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SPLITTING SCHEMES FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. This paper concerns splitting methods for solving backward stochastic differential equations (BSDEs). By splitting the original d-dimensional BSDE into d BSDEs and approximating these split BSDEs, we propose splitting schemes for the BSDE. The splitting schemes are rigorously analyzed and first-order error estimates are theoretically obtained. Numerical tests are given to verify the theoretical results.

Key words. Backward stochastic differential equations, splitting method, splitting scheme, error estimate.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete, filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ being the natural filtration generated by a d-dimensional Brownian motion $W_t = (W_t^1, W_t^2, \cdots, W_t^d)^\top$. The general form of backward stochastic differential equation (BSDE) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is

$$Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{1}$$

where $X_t = X_0 + W_t$ is a forward diffusion process with $X_0 \in \mathcal{F}_0$ being the initial condition; $T > 0$ is the deterministic terminal time; $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times d} \to \mathbb{R}^p$ and $\varphi : \mathbb{R}^d \to \mathbb{R}^p$ are the generator and the terminal function of the BSDE, respectively. We note that the integral term with respect to W_s of the BSDE is the Itô-type integral. The pair of processes $(Y_t, Z_t) : [0, T] \times \Omega \to \mathbb{R}^p \times \mathbb{R}^{p\times d}$ is called an L^2-adapted solution of the BSDE (1) if it is $\mathcal{F}_t$-adapted, square integrable, and satisfies (1).

In 1990, under certain standard conditions, Pardoux and Peng [26] originally proved the existence and uniqueness of the solutions of general nonlinear BSDEs. In 1991, Peng [29] found the nonlinear Feynman-Kac formula, that is, under some regularity conditions, the solution (Y, Z) of (1) can be represented as

$$Y_t = u(t, X_t), \quad Z_t = \nabla_x u(t, X_t), \quad t \in [0, T), \tag{2}$$

where $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^p$ is the classical solution to the following second-order parabolic partial differential equation (PDE)

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} + f(t, x, u, \nabla_x u) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d \tag{3}$$

with the terminal condition $u(T, x) = \varphi(x)$ for $x \in \mathbb{R}^d$. The representation (2) deeply connects the BSDE and the parabolic PDE, which enables us to develop numerical schemes for the BSDE (1) by solving the associated parabolic PDE (3), and vice versa. Since then, significant efforts have been made to study BSDEs

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due to their important applications in various fields, such as mathematical finance, PDEs, stochastic control, risk measurement, game theory, deep learning and so on (see, e.g., [9, 23, 28, 29, 13] and references therein).

It is usually difficult to obtain the analytical solutions of BSDEs, and thus numerical methods for solving BSDEs are in high demand. Recently, a lot of work has been put into developing effective numerical methods for solving BSDEs. One of the most widely used approaches is to discretize BSDEs directly, leading to various discretization schemes for solving BSDEs [1, 2, 6, 16, 21, 36, 3]. Popular temporal discretization strategies include Euler-type methods [14, 15, 35], the generalized θ-schemes [31, 36, 38], Runge-Kutta schemes [5], the multistep schemes [4, 19, 20, 30, 37, 39, 40], strong stability preserving multistep schemes [10, 11], and extrapolation methods [32, 33], etc. Additionally, there are several numerical schemes for solving BSDEs proposed based on the nonlinear Feynman-Kac formula and the numerical solutions of parabolic PDEs associated with BSDEs, such as in the papers [7, 22, 24].

Parabolic equations have found wide applications in various fields, such as the heat transfer in a superconductor, the chemical reaction from chemical engineering and the modeling of economic processes, etc. Thus various numerical methods have been developed for solving parabolic PDEs [8, 25]. For multi-dimensional PDEs, splitting methods, including alternating direction implicit (ADI) [8, 27] and locally one-dimensional (LOD) [34] methods, demonstrated significant advantages due to their low computational complexity and high computational efficiency. The LOD method, also known as fractional step methods, and the ADI method solve a multi-dimensional equation by converting the multi-dimensional equation to a succession of one-dimensional equations, while providing accurate numerical solutions. These methods have been extended to solve nonlinear PDE problems and other physical problems. However, up to now, there are still very few research on splitting methods for solving BSDEs.

To fill the gap, in this paper, we shall propose splitting schemes for solving BSDEs. To obtain the splitting schemes, on each time subinterval $[t_n, t_{n+1}]$ of a given time partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, first we split the BSDE (1) into d BSDEs with their solutions $\{(\bar{Y}_t^i, \bar{Z}_t^i), i = 1, \cdots, d\}$, then, by approximating these split BSDEs, we construct our splitting schemes for solving the original BSDE (1). The main advantage of the schemes is that only one-dimensional approximations are required to calculate the conditional mathematical expectations, which may reduce computational cost. We rigorously provide the theoretical error estimates, which show the first-order convergence rate of the schemes. Our numerical tests also validate our theoretical results, and show the accuracy and effectiveness of our splitting schemes.

The rest of this paper is organized as follows. In Section 2, we propose splitting schemes for solving BSDEs by splitting the original d-dimensional BSDE into d BSDEs and approximating these split BSDEs. In Section 3, we rigorously prove the first-order convergence rate in time for the splitting schemes. Several numerical tests are presented to show the accuracy and effectiveness of our splitting schemes in Section 4. In Section 5, the conclusions are given.

2. Splitting methods for BSDEs

First, we introduce some notations. Use $\Delta W_{t,s}$ to denote the increment $W_s - W_t = (\Delta W_{t,s}^1, \cdots, \Delta W_{t,s}^d)^\top$ of the Brownian motion W_s for $s \geq t$, where $\Delta W_{t,s}^i =$

$W_s^i - W_t^i$. And for the d-dimensional process X_t, we use $\mathbb{E}_t^x[\cdot]$ to denote the conditional mathematical expectation operator $\mathbb{E}[\cdot|\mathcal{F}_t, X_t = x]$, and $\mathbb{E}_t^{i,x}[\cdot]$ to denote the conditional mathematical expectation operator $\mathbb{E}[\cdot|\mathcal{F}_t, X_t^i = x^i]\big|_{X_t^j = x^j, j\neq i}$, where x is an arbitrarily fixed point in $\mathbb{R}^d$. More clearly, for a given function $g : \mathbb{R}^d \to \mathbb{R}^q$ with q being a positive integer, we have

$$\mathbb{E}_t^x[g(X_s)] = \mathbb{E}[g(X_s)|\mathcal{F}_t, X_t = x], \tag{4}$$

$$\mathbb{E}_t^{i,x}[g(X_s)] = \mathbb{E}[g(\bar{x}^1, \cdots, \bar{x}^{i-1}, X_s^i, \bar{x}^{i+1}, \cdots, \bar{x}^d)|\mathcal{F}_t, X_t^i = x^i]\big|_{X_t^j = x^j, \bar{x}^j = X_s^j, j\neq i}. \tag{5}$$

It is worth to note that the main difference between $\mathbb{E}_t^x[\cdot]$ and $\mathbb{E}_t^{i,x}[\cdot]$ is that the conditional mathematical expectation $\mathbb{E}_t^{i,x}[g(X_s)]$ is only taken for the ith component X_s^i of X_s.

2.1. Splitting approximations. For the time interval $[0, T]$, we consider the following partition:

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T. \tag{6}$$

where N is a positive integer. Let $\Delta t_n = t_{n+1} - t_n, n = 0, 1, \cdots, N-1$, and $\Delta t = \max_{0\leq n\leq N-1} \Delta t_n$. For the sake of error estimates, we assume that the time partition satisfies the regularity constraint:

$$\frac{\max_{0\leq n\leq N-1} \Delta t_n}{\min_{0\leq n\leq N-1} \Delta t_n} \leq c_0, \tag{7}$$

where $c_0 \geq 1$ is a constant. For the uniform time partition, the time step size $\Delta t_n = \Delta t = \frac{T}{N}$.

