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AN ENERGY-DISSIPATION FINITE ELEMENT PRESSURE-CORRECTION SCHEME FOR THE HYDRODYNAMICS OF SMECTIC-A LIQUID CRYSTALS

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Abstract. In this paper, we develop a new linear, fully-decoupled, unconditional energy-stable BDF2-SAV-FEM scheme for solving the smectic-A liquid crystals, based on the finite element method (FEM) for spatial discretization and two-step backward differentiation formula (BDF2) for temporal discretization. To decouple the computations of the layer function and velocity field, we introduce an additional stabilization term into the constitutive equation. The nonlinear energy potential and the Navier-Stokes equations are treated by the scalar auxiliary variable (SAV) method and the rotational pressure-correction method, respectively. The unique solvability, unconditional energy stability, and error estimations of the proposed numerical scheme have been demonstrated. Several numerical experiments are carried out to validate our theoretical analysis.

Key words. Liquid crystals flows, unconditional energy stability, second-order, finite element method.

1. Introduction

Liquid crystals (LCs) are a novel intermediate state that appears in nature. Some substances in a molten state or after being dissolved by solvents, although losing the rigidity of solid substances, gain the fluidity of liquids and retain the anisotropic ordered arrangement of some crystalline substance molecules, forming an intermediate state that combines the properties of both crystals and liquids. This oriented ordered fluid that exists during the transition from solid to liquid is called liquid crystals (see [5, 11, 22]). LCs are widely used in daily life for a variety of applications, such as using them to indicate temperature and alarm toxic gases based on its color-changing characteristics, and using the optical properties of liquid crystals to make liquid crystal displays. Divide liquid crystals into thermotropic liquid crystals and lyotropic liquid crystals due to different conditions of liquid crystals production. Thermotropic liquid crystals include two essential types: the nematic phase and the smectic phase. The molecular arrangement of the smectic phase is arranged in layers, and the long axes of molecules within the layer are parallel to each other and perpendicular to the layer. There are many different smectic phases, which are characterized by different types and degrees of positional and orientational order(see [11, 48–50]). For example, in the smectic-A phase, the director of liquid crystals molecules is perpendicular to the smectic plane, which means that the direction is consistent with the layers' normal vector. But in the smectic-C phase, the arrangement of molecules deviates from the normal vector of the layer. This paper mainly studies the numerical approximation of the smectic-A phase.

Recently, there has been a lot of interest in liquid crystals with many of these works focused on studying the nematic phase, see [1, 17, 35, 43, 55, 57] and references therein. One of the most famous continuum theories for nematic liquid crystals

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is the Ericksen-Leslie model, which is derived by Ericksen and Leslie using the variational method for the Oseen-Frank energy [14, 15, 24, 25, 29]. For smectic-A liquid crystals, de Gennes and Prost proposed the first mathematical model in [11] by coupling two order parameters that, respectively, characterize the layer structure and the average direction of molecular alignment. In [2], the authors modified the model of de Gennes and Prost by adding a second-order gradient term for the smectic order parameter and studied the transition of nematic to smectic-A and smectic-C phases. In [23], the authors investigated the smectic-A liquid crystals by using the de Gennes energy to simulate the chevron pattern formed under the effect of an external magnetic field. In [12], by assuming that the director field is completely equal to the layer's gradient, the author succeeded to derive the simplified model of the smectic-A phase while reducing the free energy to one order parameter.

The mathematical analysis and numerical methods for the smectic-A liquid crystals are available in the literature. There are a lot of works for the theoretical analysis [9, 10, 26, 31, 53, 54]. For examples, the energy dissipative relation for the density-dependent system of the smectic-A liquid crystals were obtained and the existence of global weak solutions in the system was demonstrated in [26]. The authors studied a hydrodynamic system of smectic-A liquid crystals and analyzed the well-posedness as well as asymptotic behavior of strong solutions in [53]. From the view of computation, in [22] an unconditionally energy stable numerical scheme for the model of smectic-A liquid crystals was developed, which was first-order, linear, and decoupled time-marching. In [20], an energy-stable and second-order time-accurate numerical scheme was proposed for the smectic-A liquid crystals model, and numerical simulations were presented for 2D domains. The authors in [5] developed two linear, second-order time marching schemes for the system, one is the Crank-Nicolson scheme, another is the BDF2 scheme with the Adam-Bashforth explicit interpolation, further proved the well-posedness and unconditionally energy stability of numerical methods rigorously and used numerical experiments to validate the stability and accuracy of the schemes, but the schemes are partially decoupled and the error estimates of two schemes are missing. As far as we know, there fails to be any investigation on the convergence analysis of numerical methods for the smectic-A liquid crystals in the literature.

The discussed smectic-A phase model above is a highly nonlinear system that couples a constitutive equation for the layer function and the incompressible Navier-Stokes equations. For this system, there are several numerical difficulties: (i) the coupling of layer function and fluid velocity field; (ii) the existence of nonlinear terms; (iii) the coupling of fluid velocity field and pressure. This study is primarily focused on developing a second-order time-accurate fully discrete finite element scheme that can provide unconditional energy stability, full decoupling, and linearity. To overcome the first difficulty, we add an artificial stabilization term into the constitutive equation, which plays a significant role in decoupling the computations of the velocity field $\boldsymbol{u}$ and layer function ϕ, and use the implicit-explicit (IMEX) approach to handle other nonlinear coupling terms [36, 42, 44, 58]. For the nonlinear term $f(\nabla\phi)$, which brings a lot of difficulties when proving unconditionally energy stable and error estimates of the numerical scheme, the SAV method is applied to linearize this term [45–47]. Moreover, We decouple the velocity and pressure in the Navier-Stokes equations by applying the rotational pressure-correction method. Compared with the projection method [5, 22], the rotational pressure-correction method removes the need for artificial pressure boundary conditions. Eventually, the FEM is adopted for the spatial discretization. Combining these approaches allows us to construct an effective fully discrete scheme. The essential contribution of this work is the rigorous proof of the unconditional energy stability and error estimates of the designed second-order time-accurate scheme. Finally, the stability, accuracy and effectiveness of the proposed scheme are demonstrated using several numerical experiments.

The outline of this paper is as follows. In Section 2, we introduce the model system and the SAV method, deriving the unconditional energy stability at the continuous level. To solve the coupled system, a linear decoupling fully discrete scheme is developed in Section 3, whose unique solvability and unconditional energy stability at the discrete level are proved. In Section 4, we rigorously prove the error estimates of the proposed scheme. In Section 5, we present some numerical results to confirm our theoretical analysis. The final section provides a brief conclusion.

2. Preliminary

2.1. The smectic-A liquid crystals. The hydrodynamic system of the smectic-A phase liquid crystals (see [5, 12]) is derived based on the generalized Fick's law, which states that the mass flux is proportional to the gradient of the chemical potential [27]. In a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$, the smectic-A liquid crystals read as

$$\phi_t + \boldsymbol{u} \cdot \nabla\phi = -Mw, \tag{1a}$$

$$w = \frac{\delta E}{\delta \phi}, \tag{1b}$$

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p - w\nabla\phi = \boldsymbol{0}, \tag{1c}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1d}$$

where ϕ is the layer function, $\boldsymbol{u}$ is the fluid velocity field, p denotes the pressure, the parameter $\nu > 0$ represents the fluid viscosity and $M > 0$ is the elastic relaxation time. The regularized energy $E = E(\phi)$ is defined as

$$E(\phi) = \kappa \int_\Omega \left(\frac{1}{2}|\Delta\phi|^2 + F(\nabla\phi) \right) d\boldsymbol{x}, \qquad F(\nabla\phi) = \frac{1}{4\varepsilon^2}(|\nabla\phi|^2 - 1)^2,$$

where κ is the splay, twist, and bend elastic constant. For simplicity, we take $\kappa = 1$ below. And $\varepsilon \ll 1$ is a penalization parameter. The boundary and initial conditions are given by

$$\begin{cases} \boldsymbol{u} \mid_{\partial\Omega} = \boldsymbol{0}, \quad \frac{\partial\phi}{\partial\boldsymbol{n}} \mid_{\partial\Omega} = 0, \quad \frac{\partial(\Delta\phi)}{\partial\boldsymbol{n}} \mid_{\partial\Omega} = 0, \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0, \quad \phi(\boldsymbol{x}, 0) = \phi_0, \forall \boldsymbol{x} \in \Omega, \end{cases} \tag{2}$$

where $\boldsymbol{n}$ is the unit outward normal on the boundary $\partial\Omega$.

Introducing an auxiliary function $\psi = -\Delta\phi$, then the system (1) can be rewritten as:

$$\phi_t + \boldsymbol{u} \cdot \nabla\phi + Mw = 0, \tag{3a}$$

$$w + \Delta\psi + \nabla \cdot f(\nabla\phi) = 0, \tag{3b}$$

$$\psi + \Delta\phi = 0, \tag{3c}$$

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p - w\nabla\phi = \boldsymbol{0}, \tag{3d}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{3e}$$

where $f(\nabla\phi) = F'(\nabla\phi) = \frac{1}{\varepsilon^2}\nabla\phi(|\nabla\phi|^2 - 1)$. The boundary and initial conditions of the system (3) are

$$\begin{cases} \boldsymbol{u} \mid_{\partial\Omega} = \boldsymbol{0}, \quad \frac{\partial\phi}{\partial\boldsymbol{n}} \mid_{\partial\Omega} = 0, \quad \frac{\partial\psi}{\partial\boldsymbol{n}} \mid_{\partial\Omega} = 0, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0, \quad \phi(\boldsymbol{x},0) = \phi_0, \quad \forall \boldsymbol{x} \in \Omega. \end{cases} \tag{4}$$

Lemma 2.1. *Defined the energy $\mathcal{E}$ of the system (3) as*

$$\mathcal{E}(\boldsymbol{u},\phi) = \int_\Omega (\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\psi|^2 + F(\nabla\phi))d\boldsymbol{x}, \tag{5}$$

then, the system (3) satisfies the energy law as

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{u},\phi) = -\int_\Omega (M|w|^2 + \nu|\nabla\boldsymbol{u}|^2)d\boldsymbol{x} \le 0. \tag{6}$$

Proof. Take the time derivative for (3c) and we have

$$\psi_t + \Delta\phi_t = 0. \tag{7}$$

By taking L^2 inner-product of (3a), (3b), (7), and (3d) with w, ϕ_t, ψ, and $\boldsymbol{u}$, respectively. Using (3e), (4), and the fact that $((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{u}) = 0$, we have

$$\frac{d}{dt}\int_\Omega (\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\psi|^2 + F(\nabla\phi))d\boldsymbol{x} = -\int_\Omega (M|w|^2 + \nu|\nabla\boldsymbol{u}|^2)d\boldsymbol{x} \le 0. \tag{8}$$

From the definition of free energy $\mathcal{E}$ in (5), we have

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{u},\phi) = -\int_\Omega (M|w|^2 + \nu|\nabla\boldsymbol{u}|^2)d\boldsymbol{x} \le 0.$$

□

2.2. The SAV method. We adopt the SAV approach to construct the linearity and energy stability scheme. We define the new scalar variable $R(t)$ as

$$R(t) = \sqrt{E_0(\phi)} = \left(\int_\Omega F(\nabla\phi)d\boldsymbol{x} + \epsilon\right)^{1/2}, \quad F(\nabla\phi) = \frac{1}{4\varepsilon^2}(|\nabla\phi|^2 - 1)^2. \tag{9}$$

Here $\epsilon \ge 0$ is a fixed constant that is used to guarantee the energy variable $E_0(\phi)$ can always be nonnegative. Hence, the total energy (5) can be rewritten into quadratic form as

$$\mathcal{E}(\boldsymbol{u},\phi,R) = \int_\Omega (\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\psi|^2)d\boldsymbol{x} + |R(t)|^2 - \epsilon. \tag{10}$$

Taking the time derivative for (9) and we can get

$$R_t = \frac{1}{2\sqrt{E_0(\phi)}}\int_\Omega f(\nabla\phi)\cdot\nabla\phi_t d\boldsymbol{x}.$$

Moreover, we can rewrite the system (3) as

$$\phi_t + \boldsymbol{u}\cdot\nabla\phi + Mw = 0, \tag{11a}$$

$$w + \Delta\psi + \nabla\cdot\frac{R(t)}{\sqrt{E_0(\phi)}}f(\nabla\phi) = 0, \tag{11b}$$

$$\psi + \Delta\phi = 0, \tag{11c}$$

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p - w\nabla\phi = \boldsymbol{0}, \tag{11d}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{11e}$$

$$R_t = \frac{1}{2\sqrt{E_0(\phi)}}\int_\Omega f(\nabla\phi)\cdot\nabla\phi_t d\boldsymbol{x}, \tag{11f}$$

where $\frac{R(t)}{\sqrt{E_0(\phi)}} = 1$ is used here. Note that the initial condition of (11f) is $R(0) = \sqrt{E_0(\phi_0)}$.

Before giving the variational formulation of the SAV scheme (11), we first define some notations. We use $W^{s,p}(\Omega)$ and $L^p(\Omega)$ to represent the usual Sobolev spaces and Lebesgue spaces, respectively, for $0 \le s \le \infty$, $1 \le p \le \infty$. They are equipped with the norms $\|\cdot\|_{W^{s,p}}$ and $\|\cdot\|_{L^p}$. In particular, when $p = 2$ we represent the Hilbert spaces $W^{s,2}(\Omega)$ by $H^s(\Omega)$ with norm $\|\cdot\|_{H^s}$. The norm and inner-product of $L^2(\Omega)$ represent by $\|\cdot\|$ and $(\cdot,\cdot)$. We define the following spaces:

$$\begin{aligned} &Y = H^1(\Omega), \quad Q = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \, d\boldsymbol{x} = 0\}, \\ &\boldsymbol{X} = [H_0^1(\Omega)]^d = \{\boldsymbol{v} \in [H^1(\Omega)]^d : \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\}. \end{aligned}$$

Taking L^2-inner product of (11a), (11b), (11c), (11d), and (11e) with $\rho \in Y$, $\sigma \in Y$, $\varphi \in Y$, $\boldsymbol{v} \in \boldsymbol{X}$, and $q \in Q$, respectively, we have the variational formulation of the SAV scheme (11): find $(\phi, w, \psi, \boldsymbol{u}, p, R) \in Y \times Y \times Y \times \boldsymbol{X} \times Q \times \mathbb{R}$ such that

$$(\phi_t, \rho) + (\boldsymbol{u} \cdot \nabla\phi, \rho) + M(w, \rho) = 0, \qquad \forall \rho \in Y, \tag{12a}$$

$$(w, \sigma) - (\nabla\psi, \nabla\sigma) - (\frac{R(t)}{\sqrt{E_0(\phi)}} f(\nabla\phi), \nabla\sigma) = 0, \qquad \forall \sigma \in Y, \tag{12b}$$

$$(\psi, \varphi) - (\nabla\phi, \nabla\varphi) = 0, \qquad \forall \varphi \in Y, \tag{12c}$$

$$(\boldsymbol{u}_t, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) - (w\nabla\phi, \boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{12d}$$

$$(\nabla\cdot\boldsymbol{u}, q) = 0, \qquad \forall q \in Q, \tag{12e}$$

$$R_t = \frac{1}{2\sqrt{E_0(\phi)}} \int_\Omega f(\nabla\phi) \cdot \nabla\phi_t d\boldsymbol{x}. \tag{12f}$$

Note that the term $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w})$ in (12d) which equals to zero when $\boldsymbol{v} = \boldsymbol{w}$ for $\nabla \cdot \boldsymbol{u} = 0$.

Lemma 2.2. *The system* (12) *satisfies the energy law as*

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{u}, \phi, R) = -\int_\Omega (M|w|^2 + \nu|\nabla\boldsymbol{u}|^2) d\boldsymbol{x} \le 0. \tag{13}$$

Proof. Taking $\rho = w$, $\sigma = \phi_t$, and $\boldsymbol{v} = \boldsymbol{u}$ in (12a), (12b), and (12d), respectively, taking L^2 inner-product of (7) with ψ and multiplying (12f) by $2R$, we can derive

$$\begin{aligned} &(\phi_t, w) + (\boldsymbol{u} \cdot \nabla\phi, w) + M\|w\|^2 = 0, \\ &(w, \phi_t) - (\nabla\psi, \nabla\phi_t) - (f(\nabla\phi), \nabla\phi_t) = 0, \\ &(\psi_t, \psi) - (\nabla\phi_t, \nabla\psi) = 0, \\ &(\boldsymbol{u}_t, \boldsymbol{u}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) + \nu\|\nabla\boldsymbol{u}\|^2 - (p, \nabla\cdot\boldsymbol{u}) - (w\nabla\phi, \boldsymbol{u}) = 0, \\ &\frac{d}{dt}|R|^2 = \frac{R}{\sqrt{E_0(\phi)}} \int_\Omega f(\nabla\phi) \cdot \nabla\phi_t d\boldsymbol{x}. \end{aligned}$$

Using (12e) and $b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) = 0$, we can get

$$\frac{d}{dt}\left(\int_\Omega (\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\psi|^2) d\boldsymbol{x} + |R|^2\right) = -\int_\Omega (M|w|^2 + \nu|\nabla\boldsymbol{u}|^2) d\boldsymbol{x} \le 0.$$

From the definition (10), we have

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{u}, \phi, R) = \frac{d}{dt}\left(\int_\Omega (\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\psi|^2) d\boldsymbol{x} + |R|^2\right) \le 0.$$

$\square$

3. Well-posedness and energy stability of the scheme

3.1. The BDF2-SAV-FEM scheme. Let $\mathcal{T}_h = \{K\}$ be a regular mesh of Ω consisting of elements K, and K is a triangle with the mesh parameter $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K denotes diameters of K. Let $\boldsymbol{X}_h \subset \boldsymbol{X}, Y_h \subset Y, Q_h \subset Q$ be the finite-element spaces and define as:

$$\begin{aligned}\boldsymbol{X}_h &= \{\boldsymbol{v}_h \in \boldsymbol{X}, \boldsymbol{v}_h|_K \in [\mathbb{P}_k(K)]^d\},\\ Y_h &= \{\phi_h \in Y, \phi_h|_K \in \mathbb{P}_k(K)\},\\ Q_h &= \{q_h \in Q, q_h|_K \in \mathbb{P}_{k-1}(K)\},\end{aligned}$$

where $\mathbb{P}_k(K)$ is the space of polynomials degree not great than $k \geq 1$ on K.

