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FORCE CONVERGENCE FOR THE DE GENNES-CAHN-HILLIARD ENERGY

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Abstract. The degenerate de Gennes-Cahn-Hilliard (dGCH) equation is a recent phase field model that may more accurately approximate surface diffusion. After establishing the Gamma convergence of the dGCH energy in [10], in this paper, we study the convergence of boundary force. This is done by carefully crafting a nonlinear transformation that transforms the dGCH energy into a Cahn-Hilliard-type energy with a non-smooth potential. We carry out explicit computations and analysis to this new system, which in turn enables us to establish the convergence of boundary force for the dGCH energy.

Key words. de Gennes-Cahn-Hilliard energy, Gamma convergence, force convergence.

1. Introduction

In material science applications, especially in solid-state dewetting, motion by surface diffusion is governed by the surface Laplacian of mean curvature [24]. In general, diffuse interface approximations based on fourth-order nonlinear Cahn-Hilliard equations are widely used, and they formally converge to sharp interface models as the interface thickness approaches zero, see the classical paper [20] and later developments in various settings [1, 3, 4, 5, 13, 18, 21, 25, 27, 28]. While it was formally shown that the Cahn-Hilliard equation with a degenerate mobility may converge to surface diffusion for the case of a double barrier potential or the case of a logarithmic potential with the temperature also approaching zero [3], it is not so in the case of a polynomial potential. In fact, there are unintended bulk diffusions if the degeneracy in the mobility is not strong enough [4, 5, 15, 16, 26]. See also related theoretical and numerical results [6, 7].

The diffuse interface model for surface diffusion proposed in [21], known as the doubly degenerate Cahn-Hilliard (DDCH) equation, introduces an additional degeneracy similar to approaches used in classical phase-field models for solidification [14], improving the accuracy of surface diffusion approximations without altering the asymptotic limit [2, 27]. However, this model is non-variational, lacking a known free energy dissipation, which complicates the analysis of solution properties, numerical stability, and its extension to complex multi-physics applications [21]. In [23], the variational model was proposed together with the free energy $E^{\varepsilon}_{\mathrm{dGCH}}$. This model has also recently attracted attention, in particular about questions related to its Gamma convergence [10] and the characterization of the minimizers [9]. In this work, we are concerned with force convergence of $E^{\varepsilon}_{\mathrm{dGCH}}$.

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The degenerate de Gennes-Cahn-Hilliard (dGCH) equation

$$\partial_t u = \frac{1}{\varepsilon}\nabla\cdot(M_0(u)\nabla\mu), \tag{1}$$

$$\mu = -\varepsilon\nabla\cdot\left(\frac{\nabla u}{g(u)}\right) + \frac{1}{\varepsilon g(u)}W'(u) - \frac{g'(u)}{g^2(u)}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right) \tag{2}$$

is a variational diffuse interface model that may more accurately approximate surface diffusion [23]. It can formally be interpreted as a weighted H^{-1} gradient flow for the dGCH energy

$$E^\varepsilon_{\mathrm{dGCH}}(u) := \int_\Omega \frac{1}{g(u)}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right)dx, \qquad \text{for all } u\in H^1(\Omega). \tag{3}$$

Here u is the relative concentration of the two phases, and $\mu = \delta_u E^\varepsilon_{\mathrm{dGCH}}$ is the chemical potential, defined by the variational derivative of $E^\varepsilon_{\mathrm{dGCH}}$ with respect to u. The double well potential W is taken to be smooth with two equal minima at $u^\pm$, corresponding to the two "pure" phases. In this paper we concentrate on the following smooth quartic potential

$$W(u) = (u-u^+)^2(u-u^-)^2. \tag{4}$$

The parameter $\varepsilon > 0$ is proportional to the thickness of the transition region between the two phases. The mobility $M_0(u)$ is degenerate at $u^\pm$.

This energy functional included a singularity due to the de Gennes coefficient $\frac{1}{g}$, where g is a function that is degenerate at $u^\pm$ (see, e.g., [11, 19]). A natural choice is the form

$$g(u) = |(u-u^-)(u-u^+)|^p, \quad p>0. \tag{5}$$

Intuitively, the singularities at $u^\pm$ may help to keep solutions confined in $[u^-, u^+]$. But the validity of this argument remains open.