Let (Y_t, Z_t) be the adapted solution of the BSDE (1). Then for $t \in [t_n, t_{n+1}]$ and $n = 0, 1, \cdots, N-1$, we can rewrite (1) as

$$Y_t = Y_{t_{n+1}} + \sum_{i=1}^d \int_t^{t_{n+1}} f^i(s, X_s, Y_s, Z_s)ds - \int_t^{t_{n+1}} Z_s dW_s, \tag{8}$$

where $f^i : [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^{p\times d} \to \mathbb{R}^p$ are appropriate deterministic functions satisfing $\sum_{i=1}^d f^i = f$.

Now based on (8), we introduce a family of intermediate processes $\{(\bar{Y}_t^i, \bar{Z}_t^i), i = 1, \cdots, d\}$ for $t \in [0, T]$, which will be utilized to construct splitting approximations of (8). The processes $(\bar{Y}_t^i, \bar{Z}_t^i)$ on $[t_n, t_{n+1}]$, $i = 1, \cdots, d$, are defined by the BSDEs

$$\bar{Y}_t^i = \bar{Y}_{t_{n+1}}^i + \int_t^{t_{n+1}} \bar{f}_s^i ds - \int_t^{t_{n+1}} \bar{Z}_s^i dW_s \tag{9}$$

for $i = 1, \cdots, d$ with the terminal conditions $\bar{Y}_{t_N}^1 = \varphi(X_T)$ and $\bar{Y}_{t_{n+1}}^1 = \bar{Y}_{t_{n+1}}^d(X_{t_{n+1}})$ for $0 \leq n < N-1$, and

$$\bar{Y}_{t_{n+1}}^i = \bar{Y}_{t_n}^{i-1}(X_{t_{n+1}}), \quad i = 2, 3, \cdots, d, \tag{10}$$

where φ is the terminal condition defined in (1), and $\bar{f}_s^i = f^i(s, X_s, \bar{Y}_s^i, \bar{Z}_s^i)$ with $\bar{Z}_s^i = (\bar{Z}_{s,1}^i, \cdots, \bar{Z}_{s,d}^i)$. We call the BSDEs (9) the splitting approximations of the BSDE (8).

Let $t = t_n, n = 0, 1, \cdots, N-1$, in (9), we obtain

$$\bar{Y}_{t_n}^i = \bar{Y}_{t_{n+1}}^i + \int_{t_n}^{t_{n+1}} \bar{f}_s^i ds - \int_{t_n}^{t_{n+1}} \bar{Z}_s^i dW_s \tag{11}$$

for $i = 1, \cdots, d$. Then taking the conditional mathematical expectation $\mathbb{E}_{t_n}^x[\cdot]$ on both sides of (11), we derive

$$\bar{Y}_{t_n}^i = \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+1}}^i] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[\bar{f}_s^i]ds. \tag{12}$$

The integrand $\mathbb{E}_{t_n}^x[\bar{f}_s^i]$ on the right-hand side of (12) is a deterministic function of s under the filtration $\mathcal{F}_{t_n}$. By using the left rectangle rule to approximate the integral term and using the conditional mathematical expectation $\mathbb{E}_{t_n}^{i,x}[\cdot]$ to approximate $\mathbb{E}_{t_n}^x[\cdot]$ in (12), we deduce

$$\bar{Y}_{t_n}^i = \mathbb{E}_{t_n}^{i,x}[\bar{Y}_{t_{n+1}}^i] + \Delta t_n \bar{f}_{t_n}^i + R_{\bar{y}}^i, \tag{13}$$

where $R_{\bar{y}}^i = R_{\bar{f}}^i + R_{Ey}^i$ with

$$\begin{aligned} R_{\bar{f}}^i &= \int_{t_n}^{t_{n+1}} \Big\{\mathbb{E}_{t_n}^x[\bar{f}_s^i] - \bar{f}_{t_n}^i\Big\}ds, \\ R_{Ey}^i &= \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+1}}^i] - \mathbb{E}_{t_n}^{i,x}[\bar{Y}_{t_{n+1}}^i]. \end{aligned}$$

Let $\Delta W_{n+1} = \Delta W_{t_n,t_{n+1}} = (\Delta W_{n+1}^1, \cdots, \Delta W_{n+1}^d)^\top$. Multiplying (11) by ΔW_{n+1}^j, taking the conditional mathematical expectation $\mathbb{E}_{t_n}^x[\cdot]$ on both sides of the derived equations, and using the Itô isometry formula, we obtain the equations

$$0 = \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+1}}^i \Delta W_{n+1}^j] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[\bar{f}_s^i \Delta W_{n+1}^j]ds - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[\bar{Z}_{s,j}^i]ds, \tag{14}$$

where $\bar{Z}_{s,j}^i$ is the jth column of $\bar{Z}_s^i$ for $j = 1, \cdots, d$. By using the left rectangle rule to the two integrals and the approximation $\mathbb{E}_{t_n}^{j,x}[\cdot]$ of the $\mathbb{E}_{t_n}^x[\cdot]$ on the right-hand side of (14) again, we deduce

$$\Delta t_n \bar{Z}_{t_n,j}^i = \mathbb{E}_{t_n}^{j,x}[\bar{Y}_{t_{n+1}}^i \Delta W_{n+1}^j] + R_{\bar{z}}^j, \tag{15}$$

where $R_{\bar{z}}^j = R_{\bar{z}_1}^j + R_{\bar{z}_2}^j + R_{Ez}^j$ with

$$\begin{aligned} R_{\bar{z}_1}^j &= \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[\bar{f}_s^i \Delta W_{n+1}^j]ds, \\ R_{\bar{z}_2}^j &= -\int_{t_n}^{t_{n+1}} \left\{\mathbb{E}_{t_n}^x[\bar{Z}_{s,j}^i] - \bar{Z}_{t_n,j}^i\right\} ds, \\ R_{Ez}^j &= \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+1}}^i \Delta W_{n+1}^j] - \mathbb{E}_{t_n}^{j,x}[\bar{Y}_{t_{n+1}}^i \Delta W_{n+1}^j]. \end{aligned}$$

Based on the approximation equations (13) and (15), we will propose our splitting schemes for solving the BSDE (1).

2.2. Splitting schemes. Let (Y^n, Z^n) be the numerical approximation of the analytical solution (Y_t, Z_t) of the BSDE (1) at time $t = t_n$, $n = N, N-1, \cdots, 0$. By removing the error terms $R_{\bar{y}}^i$ and $R_{\bar{z}}^j$ in (13) and (15), respectively, we propose the time semi-discrete splitting schemes for solving the BSDE (1) as follows.

Scheme 1. *Given random variables Y^N and Z^N, for $n = N-1, \cdots, 1, 0$, solve $Y^n = Y^n(x)$ and $Z^n = Z^n(x)$ for $x \in \mathbb{R}^d$ by*

(1) *Let $\bar{Y}^{n+1,1} = Y^{n+1}$;*

(2) *For* $i = 1, 2, \cdots, d$,

$$\Delta t_n \bar{Z}_j^{n,i} = \mathbb{E}_{t_n}^{j,x}[\bar{Y}^{n+1,i} \Delta W_{n+1}^j], \quad j = 1, 2, \cdots, d,$$
$$\bar{Y}^{n,i} = \mathbb{E}_{t_n}^{i,x}[\bar{Y}^{n+1,i}] + \Delta t_n f^i(t_n, x, \bar{Y}^{n,i}, \bar{Z}^{n,i}),$$
$$\bar{Y}^{n+1,i+1} = \bar{Y}^{n,i}$$

with $\bar{Z}^{n,i} = (\bar{Z}_1^{n,i}, \bar{Z}_2^{n,i}, \cdots, \bar{Z}_d^{n,i})$;

(3) *Let* $Y^n = \bar{Y}^{n,d}$ *and* $Z^n = \bar{Z}^{n,d}$.