Now we approximate $(\phi, w, \psi, \boldsymbol{u}, p, R) \in Y \times Y \times Y \times \boldsymbol{X} \times Q \times \mathbb{R}$ by finding $(\phi_h, w_h, \psi_h, \boldsymbol{u}_h, p_h, R_h) \in Y_h \times Y_h \times Y_h \times \boldsymbol{X}_h \times Q_h \times \mathbb{R}$, respectively, then we have the semi-discrete SAV-FEM scheme:

(14)
$$\begin{cases}((\phi_h)_t, \rho_h) + (\boldsymbol{u}_h \cdot \nabla\phi_h, \rho_h) + M(w_h, \rho_h) = 0, \qquad \forall \rho_h \in Y_h,\\ -(w_h, \sigma_h) + (\nabla\psi_h, \nabla\sigma_h) + (\frac{R_h(t)}{\sqrt{E_0(\phi_h)}} f(\nabla\phi_h), \nabla\sigma_h) = 0, \qquad \forall \sigma_h \in Y_h,\\ (\psi_h, \varphi_h) - (\nabla\phi_h, \nabla\varphi_h) = 0, \qquad \forall \varphi_h \in Y_h,\\ ((\boldsymbol{u}_h)_t, \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + \nu(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}_h) - (p_h, \nabla\cdot\boldsymbol{v}_h) - (w_h\nabla\phi_h, \boldsymbol{v}_h) = 0,\\ \qquad\qquad\qquad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h,\\ (\nabla\cdot\boldsymbol{u}_h, q_h) = 0, \qquad \forall q_h \in Q_h,\\ (R_h)_t = \frac{1}{2\sqrt{E_0(\phi_h)}}(f(\nabla\phi_h), \nabla(\phi_h)_t).\end{cases}$$

Let δt be the time step size satisfying $0 < \delta t < 1$, $t_n = n\delta t$, $n = 1, 2, \cdots, N$ with ending time $T = N\delta t$. We now consider the BDF2 scheme for discretization (14) in time. To develop a linear, fully-decoupled, fully-discrete BDF2-SAV-FEM scheme, we present the calculations in the following four steps. In the initial time t_0, we define the initial data as $\phi_h^0 = \mathcal{Q}_h\phi_0$, $\boldsymbol{u}_h^0 = \mathcal{P}_h\boldsymbol{u}_0$, $p_h^0 = \mathcal{R}_h p_0$, $R_h^0 = \sqrt{E_0(\phi_0)}$, $\tilde{\boldsymbol{u}}_h^0 = \boldsymbol{u}_h^0$, $z_h^0 = p_h^0$, and $\phi_h^{-1} = \phi_h^0, \boldsymbol{u}_h^{-1} = \boldsymbol{u}_h^0$ as in [56]. At the domain boundary, we set $\boldsymbol{u}_h^{n+1}|_{\partial\Omega} = \boldsymbol{0}$, $\frac{\partial}{\partial\boldsymbol{n}}\phi_h^{n+1}|_{\partial\Omega} = 0$, $\frac{\partial}{\partial\boldsymbol{n}}\psi_h^{n+1}|_{\partial\Omega} = 0$.

Step 1: Find $(\phi_h^{n+1}, w_h^{n+1}, \psi_h^{n+1}, R_h^{n+1}) \in (Y_h, Y_h, Y_h, \mathbb{R})$ satisfying

$$(15)\quad \begin{aligned}&(\frac{3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1}}{2\delta t}, \rho_h) + (\rho_h\nabla\hat{\phi}_h^n, \hat{\boldsymbol{u}}_h^n) + M(w_h^{n+1}, \rho_h)\\ &\qquad + \beta\delta t^2(w_h^{n+1}\nabla\hat{\phi}_h^n, \rho_h\nabla\hat{\phi}_h^n) = 0, \quad \forall \rho_h \in Y_h,\end{aligned}$$

$$(16)\quad -(w_h^{n+1}, \sigma_h) + (\nabla\psi_h^{n+1}, \nabla\sigma_h) + (\frac{R_h^{n+1}}{\sqrt{E_0(\hat{\phi}_h^n)}} f(\nabla\hat{\phi}_h^n), \nabla\sigma_h) = 0, \quad \forall \sigma_h \in Y_h,$$

$$(17)\quad (\psi_h^{n+1}, \varphi_h) - (\nabla\phi_h^{n+1}, \nabla\varphi_h) = 0, \quad \forall \varphi_h \in Y_h,$$

$$(18)\quad \frac{3R_h^{n+1} - 4R_h^n + R_h^{n-1}}{2\delta t} = \frac{1}{2\sqrt{E_0(\hat{\phi}_h^n)}}(f(\nabla\hat{\phi}_h^n), \frac{3\nabla\phi_h^{n+1} - 4\nabla\phi_h^n + \nabla\phi_h^{n-1}}{2\delta t}).$$

Step 2: Find $\tilde{\boldsymbol{u}}_h^{n+1} \in \boldsymbol{X}_h$ such that

(19)
$$\begin{aligned}&(\frac{3\tilde{\boldsymbol{u}}_h^{n+1} - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}}{2\delta t}, \boldsymbol{v}_h) + b(\hat{\boldsymbol{u}}_h^n, \tilde{\boldsymbol{u}}_h^{n+1}, \boldsymbol{v}_h) + \nu(\nabla\tilde{\boldsymbol{u}}_h^{n+1}, \nabla\boldsymbol{v}_h)\\ &\qquad - (p_h^n, \nabla\cdot\boldsymbol{v}_h) - (w_h^{n+1}\nabla\hat{\phi}_h^n, \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h.\end{aligned}$$

Step 3: Find $(z_h^{n+1}, p_h^{n+1}, \boldsymbol{u}_h^{n+1}) \in (Q_h, Q_h, \boldsymbol{X}_h)$ such that

$$(\nabla z_h^{n+1}, \nabla q_h) = -\frac{3}{2\delta t}(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, q_h), \quad \forall q_h \in Q_h, \tag{20}$$

$$(p_h^{n+1} - p_h^n, q_h) + \nu(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, q_h) = (z_h^{n+1}, q_h), \quad \forall q_h \in Q_h, \tag{21}$$

$$(\frac{3\boldsymbol{u}_h^{n+1} - 3\tilde{\boldsymbol{u}}_h^{n+1}}{2\delta t}, \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, z_h^{n+1}) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h. \tag{22}$$

We now make a few remarks on the above scheme. First, we introduce the linearly extrapolated forms $\hat{\phi}_h^n := 2\phi_h^n - \phi_h^{n-1}$ and $\hat{\boldsymbol{u}}_h^n := 2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}$ for the layer function ϕ and velocity $\boldsymbol{u}$, respectively.

Second, we emphasize that the fluid velocity field $\boldsymbol{u}_h^{n+1}$ satisfies the discrete divergence free condition (see [51]). To be specific, take $\boldsymbol{v}_h = \nabla q_h$ in (22), combine with (20), and use the integration by parts we can get

$$(\nabla \cdot \boldsymbol{u}_h^{n+1}, q_h) = 0, \quad \forall q_h \in Q_h. \tag{23}$$

In the following, we denote by C a generic constant that is independent of the size h and the step size δt. Generally, to describe that there exists a generic constant C such that $a \leq Cb$, we employ the expression $a \lesssim b$. To derive the well-posedness, energy stability, and error estimate of our scheme below, we give some following lemmas and elliptic projection operators.

Remark 3.1. *To decouple the computation of the layer function ϕ and velocity $\boldsymbol{u}$, a second-order artificial stabilization term $\beta\delta t^2(w_h^{n+1}\nabla\hat{\phi}_h^n, \rho_h\nabla\hat{\phi}_h^n)$ (where β is a artificial parameter) has been added in (15) to impose the stability of the scheme. And [33, 34, 59] provide similar methods.*

Remark 3.2. *The pressure gradient is explicitly calculated in the suggested numerical scheme. Consequently, since the pressure gradient is no longer coupled with the divergence-free constraint, a Stokes-style solver is avoided. Stokes-style solvers can be avoided in some works [4, 38].*

Remark 3.3. *We utilize a second-order rotational pressure-correction scheme in (20)-(22) to solve the Navier-Stokes equations, which can decouple the calculation of the pressure p and the velocity $\boldsymbol{u}$. Compared to the standard forms of pressure-correction projection method [3, 4, 7, 38], the rotational pressure-correction in the proposed scheme eliminate a portion of artificial pressure boundary conditions.*

Lemma 3.4. *The following inverse inequality holds (see [40])*

$$\|v_h\|_{W^{m,s}} \lesssim h^{n-m+\frac{d}{s}-\frac{d}{q}}\|v_h\|_{W^{n,q}}, \tag{24}$$

where d is the dimension of the space and C is a positive constant that is independent of h, with $0 \leq n \leq m \leq 1,\ 1 \leq q \leq s \leq \infty$.

Lemma 3.5. *(Inf-sup condition [18]) The spaces $\boldsymbol{X}_h$ and Q_h satisfy the inf-sup condition, which yields*

$$\inf_{q\in Q_h}\sup_{\boldsymbol{v}\in\boldsymbol{X}_h}\frac{(q, \nabla\cdot\boldsymbol{v})}{\|\nabla\boldsymbol{v}\|} \gtrsim \|q\|. \tag{25}$$

Defined the elliptic projection operators $\mathcal{Q}_h : Y \to Y_h$ and $\mathcal{R}_h : Q \cap H^1(\Omega) \to Q_h$ as (see [37]): for any $\psi \in Y$ and $p \in Q \cap H^1(\Omega)$

$$(\nabla(\psi - \mathcal{Q}_h\psi), \nabla\varphi_h) = 0, \quad \forall\varphi_h \in Y_h, \tag{26}$$

$$(\nabla(p - \mathcal{R}_h p), \nabla q_h) = 0, \quad \forall q_h \in Q_h. \tag{27}$$

Defined the operator $\mathcal{P}_h : \boldsymbol{X} \to \boldsymbol{X}_h$ as (see [19]): for any $\boldsymbol{u} \in \boldsymbol{X}$

$$(\nabla \cdot (\boldsymbol{u} - \mathcal{P}_h \boldsymbol{u}), q_h) = 0, \quad \forall q_h \in Q_h. \tag{28}$$

The operators $\mathcal{Q}_h$, $\mathcal{P}_h$, $\mathcal{R}_h$ have the classical approximation properties (see [30, 36, 37]):

$$\begin{aligned}
&\|\phi - \mathcal{Q}_h\phi\| + h\|\nabla(\phi - \mathcal{Q}_h\phi)\| \lesssim h^{k+1}\|\phi\|_{H^{k+1}}, \quad \forall \phi \in H^{k+1}(\Omega) \cap Y,\\
&\|\boldsymbol{u} - \mathcal{P}_h\boldsymbol{u}\| + h\|\nabla(\boldsymbol{u} - \mathcal{P}_h\boldsymbol{u})\| \lesssim h^{k+1}\|\boldsymbol{u}\|_{H^{k+1}}, \quad \forall \boldsymbol{u} \in [H^{k+1}(\Omega)]^d \cap \boldsymbol{X},\\
&\|\nabla^i p - \nabla^i \mathcal{R}_h p\| \lesssim h^{k-i}\|p\|_{H^k}, \quad i = 0, 1, \quad \forall p \in H^k(\Omega) \cap Q.
\end{aligned} \tag{29}$$

3.2. Well-posedness. We will prove the well-posedness of the scheme (15)-(22).

Theorem 3.6. *The numerical solution of the scheme* (15)-(22) *is unique.*

Proof. We rewrite the formula (15)-(18) as

(30)
$$\begin{aligned}
&(\phi_h^{n+1}, \rho_h) + M\frac{2\delta t}{3}(w_h^{n+1}, \rho_h) + \frac{2\beta\delta t^2}{3}(w_h^{n+1}\nabla\hat{\phi}_h^n, \rho_h\nabla\hat{\phi}_h^n)\\
&\qquad = \frac{1}{3}(4\phi_h^n - \phi_h^{n-1}, \rho_h) - \frac{2\delta t}{3}(\rho_h\nabla\hat{\phi}_h^n, \hat{\boldsymbol{u}}_h^n),
\end{aligned}$$

(31)
$$-(w_h^{n+1}, \sigma_h) + (\nabla\psi_h^{n+1}, \nabla\sigma_h) + (\frac{R_h^{n+1}}{\sqrt{E_0(\hat{\phi}_h^n)}}f(\nabla\hat{\phi}_h^n), \nabla\sigma_h) = 0,$$

(32)
$$(\psi_h^{n+1}, \varphi_h) - (\nabla\phi_h^{n+1}, \nabla\varphi_h) = 0,$$

(33)
$$\begin{aligned}
&2R_h^{n+1}D - \frac{D}{\sqrt{E_0(\hat{\phi}_h^n)}}(f(\nabla\hat{\phi}_h^n), \nabla\phi_h^{n+1})\\
&\qquad = \frac{D}{3\sqrt{E_0(\hat{\phi}_h^n)}}(f(\nabla\hat{\phi}_h^n), -4\nabla\phi_h^n + \nabla\phi_h^{n-1}) + \frac{2}{3}(4R_h^nD - R_h^{n-1}D).
\end{aligned}$$

Here we have multiplied $D \in \mathbb{R}$ with both side of (18).

Defined a bilinear form $\mathcal{A}_1(\cdot,\cdot) : (Y_h, Y_h, Y_h, \mathbb{R}) \times (Y_h, Y_h, Y_h, \mathbb{R}) \to \mathbb{R}$ and a linear form $\mathcal{F}_1(\cdot) : (Y_h, Y_h, Y_h, \mathbb{R}) \to \mathbb{R}$ as

$$\begin{aligned}
\mathcal{A}_1(w, \phi, \psi, R; \rho, \sigma, \varphi, D) &= (\phi, \rho) + M\frac{2\delta t}{3}(w, \rho)) + \frac{2\beta\delta t^2}{3}(w\nabla\hat{\phi}_h^n, \rho\nabla\hat{\phi}_h^n) - (w, \sigma)\\
&\quad + (\nabla\psi, \nabla\sigma) + (\frac{R}{\sqrt{E_0(\hat{\phi}_h^n)}}f(\nabla\hat{\phi}_h^n), \nabla\sigma) + (\psi, \varphi)\\
&\quad - (\nabla\phi, \nabla\varphi) + 2RD - \frac{D}{\sqrt{E_0(\hat{\phi}_h^n)}}(f(\nabla\hat{\phi}_h^n), \nabla\phi),
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_1(\rho, \sigma, \varphi, D) &= \frac{1}{3}(4\phi_h^n - \phi_h^{n-1}, \rho) - \frac{2\delta t}{3}(\rho\nabla\hat{\phi}_h^n, \hat{\boldsymbol{u}}_h^n)\\
&\quad + \frac{D}{3\sqrt{E_0(\hat{\phi}_h^n)}}(f(\nabla\hat{\phi}_h^n), -4\nabla\phi_h^n + \nabla\phi_h^{n-1}) + \frac{2}{3}(4R_h^nD - R_h^{n-1}D).
\end{aligned}$$

Therefore, the scheme (30)-(33) can be rewritten as:

$$\mathcal{A}_1(w_h^{n+1}, \phi_h^{n+1}, \psi_h^{n+1}, R_h^{n+1}; \rho_h, \sigma_h, \varphi_h, D) = \mathcal{F}_1(\rho_h, \sigma_h, \varphi_h, D). \tag{34}$$

Using the Cauchy-Schwarz inequality and Poincaré's inequality [13], we have

$$\begin{aligned}
&\mathcal{A}_1(w, \phi, \psi, R; \rho, \sigma, \varphi, D)\\
\leq &\|\phi\|\|\rho\| + M\frac{2\delta t}{3}\|w\|\|\rho\| + \frac{2\beta\delta t^2}{3}\|w\|\|\rho\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2 + \|w\|\|\sigma\|\\
&+ \|\nabla\psi\|\|\nabla\sigma\| + |R|\|\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}^n)}}\|\|\nabla\sigma\| + \|\psi\|\|\varphi\| + \|\nabla\phi\|\|\nabla\varphi\|\\
&+ 2|R||D| + |D|\|\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}\|\|\nabla\phi\|\\
&\leq C_1(\|\nabla\phi\| + \|w\| + \|\nabla\psi\| + |R|)(\|\rho\| + \|\nabla\sigma\| + \|\nabla\varphi\| + |D|),
\end{aligned}$$

where C_1 depends on $\delta t, M, \beta, \|\nabla\hat{\phi}_h^n\|_{L^\infty}$. And the boundedness of $\|\nabla\hat{\phi}_h^n\|_{L^\infty}$ can be obtained from the Step 3 of Lemma 4.2's proof. Taking $(\rho, \sigma, \varphi) = (w, \phi, \psi)$, we derive

$$\begin{aligned}
\mathcal{A}_1(w, \phi, \psi, R; w, \phi, \psi, R) &= M\frac{2\delta t}{3}\|w\|^2 + \frac{2\beta\delta t^2}{3}\|w\nabla\hat{\phi}_h^n\|^2 + \|\psi\|^2 + 2|R|^2\\
&\geq C_2(\|w\|^2 + \|\psi\|^2 + |R|^2),
\end{aligned}$$

where C_2 depends on $\delta t, M, \beta$. Then from the Lax-Milgram theorem, we know that (15)-(18) has a unique solution $(w_h^{n+1}, \phi_h^{n+1}, \psi_h^{n+1})$. For (19), we define a bilinear form $\mathcal{A}_2(\cdot,\cdot) : \boldsymbol{X}_h \times \boldsymbol{X}_h \to \mathbb{R}$ and a linear form $\mathcal{F}_2(\cdot) : \boldsymbol{X}_h \to \mathbb{R}$ as

$$\begin{aligned}
A_2(\boldsymbol{u}, \boldsymbol{v}) &= \frac{3}{2\delta t}(\boldsymbol{u}, \boldsymbol{v}) + b(\hat{\boldsymbol{u}}_h^n, \boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}),\\
\mathcal{F}_2(\boldsymbol{v}) &= \frac{1}{2\delta t}(4\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{v}) + (p_h^n, \nabla\cdot\boldsymbol{v}) + (w_h^{n+1}\nabla\hat{\phi}_h^n, \boldsymbol{v}).
\end{aligned}$$