To theoretically explore the systemic properties of the dGCH model, as a first step, we have studied the sharp interface limit of the dGCH energy as the thickness of the transition region goes to zero, by establishing its Gamma limit under the strong $L^1(\Omega)$ topology [10]. To be more precise, by extending the definition of $E^\varepsilon_{\mathrm{dGCH}}$ to all $u\in L^1(\Omega)$ by defining

$$E^\varepsilon_{\mathrm{dGCH}}(u) := \begin{cases} \int_\Omega \frac{1}{g(u)}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right)dx, & \text{if } u\in H^1(\Omega),\\ \infty & \text{otherwise,}\end{cases} \tag{6}$$

we proved that for $0<p\le 1$, $E^\varepsilon_{\mathrm{dGCH}}$ Γ-converges to

$$E^0_{\mathrm{dGCH}}(u) := \begin{cases} \sigma(p)\mathrm{Per}(A) & \text{if } u = u^- + (u^+-u^-)\chi_A \in BV(\Omega),\\ \infty & \text{otherwise}\end{cases} \tag{7}$$

as $\varepsilon\to 0$. Here χ_A is the characteristic function of a set A of finite perimeter, $\mathrm{Per}(A)$ is the perimeter of A, and

$$\sigma(p) = \sqrt{2}\int_{u^-}^{u^+} |(s-u^-)(s-u^+)|^{1-p}\,ds.$$

In other words, the Γ-limit of $E^\varepsilon_{\mathrm{dGCH}}$ is a multiple of the perimeter of the set A on which u takes the value u^+.

In this paper, we will introduce a nonlinear transformation to build a connection between the dGCH energy and a Cahn-Hilliard energy with a non-smooth double well potential, and state and prove Γ-convergence and force convergence of the new energy. Then, through the inverse transformation and an approximation argument, we establish the force convergence for the dGCH energy. In Section 2 we will describe the transformation and state the main results. In Section 3 we will prove the main results. We hope these convergence results will help us understand the relation between the dGCH equation and its sharp-interface limit and shed light on the relation between phase-field models and surface diffusion.

2. A nonlinear transformation and the main results

To overcome the technical difficulties and cumbersomeness caused by the de Gennes factor, we introduce the following nonlinear transformation

$$h(t) := \int_{\frac{u^- + u^+}{2}}^{t} \frac{1}{\sqrt{g(s)}} ds. \tag{8}$$

Since g equals 0 at $u^\pm$, $h(t)$ takes the form of a singular integral. As such, there may be issues regarding the existence of $h(t)$. If $p \geq 2$, then $h(t)$ blows up at $t = u^\pm$ but for $0 < p < 2$, $h(t)$ is a continuous and strictly increasing function for all $t \in \mathbb{R}$. Hence, h is invertible, and its inverse $h^{-1}(\cdot)$ is continuous and strictly increasing. For explicit formulas and more properties of $h(t)$, see the third author's dissertation [22]. A version of this transformation was used to prove the properties of the minimizers for the dGCH energy for $p = 1$ under strong anchoring conditions [9].

Lemma 1. *Suppose $0 < p < 2$. There exist positive constants C_1, C_2, C_3, C_4 such that for all $t \in \mathbb{R}$ we have*

$$|h(t)| \leq C_1|t| + C_2 \qquad \text{for all } t \in \mathbb{R}, \tag{9}$$

$$|h^{-1}(t)| \geq C_3|t| - C_4 \qquad \text{for all } t \in \mathbb{R}. \tag{10}$$

Proof. Notice there exist positive constants T and C such that $\frac{1}{\sqrt{g(t)}} \leq C$ for all $|t| \geq T$ and $\int_{-T}^{T} \frac{1}{\sqrt{g(t)}}\, dt < \infty$. Then

$$|h(t)| \leq \int_{-T}^{T} \frac{1}{\sqrt{g(t)}}\, dt \qquad \text{for } |t| < T,$$

$$|h(t)| \leq C(|t| - T) + \int_{-T}^{T} \frac{1}{\sqrt{g(t)}}\, dt \qquad \text{for } |t| \geq T.$$

This gives (9). Eq. (10) is a consequence of (9). □

Let $w(x) = h(u(x))$. Then for all $u \in C^1(\Omega)$ we have

$$\nabla w = \frac{\nabla u}{\sqrt{g(u)}} \qquad \text{for all } x \text{ such that } u(x) \neq u^\pm.$$