Remark 1. *The order of the computations in* (2) *of Scheme 1 can be any a prior fixed permutation of the index* $(1, 2, \ldots, d)$.

In Scheme 1, we have solved the jth column $\bar{Z}_j^{n,i}$ of $\bar{Z}^{n,i}$ for $j = 1, \cdots, i-1$ to determine $\bar{Z}^{n,i}$. Thus in (2) of Scheme 1 the computations of $\bar{Z}_j^{n,i}$ for j can be changed to $j = i, i+1, \cdots, d$. Based on this observation, we propose the following Scheme 2.

Scheme 2. *Given random variables* Y^N *and* Z^N, *for* $n = N-1, \cdots, 1, 0$, *solve* $Y^n = Y^n(x)$ *and* $Z^n = Z^n(x)$ *for* $x \in \mathbb{R}^d$ *by*

(1) *Let* $\bar{Y}^{n+1,1} = Y^{n+1}$;

(2) *For* $i = 1, 2, \cdots, d$,

$$\Delta t_n \bar{Z}_j^{n,i} = \mathbb{E}_{t_n}^{j,x}[\bar{Y}^{n+1,i} \Delta W_{n+1}^j], \quad j = i, i+1, \cdots, d,$$
$$\bar{Y}^{n,i} = \mathbb{E}_{t_n}^{i,x}[\bar{Y}^{n+1,i}] + \Delta t_n f^i(t_n, x, \bar{Y}^{n,i}, \bar{Z}^{n,i}),$$
$$\bar{Y}^{n+1,i+1} = \bar{Y}^{n,i}$$

with $\bar{Z}^{n,i} = (\bar{Z}_1^{n,1}, \bar{Z}_2^{n,2}, \cdots, \bar{Z}_i^{n,i}, \bar{Z}_{i+1}^{n,i}, \cdots, \bar{Z}_d^{n,i})$;

(3) *Let* $Y^n = \bar{Y}^{n,d}$ *and* $Z^n = \bar{Z}^{n,d}$.

In the above two schemes, one of the simplest splittings is $f^i = 0$ $(1 \le i \le d-1)$ and $f^d = f$. For this simplest splitting, we solve $\bar{Y}^{n,i}$ without calculating the values of $f^i(t_n, X_{t_n}, \bar{Y}^{n,i}, Z^{n,i})$ for $1 \le i \le d-1$. In this case, to make the above two schemes work, only the ith column $\bar{Z}_i^{n,i}$ of $\bar{Z}^{n,i}$ $(i = 1, \cdots, d)$ are needed. Based on these observations, to be more efficient, we propose the following Scheme 3.

Scheme 3. *Given random variables* Y^N *and* Z^N, *for* $n = N-1, \cdots, 1, 0$, *solve* $Y^n = Y^n(x)$ *and* $Z^n = Z^n(x)$ *for* $x \in \mathbb{R}^d$ *by*

(1) *Let* $\bar{Y}^{n+1,1} = Y^{n+1}$;

(2) *For* $i = 1, \cdots, d-1$,

$$\Delta t_n \bar{Z}_i^{n,i} = \mathbb{E}_{t_n}^{i,x}[\bar{Y}^{n+1,i} \Delta W_{n+1}^i],$$
$$\bar{Y}^{n,i} = \mathbb{E}_{t_n}^{i,x}[\bar{Y}^{n+1,i}],$$
$$\bar{Y}^{n+1,i+1} = \bar{Y}^{n,i},$$

and for $i = d$,

$$\Delta t_n \bar{Z}_d^{n,d} = \mathbb{E}_{t_n}^{d,x}[\bar{Y}^{n+1,d} \Delta W_{n+1}^d],$$
$$\bar{Y}^{n,d} = \mathbb{E}_{t_n}^{d,x}[\bar{Y}^{n+1,d}] + \Delta t_n f(t_n, x, \bar{Y}^{n,d}, \bar{Z}^{n,d})$$

with $\bar{Z}^{n,d} = (\bar{Z}_1^{n,1}, \bar{Z}_2^{n,2}, \cdots, \bar{Z}_d^{n,d})$;

(3) *Let* $Y^n = \bar{Y}^{n,d}$ *and* $Z^n = \bar{Z}^{n,d}$.

Remark 2. *Among the above three schemes, Scheme 3 is the most efficient. Furthermore, both Schemes 1 and 2 depend on the splitting of the generator f. The errors produced by Scheme 1 and 2 with the splitting $f^1 = f$ and $f^i = 0$ $(2 \le i \le d)$ are identical. The numerical solutions obtained by Schemes 2 and 3 are identical when the splitting $f^i = 0$ $(1 \le i \le d-1)$ and $f^d = f$ is used. Our numerical tests will show all these observations.*

Remark 3. (1) *According to the nonlinear Feynman-Kac formula, our schemes can be used for solving nonlinear PDEs. Note that existing splitting methods for solving PDEs, such as LOD methods, are challenging for solving nonlinear PDEs, especially for the function f that depends on $\nabla_x u$. However, our methods overcome this difficulty.*

(2) *All of the above schemes can be extended to the forward backward stochastic differential equations (FBSDEs) combining the BSDE* (1) *with the forward diffusion process X_t defined by*

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

where $X_t : [0,T] \times \Omega \to \mathbb{R}^q$, the drift coefficient $b : [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$ and the diffusion coefficient $\sigma : [0,T] \times \mathbb{R}^q \to \mathbb{R}^{q\times d}$.

3. Error analysis

In this section, we will rigorously prove that Scheme 3 has first-order convergence in time for solving BSDEs. For simplicity, we only consider two-dimensional scalar BSDEs (i.e. $p = 1$, $d = 2$). All the error estimates can apply to the general multi-dimensional BSDEs.

3.1. Error equations. Let (Y_t, Z_t) and $(Y^n, Z^n), n = 0, 1, \cdots, N$, be the analytical solution of the BSDE (1) and the approximate solution of Scheme 3, respectively. For simplicity, we use $\mathbb{E}^x_{t_n}[\cdot]$ and $\mathbb{E}^{i,x}_{t_n}[\cdot]$ to denote $\mathbb{E}^{X_{t_n}}_{t_n}[\cdot]$ and $\mathbb{E}^{i,X_{t_n}}_{t_n}[\cdot]$, respectively. We denote $Y_{t_n} - Y^n$ by e^n_y, and $Z_{t_n} - Z^n$ by e^n_z, where

$$e^n_z = (e^{n,1}_z, e^{n,2}_z) = (Z^1_{t_n} - Z^{n,1}, Z^2_{t_n} - Z^{n,2}).$$

By the BSDE (1), we have

$$Y_{t_n} = Y_{t_{n+1}} + \int_{t_n}^{t_{n+1}} f(s, X_s, Y_s, Z_s)ds - \int_{t_n}^{t_{n+1}} Z_s dW_s \tag{16}$$

for $n = 0, 1, \cdots, N-1$. By taking the conditional mathematical expectation $\mathbb{E}^x_{t_n}[\cdot]$ on both sides of (16) and using the left rectangle rule to approximate the integral term in the derived equation, we deduce the equation

$$Y_{t_n} = \mathbb{E}^x_{t_n}[Y_{t_{n+1}}] + \Delta t_n f(t_n, x, Y_{t_n}, Z_{t_n}) + R^n_y, \tag{17}$$

where

$$R^n_y = \int_{t_n}^{t_{n+1}} \left\{ \mathbb{E}^x_{t_n}[f(s, X_s, Y_s, Z_s)] - f(t_n, x, Y_{t_n}, Z_{t_n}) \right\} ds. \tag{18}$$

