcal{H}^{n+1}\|_h^2 \quad \le \quad & e^{T\beta_1}\big(\|\varepsilon^{\frac{1}{2}}\mathcal{E}^0\|_e^2 + \|\mu^{\frac{1}{2}}\mathcal{H}^0\|_h^2\big)\\ + \quad & O_{eh}\Big((\Delta t)^6 + (\Delta x)^4 + (\Delta y)^4 + (\Delta z)^4\Big).\end{aligned}$$

Thus, the error estimate (80) with $M = 3$ is obtained. Similarly, the other estimates (80) with $4 \le m \le 8$ are proved. This completes the proof of Theorem 4.4. □

5. Numerical experiments

In this section free-divergence property, stability, approximate energy conservation and error estimate of the solutions to the HAIT-FDTD schemes are tested by using a model problem with a known exact solution.

5.1. The model problem. We consider to solve the problem (1)-(5) with ε=μ=1 and $\Omega = [0,1]^3$, the exact solution: $\mathbf{E}(t) = (E_x, E_y, E_z)$ and $\mathbf{H}(t) = (H_x, H_y, H_z)$ as follows:

$$E_x = u_0(t)\cos(\pi x)\sin(\pi y)\sin(\pi z),\quad E_y = -2u_0(t)\sin(\pi x)\cos(\pi y)\sin(\pi z),$$
$$E_z = u_0(t)\sin(\pi x)\sin(\pi y)\cos(\pi z),\quad H_x = -\sqrt{3}\,v_0(t)\sin(\pi x)\cos(\pi y)\cos(\pi z),$$
$$H_y = 0,\ H_z = \sqrt{3}\,v_0(t)\cos(\pi x)\cos(\pi y)\sin(\pi z), u_0 = \cos(\sqrt{3}\pi t), v_0 = \sin(\sqrt{3}\pi t),$$

and the initial conditions are obtained by setting $t = 0$ in the solution expressions.

It is easy to check that the L^2 norm of the exact solution, denoted by EnL_2, is a constant, $\mathrm{EnL}_2(t) = \left(\|\varepsilon^{1/2}\mathbf{E}(t)\|^2 + \|\mu^{1/2}\mathbf{H}(t)\|^2\right)^{1/2} = \sqrt{3}/2,\ \ \forall\, t \geq 0.$

5.2. Test on free divergence. We compute error of divergence of fields by using the formulas:

$$\mathrm{Div}\mathbf{E}^n =: \max\{\varepsilon(\delta_x E_x^n + \delta_y E_y^n + \delta_z E_z^n)_{i,j,k}|_{i=1}^{I-1}|_{j=1}^{J-1}|_{k=1}^{K-1}\},$$
$$\mathrm{Div}\mathbf{H}^n =: \max\{\mu(\delta_x H_x^n + \delta_y H_y^n + \delta_z H_z^n)_{\bar{i},\bar{j},\bar{k}}|_{i=1}^{I-1}|_{j=1}^{J-1}|_{k=1}^{K-1}\}.$$

Table 1 gives the maximum values of $|\nabla^h \cdot \varepsilon\mathbf{E}^n|$ and $|\nabla^h \cdot \mu\mathbf{H}^n|$ computed by the HAIT-FDTD(M) schemes with $3 \leq M \leq 14$ and $\Delta x = \Delta y = \Delta z = \Delta t = 0.02$, $n\Delta t = 1$ for the model problem. From the data in the table we see that the errors of the divergences for HAIT-FDTD($M \geq 5$) are very close to zero, showing the consistence with Theorem 3.1. However, the values for the two cases: $M = 3$ and $M = 4$ are far away from zero. This inconsistence is caused by that the CFL number in the experiment, denoted by C_{cfl}: $(C_{cfl})^2 = O_{cfl} = 3$, breaking the stability conditions for HAIT-FDTD(M) with $M = 3(r_3 = 3/4)$ and $M = 4(r_4 = 2)$ (see Theorem 3.6).

TABLE 1. Maximum of $|\nabla^h \cdot \varepsilon\mathbf{E}^n|$ and $|\nabla^h \cdot \mu\mathbf{H}^n|$ over the grid points with $n\Delta t = 1$ for HAIT-FDTD(M): $3 \leq M \leq 14$.