Hence formally the dGCH energy $E^\varepsilon_{\text{dGCH}}(u)$ can be written as a new energy functional $F^\varepsilon_{\text{dGCH}}(w)$ defined by

$$F^\varepsilon_{\text{dGCH}}(w) := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla w|^2 + \frac{W(h^{-1}(w))}{\varepsilon g(h^{-1}(w))} \right) dx. \tag{11}$$

To clean the integrand, let

$$\hat{W}(t) := \frac{W(h^{-1}(t))}{g(h^{-1}(t))} = |h^{-1}(t) - u^+|^{2-p} \cdot |h^{-1}(t) - u^-|^{2-p}, \tag{12}$$

then

$$F^{\varepsilon}_{\mathrm{dGCH}}(w) := \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla w|^2 + \frac{\hat{W}(w)}{\varepsilon} \right) dx. \tag{13}$$

Since $\hat{W}$ is non-negative, $F^{\varepsilon}_{\mathrm{dGCH}}$ is well defined for all $w \in H^1(\Omega)$.

A couple of remarks should be made. First, $F^{\varepsilon}_{\mathrm{dGCH}}(w)$ has the same form as the classical Cahn-Hilliard equation, and the potential function $\hat{W}$ has two equal global minimizers at $w^{\pm} = h(u^{\pm})$, but is not C^2 at $w^{\pm}$. The other thing is that under the change of variables $w = h(u)$, if the chain rule holds, then $\nabla w = \frac{\nabla u}{\sqrt{g(u)}}$ and $F^{\varepsilon}_{\mathrm{dGCH}}(h(u)) = E^{\varepsilon}_{\mathrm{dGCH}}(u)$. However, since h is not Lipschitz, we cannot claim $h(u)$ is $H^1(\Omega)$ and do not expect the chain rule holds for all $u \in H^1(\Omega)$.

2.1. Γ-Convergence for $F^{\varepsilon}_{\mathrm{dGCH}}$ ($0 < p \leq 3/2$). Since the proper topological setting for the Γ-convergence of Cahn-Hilliard type energies is $L^1(\Omega)$, we need to extend the definition of $F^{\varepsilon}_{\mathrm{dGCH}}(w)$ to $L^1(\Omega)$:

$$F^{\varepsilon}_{\mathrm{dGCH}}(w) := \begin{cases} \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla w|^2 + \frac{1}{\varepsilon} \hat{W}(w) \right) dx, & \text{if } w \in H^1(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{14}$$

Theorem 2. *Suppose $0 < p \leq 3/2$. The Γ-limit of $F^{\varepsilon}_{\mathrm{dGCH}}$ is a multiple of the perimeter of the set A on which w takes the value w^+. To be precise, the Γ-limit of $F^{\varepsilon}_{\mathrm{dGCH}}$ under the strong $L^1(\Omega)$ topology is*

$$F^{0}_{\mathrm{dGCH}}(w) := \begin{cases} \sigma(p)\mathrm{Per}(A) & \textit{if } w = w^- + (w^+ - w^-)\chi_A \in BV(\Omega), \\ \infty & \textit{otherwise.} \end{cases} \tag{15}$$

Here χ_A is the characteristic function of a set A of finite perimeter, $\mathrm{Per}(A)$ *is the perimeter of A, and $\sigma(p) = \int_{w^-}^{w^+} \sqrt{2\hat{W}(s)}\, ds = \sqrt{2} \int_{u^-}^{u^+} |(t - u^-)(t - u^+)|^{1-p/2}\, dt$.*

Proof. By (10) and (12) we see that for $0 < p \leq 3/2$, $\hat{W}(w)$ grows at least linearly in w as $|w| \to \infty$. Then $F^{\varepsilon}_{\mathrm{dGCH}}$ falls into the category of situations considered in [12] (see also [17]) and we immediately obtain the Γ-limit of $F^{\varepsilon}_{\mathrm{dGCH}}$. □

Remark 3. *The Γ-convergence result for $F^{\varepsilon}_{\mathrm{dGCH}}$ in Theorem 2 is stronger than what we obtained for $E^{\varepsilon}_{\mathrm{dGCH}}$ in* [10] *in the sense that the former holds for $0 < p \leq 3/2$ while the latter holds for $0 < p \leq 1$. We want to mention that due to technical reasons, the approach in* [10] *does not apply for $p > 1$.*

2.2. Force Convergence for $F^{\varepsilon}_{\mathrm{dGCH}}$. Since $F^{\varepsilon}_{\mathrm{dGCH}}$ a Cahn-Hilliard energy with a non-smooth potential, it is expected that results about force convergence in [8] can carry over. We include them for completeness and for comparison with the force convergence results for $E^{\varepsilon}_{\mathrm{dGCH}}$ that will be stated in Section 2.3.