Then (19) can be expressed as

$$A_2(\tilde{\boldsymbol{u}}_h^{n+1}, \boldsymbol{v}_h) = \mathcal{F}_2(\boldsymbol{v}_h). \tag{35}$$

With the similar process, we can get that $\mathcal{A}_2(\cdot,\cdot)$ is bounded and coercive:

$$\begin{aligned}
A_2(\boldsymbol{u}, \boldsymbol{v}) &\leq \frac{3}{2\delta t}\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|\nabla\boldsymbol{u}\|\|\boldsymbol{v}\| + \nu\|\nabla\boldsymbol{u}\|\|\nabla\boldsymbol{v}\|\\
&\leq C_3\|\nabla\boldsymbol{u}\|\|\nabla\boldsymbol{v}\|,\\
A_2(\boldsymbol{u}, \boldsymbol{u}) &= \frac{3}{2\delta t}\|\boldsymbol{u}\|^2 + \nu\|\nabla\boldsymbol{u}\|^2 \geq C_4\|\nabla\boldsymbol{u}\|^2,
\end{aligned}$$

where C_3 depends on $\delta t, \nu, \|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}$ and C_4 depends on $\delta t, \nu$. Thus from the Lax-Milgram theorem, we note that (19) has a unique solution $\tilde{\boldsymbol{u}}_h^{n+1} \in \boldsymbol{X}_h$. Using a similar way to deal with (20)-(22), we can demonstrate the existence and the uniqueness of $(z_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in (Q_h, \boldsymbol{X}_h, Q_h)$. The proof is finished. $\square$

3.3. Unconditionally energy stability. Here we introduce some auxiliary variables to demonstrate the energy stability [28,59]. The following auxiliary functions $S_h^n \in Q_h$ and $d_h^n \in Q_h$ are defined by

$$S_h^n = \nu\sum_{j=1}^{n}\nabla\cdot\tilde{\boldsymbol{u}}_h^j, \quad d_h^n = p_h^n + S_h^n, \quad S^n = 0, \quad d^n = p^n, \tag{36}$$

with $S_h^0 = 0$ and $d_h^0 = 0$. We can rewrite (21) as

$$p_h^{n+1} = p_h^n + z_h^{n+1} - \nu \nabla \cdot \tilde{\boldsymbol{u}}_h^n. \tag{37}$$

Theorem 3.7. *The scheme (15)-(22) is unconditionally energy stable, i.e., it obeys the following discrete law of energy dissipation*

$$\mathcal{E}_h(\boldsymbol{u}_h^{n+1}, \phi_h^{n+1}) \leq \mathcal{E}_h(\boldsymbol{u}_h^n, \phi_h^n), \tag{38}$$

where the discrete energy $\mathcal{E}_h$ is defined as

$$\begin{aligned}\mathcal{E}_h(\boldsymbol{u}_h^n, \phi_h^n) = &\frac{1}{2}\|\psi_h^n\|^2 + \frac{1}{2}\|2\psi_h^n - \psi_h^{n-1}\|^2 + |R_h^n|^2 + |2R_h^n - R_h^{n-1}|^2 \\ &+ \frac{1}{2}\|\boldsymbol{u}_h^n\|^2 + \frac{1}{2}\|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2 + \frac{2\delta t^2}{3}\|\nabla d_h^n\|^2 + \frac{\delta t}{\nu}\|S_h^n\|^2.\end{aligned}$$

Proof. Firstly, let us recall the following two facts

$$a(a-b) = \frac{1}{2}(|a|^2 - |b|^2 + |a-b|^2), \tag{39}$$

$$a(3a - 4b + c) = \frac{1}{2}(|a|^2 - |b|^2 + |2a-b|^2 - |2b-c|^2 + |a - 2b + c|^2). \tag{40}$$

Setting $\rho_h = 2\delta t w_h^{n+1}$ and $\sigma_h = 3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1}$ in (15) and (16), respectively, we have

$$\begin{aligned}&(3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1}, w_h^{n+1}) + 2\delta t(w_h^{n+1}\nabla\hat{\phi}_h^n, \hat{\boldsymbol{u}}_h^n) \\ &\qquad + 2M\delta t\|w_h^{n+1}\|^2 + 2\beta\delta t^3\|w_h^{n+1}\nabla\hat{\phi}_h^n\|^2 = 0,\end{aligned} \tag{41}$$

$$\begin{aligned}&-(w_h^{n+1}, 3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1}) + (\nabla\psi_h^{n+1}, \nabla(3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1})) \\ &\qquad + (\frac{R_h^{n+1}}{\sqrt{E_0(\phi_h^n)}} f(\nabla\phi_h^n), \nabla(3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1})) = 0.\end{aligned} \tag{42}$$

Subtract (17) from the previous time step and the two previous time steps by the appropriate treatments, one can easily get

$$(3\psi_h^{n+1} - 4\psi_h^n + \psi_h^{n-1}, \varphi_h) - (\nabla(3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1}), \nabla\varphi_h) = 0. \tag{43}$$

Take $\varphi_h = \psi_h^{n+1}$ in (43) and use (40), we have

$$\begin{aligned}&\frac{1}{2}(\|\psi_h^{n+1}\|^2 - \|\psi_h^n\|^2 + \|2\psi_h^{n+1} - \psi_h^n\|^2 \\ &- \|2\psi_h^n - \psi_h^{n-1}\|^2 + \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2) \\ &\quad - (\nabla(3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1}), \nabla\psi_h^{n+1}) = 0.\end{aligned} \tag{44}$$

Multiplying (18) by $4\delta t R_h^{n+1}$, and using (40) lead to

$$\begin{aligned}&(|R_h^{n+1}|^2 - |R_h^n|^2 + |2R_h^{n+1} - R_h^n|^2 \\ &- |2R_h^n - R_h^{n-1}|^2 + |R_h^{n+1} - 2R_h^n + R_h^{n-1}|^2) \\ =&\frac{R_h^{n+1}}{\sqrt{E_0(\phi_h^n)}}(f(\nabla\phi_h^n), 3\nabla\phi_h^{n+1} - 4\nabla\phi_h^n + \nabla\phi_h^{n-1}).\end{aligned} \tag{45}$$

Adding (41)-(42), and (44)-(45), we have

$$\begin{aligned}
&2\delta t(w_h^{n+1}\nabla\hat{\phi}_h^n, \hat{\boldsymbol{u}}_h^n) + 2M\delta t\|w_h^{n+1}\|^2 + 2\beta\delta t^3\|w_h^{n+1}\nabla\hat{\phi}_h^n\|^2 \\
&\quad + \frac{1}{2}(\|\psi_h^{n+1}\|^2 - \|\psi_h^n\|^2 + \|2\psi_h^{n+1} - \psi_h^n\|^2 \\
&\quad - \|2\psi_h^n - \psi_h^{n-1}\|^2 + \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2) \\
&\quad + (|R_h^{n+1}|^2 - |R_h^n|^2 + |2R_h^{n+1} - R_h^n|^2 \\
&\quad - |2R_h^n - R_h^{n-1}|^2 + |R_h^{n+1} - 2R_h^n + R_h^{n-1}|^2) = 0.
\end{aligned} \tag{46}$$

By setting $\boldsymbol{v}_h = 2\delta t\tilde{\boldsymbol{u}}_h^{n+1}$ in (19) and using the fact that $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0$, we can obtain

$$\begin{aligned}
&(3\tilde{\boldsymbol{u}}_h^{n+1} - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \tilde{\boldsymbol{u}}_h^{n+1}) + 2\nu\delta t\|\nabla\tilde{\boldsymbol{u}}_h^{n+1}\|^2 \\
&\quad - 2\delta t(p_h^n, \nabla\cdot\tilde{\boldsymbol{u}}_h^{n+1}) - 2\delta t(w_h^{n+1}\nabla\hat{\phi}_h^n, \tilde{\boldsymbol{u}}_h^{n+1}) = 0.
\end{aligned} \tag{47}$$

Taking $\boldsymbol{v}_h = 3\boldsymbol{u}_h^{n+1} - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}$ in (22), using $(\nabla\cdot\boldsymbol{u}_h^{n+1}, q_h) = 0, \forall q_h \in Q_h$ we can get

$$\frac{3}{2\delta t}(\boldsymbol{u}_h^{n+1} - \tilde{\boldsymbol{u}}_h^{n+1}, 3\boldsymbol{u}_h^{n+1} - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}) = (\nabla\cdot(3\boldsymbol{u}_h^{n+1} - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}), z_h^{n+1}) = 0. \tag{48}$$

Then from (48), we arrive at

$$\begin{aligned}
&(3\tilde{\boldsymbol{u}}_h^{n+1} - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \tilde{\boldsymbol{u}}_h^{n+1}) \\
&= (3\boldsymbol{u}_h^{n+1} - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^{n+1}) + (3\boldsymbol{u}_h^{n+1} - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}, \tilde{\boldsymbol{u}}_h^{n+1} - \boldsymbol{u}_h^{n+1}) \\
&\quad + (3\tilde{\boldsymbol{u}}_h^{n+1} - 3\boldsymbol{u}_h^{n+1}, \tilde{\boldsymbol{u}}_h^{n+1}) \\
&= \frac{1}{2}(\|\boldsymbol{u}_h^{n+1}\|^2 - \|\boldsymbol{u}_h^n\|^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 \\
&\quad - \|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2 + \|\boldsymbol{u}_h^{n+1} - 2\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}\|^2) \\
&\quad + \frac{3}{2}(\|\tilde{\boldsymbol{u}}_h^{n+1}\|^2 - \|\boldsymbol{u}_h^{n+1}\|^2 + \|\tilde{\boldsymbol{u}}_h^{n+1} - \boldsymbol{u}_h^{n+1}\|^2).
\end{aligned} \tag{49}$$

From (49) and (47), we can get

$$\begin{aligned}
&\frac{1}{2}(\|\boldsymbol{u}_h^{n+1}\|^2 - \|\boldsymbol{u}_h^n\|^2 + \|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 - \|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2 + \|\boldsymbol{u}_h^{n+1} - 2\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}\|^2) \\
&\quad + \frac{3}{2}(\|\tilde{\boldsymbol{u}}_h^{n+1}\|^2 - \|\boldsymbol{u}_h^{n+1}\|^2 + \|\tilde{\boldsymbol{u}}_h^{n+1} - \boldsymbol{u}_h^{n+1}\|^2) + 2\nu\delta t\|\nabla\tilde{\boldsymbol{u}}_h^{n+1}\|^2 \\
&\quad - 2\delta t(p_h^n, \nabla\cdot\tilde{\boldsymbol{u}}_h^{n+1}) - 2\delta t(w_h^{n+1}\nabla\hat{\phi}_h^n, \tilde{\boldsymbol{u}}_h^{n+1}) = 0.
\end{aligned} \tag{50}$$

Taking $q_h = \frac{8\delta t^2}{9} z_h^{n+1}$ in (20), we have

$$\frac{8\delta t^2}{9}\|\nabla z_h^{n+1}\|^2 = -\frac{4\delta t}{3}(\nabla\cdot\tilde{\boldsymbol{u}}_h^{n+1}, z_h^{n+1}). \tag{51}$$

Setting $\boldsymbol{v}_h = 2\delta t\tilde{\boldsymbol{u}}_h^{n+1}$ in (22), we can derive

$$\frac{3}{2}(\|\boldsymbol{u}_h^{n+1}\|^2 - \|\tilde{\boldsymbol{u}}_h^{n+1}\|^2 - \|\boldsymbol{u}_h^{n+1} - \tilde{\boldsymbol{u}}_h^{n+1}\|^2) - 2\delta t(\nabla\cdot\tilde{\boldsymbol{u}}_h^{n+1}, z_h^{n+1}) = 0. \tag{52}$$

We rewrite (22) as

$$(\tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n, \boldsymbol{v}_h) - (\boldsymbol{u}_h^{n+1} - \hat{\boldsymbol{u}}_h^n, \boldsymbol{v}_h) = -\frac{2\delta t}{3}(\nabla\cdot v_h^{n+1}, z_h^{n+1}). \tag{53}$$

Choosing $\boldsymbol{v}_h = \tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n$ in (53) and using the fact that $(\nabla \cdot \hat{\boldsymbol{u}}_h^n, z_h^{n+1}) = 0$, we get

$$\|\tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n\|^2 - (\boldsymbol{u}_h^{n+1} - \hat{\boldsymbol{u}}_h^n, \tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n) = -\frac{2\delta t}{3}(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, z_h^{n+1}). \tag{54}$$

To deal with the term $-2\delta t(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, p_h^n)$ in (50), applying the definition of d_h^n, we have

$$-2\delta t(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, p_h^n) = -2\delta t(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, d_h^n) + 2\delta t(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, S_h^n). \tag{55}$$

Using the definition of S_h^n, we obtain

$$S_h^{n+1} - S_h^n = \nu \nabla \cdot \tilde{\boldsymbol{u}}_h^n. \tag{56}$$

From (36)-(37) and (56), it is easy to check that

$$d_h^{n+1} - d_h^n = p_h^{n+1} - p_h^n + S_h^{n+1} - S_h^n = z_h^{n+1}. \tag{57}$$

Then by setting $q_h = \frac{4\delta t^2}{3} d_h^n$ in (20) and applying (57), we derive

$$\begin{aligned}
-2\delta t(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, d_h^n) &= \frac{4\delta t^2}{3}(\nabla z_h^{n+1}, \nabla d_h^n) \\
&= \frac{4\delta t^2}{3}(\nabla(d_h^{n+1} - d_h^n), \nabla d_h^n) \\
&= \frac{2\delta t^2}{3}(\|\nabla d_h^{n+1}\|^2 - \|\nabla d_h^n\|^2 - \|\nabla(d_h^{n+1} - d_h^n)\|^2) \\
&= \frac{2\delta t^2}{3}(\|\nabla d_h^{n+1}\|^2 - \|\nabla d_h^n\|^2 - \|\nabla z_h^{n+1}\|^2).
\end{aligned} \tag{58}$$

For the second term $2\delta t(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, S_h^n)$ in the right-hand side of (55), applying (56) and (39), we derive

$$\begin{aligned}
2\delta t(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, S_h^n) &= \frac{2\delta t}{\nu}(S_h^{n+1} - S_h^n, S_h^n) \\
&= \frac{\delta t}{\nu}(\|S_h^{n+1}\|^2 - \|S_h^n\|^2 - \|S_h^{n+1} - S_h^n\|^2).
\end{aligned} \tag{59}$$

By substituting (58) and (59) into (55), we get

$$\begin{aligned}
-2\delta t(\nabla \cdot \tilde{\boldsymbol{u}}_h^{n+1}, p_h^n) &= \frac{2\delta t^2}{3}(\|\nabla d_h^{n+1}\|^2 - \|\nabla d_h^n\|^2 - \|\nabla z_h^{n+1}\|^2) \\
&\quad + \frac{\delta t}{\nu}(\|S_h^{n+1}\|^2 - \|S_h^n\|^2 - \|S_h^{n+1} - S_h^n\|^2).
\end{aligned} \tag{60}$$

Combining (46), (50)-(52), (54) and (60), we can obtain

$$\begin{aligned}
&\frac{1}{2}(\|\psi_h^{n+1}\|^2 - \|\psi_h^n\|^2) + \frac{1}{2}(\|2\psi_h^{n+1} - \psi_h^n\|^2 - \|2\psi_h^n - \psi_h^{n-1}\|^2) \\
&\quad + (|R_h^{n+1}|^2 - |R_h^n|^2) + (|2R_h^{n+1} - R_h^n|^2 - |2R_h^n - R_h^{n-1}|^2) \\
&\quad + \frac{1}{2}(\|\boldsymbol{u}_h^{n+1}\|^2 - \|\boldsymbol{u}_h^n\|^2) + \frac{1}{2}(\|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 - \|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2) \\
&\quad + \frac{2\delta t^2}{3}(\|\nabla d_h^{n+1}\|^2 - \|\nabla d_h^n\|^2) + \frac{\delta t}{\nu}(\|S_h^{n+1}\|^2 - \|S_h^n\|^2) + \frac{2\delta t^2}{9}\|\nabla z_h^{n+1}\|^2 \\
&\quad + \frac{1}{2}\|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2 + |R_h^{n+1} - 2R_h^n + R_h^{n-1}|^2 + \frac{1}{2}\|\boldsymbol{u}_h^{n+1} - 2\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}\|^2 \\
&\quad + 2\nu\delta t\|\nabla \tilde{\boldsymbol{u}}_h^{n+1}\|^2 + \|\tilde{\boldsymbol{u}}_h^{n+1} - 2\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}\|^2 + 2M\delta t\|w_h^{n+1}\|^2 + 2\beta\delta t^3\|w_h^{n+1}\nabla\hat{\phi}_h^n\|^2
\end{aligned} \tag{61}$$

$$= 2\delta t(w_h^{n+1}\nabla\hat{\phi}_h^n, \tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n) + (\boldsymbol{u}_h^{n+1} - \hat{\boldsymbol{u}}_h^n, \tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n) + \frac{\delta t}{\nu}\|S_h^{n+1} - S_h^n\|^2.$$

For the term $(\boldsymbol{u}_h^{n+1} - \hat{\boldsymbol{u}}_h^n, \tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n)$, by using the definition of $\hat{\boldsymbol{u}}_h^n$, the Cauchy-Schwarz inequality and the average inequality, we derive

(62)
$$(\boldsymbol{u}_h^{n+1} - \hat{\boldsymbol{u}}_h^n, \tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n) \le \frac{1}{2}\|\boldsymbol{u}_h^{n+1} - 2\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}\|^2 + \frac{1}{2}\|\tilde{\boldsymbol{u}}_h^{n+1} - 2\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}\|^2.$$

For the term $\frac{\delta t}{\nu}\|S_h^{n+1} - S_h^n\|^2$, using the definition of S_h^n and $\|\nabla\cdot\boldsymbol{v}\|^2 \le \|\nabla\boldsymbol{v}\|^2$ (see [21]), we obtain

$$\frac{\delta t}{\nu}\|S_h^{n+1} - S_h^n\|^2 = \nu\delta t\|\nabla\cdot\tilde{\boldsymbol{u}}_h^{n+1}\|^2 \le 2\nu\delta t\|\nabla\tilde{\boldsymbol{u}}_h^{n+1}\|^2. \tag{63}$$