Multiplying (16) by $\Delta W^\top_{n+1}$, taking the conditional mathematical expectation $\mathbb{E}^x_{t_n}[\cdot]$ on both sides of the derived equation, and using the left rectangle rule to the two integrals, we deduce

$$\Delta t_n Z_{t_n} = \mathbb{E}^x_{t_n}[Y_{t_{n+1}} \Delta W^\top_{n+1}] + R^n_z, \tag{19}$$

where

$$R^n_z = R^n_{z_1} + R^n_{z_2} \tag{20}$$

with

$$R_{z_1}^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f(s, X_s, Y_s, Z_s)\Delta W_{n+1}^\top]ds,$$
$$R_{z_2}^n = -\int_{t_n}^{t_{n+1}} \left\{\mathbb{E}_{t_n}^x[Z_s] - Z_{t_n}\right\} ds.$$

Here $Z_{t_n} = (Z_{t_n}^1, Z_{t_n}^2)$ and $R_z^n = (R_z^{n,1}, R_z^{n,2})$.

By Scheme 3, the equations in (2) of Scheme 3 are written in the equivalent form

$$\Delta t_n \bar{Z}_1^{n,1} = \mathbb{E}_{t_n}^{1,x}[Y^{n+1}\Delta W_{n+1}^1], \tag{21}$$
$$\Delta t_n \bar{Z}_2^{n,2} = \mathbb{E}_{t_n}^{2,x}[\mathbb{E}_{t_n}^{1,x}[Y^{n+1}]\Delta W_{n+1}^2], \tag{22}$$
$$Y^n = \mathbb{E}_{t_n}^{2,x}[\mathbb{E}_{t_n}^{1,x}[Y^{n+1}]] + \Delta t_n f(t_n, x, Y^n, Z^n). \tag{23}$$

Note that the components of X_{t_n} are independent of each other. By using Fubini's theorem [12], (23) can be rewritten as

$$Y^n = \mathbb{E}_{t_n}^x[Y^{n+1}] + \Delta t_n f(t_n, x, Y^n, Z^n). \tag{24}$$

Similarly, (22) can also be rewritten as

$$\Delta t_n \bar{Z}_2^{n,2} = \mathbb{E}_{t_n}^{2,x}[\mathbb{E}_{t_n}^{1,x}[Y^{n+1}]\Delta W_{n+1}^2] = \mathbb{E}_{t_n}^x[Y^{n+1}\Delta W_{n+1}^2]. \tag{25}$$

Now we let $\tilde{Z}_{t_n} = (\tilde{Z}_{t_n}^1, \tilde{Z}_{t_n}^2)$, where

$$\Delta t_n \tilde{Z}_{t_n}^1 = \mathbb{E}_{t_n}^{1,x}[Y_{t_{n+1}}\Delta W_{n+1}^1], \quad \Delta t_n \tilde{Z}_{t_n}^2 = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}\Delta W_{n+1}^2]. \tag{26}$$

Then by the reference equation (17) we get

$$\begin{aligned} Y_{t_n} &= \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \Delta t_n f(t_n, x, Y_{t_n}, Z_{t_n}) + R_y^n \\ &= \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \Delta t_n f(t_n, x, Y_{t_n}, \tilde{Z}_{t_n}) + R_y^n + R_{y_1}^n, \end{aligned} \tag{27}$$

where R_y^n is defined in (18), and

$$R_{y_1}^n = \Delta t_n f(t_n, x, Y_{t_n}, Z_{t_n}) - \Delta t_n f(t_n, x, Y_{t_n}, \tilde{Z}_{t_n}). \tag{28}$$

By (19) and (26), we have

$$\Delta t_n Z_{t_n}^1 = \mathbb{E}_{t_n}^{1,x}[Y_{t_{n+1}}\Delta W_{n+1}^1] + R_{ez}^{n,1} + R_z^{n,1}, \tag{29}$$
$$\Delta t_n Z_{t_n}^2 = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}\Delta W_{n+1}^2] + R_z^{n,2}, \tag{30}$$

where $R_z^n = (R_z^{n,1}, R_z^{n,2})$ is defined in (20), and

$$R_{ez}^{n,1} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}\Delta W_{n+1}^1] - \mathbb{E}_{t_n}^{1,x}[Y_{t_{n+1}}\Delta W_{n+1}^1]. \tag{31}$$

Let $e_{\tilde{f}}^n = f(t_n, x, Y_{t_n}, \tilde{Z}_{t_n}) - f(t_n, x, Y^n, Z^n)$. Subtracting (24), (21) and (22) from (27), (29) and (30), respectively, we obtain the error equations

$$e_y^n = \mathbb{E}_{t_n}^x[e_y^{n+1}] + \Delta t_n e_{\tilde{f}}^n + R_y^n + R_{y_1}^n, \tag{32}$$
$$\Delta t_n e_z^{n,1} = \mathbb{E}_{t_n}^{1,x}[e_y^{n+1}\Delta W_{n+1}^1] + R_{ez}^{n,1} + R_z^{n,1}, \tag{33}$$
$$\Delta t_n e_z^{n,2} = \mathbb{E}_{t_n}^x[e_y^{n+1}\Delta W_{n+1}^2] + R_z^{n,2}. \tag{34}$$

3.2. General error estimates. In this subsection, based on the error equations (32), (33) and (34), we will theoretically derive the error estimates of Scheme 3 according to the error terms R_y^n, $R_{y_1}^n$, $R_{ez}^{n,1}$, $R_z^{n,1}$, and $R_z^{n,2}$. The general estimate results are given in the following theorem, which will be used in deriving our error estimates of Scheme 3.

Theorem 3.1. *Let (Y_t, Z_t), $t \in [0,T]$ be the exact solution of the BSDE (1), and (Y^n, Z^n), $n = 0, 1, \cdots, N$, be the approximate solution obtained by Scheme 3. Assume that the generator $f(t, X, Y, Z)$ of the BSDE (1) is Lipschitz continuous with respect to X, Y and Z, and the Lipschitz constant is L. Let c_0 be the time partition regularity parameter defined in (7). Then for sufficiently small time step Δt, we have*

$$\begin{aligned}
&\mathbb{E}[|e_y^n|^2] + \Delta t \sum_{i=n}^{N-1} \Big(\frac{1+C\Delta t}{1-C\Delta t}\Big)^{i-n} (\mathbb{E}[|e_z^{i,1}|^2] + \mathbb{E}[|e_z^{i,2}|^2]) \\
\leq & C'\mathbb{E}[|e_y^N|^2] + \sum_{i=n}^{N-1} \Big(\frac{1+C\Delta t}{1-C\Delta t}\Big)^{i-n} \frac{C}{\Delta t(1-C\Delta t)} \Big(\mathbb{E}[|R_y^i|^2] \\
&+ \mathbb{E}[|R_{y_1}^i|^2] + 2\mathbb{E}[|R_{ez}^{i,1}|^2] + 2\mathbb{E}[|R_z^{i,1}|^2] + \mathbb{E}[|R_z^{i,2}|^2]\Big)
\end{aligned} \tag{35}$$

for $n = N-1, \cdots, 1, 0$, where C' and C are two positive constants depending on c_0, T and L, and R_y^i, $R_{y_1}^n$, $R_{ez}^{i,1}$ and $R_z^i = (R_z^{i,1}, R_z^{i,2})$ are defined in (18), (28), (31) and (20), respectively.