Before going further, we first go over some terminology. The divergence of an n dimensional tensor eld $T = (T_{i,j})$, denoted $\nabla \cdot T$ or div T, is the vector eld with components $\sum_{j=1}^{n} \partial_j T_{i,j}$ for $i = 1, 2, ...n$. Let $V : \Omega \to \mathbb{R}^n$ be a dierentiable vector eld with components V_i ($i = 1, 2, \cdots, n$), then the gradient ∇V is the matrix-valued function with the (i, j) components $\partial_j V_i$. For any $n \times n$ matrices A and B, we denote the Frobenius inner product by $A : B = \sum_{i,j=1}^{n} A_{i,j} B_{i,j}$ and the Frobenius norm by $|A| = \sqrt{A : A} = \left(\sum_{i,j=1}^{n} A_{i,j}^2 \right)^{1/2}$.

The convergence of variational forces for phase field models to the variational forces for the corresponding sharp interface models was studies in [8]. Using the same setting, the variational force for the phase field energy $F^{\varepsilon}_{\text{dGCH}}(w)$ is defined by the negative of the variational derivative $\delta_w F^{\varepsilon}_{\text{dGCH}}$ multiplied by $-\nabla w$. This gives

$$\hat{f}_{\varepsilon}(w) := \left[-\varepsilon\Delta w + \frac{1}{\varepsilon}\hat{W}'(w)\right]\nabla w, \quad \text{if } w \in H^2(\Omega). \tag{16}$$

From [8], defining the tensor for the free energy $F^{\varepsilon}_{\text{dGCH}}(w)$ as follows:

$$\hat{T}_{\varepsilon}(w) = \left[\frac{\varepsilon}{2}|\nabla w|^2 + \frac{1}{\varepsilon}\hat{W}(w)\right] I - \varepsilon\nabla w \otimes \nabla w, \quad \text{if } w \in H^1(\Omega), \tag{17}$$

then $\hat{f}(w)$ can be written as the divergence of $\hat{T}_{\varepsilon}(w)$ for all $w \in H^2(\Omega)$, i.e., $\hat{f}_{\varepsilon}(w) = \nabla \cdot \hat{T}_{\varepsilon}(w)$ if $w \in H^2(\Omega)$.

For the sharp interface energy $F^0_{\text{dGCH}}(A)$, where A is a set of finite perimeter, let H be the mean curvature of ∂A and ν be the outer normal of ∂A. Then its variational force is

$$\hat{f}_0[\partial A] = -(n-1)\sigma(p)H\nu. \tag{18}$$

The corresponding sharp interface tensor is

$$\hat{T}_0[\partial A] = \sigma(p)(I - \nu \otimes \nu).$$

The results in [8] about force convergence for the Cahn-Hilliard energy immediately carry over, and we have the following lemmas and theorems.

Lemma 4. *For any $V \in C^1_c(\Omega, \mathbb{R}^n)$,*

$$\int_{\Omega} \hat{f}_{\varepsilon}(w) \cdot V = -\int_{\Omega} \hat{T}_{\varepsilon}(w) : \nabla V \quad \textit{if } w \in H^2(\Omega),. \tag{19}$$

Assume that the boundary of A is C^2. Let ν be the unit normal pointing from A to A^c. For $V \in C^1(\Omega, \mathbb{R}^n)$,

$$\int_{\partial A} \hat{f}_0[\partial A] \cdot V dS = -\sigma(p)\int_{\partial A} (I - \nu \otimes \nu) : \nabla V dS. \tag{20}$$

Theorem 5. *Let ε_k be a sequence of strictly decreasing positive numbers that converges to 0. Assume A is a set of finite perimeter such that $\bar{A} \subset \Omega$ and that $w_k \in H^1(\Omega)$ is a sequence that converges to $w = w^- + (w^+ - w^-)\chi_A$ a.e. in Ω. In addition, assume $\lim_{k\to\infty} F^{\varepsilon_k}_{\text{dGCH}}(w_k) = F^0_{\text{dGCH}}(\partial A)$. Then $\hat{f}_{\varepsilon_k}(w_k)$ weakly converges to $\hat{f}_0[\partial A]$ in the following sense: for any $V \in C^1_c(\Omega, \mathbb{R}^n)$,*

$$\lim_{k\to\infty} \int_{\Omega} \hat{T}_{\varepsilon_k}(w_k) : \nabla V = \sigma(p)\int_{\partial^* A} (I - \nu \otimes \nu) : \nabla V dS. \tag{21}$$

Here $\partial^* A$ is the reduced boundary of A.