To bound the term $2\delta t(w_h^{n+1}\nabla\hat{\phi}_h^n, \tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n)$, from the Cauchy-Schwarz inequality and Young's inequality, we get

$$\begin{aligned} 2\delta t(w_h^{n+1}\nabla\hat{\phi}_h^n, \tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n) &\le 2\delta t\|w_h^{n+1}\nabla\phi_h^n\|\|\tilde{\boldsymbol{u}}_h^{n+1} - \hat{\boldsymbol{u}}_h^n\| \\ &\le 2\delta t^2\|w_h^{n+1}\nabla\phi_h^n\|^2 + \frac{1}{2}\|\tilde{\boldsymbol{u}}_h^{n+1} - 2\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}\|^2. \end{aligned} \tag{64}$$

Combining (61)-(64), and choosing the artificial parameter $\beta = \frac{2N}{t_n}$, then one can deduce

(65)
$$\begin{aligned} &\frac{1}{2}(\|\psi_h^{n+1}\|^2 - \|\psi_h^n\|^2) + \frac{1}{2}(\|2\psi_h^{n+1} - \psi_h^n\|^2 - \|2\psi_h^n - \psi_h^{n-1}\|^2) + (|R_h^{n+1}|^2 - |R_h^n|^2) \\ &\quad + (|2R_h^{n+1} - R_h^n|^2 - |2R_h^n - R_h^{n-1}|^2) + \frac{1}{2}(\|\boldsymbol{u}_h^{n+1}\|^2 - \|\boldsymbol{u}_h^n\|^2) \\ &\quad + \frac{1}{2}(\|2\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n\|^2 - \|2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|^2) + \frac{2\delta t^2}{3}(\|\nabla d_h^{n+1}\|^2 - \|\nabla d_h^n\|^2) \\ &\quad + \frac{\delta t}{\nu}(\|S_h^{n+1}\|^2 - \|S_h^n\|^2) + \frac{2\delta t^2}{9}\|\nabla z_h^{n+1}\|^2 + \frac{1}{2}\|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2 \\ &\quad + |R_h^{n+1} - 2R_h^n + R_h^{n-1}|^2 + 2M\delta t\|w_h^{n+1}\|^2 \le 0. \end{aligned}$$

Then the desired result follows immediately. □

4. Error estimates

We can now prove the estimated errors for the system (15)-(22) as follows. Let us assume that the exact solution satisfies the following assumptions on regularity:

$$\begin{cases} \boldsymbol{u} \in L^\infty(0,T;[H^{k+1}(\Omega)]^d \cap [W^{1,\infty}(\Omega)]^d), \boldsymbol{u}_t \in L^\infty(0,T;[H^{k+1}(\Omega)]^d), \\ \boldsymbol{u}_{tt} \in L^\infty(0,T;[L^2(\Omega)]^d), \boldsymbol{u}_{ttt} \in L^\infty(0,T;[L^\infty(\Omega)]^d), \\ \phi \in L^\infty(0,T;H^{k+1}(\Omega) \cap W^{1,\infty}(\Omega)), \phi_t \in L^\infty(0,T;H^{k+1}(\Omega)), \\ \nabla\phi_t \in L^\infty(0,T;L^3(\Omega)), \phi_{tt} \in L^\infty(0,T;H^2(\Omega)), \\ \phi_{ttt} \in L^\infty(0,T;W^{1,\infty}(\Omega) \cap H^1(\Omega)), \\ \psi, \psi_t \in L^\infty(0,T;H^{k+1}(\Omega)), w \in L^\infty(0,T;L^\infty(\Omega)), \\ p, p_t \in L^\infty(0,T;H^k(\Omega) \cap H^1(\Omega)). \end{cases} \tag{66}$$

From Lemma 2.2, Theorem 3.4, and the assumptions (66), applying the Sobolev embedding theorem, it is easily get that $f(\nabla\phi)$ satisfies the following condition [56]:

$$|f(\nabla\phi)|,\ |f'(\nabla\phi)|,\ |f''(\nabla\phi)|, |f(\nabla\phi_h^n)| \le C. \tag{67}$$

For convenience, we introduce the notions for any $n \geq 0$:

$$
\begin{aligned}
&e_\phi^n = \phi^n - \phi_h^n = \xi_\phi^n + \eta_\phi^n, && \xi_\phi^n = \phi^n - \mathcal{Q}_h\phi^n, && \eta_\phi^n = \mathcal{Q}_h\phi^n - \phi_h^n,\\
&e_w^n = w^n - w_h^n = \xi_w^n + \eta_w^n, && \xi_w^n = w^n - \mathcal{Q}_h w^n, && \eta_w^n = \mathcal{Q}_h w^n - w_h^n,\\
&e_\psi^n = \psi^n - \psi_h^n = \xi_\psi^n + \eta_\psi^n, && \xi_\psi^n = \psi^n - \mathcal{Q}_h\psi^n, && \eta_\psi^n = \mathcal{Q}_h\psi^n - \psi_h^n,\\
&e_{\boldsymbol{u}}^n = \boldsymbol{u}^n - \boldsymbol{u}_h^n = \xi_{\boldsymbol{u}}^n + \eta_{\boldsymbol{u}}^n, && \xi_{\boldsymbol{u}}^n = \boldsymbol{u}^n - \mathcal{P}_h\boldsymbol{u}^n, && \eta_{\boldsymbol{u}}^n = \mathcal{P}_h\boldsymbol{u}^n - \boldsymbol{u}_h^n,\\
&\tilde{e}_{\boldsymbol{u}}^n = \boldsymbol{u}^n - \tilde{\boldsymbol{u}}_h^n = \xi_{\boldsymbol{u}}^n + \tilde{\eta}_{\boldsymbol{u}}^n, && \xi_{\boldsymbol{u}}^n = \boldsymbol{u}^n - \mathcal{P}_h\boldsymbol{u}^n, && \tilde{\eta}_{\boldsymbol{u}}^n = \mathcal{P}_h\boldsymbol{u}^n - \tilde{\boldsymbol{u}}_h^n,\\
&e_p^n = p^n - p_h^n = \xi_p^n + \eta_p^n, && \xi_p^n = p^n - \mathcal{R}_h p^n, && \eta_p^n = \mathcal{R}_h p^n - p_h^n,\\
&e_d^n = d^n - d_h^n = \xi_d^n + \eta_d^n, && \xi_d^n = d^n - \mathcal{R}_h d^n, && \eta_d^n = \mathcal{R}_h d^n - d_h^n,\\
&e_R^n = R^n - R_h^n.
\end{aligned}
$$

From the definition of d_h^n and d^n in (36), we can easily obtain $\xi_d^n = p^n - \mathcal{R}_h p^n, \eta_d^n = \mathcal{R}_h p^n - (p_h^n + S_h^n)$.

The exact solution satisfies the following equations for any $0 \leq n \leq N-1$:

(68a)
$$(\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\delta t}, \rho) + (\rho\nabla\phi^{n+1}, \boldsymbol{u}^{n+1}) + M(w^{n+1}, \rho) = (E_\phi^{n+1}, \rho),$$

(68b)
$$-(w^{n+1}, \sigma) + (\nabla\psi^{n+1}, \nabla\sigma) + (\frac{R^{n+1}}{\sqrt{E_0(\hat{\phi}^n)}} f(\nabla\hat{\phi}^n), \nabla\sigma) = (E_w^{n+1}, \sigma),$$

(68c)
$$(\psi^{n+1}, \varphi) - (\nabla\phi^{n+1}, \nabla\varphi) = 0,$$

(68d)
$$\frac{3R^{n+1} - 4R^n + R^{n-1}}{2\delta t} = \frac{1}{2\sqrt{E_0(\hat{\phi}^n)}}(f(\nabla\hat{\phi}^n), \frac{3\nabla\phi^{n+1} - 4\nabla\phi^n + \nabla\phi^{n-1}}{2\delta t}) + E_R^{n+1},$$

(68e)
$$(\frac{3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\delta t}, \boldsymbol{v}) + b(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{v}) + \nu(\nabla\boldsymbol{u}^{n+1}, \nabla\boldsymbol{v}) - (p^{n+1}, \nabla\cdot\boldsymbol{v})$$
$$- (w^{n+1}\nabla\phi^{n+1}, \boldsymbol{v}) = (E_{\boldsymbol{u}}^{n+1}, \boldsymbol{v}),$$

(68f)
$$(\nabla(p^{n+1} - p^n), \nabla q) = -\frac{3}{2\delta t}(\nabla\cdot\boldsymbol{u}^{n+1}, q) + (\nabla(p^{n+1} - p^n), \nabla q),$$

(68g)
$$(\frac{3\boldsymbol{u}^{n+1} - 3\boldsymbol{u}^{n+1}}{2\delta t}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, p^{n+1} - p^n) = -(\nabla\cdot\boldsymbol{v}, p^{n+1} - p^n),$$

where the approximation of v at time t_n is denoted by $v^n = v(t_n)$, the approximation of $\boldsymbol{u}$ at time t_n is represented by $\boldsymbol{u}^n = \boldsymbol{u}(t_n)$, and the truncation errors are defined by

$$
\begin{aligned}
E_\phi^{n+1} &= \frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\delta t} - \phi_t^{n+1},\\
E_w^{n+1} &= R^{n+1}\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}} - R^{n+1}\frac{f(\nabla\phi^{n+1})}{\sqrt{E_0(\phi^{n+1})}},\\
E_{\boldsymbol{u}}^{n+1} &= \frac{3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\delta t} - \boldsymbol{u}_t^{n+1},
\end{aligned}
$$

$$E_R^{n+1} = \frac{3R^{n+1} - 4R^n + R^{n-1}}{2\delta t} - R_t^{n+1} + \frac{1}{2\sqrt{E_0(\phi^{n+1})}}(f(\nabla\phi^{n+1}), \nabla\phi_t^{n+1})$$
$$- \frac{1}{2\sqrt{E_0(\hat{\phi}^n)}}(f(\nabla\hat{\phi}^n), \frac{3\nabla\phi^{n+1} - 4\nabla\phi^n + \nabla\phi^{n-1}}{2\delta t}).$$

Lemma 4.1. *Under the assumption (66), the truncation errors $E_{\boldsymbol{u}}^{n+1}$, E_w^{n+1} and E_ϕ^{n+1} satisfy that*

$$\max_{0\le n\le N-1}(\|E_{\boldsymbol{u}}^{n+1}\| + \|E_w^{n+1}\| + \|E_\phi^{n+1}\|) \lesssim \delta t^2. \tag{69}$$

Proof. Applying the Taylor formula, the assumption (66) and f is Lipschitz continuous (referring to [6]), we have

$$|E_\phi^{n+1}| = |\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\delta t} - \phi_t^{n+1}| \le C\delta t \int_{t_n}^{t_{n+1}} |\phi_{ttt}(s)|ds \le C\delta t^2,$$
$$|E_{\boldsymbol{u}}^{n+1}| = |\frac{3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\delta t} - \boldsymbol{u}_t^{n+1}| \le C\delta t \int_{t_n}^{t_{n+1}} |\boldsymbol{u}_{ttt}(s)|ds \le C\delta t^2,$$
$$\begin{aligned}|E_w^{n+1}| &= |R^{n+1}\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}} - R^{n+1}\frac{f(\nabla\phi^{n+1})}{\sqrt{E_0(\phi^{n+1})}}| \\ &\le |R^{n+1}|(\frac{\|f(\nabla\hat{\phi}^n) - f(\nabla\phi^{n+1})\|}{\sqrt{E_0(\hat{\phi}^n)}} \\ &\quad + \frac{\|f(\nabla\phi^{n+1})\||(E_0(\phi^{n+1}) - E_0(\hat{\phi}^n))|}{\sqrt{E_0(\hat{\phi}^n)E_0(\phi^{n+1})}(\sqrt{E_0(\hat{\phi}^n)} + \sqrt{E_0(\phi^{n+1})})}) \\ &\le C(\max_{0\le t\le T}|R(t)| + \|\nabla\phi_{tt}\|_{L^\infty(H^1)})\delta t^2.\end{aligned}$$

By a direct calculation, we deduce that

$$\begin{aligned}R_{ttt} &= \frac{3}{8\sqrt{E_0(\phi)^5}}\left(\int_\Omega f(\nabla\phi)\nabla\phi_t d\boldsymbol{x}\right)^3 \\ &\quad - \frac{3}{4\sqrt{E_0(\phi)^3}}\int_\Omega f(\nabla\phi)\nabla\phi_t d\boldsymbol{x}\int_\Omega (f'(\nabla\phi)(\nabla\phi_t)^2 + f(\nabla\phi)\nabla\phi_{tt})d\boldsymbol{x} \\ &\quad + \frac{1}{2\sqrt{E_0(\phi)}}\int_\Omega (f''(\nabla\phi)(\nabla\phi_t)^3 + 3f'(\nabla\phi)\nabla\phi_t\nabla\phi_{tt} + f(\nabla\phi)\nabla\phi_{ttt})d\boldsymbol{x}.\end{aligned}$$

From (66) and (67), we have

$$\begin{aligned}\int_0^T |R_{ttt}|^2 dt &\le C\int_0^T (\|\nabla\phi_t\|^6 + \|\nabla\phi_t\|^2(\|\nabla\phi_t\|^4 + \|\nabla\phi_{tt}\|^2) + \|\nabla\phi_t\|_{L^3}^6 \\ &\quad + \|\nabla\phi_t\|^4 + \|\nabla\phi_{tt}\|^4 + \|\nabla\phi_{ttt}\|^2)dt \le C,\end{aligned}$$

$$\begin{aligned}E_R^{n+1} &= |\frac{3R^{n+1} - 4R^n + R^{n-1}}{2\delta t} - R_t^{n+1} + \frac{1}{2\sqrt{E_0(\phi^{n+1})}}(f(\nabla\phi^{n+1}), \nabla\phi_t^{n+1}) \\ &\quad - \frac{1}{2\sqrt{E_0(\hat{\phi}^n)}}(f(\nabla\hat{\phi}^n), \frac{3\nabla\phi^{n+1} - 4\nabla\phi^n + \nabla\phi^{n-1}}{2\delta t})| \\ &\le C(\delta t\int_{t_n}^{t_{n+1}} |R_{ttt}(s)|ds + \delta t\int_{t_n}^{t_{n+1}} |\nabla\phi_{ttt}(s)|ds + \delta t^2) \le C\delta t^2.\end{aligned}$$

Combining the bounds above, we derive (69). □

Lemma 4.2. *Let $(\phi_h^{n+1}, w_h^{n+1}, \psi_h^{n+1}, \boldsymbol{u}_h^{n+1}, p_h^{n+1})$ be the solution of (15)-(22), we have*

$$\|\boldsymbol{u}_h^{n+1}\|_{L^\infty}^2 \le K_0,\ \|\nabla\phi_h^{n+1}\|_{L^\infty}^2 \le K_0^*. \tag{70}$$

Proof. Here we utilize the mathematical induction method to prove it.

Step 1: For $n=0$, we have $\|\boldsymbol{u}_h^0\|_{L^\infty}^2 \le K_0$ and $\|\nabla\phi_h^0\|_{L^\infty}^2 \le K_0^*$ immediately.

Assuming $\|\boldsymbol{u}_h^n\|_{L^\infty}^2 \le K_0$ and $\|\nabla\phi_h^n\|_{L^\infty}^2 \le K_0^*$ are valid for $n = 0,1,...,N-1$, then we shall prove $\|\boldsymbol{u}_h^{n+1}\|_{L^\infty}^2 \le K_0$ and $\|\nabla\phi_h^{n+1}\|_{L^\infty}^2 \le K_0^*$.