Proof. There are mainly three parts in the proof of the theorem: (i) the estimate of e_y^n; (ii) the estimates of e_z^n; (iii) the estimate (35) in the Theorem 3.1.

(i) the estimate of e_y^n. For $0 \leq n \leq N-1$, we easily get from (32) that

$$|e_y^n| \leq |\mathbb{E}_{t_n}^x[e_y^{n+1}]| + \Delta t_n |e_{\tilde{f}}^n| + |R_y^n| + |R_{y_1}^n|. \tag{36}$$

By the inequalities

$$(a+b)^2 \leq (1+\gamma\Delta t)a^2 + \Big(1 + \frac{1}{\gamma\Delta t}\Big)b^2, \quad (\sum_{n=1}^m a_n)^2 \leq m \sum_{n=1}^m a_n^2$$

with any positive real number γ and positive integer m, we deduce

$$\begin{aligned}
|e_y^n|^2 \leq & (1+\gamma\Delta t)|\mathbb{E}_{t_n}^x[e_y^{n+1}]|^2 + \Big(1 + \frac{1}{\gamma\Delta t}\Big)\Big\{\Delta t_n |e_{\tilde{f}}^n| + |R_y^n| + |R_{y_1}^n|\Big\}^2 \\
\leq & (1+\gamma\Delta t)|\mathbb{E}_{t_n}^x[e_y^{n+1}]|^2 + 3\Big(1 + \frac{1}{\gamma\Delta t}\Big)(\Delta t_n)^2 |e_{\tilde{f}}^n|^2 \\
&+ 3\Big(1 + \frac{1}{\gamma\Delta t}\Big)|R_y^n|^2 + 3\Big(1 + \frac{1}{\gamma\Delta t}\Big)|R_{y_1}^n|^2.
\end{aligned} \tag{37}$$

Based on the properties of the generator f, we have

$$\begin{aligned}
|e_{\tilde{f}}^n|^2 \leq & |L(|Y_{t_n} - Y^n| + \|\tilde{Z}_{t_n} - Z^n\|)|^2 \\
\leq & 2L^2(|e_y^n|^2 + |\tilde{Z}_{t_n}^1 - Z^{n,1}|^2 + |\tilde{Z}_{t_n}^2 - Z^{n,2}|^2) \\
\leq & 2L^2\Big(|e_y^n|^2 + 2|e_z^{n,1}|^2 + 2|e_z^{n,2}|^2 + \frac{4|R_{ez}^{n,1}|^2}{(\Delta t_n)^2} + \frac{4|R_z^{n,1}|^2}{(\Delta t_n)^2} + \frac{2|R_z^{n,2}|^2}{(\Delta t_n)^2}\Big),
\end{aligned} \tag{38}$$

where $\|\cdot\|$ represents the Euclidean norm. Inserting (38) into (37), we obtian

$$
\begin{aligned}
|e_y^n|^2 \le&(1+\gamma\Delta t)|\mathbb{E}_{t_n}^x[e_y^{n+1}]|^2 + 6\Big(1+\frac{1}{\gamma\Delta t}\Big)(\Delta t)^2L^2(|e_y^n|^2+2|e_z^{n,1}|^2\\
&+2|e_z^{n,2}|^2)+6\Big(1+\frac{1}{\gamma\Delta t}\Big)L^2\Big(4|R_{ez}^{n,1}|^2+4|R_z^{n,1}|^2+2|R_z^{n,2}|^2\Big)\\
&+3\Big(1+\frac{1}{\gamma\Delta t}\Big)|R_y^n|^2+3\Big(1+\frac{1}{\gamma\Delta t}\Big)|R_{y_1}^n|^2.
\end{aligned}
\tag{39}
$$

(ii) the estimates of $e_z^n=(e_z^{n,1},e_z^{n,2})$. We first estimate $e_z^{n,1}$. By (33), we easily obtain

$$
\Delta t_n e_z^{n,1} = \mathbb{E}_{t_n}^x[e_y^{n+1}\Delta W_{n+1}^1] + \mathbb{E}_{t_n}^{2,x}[R_{ez}^{n,1}] + \mathbb{E}_{t_n}^{2,x}[R_z^{n,1}].
\tag{40}
$$

Then we have

$$
|e_z^{n,1}| \le \frac{1}{\Delta t_n}|\mathbb{E}_{t_n}^x[e_y^{n+1}\Delta W_{n+1}^1]| + \frac{1}{\Delta t_n}\mathbb{E}_{t_n}^{2,x}[|R_{ez}^{n,1}|] + \frac{1}{\Delta t_n}\mathbb{E}_{t_n}^{2,x}[|R_z^{n,1}|].
\tag{41}
$$

By using the inequality $(a+b)^2\le(1+\varepsilon)a^2+\Big(1+\frac{1}{\varepsilon}\Big)b^2$ for any positive real number ε, we deduce

$$
\begin{aligned}
|e_z^{n,1}|^2 \le&(1+\varepsilon)\frac{1}{(\Delta t_n)^2}|\mathbb{E}_{t_n}^x[e_y^{n+1}\Delta W_{n+1}^1]|^2\\
&+2\Big(1+\frac{1}{\varepsilon}\Big)\Big\{\frac{\mathbb{E}_{t_n}^{2,x}[|R_{ez}^{n,1}|^2]}{(\Delta t_n)^2}+\frac{\mathbb{E}_{t_n}^{2,x}[|R_z^{n,1}|^2]}{(\Delta t_n)^2}\Big\}.
\end{aligned}
\tag{42}
$$

Furthermore, applying

$$
\begin{aligned}
|\mathbb{E}_{t_n}^x[e_y^{n+1}\Delta W_{n+1}^1]|^2 &= |\mathbb{E}_{t_n}^x[(e_y^{n+1}-\mathbb{E}_{t_n}^x[e_y^{n+1}])\Delta W_{n+1}^1]|^2\\
&\le \Delta t_n(\mathbb{E}_{t_n}^x[|e_y^{n+1}|^2]-|\mathbb{E}_{t_n}^x[e_y^{n+1}]|^2)
\end{aligned}
\tag{43}
$$

into (42), we obtain

$$
\begin{aligned}
|e_z^{n,1}|^2 \le&(1+\varepsilon)\frac{1}{\Delta t_n}(\mathbb{E}_{t_n}^x[|e_y^{n+1}|^2]-|\mathbb{E}_{t_n}^x[e_y^{n+1}]|^2)\\
&+2\Big(1+\frac{1}{\varepsilon}\Big)\Big\{\frac{\mathbb{E}_{t_n}^{2,x}[|R_{ez}^{n,1}|^2]}{(\Delta t_n)^2}+\frac{\mathbb{E}_{t_n}^{2,x}[|R_z^{n,1}|^2]}{(\Delta t_n)^2}\Big\}.
\end{aligned}
\tag{44}
$$

Dividing both sides of the inequality (44) by $\frac{1+\varepsilon}{\Delta t}$, we deduce

$$
\begin{aligned}
\frac{\Delta t}{1+\varepsilon}|e_z^{n,1}|^2 \le& c_0(\mathbb{E}_{t_n}^x[|e_y^{n+1}|^2]-|\mathbb{E}_{t_n}^x[e_y^{n+1}]|^2)\\
&+\frac{2\Delta t}{\varepsilon}\Big\{\frac{\mathbb{E}_{t_n}^{2,x}[|R_{ez}^{n,1}|^2]}{(\Delta t_n)^2}+\frac{\mathbb{E}_{t_n}^{2,x}[|R_z^{n,1}|^2]}{(\Delta t_n)^2}\Big\}.
\end{aligned}
\tag{45}
$$