2.3. Force convergence for $E^{\varepsilon}_{\text{dGCH}}$. Following the same setting as in [8] and Section 2.2, the variational force for the phase field energy $E^{\varepsilon}_{\text{dGCH}}(u)$ is defined by the negative of the variational derivative $\delta_u E^{\varepsilon}_{\text{dGCH}}$ multiplied by $-\nabla u$. This gives

$$f_{\varepsilon}(u) = \left[-\frac{\varepsilon\Delta u}{g(u)} + \frac{\varepsilon|\nabla u|^2}{2g(u)^2} + \frac{W'(u)}{\varepsilon g(u)} - \frac{g'(u)W(u)}{\varepsilon g(u)^2}\right]\nabla u \quad \text{if } u \in H^2(\Omega).$$

The tensor for $E^\varepsilon_{\text{dGCH}}(u)$ is

$$T_\varepsilon(u) = \left[\frac{\varepsilon|\nabla u|^2}{2g(u)} + \frac{W(u)}{\varepsilon g(u)}\right] I - \frac{\varepsilon}{g(u)}\nabla u \otimes \nabla u, \quad \text{if } u \in H^1(\Omega). \tag{22}$$

Then $f_\varepsilon(u)$ is the divergence of $T_\varepsilon(u)$ for $u \in H^2(\Omega)$. To show this, we need only carry out the following component wise computation. Using the summation convention and Kronecker delta notation δ_{ij}, the ith component of $\nabla \cdot T_\varepsilon(u)$ is

$$\begin{aligned}
\partial_j (T_\varepsilon(u))_{ij} =& \partial_j \left\{ \left[\frac{\varepsilon}{2}\sum_{k=1}^n \partial_k u \partial_k u + \frac{1}{\varepsilon} W(u)\right] \frac{\delta_{ij}}{g(u)} - \frac{\varepsilon}{g(u)}\partial_i \partial_j u \right\} \\
=& \left[\varepsilon \sum_{k=1}^n \partial_{jk} u \partial_k u + \frac{1}{\varepsilon} W'(u)\partial_j u\right] \frac{\delta_{ij}}{g(u)} \\
&+ \left[\frac{\varepsilon}{2}\sum_{k=1}^n \partial_k u \partial_k u + \frac{1}{\varepsilon} W(u)\right] \frac{-\delta_{ij} g'(u)}{g(u)^2}\partial_j u \\
&- \frac{\varepsilon}{g(u)}\partial_i u \partial_{jj} u - \frac{\varepsilon}{g(u)}\partial_{ij} u \partial_j u + \frac{\varepsilon g'(u)}{g(u)^2}\partial_i u (\partial_j u)^2 \\
=& \left[-\frac{\varepsilon \Delta u}{g(u)} + \frac{\varepsilon |\nabla u|^2}{2g(u)^2} + \frac{W'(u)}{\varepsilon g(u)} - \frac{g'(u) W(u)}{\varepsilon g(u)^2}\right]\partial_i u.
\end{aligned}$$

This is the ith component of the force.

The sharp interface force for E^0_{dGCH} is

$$f_0[\partial A] = -(n-1)\sigma(p) H \nu, \tag{23}$$

which has the same form as that for F^0_{dGCH}, which is not surprising since for a given set A, if we let $u = u^- + (u^+ - u^-)\chi_A$ and $w = h(u)$, then $w = w^- + (w^+ - w^-)\chi_A$ and $E^0_{\text{dGCH}}(u) = F^0_{\text{dGCH}}(w)$.

It is natural for us to expect a weak convergence result about $f_\varepsilon(u)$ to $f_0[\partial A]$, in the same format as Theorem 5.