Step 2: Subtracting (15)-(22) from (68), the error equations become

(71a)
$$\begin{aligned}&(\frac{3e_\phi^{n+1}-4e_\phi^n+e_\phi^{n-1}}{2\delta t},\rho_h) + (\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n,\rho_h) + M(e_w^{n+1},\rho_h)\\ &- \beta\delta t^2(w_h^{n+1}\nabla\hat{\phi}_h^n, \rho_h\nabla\hat{\phi}_h^n) = (E_\phi^{n+1},\rho_h),\end{aligned}$$

(71b)
$$\begin{aligned}&-(e_w^{n+1},\sigma_h) + (\nabla e_\psi^{n+1},\nabla\sigma_h) + (\frac{R^{n+1}}{\sqrt{E_0(\hat{\phi}^n)}}f(\nabla\hat{\phi}^n),\nabla\sigma_h)\\ &-(\frac{R_h^{n+1}}{\sqrt{E_0(\hat{\phi}_h^n)}}f(\nabla\hat{\phi}_h^n),\nabla\sigma_h) = (E_w^{n+1},\sigma_h),\end{aligned}$$

(71c)
$$(e_\psi^{n+1},\varphi_h) - (\nabla e_\phi^{n+1},\nabla\varphi_h) = 0,$$

(71d)
$$\begin{aligned}\frac{3e_R^{n+1}-4e_R^n+e_R^{n-1}}{2\delta t} &= \frac{1}{2\sqrt{E_0(\hat{\phi}^n)}}(f(\nabla\hat{\phi}^n), \frac{3\nabla\phi^{n+1}-4\nabla\phi^n+\nabla\phi^{n-1}}{2\delta t})\\ &- \frac{1}{2\sqrt{E_0(\hat{\phi}_h^n)}}(f(\nabla\hat{\phi}_h^n), \frac{3\nabla\phi_h^{n+1}-4\nabla\phi_h^n+\nabla\phi_h^{n-1}}{2\delta t}) + E_R^{n+1},\end{aligned}$$

(71e)
$$\begin{aligned}&(\frac{3\tilde{e}_{\boldsymbol{u}}^{n+1}-4e_{\boldsymbol{u}}^n+e_{\boldsymbol{u}}^{n-1}}{2\delta t},\boldsymbol{v}_h) + b(\boldsymbol{u}^{n+1},\boldsymbol{u}^{n+1},\boldsymbol{v}_h) - b(\hat{\boldsymbol{u}}_h^n,\tilde{\boldsymbol{u}}_h^{n+1},\boldsymbol{v}_h) + \nu(\nabla\tilde{e}_{\boldsymbol{u}}^{n+1},\nabla\boldsymbol{v}_h)\\ &-(p^{n+1}-p_h^n,\nabla\cdot\boldsymbol{v}_h) - (w^{n+1}\nabla\phi^{n+1} - w_h^{n+1}\nabla\hat{\phi}_h^n,\boldsymbol{v}_h) = (E_{\boldsymbol{u}}^{n+1},\boldsymbol{v}_h),\end{aligned}$$

(71f)
$$(\nabla e_d^{n+1}-\nabla e_d^n,\nabla q_h) = -\frac{3}{2\delta t}(\nabla\cdot\tilde{e}_{\boldsymbol{u}}^{n+1},q_h) + (\nabla p^{n+1}-\nabla p^n,\nabla q_h),$$

(71g)
$$(\frac{3e_{\boldsymbol{u}}^{n+1}-3\tilde{e}_{\boldsymbol{u}}^{n+1}}{2\delta t},\boldsymbol{v}_h) - (\nabla\cdot\boldsymbol{v}_h, e_d^{n+1}-e_d^n) = -(\nabla\cdot\boldsymbol{v}_h, p^{n+1}-p^n).$$

Adding and subtracting the projection operators and using (27) and (28), we obtain

(72a)
$$\begin{aligned}&(\frac{3\eta_\phi^{n+1}-4\eta_\phi^n+\eta_\phi^{n-1}}{2\delta t},\rho_h) + M(\eta_w^{n+1},\rho_h)\\ &= -M(\xi_w^{n+1},\rho_h) - (\frac{3\xi_\phi^{n+1}-4\xi_\phi^n+\xi_\phi^{n-1}}{2\delta t},\rho_h)\end{aligned}$$

$$+ (E_\phi^{n+1}, \rho_h) - (\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, \rho_h) + \beta\delta t^2 (w_h^{n+1}\nabla\hat{\phi}_h^n, \rho_h\nabla\hat{\phi}_h^n),$$

(72b)

$$-(\eta_w^{n+1}, \sigma_h) + (\nabla\eta_\psi^{n+1}, \nabla\sigma_h) = (\xi_w^{n+1}, \sigma_h) - (\nabla\xi_\psi^{n+1}, \nabla\sigma_h) + (E_w^{n+1}, \sigma_h)$$
$$- (\frac{R^{n+1}}{\sqrt{E_0(\hat{\phi}^n)}} f(\nabla\hat{\phi}^n), \nabla\sigma_h) + (\frac{R_h^{n+1}}{\sqrt{E_0(\hat{\phi}_h^n)}} f(\nabla\hat{\phi}_h^n), \nabla\sigma_h),$$

(72c)

$$(\eta_\psi^{n+1}, \varphi_h) - (\nabla\eta_\phi^{n+1}, \nabla\varphi_h) = -(\xi_\psi^{n+1}, \varphi_h) + (\nabla\xi_\phi^{n+1}, \nabla\varphi_h),$$

(72d)

$$\frac{3e_R^{n+1} - 4e_R^n + e_R^{n-1}}{2\delta t} = \frac{1}{2}(\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}} - \frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, \frac{3\nabla\phi^{n+1} - 4\nabla\phi^n + \nabla\phi^{n-1}}{2\delta t})$$
$$+ \frac{1}{2}(\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, \frac{3\nabla\eta_\phi^{n+1} - 4\nabla\eta_\phi^n + \nabla\eta_\phi^{n-1}}{2\delta t})$$
$$+ \frac{1}{2}(\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, \frac{3\nabla\xi_\phi^{n+1} - 4\nabla\xi_\phi^n + \nabla\xi_\phi^{n-1}}{2\delta t}) + E_R^{n+1},$$

(72e)

$$(\frac{3\tilde{\eta}_{\boldsymbol{u}}^{n+1} - 4\eta_{\boldsymbol{u}}^n + \eta_{\boldsymbol{u}}^{n-1}}{2\delta t}, \boldsymbol{v}_h) + \nu(\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}, \nabla\boldsymbol{v}_h)$$
$$= -(\frac{3\xi_{\boldsymbol{u}}^{n+1} - 4\xi_{\boldsymbol{u}}^n + \xi_{\boldsymbol{u}}^{n-1}}{2\delta t}, \boldsymbol{v}_h) - \nu(\nabla\xi_{\boldsymbol{u}}^{n+1}, \nabla\boldsymbol{v}_h)$$
$$+ (E_{\boldsymbol{u}}^{n+1}, \boldsymbol{v}_h) - b(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{v}_h) + b(\hat{\boldsymbol{u}}_h^n, \tilde{\boldsymbol{u}}_h^{n+1}, \boldsymbol{v}_h) + (p^{n+1} - p_h^n, \nabla\cdot\boldsymbol{v}_h)$$
$$+ (w^{n+1}\nabla\phi^{n+1} - w_h^{n+1}\nabla\hat{\phi}_h^n, \boldsymbol{v}_h),$$

(72f)

$$(\nabla\eta_d^{n+1} - \nabla\eta_d^n, \nabla q_h) = -\frac{3}{2\delta t}(\nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}, q_h) + (\nabla p^{n+1} - \nabla p^n, \nabla q_h),$$

(72g)

$$(\frac{3\eta_{\boldsymbol{u}}^{n+1} - 3\tilde{\eta}_{\boldsymbol{u}}^{n+1}}{2\delta t}, \boldsymbol{v}_h) - (\nabla\cdot\boldsymbol{v}_h, \eta_d^{n+1} - \eta_d^n)$$
$$= (\nabla\cdot\boldsymbol{v}_h, \xi_d^{n+1} - \xi_d^n) - (\nabla\cdot\boldsymbol{v}_h, p^{n+1} - p^n).$$

By taking $\rho_h = 2\delta t\eta_\phi^{n+1}$ and $\rho_h = 2\delta t\eta_w^{n+1}$ in (72a), respectively, we derive

(73)

$$\frac{1}{2}(\|\eta_\phi^{n+1}\|^2 - \|\eta_\phi^n\|^2 + \|2\eta_\phi^{n+1} - \eta_\phi^n\|^2 - \|2\eta_\phi^n - \eta_\phi^{n-1}\|^2 + \|\eta_\phi^{n+1} - 2\eta_\phi^n + \eta_\phi^{n-1}\|^2)$$
$$+ 2M\delta t(\eta_w^{n+1}, \eta_\phi^{n+1}) = -2M\delta t(\xi_w^{n+1}, \eta_\phi^{n+1}) - (3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}, \eta_\phi^{n+1})$$
$$+ 2\delta t(E_\phi^{n+1}, \eta_\phi^{n+1}) - 2\delta t(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, \eta_\phi^{n+1})$$
$$+ 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, \eta_\phi^{n+1}\nabla\hat{\phi}_h^n),$$

and

$$(74)\quad \begin{aligned} &(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}, \eta_w^{n+1}) + 2M\delta t\|\eta_w^{n+1}\|^2 = -2M\delta t(\xi_w^{n+1}, \eta_w^{n+1}) \\ &\quad - (3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}, \eta_w^{n+1}) + 2\delta t(E_\phi^{n+1}, \eta_w^{n+1}) \\ &\quad - 2\delta t(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, \eta_w^{n+1}) + 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, \eta_w^{n+1}\nabla\hat{\phi}_h^n). \end{aligned}$$

Setting $\sigma_h = 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}$ in (72b), we obtain

$$
\begin{aligned}
&-(\eta_w^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}) + (\nabla\eta_\psi^{n+1}, \nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}))\\
&= (\xi_w^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}) - (\nabla\xi_\psi^{n+1}, \nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}))\\
&\quad + (E_w^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})\\
&\quad - R^{n+1}(\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}} - \frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, \nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}))\\
&\quad - e_R^{n+1}(\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, \nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})).
\end{aligned} \tag{75}
$$

Taking $\varphi_h = -(3\eta_\phi^{n+1} - 4\eta_\phi^n - \eta_\phi^{n-1})$ in (72c), we derive

$$
\begin{aligned}
&\frac{1}{2}(\|\nabla\eta_\phi^{n+1}\|^2 - \|\nabla\eta_\phi^n\|^2 + \|2\nabla\eta_\phi^{n+1} - \nabla\eta_\phi^n\|^2 - \|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2\\
&\quad + \|\nabla\eta_\phi^{n+1} - 2\nabla\eta_\phi^n + \nabla\eta_\phi^{n-1}\|^2) - (\eta_\psi^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})\\
&= (\xi_\psi^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}) - (\nabla\xi_\phi^{n+1}, \nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})).
\end{aligned} \tag{76}
$$

From (43) and (72c), we have

$$
\begin{aligned}
&(3\eta_\psi^{n+1} - 4\eta_\psi^n + \eta_\psi^{n-1}, \varphi_h) - (\nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}), \nabla\varphi_h)\\
&= -(3\xi_\psi^{n+1} - 4\xi_\psi^n + \xi_\psi^{n-1}, \varphi_h) + (\nabla(3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}), \nabla\varphi_h).
\end{aligned} \tag{77}
$$

Taking $\varphi_h = \eta_\psi^{n+1}$ in (77), we derive

$$
\begin{aligned}
&\frac{1}{2}(\|\eta_\psi^{n+1}\|^2 - \|\eta_\psi^n\|^2 + \|2\eta_\psi^{n+1} - \eta_\psi^n\|^2 - \|2\eta_\psi^n - \eta_\psi^{n-1}\|^2 + \|\eta_\psi^{n+1} - 2\eta_\psi^n + \eta_\psi^{n-1}\|^2)\\
&\quad - (\nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}), \nabla\eta_\psi^{n+1}) = -(3\xi_\psi^{n+1} - 4\xi_\psi^n + \xi_\psi^{n-1}, \eta_\psi^{n+1})\\
&\quad + (\nabla(3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}), \nabla\eta_\psi^{n+1}).
\end{aligned} \tag{78}
$$

Multiplying (72d) by $4\delta t e_R^{n+1}$ leads to

$$
\begin{aligned}
&(\|e_R^{n+1}\|^2 - \|e_R^n\|^2 + \|2e_R^{n+1} - e_R^n\|^2 - \|2e_R^n - e_R^{n-1}\|^2 + \|e_R^{n+1} - 2e_R^n + e_R^{n-1}\|^2)\\
&= e_R^{n+1}(\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}} - \frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, 3\nabla\phi^{n+1} - 4\nabla\phi^n + \nabla\phi^{n-1})\\
&\quad + e_R^{n+1}(\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, 3\nabla\eta_\phi^{n+1} - 4\nabla\eta_\phi^n + \nabla\eta_\phi^{n-1})\\
&\quad + e_R^{n+1}(\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, 3\nabla\xi_\phi^{n+1} - 4\nabla\xi_\phi^n + \nabla\xi_\phi^{n-1}) + 4\delta t(E_R^{n+1}, e_R^{n+1}).
\end{aligned} \tag{79}
$$

Taking $\boldsymbol{v}_h = 2\delta t \tilde{\eta}_{\boldsymbol{u}}^{n+1}$ in (72e), from (36) and (49), we obtain

$$
\begin{aligned}
&\frac{1}{2}(\|\eta_{\boldsymbol{u}}^{n+1}\|^2 - \|\eta_{\boldsymbol{u}}^{n}\|^2 + \|2\eta_{\boldsymbol{u}}^{n+1} - \eta_{\boldsymbol{u}}^{n}\|^2 \\
&- \|2\eta_{\boldsymbol{u}}^{n} - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \|\eta_{\boldsymbol{u}}^{n+1} - 2\eta_{\boldsymbol{u}}^{n} + \eta_{\boldsymbol{u}}^{n-1}\|^2) \\
&\quad + \frac{3}{2}(\|\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2 - \|\eta_{\boldsymbol{u}}^{n+1}\|^2 + \|\tilde{\eta}_{\boldsymbol{u}}^{n+1} - \eta_{\boldsymbol{u}}^{n+1}\|^2) \\
&\quad + (3\eta_{\boldsymbol{u}}^{n+1} - 4\eta_{\boldsymbol{u}}^{n} + \eta_{\boldsymbol{u}}^{n-1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1} - \eta_{\boldsymbol{u}}^{n+1}) + 2\nu\delta t\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2 \\
&= -(3\xi_{\boldsymbol{u}}^{n+1} - 4\xi_{\boldsymbol{u}}^{n} + \xi_{\boldsymbol{u}}^{n-1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) - 2\nu\delta t(\nabla\xi_{\boldsymbol{u}}^{n+1}, \nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}) \\
&\quad + 2\delta t(E_{\boldsymbol{u}}^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) - 2\delta t b(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) + 2\delta t b(\hat{\boldsymbol{u}}_h^{n}, \tilde{\boldsymbol{u}}_h^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) \\
&\quad + 2\delta t(w^{n+1}\nabla\phi^{n+1} - w_h^{n+1}\nabla\hat{\phi}_h^{n}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) + 2\delta t(p^{n+1} - p^{n}, \nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}) \\
&\quad + 2\delta t(\xi_d^{n}, \nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}) + 2\delta t(\eta_d^{n}, \nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}) + 2\delta t(S_h^{n}, \nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}).
\end{aligned}
\tag{80}
$$

Setting $q_h = \frac{4\delta t^2}{3}\eta_d^{n+1}$ in (72f), we have

$$
\begin{aligned}
&\frac{2\delta t^2}{3}(\|\nabla\eta_d^{n+1}\|^2 - \|\nabla\eta_d^{n}\|^2 + \|\nabla\eta_d^{n+1} - \nabla\eta_d^{n}\|^2) \\
&= -2\delta t(\nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}, \eta_d^{n+1}) + \frac{4\delta t^2}{3}(\nabla p^{n+1} - \nabla p^{n}, \nabla\eta_d^{n+1}).
\end{aligned}
\tag{81}
$$

Due to $(\nabla\cdot e_{\boldsymbol{u}}^{n+1}, q_h) = 0$ and $(\nabla\cdot\xi_{\boldsymbol{u}}^{n+1}, q_h) = 0$, we can derive the fact that $(\nabla\cdot\eta_{\boldsymbol{u}}^{n+1}, q_h) = 0$. Using it and taking $\boldsymbol{v}_h = 2\delta t\tilde{\eta}_{\boldsymbol{u}}^{n+1}, \boldsymbol{v}_h = \frac{2\delta t}{3}(3\eta_{\boldsymbol{u}}^{n+1} - 4\eta_{\boldsymbol{u}}^{n} + \eta_{\boldsymbol{u}}^{n-1})$ in (72g), we get

$$
\begin{aligned}
&\frac{3}{2}(\|\eta_{\boldsymbol{u}}^{n+1}\|^2 - \|\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2 - \|\tilde{\eta}_{\boldsymbol{u}}^{n+1} - \eta_{\boldsymbol{u}}^{n+1}\|^2) - 2\delta t(\nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}, \eta_d^{n+1} - \eta_d^{n}) \\
&= 2\delta t(\nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}, \xi_d^{n+1} - \xi_d^{n}) - 2\delta t(\nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}, p^{n+1} - p^{n}),
\end{aligned}
\tag{82}
$$

and

$$
\begin{aligned}
&(\eta_{\boldsymbol{u}}^{n+1} - \tilde{\eta}_{\boldsymbol{u}}^{n+1}, 3\eta_{\boldsymbol{u}}^{n+1} - 4\eta_{\boldsymbol{u}}^{n} + \eta_{\boldsymbol{u}}^{n-1}) \\
&= \frac{2\delta t}{3}(\nabla\cdot(3\eta_{\boldsymbol{u}}^{n+1} - 4\eta_{\boldsymbol{u}}^{n} + \eta_{\boldsymbol{u}}^{n-1}), \eta_d^{n+1} - \eta_d^{n}) \\
&\quad + \frac{2\delta t}{3}(\nabla\cdot(3\eta_{\boldsymbol{u}}^{n+1} - 4\eta_{\boldsymbol{u}}^{n} + \eta_{\boldsymbol{u}}^{n-1}), \xi_d^{n+1} - \xi_d^{n}) \\
&\quad - \frac{2\delta t}{3}(\nabla\cdot(3\eta_{\boldsymbol{u}}^{n+1} - 4\eta_{\boldsymbol{u}}^{n} + \eta_{\boldsymbol{u}}^{n-1}), p^{n+1} - p^{n}) = 0.
\end{aligned}
\tag{83}
$$