Similarly, the estimate of $e_z^{n,2}$ is

$$
\frac{\Delta t}{1+\varepsilon}|e_z^{n,2}|^2 \le c_0(\mathbb{E}_{t_n}^x[|e_y^{n,1}|^2]-|\mathbb{E}_{t_n}^x[e_y^{n,1}]|^2)+\frac{\Delta t}{\varepsilon}\frac{|R_z^{n,2}|^2}{(\Delta t_n)^2}.
\tag{46}
$$

(iii) the estimate of (35)**.** Multiplying (39) by c_0, (45) and (46) by $\frac{1}{2}$, and adding the derived inequalities to obtain

(47)
$$\begin{aligned}
&c_0|e_y^n|^2+\frac{\Delta t}{2(1+\varepsilon)}(|e_z^{n,1}|^2+|e_z^{n,2}|^2)\\
\le& c_0(1+\gamma\Delta t)\mathbb{E}_{t_n}^x[|e_y^{n+1}|^2]+6c_0\Big(1+\frac{1}{\gamma\Delta t}\Big)(\Delta t)^2L^2(|e_y^n|^2+2|e_z^{n,1}|^2+2|e_z^{n,2}|^2)\\
&+3c_0\Big(1+\frac{1}{\gamma\Delta t}\Big)|R_y^n|^2+3c_0\Big(1+\frac{1}{\gamma\Delta t}\Big)|R_{y_1}^n|^2\\
&+6c_0\Big(1+\frac{1}{\gamma\Delta t}\Big)L^2\Big(4|R_{ez}^{n,1}|^2+4|R_z^{n,1}|^2+2|R_z^{n,2}|^2\Big)\\
&+\frac{\Delta t}{\varepsilon}\Big\{\frac{\mathbb{E}_{t_n}^{2,x}[|R_{ez}^{n,1}|^2]}{(\Delta t_n)^2}+\frac{\mathbb{E}_{t_n}^{2,x}[|R_z^{n,1}|^2]}{(\Delta t_n)^2}\Big\}+\frac{\Delta t}{2\varepsilon}\frac{|R_z^{n,2}|^2}{(\Delta t_n)^2},
\end{aligned}$$

which can be further simplified to

$$\begin{aligned}
&c_0(1-C_1\Delta t)\mathbb{E}[|e_y^n|^2]+C_3\Delta t(\mathbb{E}[|e_z^{n,1}|^2]+\mathbb{E}[|e_z^{n,2}|^2])\\
\le& c_0(1+C_2\Delta t)\mathbb{E}[|e_y^{n+1}|^2]+\frac{C_4}{\Delta t}\Big(\mathbb{E}[|R_y^n|^2]+\mathbb{E}[|R_{y_1}^n|^2]\Big)\\
&+\frac{C_5}{\Delta t}\Big(2\mathbb{E}[|R_{ez}^{n,1}|^2]+2\mathbb{E}[|R_z^{n,1}|^2]+\mathbb{E}[|R_z^{n,2}|^2]\Big),
\end{aligned}\tag{48}$$

where

$$\begin{aligned}
&C_1=\frac{6L^2(1+\gamma\Delta t)}{\gamma},\quad C_2=\gamma,\quad C_3=\frac{1}{2(1+\varepsilon)}-\frac{8c_0L^2(1+\gamma\Delta t)}{\gamma},\\
&C_4=\frac{3c_0(1+\gamma\Delta t)}{\gamma},\quad C_5=\frac{8c_0L^2(1+\gamma\Delta t)}{\gamma}+\frac{(c_0)^2}{2\varepsilon}.
\end{aligned}$$

Now we choose $\varepsilon=1$, γ large enough, and Δt_0 sufficiently small such that for $0\le\Delta t\le\Delta t_0$, then $C_1\le C$, $C_2\le C$, $C_4\le C$, $C_5\le C$, and $1-C\Delta t\ge 0$, where C is a positive constant depending on c_0 and L. Then for $0\le\Delta t\le\Delta t_0$, we have

$$\begin{aligned}
&c_0\mathbb{E}[|e_y^n|^2]+C_3\Delta t(\mathbb{E}[|e_z^{n,1}|^2]+\mathbb{E}[|e_z^{n,2}|^2])\\
\le& c_0\frac{1+C\Delta t}{1-C\Delta t}\mathbb{E}[|e_y^{n+1}|^2]+\frac{C}{\Delta t(1-C\Delta t)}\Big(\mathbb{E}[|R_y^n|^2]+\mathbb{E}[|R_{y_1}^n|^2]\\
&+2\mathbb{E}[|R_{ez}^{n,1}|^2]+2\mathbb{E}[|R_z^{n,1}|^2]+\mathbb{E}[|R_z^{n,2}|^2]\Big),
\end{aligned}\tag{49}$$

which is written in the equivalent form

$$\begin{aligned}
&c_0\mathbb{E}[|e_y^n|^2]+C_3\Delta t\sum_{i=n}^{N-1}\Big(\frac{1+C\Delta t}{1-C\Delta t}\Big)^{i-n}(\mathbb{E}[|e_z^{i,1}|^2]+\mathbb{E}[|e_z^{i,2}|^2])\\
\le& c_0\Big(\frac{1+C\Delta t}{1-C\Delta t}\Big)^{N-n}\mathbb{E}[|e_y^N|^2]+\sum_{i=n}^{N-1}\Big(\frac{1+C\Delta t}{1-C\Delta t}\Big)^{i-n}\frac{C}{\Delta t(1-C\Delta t)}\Big(\mathbb{E}[|R_y^i|^2]\\
&+\mathbb{E}[|R_{y_1}^i|^2+2\mathbb{E}[|R_{ez}^{i,1}|^2]+2\mathbb{E}[|R_z^{i,1}|^2]+\mathbb{E}[|R_z^{i,2}|^2]\Big),
\end{aligned}\tag{50}$$

which leads to the inequality (35). The proof is completed. □

3.3. Error estimates. Under certain conditions, we will first estimate the error terms R_y^n, $R_{y_1}^n$, $R_{ez}^{n,1}$ and $R_z^n=(R_z^{n,1},R_z^{n,2})$, and then present the error estimates of Scheme 3. For simplicity, we make the following assumption.

Assumption 3.1. *The functions f and φ of the BSDE* (1) *are bounded and smooth enough, and their derivatives are also bounded.*

Before giving the error estimates of the terms R_y^n, $R_{y_1}^n$, $R_{ez}^{n,1}$ and $R_z^n = (R_z^{n,1}, R_z^{n,2})$, we introduce the following lemma.

Lemma 3.1. *Let W_s be a d-dimensional Brownian motion. Assume that the function $g(s,x)$ is bounded and smooth enough with bounded derivatives. Then we have*

$$|\mathbb{E}_t^x[g(s,X_s)\Delta W_{t,s}^i] - \mathbb{E}_t^{i,x}[g(s,X_s)\Delta W_{t,s}^i]| \le C(s-t)^2, \quad s \ge t \tag{51}$$

for $i = 1, \cdots, d$, where $C > 0$ is a generic constant depending on the bounds of derivatives of g.