Theorem 6. *Let ε_k be a sequence of strictly decreasing positive numbers that converges to 0. Assume $A \subset \bar{A} \subset \Omega$ has finite perimeter and that $u_k \in H^1(\Omega)$ is a sequence that converges to $u = u^- + (u^+ - u^-)\chi_A$ a.e. in Ω. In addition, assume $\lim_{k\to\infty} E^{\varepsilon_k}_{\text{dGCH}}(u_k) = E^0_{\text{dGCH}}(u)$. Then $f_{\varepsilon_k}(u_k)$ weakly converges to $f_0[\partial A]$ in the following sense: for any $V \in C^1_c(\Omega, \mathbb{R}^n)$,*

$$\lim_{k\to\infty} \int_\Omega T_{\varepsilon_k}(u_k) : \nabla V = \sigma(p) \int_{\partial^* A} (I - \nu \otimes \nu) : \nabla V dS. \tag{24}$$

We will prove this theorem in Section 3.

3. Proof of force convergence for $E^\varepsilon_{\text{dGCH}}$

3.1. A lemma about approximation. Since h is not Lipschitz, for any $u \in H^1(\Omega)$ we cannot claim $w := h(u)$ is in $H^1(\Omega)$ and hence cannot claim $F^\varepsilon_{\text{dGCH}}(w) = E^\varepsilon_{\text{dGCH}}(u)$. However, we can show that this is *almost* true. Let α_j be a strictly decreasing positive sequence that converges to 0. Define

$$h_j(t) = \int_{\frac{u^- + u^+}{2}}^t \frac{1}{\sqrt{g(s) + \alpha_j^2}}\, ds. \tag{25}$$

Lemma 7. *For any $u \in H^1(\Omega)$ such that $E^{\varepsilon}_{\mathrm{dGCH}}(u) < \infty$, define $w = h(u)$ and $w_j = h_j(u)$. Then $w_j \in H^1(\Omega)$ and $w_j \to w$ a.e. in Ω and strongly in $L^q(\Omega)$ for any $1 \le q < 2^* := 2n/(n-2)$ if $n \ge 3$ and any $1 \le q < \infty$ if $n = 2$. In addition, define*

$$\Omega_+(u) := \{x \in \Omega : u > (u^- + u^+)/2\}, \tag{26}$$

$$\Omega_-(u) := \{x \in \Omega : u < (u^- + u^+)/2\}, \tag{27}$$

$$\Omega_0(u) := \{x \in \Omega : u = (u^- + u^+)/2\}. \tag{28}$$

Then

(a) *In Ω_+, $0 < w_j < w$ and w_j is monotonically increasing and converges to w point wise.*
(b) *In Ω_-, $w < w_j < 0$ and w_j is monotonically decreasing and converges to w point wise.*
(c) *In Ω_0, $w_j = w = 0$.*
(d) $\int_\Omega |\nabla w_j|^2\,dx \to \int_\Omega \frac{|\nabla u|^2}{g(u)}\,dx, \quad \int_\Omega \hat{W}(w_j)\,dx \to \int_\Omega \hat{W}(w)\,dx.$
(e) $F^{\varepsilon}_{\mathrm{dGCH}}(w_j) \to E^{\varepsilon}_{\mathrm{dGCH}}(u)$ *as $j \to \infty$.*
(f) *For any $V \in C^1_c(\Omega;\mathbb{R}^n)$ we have*

$$\int_\Omega \hat{T}_\varepsilon(w_j) : \nabla V\,dx \to \int_\Omega T_\varepsilon(u) : \nabla V\,dx \quad \text{as } j \to \infty. \tag{29}$$

Proof. (a-c) are trivial. In addition, by (9),

$$|w_j| = |h_j(u)| \le |h(u)| \le C_1|u| + C_2.$$

So $w_j \in L^2(\Omega)$ and $\|w_j\|_{L^2(\Omega)} \le C_1\|u\|_{L^2(\Omega)} + C_2|\Omega|^{1/2}$. Since $|h_j'| \le \frac{1}{\alpha_j}$, we see that w_j is weakly differentiable and $\nabla w_j = \frac{\nabla u}{\sqrt{g(u)+\alpha_j^2}}$. Then

$$\int_\Omega |\nabla w_j|^2\,dx = \int_\Omega \frac{|\nabla u|^2}{g(u)+\alpha_j^2}\,dx \le \int_\Omega \frac{|\nabla u|^2}{g(u)}\,dx < \infty. \tag{30}$$

Hence $w_j \in H^1(\Omega)$ and is bounded in $H^1(\Omega)$. By the Compact Embedding Theorem, for any $1 \le q < 2^* := 2n/(n-2)$ if $n \ge 3$ and any $1 \le q < \infty$ if $n = 2$, there exists a subsequence w_{j_l} that strongly converges to a function $\tilde{w}$ in $L^q(\Omega)$. But since $w_j \to w$ a.e., we have $\tilde{w} = w$ a.e. and hence the whole sequence w_j strongly converges to w in $L^q(\Omega)$.