Combining (73)-(76) and (78)-(83), we derive

(84)

$$
\begin{aligned}
&\frac{1}{2}(\|\eta_{\phi}^{n+1}\|^2 - \|\eta_{\phi}^{n}\|^2 + \|2\eta_{\phi}^{n+1} - \eta_{\phi}^{n}\|^2 - \|2\eta_{\phi}^{n} - \eta_{\phi}^{n-1}\|^2 + \|\eta_{\phi}^{n+1} - 2\eta_{\phi}^{n} + \eta_{\phi}^{n-1}\|^2) \\
&\quad + \frac{1}{2}(\|\eta_{\psi}^{n+1}\|^2 - \|\eta_{\psi}^{n}\|^2 + \|2\eta_{\psi}^{n+1} - \eta_{\psi}^{n}\|^2 - \|2\eta_{\psi}^{n} - \eta_{\psi}^{n-1}\|^2 + \|\eta_{\psi}^{n+1} - 2\eta_{\psi}^{n} + \eta_{\psi}^{n-1}\|^2) \\
&\quad + (\|e_R^{n+1}\|^2 - \|e_R^{n}\|^2 + \|2e_R^{n+1} - e_R^{n}\|^2 - \|2e_R^{n} - e_R^{n-1}\|^2 + \|e_R^{n+1} - 2e_R^{n} + e_R^{n-1}\|^2) \\
&\quad + \frac{1}{2}(\|\eta_{\boldsymbol{u}}^{n+1}\|^2 - \|\eta_{\boldsymbol{u}}^{n}\|^2 + \|2\eta_{\boldsymbol{u}}^{n+1} - \eta_{\boldsymbol{u}}^{n}\|^2 - \|2\eta_{\boldsymbol{u}}^{n} - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \|\eta_{\boldsymbol{u}}^{n+1} - 2\eta_{\boldsymbol{u}}^{n} + \eta_{\boldsymbol{u}}^{n-1}\|^2) \\
&\quad + \frac{1}{2}(\|\nabla\eta_{\phi}^{n+1}\|^2 - \|\nabla\eta_{\phi}^{n}\|^2 + \|2\nabla\eta_{\phi}^{n+1} - \nabla\eta_{\phi}^{n}\|^2 - \|2\nabla\eta_{\phi}^{n} - \nabla\eta_{\phi}^{n-1}\|^2 \\
&\quad + \|\nabla\eta_{\phi}^{n+1} - 2\nabla\eta_{\phi}^{n} + \nabla\eta_{\phi}^{n-1}\|^2) + 2M\delta t\|\eta_w^{n+1}\|^2 + 2\nu\delta t\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2 \\
&\quad + \frac{2\delta t^2}{3}(\|\nabla\eta_d^{n+1}\|^2 - \|\nabla\eta_d^{n}\|^2 + \|\nabla\eta_d^{n+1} - \nabla\eta_d^{n}\|^2)
\end{aligned}
$$

$$
\begin{aligned}
&= -2M\delta t(\eta_w^{n+1}, \eta_\phi^{n+1}) - 2M\delta t(\xi_w^{n+1}, \eta_\phi^{n+1}) - 2M\delta t(\xi_w^{n+1}, \eta_w^{n+1}) \\
&\quad - 2\nu\delta t(\nabla\xi_{\boldsymbol{u}}^{n+1}, \nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}) + 2\delta t(E_\phi^{n+1}, \eta_\phi^{n+1}) + 2\delta t(E_\phi^{n+1}, \eta_w^{n+1}) \\
&\quad + 4\delta t(E_R^{n+1}, e_R^{n+1}) + 2\delta t(E_{\boldsymbol{u}}^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) - (3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}, \eta_\phi^{n+1}) \\
&\quad - (3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}, \eta_w^{n+1}) - (3\xi_\psi^{n+1} - 4\xi_\psi^n - \xi_\psi^{n-1}, \eta_\psi^{n+1}) \\
&\quad - (3\xi_{\boldsymbol{u}}^{n+1} - 4\xi_{\boldsymbol{u}}^n + \xi_{\boldsymbol{u}}^{n-1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) + (\eta_\psi^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}) \\
&\quad + (\xi_\psi^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}) - (\nabla\xi_\phi^{n+1}, \nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})) \\
&\quad - (\nabla\xi_\psi^{n+1}, \nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})) + (\nabla(3\xi_\phi^{n+1} - 4\xi_\phi^n - \xi_\phi^{n-1}), \nabla\eta_\psi^{n+1}) \\
&\quad + 2\delta t(S_h^n, \nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}) + \frac{4\delta t^2}{3}(\nabla p^{n+1} - \nabla p^n, \nabla\eta_d^{n+1}) \\
&\quad + 2\delta t(\nabla\cdot\tilde{\eta}_{\boldsymbol{u}}^{n+1}, \xi_d^{n+1}) - 2\delta t b(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) + 2\delta t b(\hat{\boldsymbol{u}}_h^n, \tilde{\boldsymbol{u}}_h^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) \\
&\quad + 2\delta t(w^{n+1}\nabla\phi^{n+1} - w_h^{n+1}\nabla\hat{\phi}_h^n, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) - 2\delta t(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, \eta_\phi^{n+1}) \\
&\quad + 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, \eta_\phi^{n+1}\nabla\hat{\phi}_h^n) - 2\delta t(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, \eta_w^{n+1}) \\
&\quad + 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, \eta_w^{n+1}\nabla\hat{\phi}_h^n) + (E_w^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}) \\
&\quad + (\xi_w^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1}) + e_R^{n+1}(\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, 3\nabla\xi_\phi^{n+1} - 4\nabla\xi_\phi^n + \nabla\xi_\phi^{n-1}) \\
&\quad + e_R^{n+1}(\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}} - \frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, 3\nabla\phi^{n+1} - 4\nabla\phi^n + \nabla\phi^{n-1}) \\
&\quad - R^{n+1}(\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}} - \frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}, \nabla(3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})) \\
&= I_1 + I_2 + \cdots + I_{32}.
\end{aligned}
$$

From the Cauchy-Schwarz inequality, Young's inequality [13] and Lemma 4.1, we can get

$$
\begin{aligned}
I_1 &= -2M\delta t(\eta_w^{n+1}, \eta_\phi^{n+1}) \lesssim \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2 + \delta t\|\eta_\phi^{n+1}\|^2, \\
I_2 &= -2M\delta t(\xi_w^{n+1}, \eta_\phi^{n+1}) \lesssim \delta t h^{2k+2} + \delta t\|\eta_\phi^{n+1}\|^2, \\
I_3 &= -2M\delta t(\xi_w^{n+1}, \eta_w^{n+1}) \lesssim \delta t h^{2k+2} + \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2, \\
I_4 &= -2\nu\delta t(\nabla\xi_{\boldsymbol{u}}^{n+1}, \nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}) \lesssim \delta t h^{2k} + \frac{\nu}{24}\delta t\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2, \\
I_5 &= 2\delta t(E_\phi^{n+1}, \eta_\phi^{n+1}) \lesssim \delta t^5 + \delta t\|\eta_\phi^{n+1}\|^2, \\
I_6 &= 2\delta t(E_\phi^{n+1}, \eta_w^{n+1}) \lesssim \delta t^5 + \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2, \\
I_7 &= 4\delta t(E_R^{n+1}, e_R^{n+1}) \lesssim \delta t^5 + \delta t|e_R^{n+1}|^2.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, Young's inequality, and Poincaré's inequality, we derive

$$
I_8 = 2\delta t(E_{\boldsymbol{u}}^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) \lesssim \delta t^5 + \frac{\nu}{24}\delta t\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2.
$$

Applying Cauchy-Schwarz inequality, Young's inequality and (29), we deduce

$$
I_9 = -(3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}, \eta_\phi^{n+1}) \lesssim \int_{t_n}^{t_{n+1}} \|\frac{\partial\xi_\phi}{\partial t}\|^2 dt + \delta t\|\eta_\phi^{n+1}\|^2
$$

$$
\begin{aligned}
&\lesssim h^{2k+2}\int_{t_n}^{t_{n+1}}\|\phi_t\|^2_{H^{k+1}}dt+\delta t\|\eta_\phi^{n+1}\|^2\\
&\lesssim \delta t h^{2k+2}+\delta t\|\eta_\phi^{n+1}\|^2,\\
I_{10}=-(3\xi_\phi^{n+1}-4\xi_\phi^n+\xi_\phi^{n-1},\eta_w^{n+1})&\lesssim\int_{t_n}^{t_{n+1}}\|\frac{\partial\xi_\phi}{\partial t}\|^2dt+\delta t\|\eta_w^{n+1}\|^2\\
&\lesssim h^{2k+2}\int_{t_n}^{t_{n+1}}\|\phi_t\|^2_{H^{k+1}}dt+\delta t\|\eta_w^{n+1}\|^2\\
&\lesssim \delta t h^{2k+2}+\frac{M}{24}\delta t\|\eta_w^{n+1}\|^2,\\
I_{11}=-(3\xi_\psi^{n+1}-4\xi_\psi^n-\xi_\psi^{n-1},\eta_\psi^{n+1})&\lesssim\int_{t_n}^{t_{n+1}}\|\frac{\partial\xi_\psi}{\partial t}\|^2dt+\delta t\|\eta_\psi^{n+1}\|^2\\
&\lesssim h^{2k+2}\int_{t_n}^{t_{n+1}}\|\psi_t\|^2_{H^{k+1}}dt+\delta t\|\eta_\psi^{n+1}\|^2\\
&\lesssim \delta t h^{2k+2}+\delta t\|\eta_\psi^{n+1}\|^2,\\
I_{12}=-(3\xi_{\boldsymbol{u}}^{n+1}-4\xi_{\boldsymbol{u}}^n+\xi_{\boldsymbol{u}}^{n-1},\tilde\eta_{\boldsymbol{u}}^{n+1})&\lesssim\int_{t_n}^{t_{n+1}}\|\frac{\partial\xi_{\boldsymbol{u}}}{\partial t}\|^2dt+\delta t\|\nabla\tilde\eta_{\boldsymbol{u}}^{n+1}\|^2\\
&\lesssim h^{2k+2}\int_{t_n}^{t_{n+1}}\|\boldsymbol{u}_t\|^2_{H^{k+1}}dt+\delta t\|\nabla\tilde\eta_{\boldsymbol{u}}^{n+1}\|^2\\
&\lesssim \delta t h^{2k+2}+\frac{\nu}{24}\delta t\|\nabla\tilde\eta_{\boldsymbol{u}}^{n+1}\|^2.
\end{aligned}
$$

Using (26), we have

$$
\begin{aligned}
I_{15}&=-(\nabla\xi_\phi^{n+1},\nabla(3\eta_\phi^{n+1}-4\eta_\phi^n+\eta_\phi^{n-1}))=0,\\
I_{16}&=-(\nabla\xi_\psi^{n+1},\nabla(3\eta_\phi^{n+1}-4\eta_\phi^n+\eta_\phi^{n-1}))=0,\\
I_{17}&=(\nabla(3\xi_\phi^{n+1}-4\xi_\phi^n-\xi_\phi^{n-1}),\nabla\eta_\psi^{n+1})=0.
\end{aligned}
$$

Since $\tilde\eta_{\boldsymbol{u}}^{n+1}=\mathcal{P}_h\boldsymbol{u}^{n+1}-\tilde{\boldsymbol{u}}_h^{n+1}$, using (23), (56) and (63), we get

$$
\begin{aligned}
I_{18}&=2\delta t(S_h^n,\nabla\cdot\tilde\eta_{\boldsymbol{u}}^{n+1})=-\delta t(S_h^n,\nabla\cdot\tilde{\boldsymbol{u}}_h^{n+1})\\
&=-\frac{\delta t}{\nu}(S_h^n,S_h^{n+1}-S_h^n)\\
&=-\frac{\delta t}{2\nu}(\|S_h^{n+1}\|^2-\|S_h^n\|^2)+\frac{\delta t}{2\nu}\|S_h^n-S_h^{n+1}\|^2\\
&\le-\frac{\delta t}{2\nu}(\|S_h^{n+1}\|^2-\|S_h^n\|^2)+\frac{\nu}{2}\delta t\|\nabla\tilde\eta_{\boldsymbol{u}}^{n+1}\|^2.
\end{aligned}
$$

From Taylor formula and CFL condition $\delta t\le h^2$, it is easily get that

$$
I_{19}=\frac{4\delta t^2}{3}(\nabla p^{n+1}-\nabla p^n,\nabla\eta_d^{n+1})\lesssim\frac{4\delta t^3}{3}\|\nabla p_t(t_n)\|\|\nabla\eta_d^{n+1}\|\lesssim\delta t h^4+\delta t^3\|\nabla\eta_d^{n+1}\|^2.
$$

Utilizing Young's inequality, (29) and the fact that $\|\nabla\cdot\boldsymbol{v}\|^2\le\|\nabla\boldsymbol{v}\|^2$ (see [21]), we deduce

$$
I_{20}=2\delta t(\nabla\cdot\tilde\eta_{\boldsymbol{u}}^{n+1},\xi_d^{n+1})\lesssim\delta t h^{2k}+\frac{\nu}{24}\delta t\|\nabla\tilde\eta_{\boldsymbol{u}}^{n+1}\|^2.
$$

From $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v})=0,\boldsymbol{u},\boldsymbol{v}\in[H_0^1(\Omega)]^d$, General Hölder inequality, Poincaré's inequality and (29), we derive

$$
I_{21}+I_{22}=-2\delta t b(\boldsymbol{u}^{n+1},\boldsymbol{u}^{n+1},\tilde\eta_{\boldsymbol{u}}^{n+1})+2\delta t b(\hat{\boldsymbol{u}}_h^n,\tilde{\boldsymbol{u}}_h^{n+1},\tilde\eta_{\boldsymbol{u}}^{n+1})
$$

$$
\begin{aligned}
&= -2\delta t b(\hat{\boldsymbol{u}}_h^n, \tilde{e}_{\boldsymbol{u}}^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1}) - 2\delta t b(2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}, \boldsymbol{u}^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1})\\
&\quad - 2\delta t b(\boldsymbol{u}^{n+1} - \hat{\boldsymbol{u}}^n, \boldsymbol{u}^{n+1}, \tilde{\eta}_{\boldsymbol{u}}^{n+1})\\
&\lesssim \delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|\nabla\tilde{\xi}_{\boldsymbol{u}}^{n+1}\|\|\tilde{\eta}_{\boldsymbol{u}}^{n+1}\| + \delta t\|2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}\|\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|\\
&\quad + \delta t^3 \max_{t_n\le t\le t_{n+1}}\|\boldsymbol{u}_{tt}(t)\|\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|\\
&\lesssim \delta t^5 + \delta t h^{2k} + \delta t h^{2k+2} + \delta t\|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \frac{\nu}{24}\delta t\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2.
\end{aligned}
$$

Using General Hölder inequality, Taylor formula and (29) yields

$$
\begin{aligned}
I_{23} &= 2\delta t(w^{n+1}\nabla\phi^{n+1} - w_h^{n+1}\nabla\hat{\phi}_h^n, \tilde{\eta}_{\boldsymbol{u}}^{n+1})\\
&= \delta t(w^{n+1}(\nabla\phi^{n+1} - \nabla\hat{\phi}^n) + w^{n+1}\nabla(2e_\phi^n - e_\phi^{n-1}) - e_w^{n+1}\nabla\hat{\phi}_h^n, \tilde{\eta}_{\boldsymbol{u}}^{n+1})\\
&\lesssim \delta t\|w^{n+1}\|_{L^\infty}\|\nabla\phi^{n+1} - \nabla\hat{\phi}^n\|\|\tilde{\eta}_{\boldsymbol{u}}^{n+1}\| + \delta t\|w^{n+1}\|_{L^\infty}\|\nabla(2e_\phi^n - e_\phi^{n-1})\|\|\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|\\
&\quad + \delta t\|e_w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}\|\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|\\
&\lesssim \delta t^5 + \delta t h^{2k} + \delta t h^{2k+2} + \delta t\|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2 + \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2 + \frac{\nu}{24}\delta t\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2,
\end{aligned}
$$

where we choose ν and M so that $\frac{18K_0^*}{\nu} \le \frac{M}{24}$.

$$
\begin{aligned}
I_{24} &= -2\delta t(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, \eta_\phi^{n+1})\\
&= -\delta t((\boldsymbol{u}^{n+1} - \hat{\boldsymbol{u}}^n)\nabla\phi^{n+1}, \eta_\phi^{n+1}) - \delta t((2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1})\nabla\phi^{n+1}, \eta_\phi^{n+1})\\
&\quad - \delta t(\hat{\boldsymbol{u}}_h^n\nabla(\phi^{n+1} - \hat{\phi}^n), \eta_\phi^{n+1}) - \delta t(\hat{\boldsymbol{u}}_h^n(2\nabla e_\phi^n - \nabla e_\phi^{n-1}), \eta_\phi^{n+1})\\
&\lesssim \delta t\|\boldsymbol{u}^{n+1} - \hat{\boldsymbol{u}}^n\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_\phi^{n+1}\| + \delta t\|2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_\phi^{n+1}\|\\
&\quad + \delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|\nabla(\phi^{n+1} - \hat{\phi}^n)\|\|\eta_\phi^{n+1}\| + \delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|2\nabla e_\phi^n - \nabla e_\phi^{n-1}\|\|\eta_\phi^{n+1}\|\\
&\lesssim \delta t^5 + \delta t h^{2k} + \delta t h^{2k+2} + \delta t\|\eta_\phi^{n+1}\|^2 + \delta t\|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \delta t\|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2,\\
I_{25} &= 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, \eta_\phi^{n+1}\nabla\hat{\phi}_h^n)\\
&= 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n - \mathcal{Q}_h w^{n+1}\nabla\hat{\phi}_h^n + \mathcal{Q}_h w^{n+1}\nabla\hat{\phi}_h^n, \eta_\phi^{n+1}\nabla\hat{\phi}_h^n)\\
&= -2\beta\delta t^3(\eta_w^{n+1}\nabla\hat{\phi}_h^n, \eta_\phi^{n+1}\nabla\hat{\phi}_h^n) + 2\beta\delta t^3(\mathcal{Q}_h w^{n+1}\nabla\hat{\phi}_h^n, \eta_\phi^{n+1}\nabla\hat{\phi}_h^n)\\
&\lesssim 2\beta\delta t^3\|\eta_w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2\|\eta_\phi^{n+1}\| + 2\beta\delta t^3\|\mathcal{Q}_h w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2\|\eta_\phi^{n+1}\|\\
&\lesssim \delta t^5 + \delta t\|\eta_\phi^{n+1}\|^2 + \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2,\\
I_{26} &= -2(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, \eta_w^{n+1})\\
&= -\delta t((\boldsymbol{u}^{n+1} - \hat{\boldsymbol{u}}^n)\nabla\phi^{n+1}, \eta_w^{n+1}) - \delta t((2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1})\nabla\phi^{n+1}, \eta_w^{n+1})\\
&\quad - \delta t(\hat{\boldsymbol{u}}_h^n\nabla(\phi^{n+1} - \hat{\phi}^n), \eta_w^{n+1}) - \delta t(\hat{\boldsymbol{u}}_h^n(2\nabla e_\phi^n - \nabla e_\phi^{n-1}), \eta_w^{n+1})\\
&\lesssim \delta t\|\boldsymbol{u}^{n+1} - \hat{\boldsymbol{u}}^n\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_w^{n+1}\| + \delta t\|2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_w^{n+1}\|\\
&\quad + \delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|\nabla(\phi^{n+1} - \hat{\phi}^n)\|\|\eta_w^{n+1}\| + \delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|2\nabla e_\phi^n - \nabla e_\phi^{n-1}\|\|\eta_w^{n+1}\|\\
&\lesssim \delta t^5 + \delta t h^{2k} + \delta t h^{2k+2} + \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2\\
&\quad + \delta t\|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \delta t\|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2,\\
I_{27} &= 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, \eta_w^{n+1}\nabla\hat{\phi}_h^n)\\
&= 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n - \mathcal{Q}_h w^{n+1}\nabla\hat{\phi}_h^n + \mathcal{Q}_h w^{n+1}\nabla\hat{\phi}_h^n, \eta_w^{n+1}\nabla\hat{\phi}_h^n)\\
&= -2\beta\delta t^3(\eta_w^{n+1}\nabla\hat{\phi}_h^n, \eta_w^{n+1}\nabla\hat{\phi}_h^n) + 2\beta\delta t^3(\mathcal{Q}_h w^{n+1}\nabla\hat{\phi}_h^n, \eta_w^{n+1}\nabla\hat{\phi}_h^n)
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim 2\beta\delta t^3\|\eta_w^{n+1}\|^2\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2 + 2\beta\delta t^3\|\mathcal{Q}_h w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2\|\eta_w^{n+1}\|\\
&\lesssim \delta t^5 + \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2.
\end{aligned}
$$