	$M=3$	$M=4$	$M=5$	$M=6$	$M=7$	$M=8$
$\mathrm{Div}\mathbf{E}^n$	1.85e+8	6.64e-4	2.26e-13	2.54e-13	2.40e-13	2.48e-13
$\mathrm{Div}\mathbf{H}^n$	1.34e+8	7.32e-4	2.34e-13	2.71e-13	2.91e-13	2.89e-13
	$M=9$	$M=10$	$M=11$	$M=12$	$M=13$	$M=14$
$\mathrm{Div}\mathbf{E}^n$	3.18e-13	2.90e-13	3.18e-13	3.07e-13	3.37e-13	2.81e-13
$\mathrm{Div}\mathbf{H}^n$	3.33e-13	3.48e-13	4.10e-13	3.39e-13	3.32e-13	3.44e-13

If we select $\Delta t = h/2 = 0.01$, $n_1\Delta t = 1$ for HAIT-FDTD(M=3), and $\Delta t = 0.8h = 0.016$, $n_2\Delta t = 1$ for HAIT-FDTD(M=4) ($O_{cfl} = 3/4$, both satisfy the stability conditions in Theorem 3.6), then, we have

$$\text{HAIT-FDTD(3)}:\ \mathrm{Div}\mathbf{E}^{n_1} = 2.66e-13,\quad \mathrm{Div}\mathbf{H}^{n_1} = 2.83e-13,$$
$$\text{HAIT-FDTD(4)}:\ \mathrm{Div}\mathbf{E}^{n_2} = 2.24e-13,\quad \mathrm{Div}\mathbf{H}^{n_2} = 2.72e-13,$$

which is in accordance with free-divergence in Theorem 3.1.

In order to see the variation of divergence error in long time computation, we compute $\mathrm{Div}\mathbf{E}^n$ and $\mathrm{Div}\mathbf{H}^n$ by selecting t^n=$n\Delta t$ =T=4, 8, 10 and 12. Table 2 shows us the divergence error for HAIT-FDTD(M=8,10) with Δt=Δx=Δy=Δz=0.02. From these values we see that the HAIT-FDTD(M) schemes are numerically divergence free in long time computation. The experimental results for the other schemes are similar. However, the degree M for long time computation should be chosen relatively large.

5.3. Test on stability and approximate energy conservation. We use the schemes HAIT-FDTD(M) with $3 \leq M \leq 14$ and $\Delta x = \Delta y = \Delta z = \Delta t = h = 0.02$ to solve the model problem, and then work out the energy at $t = 0$ and $t = t^n$ by using the formulas: $\mathrm{En}^n = (\|\sqrt{\varepsilon}\mathbf{E}^n\|_e^2 + \|\sqrt{\mu}\mathbf{H}^n\|_h^2)^{1/2}$. The values En^n at $t^n = n\Delta t = 1$ and the difference $\mathrm{DEn}^n = \mathrm{En}^n - \mathrm{En}^0$ for these schemes are shown in Table 3.

Table 2. Divergence error Div$\mathbf{E}^n$ and Div$\mathbf{H}^n$at $t^n = n\Delta t = T$ with $T = 4, 8, 10, 12$ for HAIT-FDTD(M)($M = 8, 10$).

HAIT-FDTD(M=8)	$T=4$	$T=8$	$T=10$	$T=12$
Div$\mathbf{E}^n$	6.10e-13	8.03e-13	9.24e-13	9.59e-13
Div$\mathbf{H}^n$	6.09e-13	8.71e-13	9.41e-13	9.68e-13
HAIT-FDTD(M=10)	$T=4$	$T=8$	$T=10$	$T=12$
Div$\mathbf{E}^n$	5.92e-13	9.04e-13	1.02e-12	1.12e-12
Div$\mathbf{H}^n$	6.59e-13	9.12e-13	1.10e-12	1.17e-12

Table 3. Energy values En^n at $t^n = n\Delta t = 1$ and their difference from En^0=0.8660254 for HAIT-FDTD(M)($3 \le M \le 14$).

	$M=3$	$M=4$	$M=5$	$M=6$	$M=7$	$M=8$
En^n	1.21e+21	3.44e+9	0.86603	0.86603	0.86603	0.86603
DEn^n	1.21e+21	3.44e+9	9.94e-8	1.48e-10	-2.10e-11	-2.68e-14
	$M=9$	$M=10$	$M=11$	$M=12$	$M=13$	$M=14$
En^n	0.86603	0.86603	0.86603	0.86603	0.86603	0.86603
DEn^n	-6.66e-15	7.99e-15	-7.88e-15	-7.88e-15	-8.11e-15	-7.99e-15

From Table 3 we see that the energy differences become smaller and smaller as M becomes larger; this is consistent with Theorem 3.4. The CFL number for these schemes is $\sqrt{3}$, showing that the stability conditions overcome the CFL stability condition ($C_{cfl} \le 1$). However, the values for the cases $M = 3$ and $M = 4$ are very large. This contradiction to Theorem 3.4 is caused by that $O_{cfl} = 3$ for the two schemes doesn't satisfy the stability conditions: $O_{cfl} \le 3/4(M = 3)$ and $O_{cfl} \le 2(M = 4)$.