(d) By the Monotone Convergence Theorem (or Dominated Convergence Theorem), taking limit as $j \to \infty$ of the first two terms of (30) gives

$$\int_\Omega |\nabla w_j|^2\,dx \to \int_\Omega \frac{|\nabla u|^2}{g(u)}\,dx \quad \text{as } j \to \infty.$$

Since h^{-1} is a continuous and strictly monotonically increasing function, in Ω_+ we have $h^{-1}(w_j) < h^{-1}(w)$. For $t > u^+$ it is easy to see that for $0 < p < 2$

$$\frac{W(t)}{g(t)} = |t - u^+|^{2-p} \cdot |t - u^-|^{2-p} = (t - u^+)^{2-p} \cdot (t - u^-)^{2-p}$$

is increasing. Let $M_1 := \max\left\{\frac{W(t)}{g(t)} : \frac{u^- + u^+}{2} \le t \le u^+\right\}$.

Since $\hat{W}(w_j) = \frac{W(h^{-1}(w_j))}{g(h^{-1}(w_j))}$, for any $x \in \Omega_+$ and any index j, either

(i) $0 < h^{-1}(w_j(x)) \le u^+$ and hence $0 \le \hat{W}(w_j) \le M_1$; or
(ii) $u^+ < h^{-1}(w_j(x)) < h^{-1}(w(x))$ and hence $0 \le \hat{W}(w_j) < \hat{W}(w)$.

Either way

$$\hat{W}(w_j) \le M_1 + \hat{W}(w) \quad \text{for all } j \text{ and all } x \in \Omega_+. \tag{31}$$

So by the Dominated Convergence Theorem, we have

$$\int_{\Omega_+} \hat{W}(w_j)\,dx \to \int_{\Omega_+} \hat{W}(w)\,dx \quad \text{as } j \to \infty.$$

A similar argument in Ω_- gives

$$\int_{\Omega_-} \hat{W}(w_j)\,dx \to \int_{\Omega_-} \hat{W}(w)\,dx \quad \text{as } j \to \infty.$$

Summing these two limits up and using $w_j = w$ in Ω_0, we obtain

$$\int_{\Omega} \hat{W}(w_j)\,dx \to \int_{\Omega} \hat{W}(w)\,dx \quad \text{as } j \to \infty.$$

(e) This is an immediate consequence of (d).

(f) Since $\hat{W}(w) = W(u)/g(u)$ we have

$$\begin{aligned}
\hat{T}_\varepsilon(w_j) &= \left(\frac{\varepsilon}{2}|\nabla w_j|^2 + \frac{1}{\varepsilon}\hat{W}(w_j)\right) I - \varepsilon \nabla w_j \otimes \nabla w_j \\
&= \left(\frac{\varepsilon|\nabla u|^2}{2(g(u)+\alpha_j^2)} + \frac{\hat{W}(w_j)}{\varepsilon}\right) I - \frac{\varepsilon \nabla u \otimes \nabla u}{g(u)+\alpha_j^2} \\
&\to \left(\frac{\varepsilon|\nabla u|^2}{2g(u)} + \frac{\hat{W}(w)}{\varepsilon}\right) I - \frac{\varepsilon \nabla u \otimes \nabla u}{g(u)} \quad \text{a.e. in } \Omega \\
&= T_\varepsilon(u) \quad \text{a.e. in } \Omega.
\end{aligned}$$

For any $V \in C_c^1(\Omega)$, there exists a constant $M_2 > 0$ such that $|\nabla V| \le M_2$ for all $x \in \Omega$. Then for $x \in \Omega_+$, we have

$$\begin{aligned}
&\left|\left(\frac{\varepsilon}{2}|\nabla w_j|^2 + \frac{1}{\varepsilon}\hat{W}(w_j)\right) I : \nabla V\right| \\
\le & M_2 \left|\frac{\varepsilon}{2}|\nabla w_j|^2 + \frac{1}{\varepsilon}\hat{W}(w_j)\right| = M_2 \left(\frac{\varepsilon|\nabla u|^2}{2(g(u)+\alpha_j^2)} + \frac{\hat{W}(w_j)}{\varepsilon}\right) \\
\le & M_2 \left(\frac{\varepsilon|\nabla u|^2}{2g(u)} + M_1 + \frac{\hat{W}(w)}{\varepsilon}\right) \quad \text{by (31).}
\end{aligned}$$