Taking $\rho_h = 2\delta t e_\psi^{n+1}$, $\rho_h = 2\delta t\xi_w^{n+1}$ and $\rho_h = 2\delta t E_w^{n+1}$ in (72a), by using Cauchy-Schwarz inequality, Young's inequality and (29), we get

$$
\begin{aligned}
&I_{13} + I_{14}\\
&= (e_\psi^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})\\
&= -2M\delta t(\eta_w^{n+1}, e_\psi^{n+1}) - 2M\delta t(\xi_w^{n+1}, e_\psi^{n+1}) - (3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}, e_\psi^{n+1})\\
&\quad - 2\delta t(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, e_\psi^{n+1}) + 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, e_\psi^{n+1}\nabla\hat{\phi}_h^n)\\
&\quad + 2\delta t(E_\phi^{n+1}, e_\psi^{n+1})\\
&\lesssim M\delta t\|\eta_w^{n+1}\|\|e_\psi^{n+1}\| + M\delta t\|\xi_w^{n+1}\|\|e_\psi^{n+1}\| + \delta t\left\|\frac{3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}}{\delta t}\right\|\|e_\psi^{n+1}\|\\
&\quad + \delta t\|\boldsymbol{u}^{n+1} - \hat{\boldsymbol{u}}^n\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_w^{n+1}\| + \delta t\|2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_w^{n+1}\|\\
&\quad + \delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|\nabla(\phi^{n+1} - \hat{\phi}^n)\|\|\eta_w^{n+1}\| + \delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|2\nabla e_\phi^n - \nabla e_\phi^{n-1}\|\|\eta_w^{n+1}\|\\
&\quad + 2\beta\delta t^3\|\eta_w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2\|e_\psi^{n+1}\| + 2\beta\delta t^3\|\mathcal{Q}_h w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2\|e_\psi^{n+1}\|\\
&\quad + \delta t\|E_\phi^{n+1}\|\|e_\psi^{n+1}\|\\
&\lesssim \delta t^5 + \delta t h^{2k} + \delta t h^{2k+2} + \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2 + \delta t\|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2\\
&\quad + \delta t\|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2 + \delta t\|\eta_\psi^{n+1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
I_{28} &= (E_w^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})\\
&= -2M\delta t(\eta_w^{n+1}, E_w^{n+1}) - 2M\delta t(\xi_w^{n+1}, E_w^{n+1}) - (3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}, E_w^{n+1})\\
&\quad - 2\delta t(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, E_w^{n+1})\\
&\quad + 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, E_w^{n+1}\nabla\hat{\phi}_h^n) + 2\delta t(E_\phi^{n+1}, E_w^{n+1})\\
&\lesssim M\delta t\|\eta_w^{n+1}\|\|E_w^{n+1}\| + M\delta t\|\xi_w^{n+1}\|\|E_w^{n+1}\| + \delta t\left\|\frac{3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}}{\delta t}\right\|\|E_w^{n+1}\|\\
&\quad + \delta t\|\boldsymbol{u}^{n+1} - \hat{\boldsymbol{u}}^n\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_w^{n+1}\| + \delta t\|2e_{\boldsymbol{u}}^n - e_{\boldsymbol{u}}^{n-1}\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_w^{n+1}\|\\
&\quad + \delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|\nabla(\phi^{n+1} - \hat{\phi}^n)\|\|\eta_w^{n+1}\| + \delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|2\nabla e_\phi^n - \nabla e_\phi^{n-1}\|\|\eta_w^{n+1}\|\\
&\quad + 2\beta\delta t^3\|\eta_w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2\|E_w^{n+1}\| + 2\beta\delta t^3\|\mathcal{Q}_h w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2\|E_w^{n+1}\|\\
&\quad + \delta t\|E_\phi^{n+1}\|\|E_w^{n+1}\|\\
&\lesssim \delta t^5 + \delta t h^{2k} + \delta t h^{2k+2} + \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2\\
&\quad + \delta t\|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \delta t\|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
I_{29} &= (\xi_w^{n+1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})\\
&= -2M\delta t(\eta_w^{n+1}, \xi_w^{n+1}) - 2M\delta t(\xi_w^{n+1}, \xi_w^{n+1}) - (3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}, \xi_w^{n+1})\\
&\quad - 2\delta t(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, \xi_w^{n+1})\\
&\quad + 2\beta\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, \xi_w^{n+1}\nabla\hat{\phi}_h^n) + 2\delta t(E_\phi^{n+1}, \xi_w^{n+1})\\
&\lesssim M\delta t\|\eta_w^{n+1}\|\|\xi_w^{n+1}\| + M\delta t\|\xi_w^{n+1}\|\|\xi_w^{n+1}\| + \delta t\left\|\frac{3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}}{\delta t}\right\|\|\xi_w^{n+1}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\delta t\|\boldsymbol{u}^{n+1}-\hat{\boldsymbol{u}}^n\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_w^{n+1}\|+\delta t\|2e_{\boldsymbol{u}}^n-e_{\boldsymbol{u}}^{n-1}\|\|\nabla\phi^{n+1}\|_{L^\infty}\|\eta_w^{n+1}\|\\
&+\delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|\nabla(\phi^{n+1}-\hat{\phi}^n)\|\|\eta_w^{n+1}\|+\delta t\|\hat{\boldsymbol{u}}_h^n\|_{L^\infty}\|2\nabla e_\phi^n-\nabla e_\phi^{n-1}\|\|\eta_w^{n+1}\|\\
&+2\beta\delta t^3\|\eta_w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2\|\xi_w^{n+1}\|+2\beta\delta t^3\|\mathcal{Q}_h w^{n+1}\|\|\nabla\hat{\phi}_h^n\|_{L^\infty}^2\|\xi_w^{n+1}\|\\
&+\delta t\|E_\phi^{n+1}\|\|\xi_w^{n+1}\|\\
\lesssim\,&\delta t^5+\delta t h^{2k}+\delta t h^{2k+2}+\frac{M}{24}\delta t\|\eta_w^{n+1}\|^2\\
&+\delta t\|2\eta_{\boldsymbol{u}}^n-\eta_{\boldsymbol{u}}^{n-1}\|^2+\delta t\|2\nabla\eta_\phi^n-\nabla\eta_\phi^{n-1}\|^2.
\end{aligned}
$$

Applying Cauchy-Schwarz inequality, Young's inequality, (29) and (67), we get

$$
\begin{aligned}
I_{30}&=e_R^{n+1}(\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}},3\nabla\xi_\phi^{n+1}-4\nabla\xi_\phi^n+\nabla\xi_\phi^{n-1})\\
&\lesssim\int_{t_n}^{t_{n+1}}\|\frac{\partial\nabla\xi_\phi}{\partial t}\|^2dt+\delta t|e_R^{n+1}|^2\\
&\lesssim h^{2k}\int_{t_n}^{t_{n+1}}\|\phi_t\|_{H^{k+1}}^2dt+\delta t|e_R^{n+1}|^2\\
&\lesssim\delta t h^{2k}+\delta t|e_R^{n+1}|^2.
\end{aligned}
$$

For the last two terms I_{31} and I_{32}, it is easy to get that

$$
\begin{aligned}
&\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}}-\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}\\
=&\frac{f(\nabla\hat{\phi}^n)-f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}^n)}}+\frac{f(\nabla\hat{\phi}_h^n)(E_0(\hat{\phi}_h^n)-E_0(\hat{\phi}^n))}{\sqrt{E_0(\hat{\phi}^n)E_0(\hat{\phi}_h^n)}(\sqrt{E_0(\hat{\phi}^n)}+\sqrt{E_0(\hat{\phi}_h^n)})}.
\end{aligned}
$$

Using the fact that $E_0(\phi)\geq B$ and (67), we can derive

$$
\begin{aligned}
|\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}}-\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}}|&\lesssim|f(\nabla\hat{\phi}^n)-f(\nabla\hat{\phi}_h^n)|+|f(\nabla\hat{\phi}_h^n)(E_0(\hat{\phi}_h^n)-E_0(\hat{\phi}^n))|\\
&\lesssim|\nabla\hat{\phi}^n-\nabla\hat{\phi}_h^n|=|2\nabla e_\phi^n-\nabla e_\phi^{n-1}|.
\end{aligned}
$$

Using the above formula, Cauchy-Schwarz inequality, Young's inequality, and (29), we derive

$$
\begin{aligned}
I_{31}&=e_R^{n+1}(\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}}-\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}},3\nabla\phi^{n+1}-4\nabla\phi^n+\nabla\phi^{n-1})\\
&=2\delta t e_R^{n+1}(\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}}-\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}},\frac{3\nabla\phi^{n+1}-4\nabla\phi^n+\nabla\phi^{n-1}}{2\delta t})\\
&\lesssim\delta t h^{2k}+\delta t|e_R^{n+1}|^2+\delta t\|2\nabla\eta_\phi^n-\nabla\eta_\phi^{n-1}\|^2.
\end{aligned}
$$

Applying (71c), taking $\rho_h=2\delta t R^{n+1}(2e_\psi^n-e_\psi^{n-1})$ in (72a), using Cauchy-Schwarz inequality, Young's inequality and (29), we have

$$
\begin{aligned}
I_{32}=&-R^{n+1}(\frac{f(\nabla\hat{\phi}^n)}{\sqrt{E_0(\hat{\phi}^n)}}-\frac{f(\nabla\hat{\phi}_h^n)}{\sqrt{E_0(\hat{\phi}_h^n)}},\nabla(3\eta_\phi^{n+1}-4\eta_\phi^n+\eta_\phi^{n-1}))\\
&\lesssim|R^{n+1}(2\nabla e_\phi^n-\nabla e_\phi^{n-1},\nabla(3\eta_\phi^{n+1}-4\eta_\phi^n+\eta_\phi^{n-1}))|
\end{aligned}
$$

$$
\begin{aligned}
&= |R^{n+1}(2e_\psi^n - e_\psi^{n-1}, 3\eta_\phi^{n+1} - 4\eta_\phi^n + \eta_\phi^{n-1})| \\
&= -2MR^{n+1}\delta t(\eta_w^{n+1}, 2e_\psi^n - e_\psi^{n-1}) - 2MR^{n+1}\delta t(\xi_w^{n+1}, 2e_\psi^n - e_\psi^{n-1}) \\
&\quad - R^{n+1}(3\xi_\phi^{n+1} - 4\xi_\phi^n + \xi_\phi^{n-1}, 2e_\psi^n - e_\psi^{n-1}) \\
&\quad - 2R^{n+1}\delta t(\boldsymbol{u}^{n+1}\nabla\phi^{n+1} - \hat{\boldsymbol{u}}_h^n\nabla\hat{\phi}_h^n, 2e_\psi^n - e_\psi^{n-1}) \\
&\quad + 2\beta R^{n+1}\delta t^3(w_h^{n+1}\nabla\hat{\phi}_h^n, (2e_\psi^n - e_\psi^{n-1})\nabla\hat{\phi}_h^n) \\
&\quad + 2R^{n+1}\delta t(E_\phi^{n+1}, 2e_\psi^n - e_\psi^{n-1}) \\
&\lesssim \delta t^5 + \delta t h^{2k} + \delta t h^{2k+2} + \frac{M}{24}\delta t\|\eta_w^{n+1}\|^2 + \delta t\|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2 \\
&\quad + \delta t\|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2 + \delta t\|2\eta_\psi^n - \eta_\psi^{n-1}\|^2.
\end{aligned}
$$

Combining the above estimates, we deduce

(85)

$$
\begin{aligned}
&\frac{1}{2}(\|\eta_\phi^{n+1}\|^2 - \|\eta_\phi^n\|^2 + \|2\eta_\phi^{n+1} - \eta_\phi^n\|^2 - \|2\eta_\phi^n - \eta_\phi^{n-1}\|^2 + \|\eta_\phi^{n+1} - 2\eta_\phi^n + \eta_\phi^{n-1}\|^2) \\
&\quad + \frac{1}{2}(\|\eta_\psi^{n+1}\|^2 - \|\eta_\psi^n\|^2 + \|2\eta_\psi^{n+1} - \eta_\psi^n\|^2 - \|2\eta_\psi^n - \eta_\psi^{n-1}\|^2 + \|\eta_\psi^{n+1} - 2\eta_\psi^n + \eta_\psi^{n-1}\|^2) \\
&\quad + (\|e_R^{n+1}\|^2 - \|e_R^n\|^2 + \|2e_R^{n+1} - e_R^n\|^2 - \|2e_R^n - e_R^{n-1}\|^2 + \|e_R^{n+1} - 2e_R^n + e_R^{n-1}\|^2) \\
&\quad + \frac{1}{2}(\|\eta_{\boldsymbol{u}}^{n+1}\|^2 - \|\eta_{\boldsymbol{u}}^n\|^2 + \|2\eta_{\boldsymbol{u}}^{n+1} - \eta_{\boldsymbol{u}}^n\|^2 - \|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \|\eta_{\boldsymbol{u}}^{n+1} - 2\eta_{\boldsymbol{u}}^n + \eta_{\boldsymbol{u}}^{n-1}\|^2) \\
&\quad + \frac{1}{2}(\|\nabla\eta_\phi^{n+1}\|^2 - \|\nabla\eta_\phi^n\|^2 + \|2\nabla\eta_\phi^{n+1} - \nabla\eta_\phi^n\|^2 - \|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2 \\
&\quad + \|\nabla\eta_\phi^{n+1} - 2\nabla\eta_\phi^n + \nabla\eta_\phi^{n-1}\|^2) + \frac{2\delta t^2}{3}(\|\nabla\eta_d^{n+1}\|^2 - \|\nabla\eta_d^n\|^2) \\
&\quad + \frac{\delta t}{2\nu}(\|S_h^{n+1}\|^2 - \|S_h^n\|^2) + \frac{M}{2}\delta t\|\eta_w^{n+1}\|^2 + \frac{\nu}{4}\delta t\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2 \\
&\lesssim \delta t^5 + \delta t h^4 + \delta t h^{2k} + \delta t h^{2k+2} + \delta t(\|\eta_\phi^{n+1}\|^2 + |e_R^{n+1}|^2 + \|\eta_\psi^{n+1}\|^2 + \delta t^2\|\nabla\eta_d^{n+1}\|^2 \\
&\quad + \|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2 + \|2\eta_\psi^n - \eta_\psi^{n-1}\|^2).
\end{aligned}
$$

Summing (85) over n from 0 to m, using $\eta_\phi^0 = 0, \eta_\psi^0 = 0, \eta_{\boldsymbol{u}}^0 = 0, \eta_d^0 = 0, S_h^0 = 0, e_R^0 = 0$ and neglecting some positive terms, we have

$$
\begin{aligned}
&\|\eta_\phi^{m+1}\|^2 + \|\eta_\psi^{m+1}\|^2 + \|e_R^{m+1}\|^2 + \|\eta_{\boldsymbol{u}}^{m+1}\|^2 + \|\nabla\eta_\phi^{m+1}\|^2 \\
&\quad + \frac{4\delta t^2}{3}\|\nabla\eta_d^{m+1}\|^2 + \frac{\delta t}{\nu}\|S_h^{m+1}\|^2 \\
&\quad + \|2\eta_\phi^{m+1} - \eta_\phi^m\|^2 + \|2\eta_\psi^{m+1} - \eta_\psi^m\|^2 + \|2e_R^{m+1} - e_R^m\|^2 + \|2\eta_{\boldsymbol{u}}^{m+1} - \eta_{\boldsymbol{u}}^m\|^2 \\
&\quad + \|2\nabla\eta_\phi^{m+1} - \nabla\eta_\phi^m\|^2 + \delta t\sum_{n=0}^m(M\|\eta_w^{n+1}\|^2 + \frac{\nu}{2}\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2) \\
&\lesssim \delta t\sum_{n=0}^m(\delta t^4 + h^4 + h^{2k} + h^{2k+2}) \\
&\quad + C_5\delta t\sum_{n=0}^m(\|\eta_\phi^{n+1}\|^2 + |e_R^{n+1}|^2 + \|\eta_\psi^{n+1}\|^2 + \delta t^2\|\nabla\eta_d^{n+1}\|^2 \\
&\quad + \|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2 + \|2\eta_\psi^n - \eta_\psi^{n-1}\|^2).
\end{aligned}
\tag{86}
$$

When $0 < \delta t \leq \delta t_0 := \frac{1}{2C_5} < \frac{1}{C_5}$, since $1 < \frac{1}{1-C_5\delta t} \leq 2$ and from (86), it can be readily seen that

$$\begin{aligned}
&\|\eta_\phi^{m+1}\|^2 + \|\eta_\psi^{m+1}\|^2 + \|e_R^{m+1}\|^2 + \|\eta_{\boldsymbol{u}}^{m+1}\|^2 + \|\nabla\eta_\phi^{m+1}\|^2 \\
&\quad + \frac{4\delta t^2}{3}\|\nabla\eta_d^{m+1}\|^2 + \frac{\delta t}{\nu}\|S_h^{m+1}\|^2 \\
&\quad + \|2\eta_\phi^{m+1} - \eta_\phi^m\|^2 + \|2\eta_\psi^{m+1} - \eta_\psi^m\|^2 + \|2e_R^{m+1} - e_R^m\|^2 + \|2\eta_{\boldsymbol{u}}^{m+1} - \eta_{\boldsymbol{u}}^m\|^2 \\
&\quad + \|2\nabla\eta_\phi^{m+1} - \nabla\eta_\phi^m\|^2 + \delta t\sum_{n=0}^m (M\|\eta_w^{n+1}\|^2 + \frac{\nu}{2}\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2) \\
&\lesssim \frac{\delta t}{1 - C_5\delta t}\sum_{n=0}^m (\delta t^4 + h^4 + h^{2k} + h^{2k+2}) \\
&\quad + \frac{C_5\delta t}{1 - C_5\delta t}\sum_{n=0}^m (\|2\eta_{\boldsymbol{u}}^n - \eta_{\boldsymbol{u}}^{n-1}\|^2 + \|2\nabla\eta_\phi^n - \nabla\eta_\phi^{n-1}\|^2 + \|2\eta_\psi^n - \eta_\psi^{n-1}\|^2).
\end{aligned}$$