Proof. Without loss of generality, we set $d = 2$. We consider $i = 1$, and the case of $i = 2$ is similar. By using the Taylor expansion formula, we deduce

$$\begin{aligned}
&g(s, x^1 + \Delta W_{t,s}^1, x^2 + \Delta W_{t,s}^2) - g(s, x^1 + \Delta W_{t,s}^1, x^2)\\
=&g'_{x^2}(s, x^1 + \Delta W_{t,s}^1, x^2)\Delta W_{t,s}^2 + \frac{1}{2} g''_{x^2x^2}(s, x^1 + \Delta W_{t,s}^1, x^2)(\Delta W_{t,s}^2)^2\\
&+ \frac{1}{6} g'''_{x^2x^2x^2}(s, x^1 + \Delta W_{t,s}^1, x^2)(\Delta W_{t,s}^2)^3\\
&+ \frac{1}{24} g^{(4)}_{x^2x^2x^2x^2}(s, x^1 + \Delta W_{t,s}^1, \xi^2)(\Delta W_{t,s}^2)^4,
\end{aligned} \tag{52}$$

where $\xi^2 \in (t,s)$. We denote (52) as $h(s, x^1 + \Delta W_{t,s}^1)$, then by the Taylor expansion formula, we get

$$\begin{aligned}
&h(s, x^1 + \Delta W_{t,s}^1)\Delta W_{t,s}^1\\
=&h(s,x^1)\Delta W_{t,s}^1 + h'_{x^1}(s,x^1)(\Delta W_{t,s}^1)^2 + \frac{1}{2} h''_{x^1x^1}(s,x^1)(\Delta W_{t,s}^1)^3\\
&+ \frac{1}{2} h'''_{x^1x^1x^1}(s,\xi^1)(\Delta W_{t,s}^1)^4,
\end{aligned} \tag{53}$$

where $\xi^1 \in (t,s)$. Noted that $\Delta W_{t,s}^1$ and $\Delta W_{t,s}^2$ are independent. Inserting (52) into (53), we deduce

$$\begin{aligned}
&|\mathbb{E}[g(s, x^1 + \Delta W_{t,s}^1, x^2 + \Delta W_{t,s}^2)\Delta W_{t,s}^1] - \mathbb{E}[g(s, x^1 + \Delta W_{t,s}^1, x^2)\Delta W_{t,s}^1]|\\
=&|\mathbb{E}[h(s, x^1 + \Delta W_{t,s}^1)\Delta W_{t,s}^1]|\\
\le&C(s-t)^2,
\end{aligned} \tag{54}$$

which leads to the inequality (51). The proof is completed. □

To get the error estimates of Scheme 3, we need the estimates of R_y^n, $R_{y_1}^n$, $R_{ez}^{n,1}$ and $R_z^n = (R_z^{n,1}, R_z^{n,2})$ in (35), which are stated in the following lemma.

Lemma 3.2. *Let R_y^n, $R_{y_1}^n$, $R_{ez}^{n,1}$ and $R_z^n = (R_z^{n,1}, R_z^{n,2})$ be the error terms defined in* (18), (28), (31) *and* (20), *respectively. Then under the Assumption 3.1, for sufficiently small time step Δt_n, we have*

$$\begin{aligned}
&|R_y^n| \le C(\Delta t_n)^2, \quad |R_{y_1}^n| \le C(\Delta t_n)^2, \quad |R_{ez}^{n,1}| \le C(\Delta t_n)^2,\\
&|R_z^{n,1}| \le C(\Delta t_n)^2, \quad |R_z^{n,2}| \le C(\Delta t_n)^2,
\end{aligned} \tag{55}$$

where C is a positive constant only depending on T, the Lipschitz constant L of f and the upper bounds of the derivatives of the functions f and φ.

Proof. Under Assumption 3.1, it is well known that the solution $u(s, X_s)$ of (3) and its derivatives are bounded [18], and by the nonlinear Feynman-Kac formula (2), the solution of (1) can be represented as $Y_s = u(s, X_s)$ and $Z_s = \nabla_x u(s, X_s)$ for $s \in [0, T]$. Then by Lemma 3.1, we easily obtain the estimate

$$|R_{ez}^{n,1}| \leq C(\Delta t_n)^2. \tag{56}$$

Under the condition of Assumption 3.1, based on the standard Itô-Taylor expansion formula [17] and the properties of the Brownian motion, for sufficiently small time step Δt_n, we have

$$|R_y^n| \leq C(\Delta t_n)^2, \quad |R_z^{n,1}| \leq C(\Delta t_n)^2, \quad |R_z^{n,2}| \leq C(\Delta t_n)^2. \tag{57}$$

From (28), we get

$$\begin{aligned} |R_{y_1}^n| \leq & \Delta t_n |f(t_n, X_{t_n}, Y_{t_n}, Z_{t_n}) - f(t_n, X_{t_n}, Y_{t_n}, \tilde{Z}_{t_n})| \\ \leq & \Delta t_n L \|Z_{t_n} - \tilde{Z}_{t_n}\| \\ \leq & \Delta t_n L (|Z_{t_n}^1 - \tilde{Z}_{t_n}^1| + |Z_{t_n}^2 - \tilde{Z}_{t_n}^2|) \\ \leq & L(|R_{ez}^{n,1}| + |R_z^{n,1}| + |R_z^{n,2}|), \end{aligned} \tag{58}$$

where $\|\cdot\|$ represents the Euclidean norm. Inserting (56) and (57) into (58), we have

$$|R_{y_1}^n| \leq C(\Delta t_n)^2. \tag{59}$$

The proof is completed. □

Now combining Theorem 3.1 and Lemma 3.2, we obtain the following theorem.

Theorem 3.2. *Under the condition of Assumption 3.1, let (Y_t, Z_t), $t \in [0, T]$ be the solution of the BSDE (1), and (Y^n, Z^n), $n = 0, 1, \cdots, N$, be the approximate solution obtained by Scheme 3. Suppose that the terminal value satisfies $\mathbb{E}[|Y_{t_N} - Y^N|^2] = O((\Delta t)^2)$, then we have the following error estimate*

$$\mathbb{E}[|Y_{t_n} - Y^n|^2] + \Delta t \sum_{i=n}^{N-1} \mathbb{E}[\|Z_{t_n} - Z^n\|^2] \leq C(\Delta t)^2, \tag{60}$$

where $\|\cdot\|$ represents the Euclidean norm, and C is a constant depending on c_0, T, the Lipschitz constant L of f and the upper bounds of f and φ.

4. Numerical tests

In this section, we will present several numerical tests to demonstrate the accuracy and effectiveness of our schemes for solving BSDEs.

In order to use the splitting schemes, space partition $\mathfrak{R}_h^d$ and the approximation of $\mathbb{E}_{t_n}^{i,x}[\cdot]$ $(i = 1, \cdots, d)$ at discrete space grid points $\boldsymbol{x}_j \in \mathfrak{R}_h^d$ are needed. For the space $\mathbb{R}^d$, we introduce the following spatial partition $\mathfrak{R}_h^d$:

$$\mathfrak{R}_h^d = \left\{ \boldsymbol{x}_j = (x_{j_1}^1, x_{j_2}^2, \cdots, x_{j_d}^d)^\top | \boldsymbol{x}_j \in \mathbb{R}^d, j_m \in \mathbb{Z}, m = 1 \cdots, d \right\}, \tag{61}$$

where $\mathbb{Z}$ is the set of all integer numbers. We use Δx^m to denote the uniform space step in the mth direction, that is, $x_k^m = k\Delta x^m$. It should be noted that the conditional mathematical expectations $\mathbb{E}_{t_n}^{i,x}[\cdot]$ are continuous and deterministic. In practice, we evaluate $\mathbb{E}_{t_n}^{i,x}[\cdot]$ by using one-dimensional Gauss-Hermite quadrature rule in the ith direction of $X_{t_n} = x$, and the values of the integrands of $\mathbb{E}_{t_n}^{i,x}[\cdot]$ at non-grid points are approximated by local cubic interpolations. Since our goal is to test the accuracy of the splitting schemes with respect to the time step size Δt, we set the number of Gauss-Hermite quadrature points to a relatively big value to ensure that the error of the Gauss-Hermite quadrature rule can be neglected. For simplicity, we use a uniform partition for the time interval $[0, T]$. The time partition number is N and the time step size Δt is given by $\Delta t = \frac{T}{N}$.