In addition,

$$|(\varepsilon \nabla w_j \otimes \nabla w_j) : \nabla V| = M_2 \left|\frac{\varepsilon \nabla u \otimes \nabla u}{g(u)+\alpha_j^2}\right| \le n^2 M_2 \varepsilon \frac{|\nabla u|^2}{g(u)}. \tag{32}$$

So on Ω_+ we have

$$\left|\hat{T}_\varepsilon(w_j) : \nabla V\right| \le M_2 \left(\frac{\varepsilon|\nabla u|^2}{2g(u)} + M_1 + \frac{\hat{W}(w)}{\varepsilon} + \varepsilon \frac{|\nabla u|^2}{g(u)}\right)$$

We can get a similar bound on Ω_-. Since $w_j = w$ on Ω_0, we can apply the Dominated Convergence Theorem to obtain (f). □

3.2. Proof of Theorem 6. Now we prove Theorem 6. Suppose ε_k converges to 0, and $A \subset \bar{A} \subset \Omega$ has finite perimeter, and $u_k \in H^1(\Omega)$ converges to $u = u^- + (u^+ - u^-)\chi_A$ a.e. in Ω, and $\lim_{k\to\infty} E^{\varepsilon_k}_{\mathrm{dGCH}}(u_k) = E^0_{\mathrm{dGCH}}(u)$. Define $w := h(u)$. Then $h(u_k) \to w$ a.e. in Ω and $F^0_{\mathrm{dGCH}}(w) = E^0_{\mathrm{dGCH}}(u) < \infty$.

For each k, by Lemma 7, there exists a sequence $\{w_{k,j}, j = 1, 2, \dots\}$ in $H^1(\Omega)$ such that $w_{k,j} \to h(u_k)$ a.e. in Ω as $j \to \infty$ and satisfies (a)–(f) of Lemma 7 for u_k. By Egorov's theorem, there exists a subset $\Omega_k \subset \Omega$ such that $|\Omega_k| < \varepsilon_k$ and $w_{k,j} \to h(u_k)$ uniformly in $\Omega \setminus \Omega_k$. Hence there exists w_{k,j_k} such that

(i) $|w_{k,j_k} - h(u_k)| \le \varepsilon_k$ for all $x \in \Omega \setminus \Omega_k$;
(ii) $|F^{\varepsilon_k}_{\mathrm{dGCH}}(w_{k,j_k}) - E^{\varepsilon_k}_{\mathrm{dGCH}}(u_k)| \le \varepsilon_k$;
(iii) $\left| \int_\Omega \hat{T}_{\varepsilon_k}(w_{k,j_k}) : \nabla V \, dx - \int_\Omega T_{\varepsilon_k}(u_k) : \nabla V \, dx \right| \le \varepsilon_k$.

Since $h(u_k) \to h(u) = w$ a.e. in Ω, from (i) we conclude that $w_{k,j_k} \to w$ a.e. in Ω. From (ii) we conclude that

$$\lim_{k\to\infty} F^{\varepsilon_k}_{\mathrm{dGCH}}(w_{k,j_k}) = \lim_{k\to\infty} E^{\varepsilon_k}_{\mathrm{dGCH}}(u_k) = E^0_{\mathrm{dGCH}}(u) = F^0_{\mathrm{dGCH}}(w).$$

Then by Theorem 5, we have

$$\lim_{k\to\infty} \int_\Omega \hat{T}_{\varepsilon_k}(w_{k,j_k}) : \nabla V \, dx = \sigma(p) \int_{\partial^* A} (I - \nu \otimes \nu) : \nabla V dS.$$

However, by (iii) we have

$$\lim_{k\to\infty} \int_\Omega \hat{T}_{\varepsilon_k}(w_{k,j_k}) : \nabla V \, dx = \lim_{k\to\infty} \int_\Omega T_{\varepsilon_k}(u_k) : \nabla V \, dx.$$

So we get

$$\lim_{k\to\infty} \int_\Omega T_{\varepsilon_k}(u_k) : \nabla V \, dx = \sigma(p) \int_{\partial^* A} (I - \nu \otimes \nu) : \nabla V dS$$

and Theorem 6 is proved.

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