Applying the discrete Gronwall's inequality [13], we derive

$$\begin{aligned}
&\|\eta_\phi^{m+1}\|^2 + \|\eta_\psi^{m+1}\|^2 + \|e_R^{m+1}\|^2 + \|\eta_{\boldsymbol{u}}^{m+1}\|^2 + \|\nabla\eta_\phi^{m+1}\|^2 \\
&\quad + \frac{4\delta t^2}{3}\|\nabla\eta_d^{m+1}\|^2 + \frac{\delta t}{\nu}\|S_h^{m+1}\|^2 \\
&\quad + \|2\eta_\phi^{m+1} - \eta_\phi^m\|^2 + \|2\eta_\psi^{m+1} - \eta_\psi^m\|^2 + \|2e_R^{m+1} - e_R^m\|^2 + \|2\eta_{\boldsymbol{u}}^{m+1} - \eta_{\boldsymbol{u}}^m\|^2 \\
&\quad + \|2\nabla\eta_\phi^{m+1} - \nabla\eta_\phi^m\|^2 + \delta t\sum_{n=0}^m (M\|\eta_w^{n+1}\|^2 + \frac{\nu}{2}\|\nabla\tilde{\eta}_{\boldsymbol{u}}^{n+1}\|^2) \\
&\lesssim \delta t^4 + h^4 + h^{2k}.
\end{aligned}$$

Finally, applying the triangle inequality, it is easy to get that

$$\begin{aligned}
&\|e_\phi^{m+1}\|^2 + \|e_\psi^{m+1}\|^2 + \|e_R^{m+1}\|^2 + \|e_{\boldsymbol{u}}^{m+1}\|^2 + \|\nabla e_\phi^{m+1}\|^2 \\
&\quad + \frac{4\delta t^2}{3}\|\nabla e_d^{m+1}\|^2 + \frac{\delta t}{\nu}\|S_h^{m+1}\|^2 \\
&\quad + \|2e_\phi^{m+1} - e_\phi^m\|^2 + \|2e_\psi^{m+1} - e_\psi^m\|^2 + \|2e_R^{m+1} - e_R^m\|^2 + \|2e_{\boldsymbol{u}}^{m+1} - e_{\boldsymbol{u}}^m\|^2 \\
&\quad + \|2\nabla e_\phi^{m+1} - \nabla e_\phi^m\|^2 + \delta t\sum_{n=0}^m (M\|e_w^{n+1}\|^2 + \frac{\nu}{2}\|\nabla\tilde{e}_{\boldsymbol{u}}^{n+1}\|^2) \\
&\leq C^\dagger(\delta t^4 + h^4 + h^{2k}).
\end{aligned} \tag{87}$$

Step 3: From (87), using Lemma 3.1, then we derive

$$\begin{aligned}
\|\boldsymbol{u}_h^{n+1}\|_{L^\infty}^2 &\leq 2\|\boldsymbol{u}^{n+1}\|_{L^\infty}^2 + 2\|e_{\boldsymbol{u}}^{n+1}\|_{L^\infty}^2 \\
&\leq 2\|\boldsymbol{u}^{n+1}\|_{L^\infty}^2 + 2C^*h^{-d}\|e_{\boldsymbol{u}}^{n+1}\|^2 \\
&\leq 2\|\boldsymbol{u}^{n+1}\|_{L^\infty}^2 + 2C^*C^\dagger h^{-d}(\delta t^4 + h^4 + h^{2k}) \\
&\leq C_0,
\end{aligned}$$

$$\begin{aligned}
\|\nabla\phi_h^{n+1}\|_{L^\infty}^2 &\leq 2\|\nabla\phi^{n+1}\|_{L^\infty}^2 + 2\|\nabla e_\phi^{n+1}\|_{L^\infty}^2 \\
&\leq 2\|\nabla\phi^{n+1}\|_{L^\infty}^2 + 2C^*h^{-d}\|\nabla e_\phi^{n+1}\|^2 \\
&\leq 2\|\nabla\phi^{n+1}\|_{L^\infty}^2 + 2C^*C^\dagger h^{-d}(\delta t^4 + h^4 + h^{2k}) \\
&\leq C_0^*,
\end{aligned}$$

for $\delta t \leq \frac{h}{\sqrt[4]{2C^*C^\dagger}}$, $h \leq h_0$, where h_0 is a small positive constant and $d = 2, 3$ represents the dimension of Ω. The above estimate implies that the induction assumption (70) could be recovered at $n = N-1$ with $C_0 \leq K_0$ and $C_0^* \leq K_0^*$, the similar skills can be found in [40, 52, 55, 56]. Thus the mathematical induction is closed. $\square$

Theorem 4.3. *For any $0 \leq m \leq N-1$, under the assumptions (66), the following inequality holds*

$$\begin{aligned}
&\|e_\phi^{m+1}\|^2 + \|e_\psi^{m+1}\|^2 + \|e_R^{m+1}\|^2 + \|e_{\boldsymbol{u}}^{m+1}\|^2 + \|\nabla e_\phi^{m+1}\|^2 \\
&\quad + \frac{4\delta t^2}{3}\|\nabla e_d^{m+1}\|^2 + \frac{\delta t}{\nu}\|S_h^{m+1}\|^2 \\
&\quad + \|2e_\phi^{m+1} - e_\phi^m\|^2 + \|2e_\psi^{m+1} - e_\psi^m\|^2 + \|2e_R^{m+1} - e_R^m\|^2 + \|2e_{\boldsymbol{u}}^{m+1} - e_{\boldsymbol{u}}^m\|^2 \\
&\quad + \|2\nabla e_\phi^{m+1} - \nabla e_\phi^m\|^2 + \delta t \sum_{n=0}^{m} (M\|e_w^{n+1}\|^2 + \frac{\nu}{2}\|\nabla \tilde{e}_{\boldsymbol{u}}^{n+1}\|^2) \\
&\lesssim \delta t^4 + h^4 + h^{2k}.
\end{aligned}$$

Proof. The process of this proof can be found in Step 2 of Lemma 4.2. $\square$

Remark 4.4. *We have provided the error estimates for variables ϕ and $\boldsymbol{u}$ under the L^2 norm above. The error estimation of pressure can be obtained by using a similar method and the inf-sup condition in Lemma 3.5. For simplicity, we omit the proof and only use numerical experiments to confirm the result.*

Remark 4.5. *A molecule beamer epitaxial (MBE) style free energy [16, 32, 39] is taken in this work. We found that the long time simulation results in the MBE model has been established theoretically, which indicate a structure with the gradient of the layer function approximately of Euclidean norm 1. For the SAV-related numerical schemes, we also can observe the phenomenon in [8, 41]. By a more careful analysis similar to the references [16, 41], it is achievable to obtain similar result of our proposed numerical scheme.*

5. Numerical Simulations

In this section, a few numerical examples are simulated on a domain $\Omega = [-1, 1]^2$ to validate our theoretical analysis of the proposed scheme. Here we adopt $P_2 - P_1$ elements for $\boldsymbol{u}, p$ and P_2 element for ϕ, respectively.

5.1. Accuracy test. To show the convergence rates of the scheme (15)-(22), we take the parameters $M = 1, \varepsilon = 1, \nu = 1$ in the smectic-A phase liquid crystals (1) and the artificial parameter $\beta = 2000$. The exact solution is set by

$$\begin{cases}
u = \pi \sin(2\pi y)\sin^2(\pi x)\sin(t), \\
v = -\pi \sin(2\pi x)\sin^2(\pi y)\sin(t), \\
\phi = 2 + \cos(\pi x)\cos(\pi y)\sin(t), \\
p = \cos(\pi x)\sin \pi y \sin(t).
\end{cases} \tag{88}$$

To show the convergence rates of spatial errors, we fix the spatial mesh size $h = 1/100$ and choose the temporal mesh sizes $\delta t = 1/20, 1/40, 1/80, 1/160, 1/320$. The temporal error in L^2 norm for the layer function ϕ, velocity field $|\boldsymbol{u}|$, and pressure p are plotted in Fig.1(a). The convergence rates are closed $O(\delta t^2)$ which corresponds to the theoretical predictions given in Section 4. Then we try to show the spatial convergence rate, we plot the spatial L^2 error for the layer function ϕ, velocity field

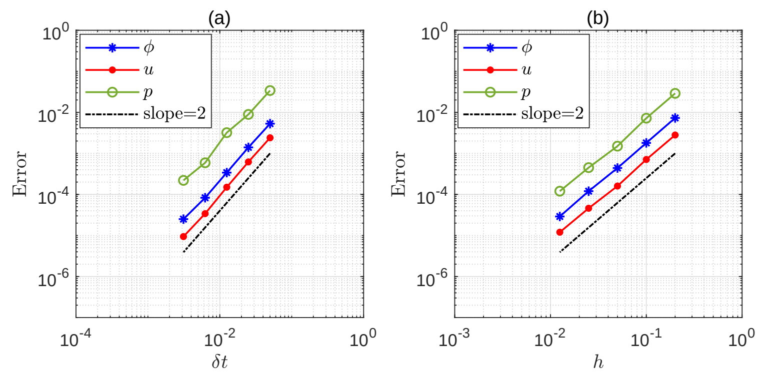

FIGURE 1. The errors of computed $\phi, \boldsymbol{u}, p$ in L^2-norm at $t = 0.5$. (a): convergence rate in the temporal. (b): convergence rate in the spatial.

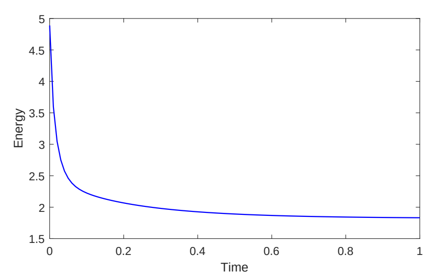

FIGURE 2. Time evolution of the total energy $\mathcal{E}_{tot}^h$.

$|\boldsymbol{u}|$, and pressure p of fixing a small enough time mesh size $\delta t = 0.001$ with different spatial step sizes $h = 1/5, 1/10, 1/20, 1/40, 1/80$ in Fig.1(b). It shows that the spatial convergence rates are close to $O(h^2)$, which coincides with our theoretical analysis.

5.2. Energy stability test. In this example, we set the parameters to be $M = 1, \varepsilon = 1, \nu = 1, \beta = 200$. The initial conditions are set as

$$\phi_0 = \cos(\pi x)\cos(\pi y), \quad \boldsymbol{u}_0 = (u_0, v_0) = \mathbf{0}, \quad p_0 = 0.$$

The temporal step size and spatial mesh size are taken as $\delta t = 0.01,\ h = 1/50$. The time evolution of the total energy for the proposed scheme is plotted in Fig.2. It is evident that the discrete energy decreases with time, i.e., our scheme is energy stable.

5.3. The dynamical evolution. We simulate the evolution of the layer function ϕ and the velocity field $\boldsymbol{u}$ in this example. The parameters are taken as $M = 0.1, \varepsilon = 0.01, \nu = 1, \beta = 200$. The initial conditions as given by

$$\phi_0 = \sin(x)\cos^2(y), \quad \boldsymbol{u}_0 = (u_0, v_0) = \mathbf{0}, \quad p_0 = 0.$$

The temporal step size and spatial mesh size are taken as $\delta t = 0.01,\ h = 1/50$. The simulations of ϕ and $\boldsymbol{u}$ eventually reach the steady states at $t = 1.8$. The dynamic evolution of the layer function ϕ and velocity field $\boldsymbol{u}$ in the simulation are shown in Fig. 3 and Fig. 4. By observing the right columns of Fig. 3, we can see that

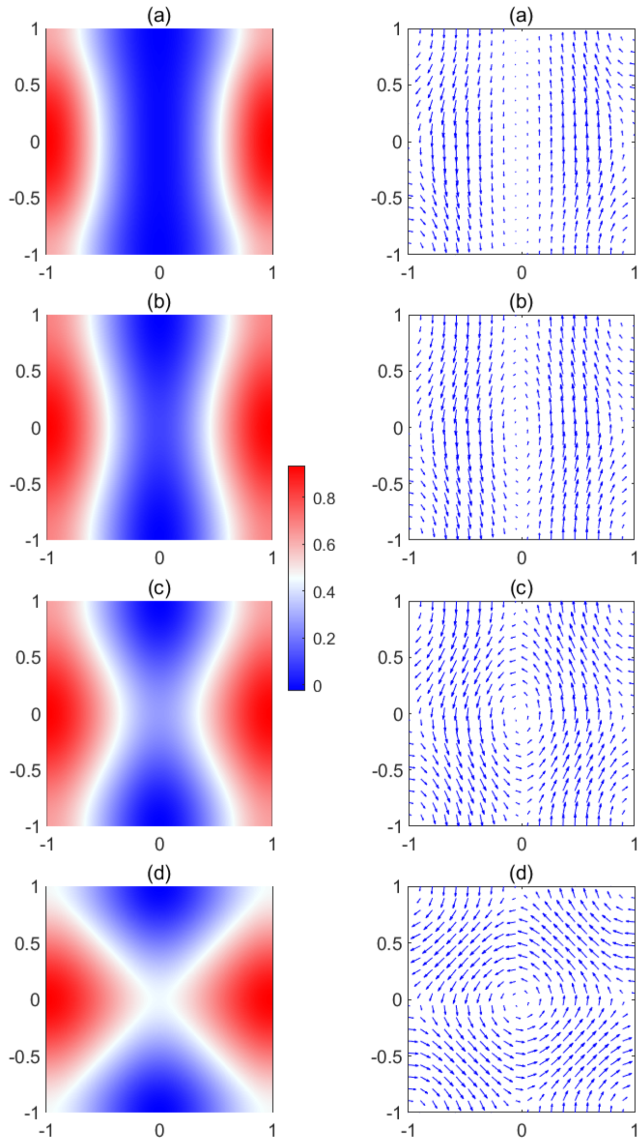

FIGURE 3. The evolution of the layer function ϕ (the left columns) and the orientation vector $\nabla\phi$ (the right columns) at different times. (a): $t = 0.01$, (b): $t = 0.3$, (c): $t = 0.8$, (d): $t = 1.8$.

the topological defects of the orientation vector $\nabla\phi$ are eventually formed. From the right columns of Fig. 4, we can see that the velocity field $\boldsymbol{u}$ eventually forms vortices as it changes with time.

6. Conclusion

In this paper, we propose a linear, decoupled, unconditionally energy-stable BDF2-SAV-FEM scheme for the smectic-A liquid crystals. To decouple velocity

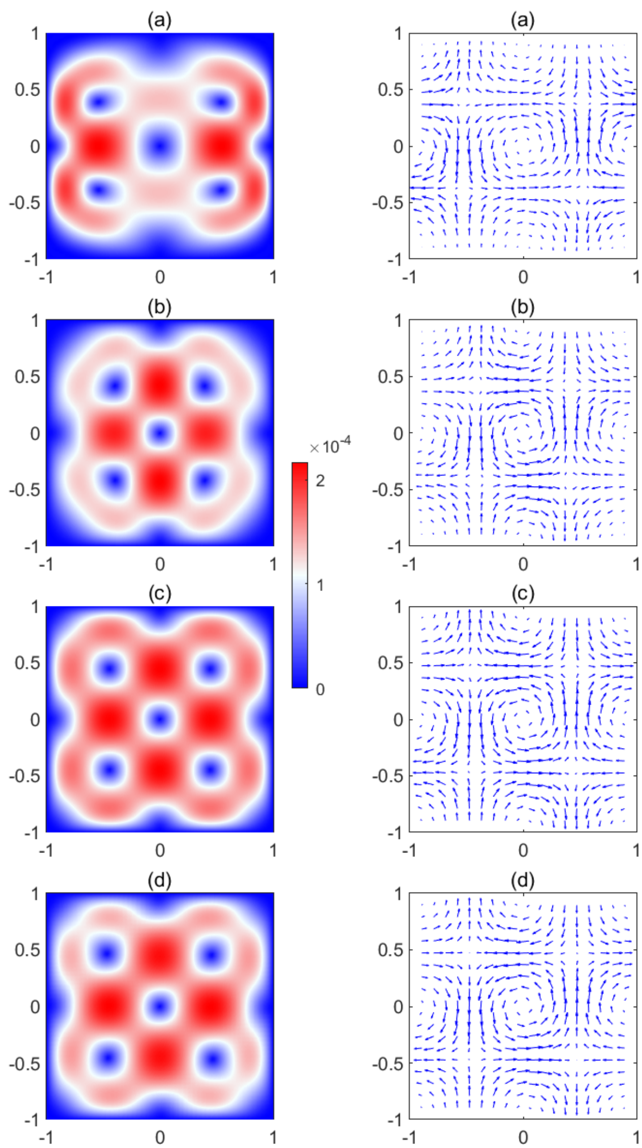

FIGURE 4. The evolution of the velocity field $|\boldsymbol{u}|$ (the left columns) and $\boldsymbol{u}$ (the right columns) at different times. (a): $t = 0.01$, (b): $t = 0.3$, (c): $t = 0.8$, (d): $t = 1.8$.

and pressure, we utilize the rotational pressure-correction method for the Navier-Stokes equations. For the constitutive equation, we introduce an additional stabilization term to achieve the explicit treatment of the coupled nonlinear term, which decouples the computations of the velocity field and the layer function. Moreover, we apply the subtle IMEX technique for the nonlinear coupled terms and the SAV method to linearize the nonlinear energy potential. In addition, the unconditional energy stability at the discrete level is derived, and we rigorously demonstrate the error estimates of the proposed scheme. Finally, some numerical experiments are carried out to demonstrate the accuracy and stability of the proposed system, and the numerical results show the good performance of the proposed system.

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Data Availability

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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