If taking $\Delta t_1 = h/2 = 0.01$, $t^{n_1} = 1$ and Δt_2 =$0.8h = 0.016$, $t^{n_2} = 1$ for HAIT-FDTD(M) with M=3 and 4 respectively, then the energy differences are DEn^{n_1}=-3.158e-5, DEn^{n_2}=-1.621e-7. This agrees with the result in Theorem 3.4 that the squared energy differences are $O(\Delta t_1)^6 = O(10^{-6})$ and $O(\Delta t_2)^6 = O(10^{-8})$. In addition, the signs of DEn^{n_1}, DEn^{n_2}, and those appeared in Table 3 with $M \ge 5$ confirm Remark 3.6 that HAIT-FDTD(M)(M = 3, 4, 7, 8) are energy-decreasing, while the schemes with M = 5, 6 are energy-increasing.

The behavior of energy conservation in long time computation is checked by computing the energy difference DEn^n with t^n=$n\Delta t$=T=4, 8, 10 and 12. Table 4 gives us the

Table 4. Energy values En^n at $t^n = 4, 8, 10, 12$ and their difference from En^0=0.8660254 for HAIT-FDTD(M)($M = 8, 10$).

HAIT-FDTD(M=8)	$T=4$	$T=8$	$T=10$	$T=12$
DEn^n	-1.903e-13	-2.359e-13	-2.184e-13	-3.286e-13
HAIT-FDTD(M=10)	$T=4$	$T=8$	$T=10$	$T=12$
DEn^n	-1.403e-13	-2.454e-14	-3.086e-14	-1.843e-14

energy difference for the schemes HAIT-FDTD(M=8,10) with Δt=Δx=Δy=Δz=0.02. The results for the other cases are similar to those in Table 4. From these data we see that HAIT-FDTD(M) also preserves approximately the energy in long time computation.

5.4. Test on error estimates and convergence rates. Denote the relative error under the discrete L^2 norm by RErr, that is

$$\text{RErr} = \text{RErr}(h) = (\|\sqrt{\varepsilon}\mathcal{E}^n\|^2 + \|\sqrt{\mu}\mathcal{H}^n\|^2)^{\frac{1}{2}}/\text{EnL}_2(t^n).$$

The convergence order is computed by Order = $\log_2\big(\text{RErr(h)}/\text{RErr(h/2)}\big)$.

TABLE 5. Relative errors and convergence orders of HAIT-FDTD(M): $3 \leq M \leq 8$.

h	M=3		M=4		M=5	
	RErr	Order	RErr	Order	RErr	Order
0.04	3.5656e-3		3.5859e-3		3.5795e-3	
0.02	8.9417e-4	1.9954	8.9543e-4	2.0017	8.9503e-4	1.9998
0.01	2.2371e-4	1.9989	2.2379e-4	2.0004	2.2377e-4	1.9999
	M=6		M=7		M=8	
	RErr	Order	RErr	Order	RErr	Order
0.04	3.5796e-3		3.5796e-3		3.5796e-3	
0.02	8.9503e-4	1.9998	8.9503e-4	1.9998	8.9503e-4	1.9998
0.01	2.2376e-4	1.9999	2.2377e-4	1.9999	2.2377e-4	1.9999

We use HAIT-FDTD(M) with $3 \leq M \leq 8$ and $\Delta x = \Delta y = \Delta z = \Delta t = h$ to solve the model problem, and compute the relative errors and the convergence orders of the solutions $\mathbf{E}^n$ and $\mathbf{H}^n$ at $t^n = n\Delta t = 1$, which are displayed in Table 5. From the numbers in this table, we see that the convergence orders are close to 2 and have more and more significant digits as h decreases, and that the relative errors at each row change very little as M increases. This confirms Theorem 4.4 that the error bound of the solutions, $O\big((\Delta t)^M + h^2\big)$, is dominated by h when M is larger than 3.

6. Conclusions and remarks

In this paper, we established a rigorous analysis of the HAIT-FDTD(M) schemes on free divergence, energy conservation, stability, and convergence. This enhances the reliability of these schemes and tells us how to choose the time step size Δt when the polynomial degree M is less than 8. The error estimate in time $O((\Delta t)^M)$ also tells us that the value of M can also be selected by making $(\Delta t)^M$ less than the machine error. Selection of a large M does not bring much increase in workload and CPU time (see [22]) since the scheme is explicit and not involved in solutions of systems of equations. The rigorous analysis here can be extended to analyze the HAIT-FDTD(M) schemes in the other media and the new high order HAIT-FDTD(M) schemes by combining higher order space discretization with the Taylor expansion in time. This will be considered in future.

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