Let (Y_0, Z_0) denotes the exact solution of (1) at $t = 0$, and (Y^0, Z^0) denotes the approximate solution of Schemes 1, 2 and 3 at $n = 0$. The splitting of the generator f is set as $f^i = \alpha^i f$, where $\sum_{i=1}^{d} \alpha^i = 1$ with $\alpha^i \in [0, 1]$ $(i = 1, \cdots, d)$. We denote the convergence rates and running time by CR and RT, respectively. For all the tests, we set the terminal time $T = 1.0$. All numerical tests are carried out in Python 3.11.7 on a laptop with an Intel Core i9-18500HX 16-Core Processor (2.06GHz) and 32.0 GB of DDR5 RAM (7500MT).

Example 1. We demonstrate the capability of our splitting schemes for the following two-dimensional BSDE

$$Y_t = Y_T + \int_t^T \Big((1 + \frac{5}{2}\sigma^2)e^{-2t}\frac{Y_t}{Y_t^2 + (Z_t\tilde{\boldsymbol{\sigma}})^2}\Big)ds - \int_t^T Z_s dW_s \tag{62}$$

with $X_t = X_0 + \sigma W_t$ and $Y_T = e^{-T}\sin(X_T^1 + 2X_T^2)$. Here $Z_t = (Z_t^1, Z_t^2)$, $\tilde{\boldsymbol{\sigma}} = (\frac{3}{\sigma}, -\frac{1}{\sigma})^\top$, and W_t is a standard two-dimensional Brownian motion. The analytical solution is given by

$$\begin{cases} Y_t = e^{-t}\sin(X_t^1 + 2X_t^2), \\ Z_t = (\sigma e^{-t}\cos(X_t^1 + 2X_t^2), 2\sigma e^{-t}\cos(X_t^1 + 2X_t^2)). \end{cases} \tag{63}$$

In this example, we set $\sigma = 0.2$. We use Schemes 1, 2 and 3 to solve the above BSDE. The errors $|Y_0 - Y^0|$, $|Z_0^1 - Z_1^0|$ and $|Z_0^2 - Z_2^0|$, CR and RT for different time partitions with different parameters of α^i $(i = 1, 2)$ are listed in Tables 1-3.

By the numerical results in Tables 1-3, we have the following conclusions.

(1) Our splitting schemes are accurate and effective for solving the BSDE (62), and the expected first-order convergence in time is observed.
(2) Scheme 3 is the most efficient. The different splittings of the generator f may affect numerical results, but all are first-order accurate. The numerical errors of Schemes 2 and 3 with the splitting $f^1 = 0, f^2 = f$ are identical. Schemes 1 and 2 produce the same errors when using the splitting $f^1 = f, f^2 = 0$. These results are consistent with the conclusions in Remark 2.

Table 1. Numerical results of Schemes 1, 2 and 3 for Example 1 with $\alpha^1 = 0, \alpha^2 = 1$.

		$N = 15$	$N = 18$	$N = 21$	$N = 24$	$N = 27$	CR
Scheme 1	$\lvert Y_0 - Y^0\rvert$	3.412E-02	2.838E-02	2.431E-02	2.128E-02	1.891E-02	1.002
	$\lvert Z_0^1 - Z_1^0\rvert$	1.163E-02	9.665E-03	8.280E-03	7.209E-03	6.415E-03	1.014
	$\lvert Z_0^2 - Z_2^0\rvert$	2.263E-02	1.892E-02	1.627E-02	1.436E-02	1.279E-02	0.969
	RT	1.337s	2.390s	4.037s	6.634s	10.182s	
Scheme 2 Scheme 3	$\lvert Y_0 - Y^0\rvert$	3.348E-02	2.778E-02	2.365E-02	2.070E-02	1.838E-02	1.021
	$\lvert Z_0^1 - Z_1^0\rvert$	1.101E-02	9.351E-03	7.693E-03	6.817E-03	6.049E-03	1.035
	$\lvert Z_0^2 - Z_2^0\rvert$	2.068E-02	1.812E-02	1.543E-02	1.363E-02	1.213E-02	0.919
	RT(Scheme 2)	0.994s	1.779s	3.039s	5.143s	8.041s	
	RT(Scheme 3)	0.637s	1.221s	2.088s	3.420s	5.181s	

TABLE 2. Numerical results of Scheme 1 for Example 1 with different α^i $(i = 1, 2)$.

		$N=15$	$N=18$	$N=21$	$N=24$	$N=27$	CR
$\alpha^1=1$, $\alpha^2=0$	$\lvert Y_0-Y^0\rvert$	4.748E-02	3.924E-02	3.344E-02	2.913E-02	2.581E-02	1.036
	$\lvert Z_0^1-Z_1^0\rvert$	1.306E-02	1.084E-02	9.281E-03	8.112E-03	7.206E-03	1.011
	$\lvert Z_0^2-Z_2^0\rvert$	2.558E-02	2.140E-02	1.840E-02	1.616E-02	1.440E-02	0.977
	RT	1.350s	2.588s	4.318s	6.830s	10.207s	
$\alpha^1=0.5$, $\alpha^2=0.5$	$\lvert Y_0-Y^0\rvert$	6.780E-03	5.656E-03	4.821E-03	4.202E-03	3.726E-03	1.021
	$\lvert Z_0^1-Z_1^0\rvert$	7.015E-03	5.837E-03	5.005E-03	4.364E-03	3.884E-03	1.006
	$\lvert Z_0^2-Z_2^0\rvert$	1.407E-02	1.175E-02	1.010E-02	8.907E-03	7.933E-03	0.973
	RT	1.313s	2.411s	4.319s	6.833s	10.264s	
$\alpha^1=0.25$, $\alpha^2=0.75$	$\lvert Y_0-Y^0\rvert$	1.249E-02	1.031E-02	8.805E-03	7.692E-03	6.821E-03	1.029
	$\lvert Z_0^1-Z_1^0\rvert$	7.899E-03	6.584E-03	5.653E-03	4.928E-03	4.391E-03	1.000
	$\lvert Z_0^2-Z_2^0\rvert$	1.568E-02	1.312E-02	1.130E-02	9.981E-03	8.895E-03	0.962
	RT	1.179s	2.202s	4.410s	6.754s	10.350s	
$\alpha^1=0.75$, $\alpha^2=0.25$	$\lvert Y_0-Y^0\rvert$	2.764E-02	2.303E-02	1.974E-02	1.726E-02	1.534E-02	1.001
	$\lvert Z_0^1-Z_1^0\rvert$	8.721E-03	7.247E-03	6.205E-03	5.418E-03	4.817E-03	1.010
	$\lvert Z_0^2-Z_2^0\rvert$	1.735E-02	1.450E-02	1.246E-02	1.095E-02	9.762E-03	0.978
	RT	1.147s	2.138s	4.325s	6.677s	10.195s	

TABLE 3. Numerical results of Scheme 2 for Example 1 with different α^i $(i = 1, 2)$.

		$N=15$	$N=18$	$N=21$	$N=24$	$N=27$	CR
$\alpha^1=1$, $\alpha^2=0$	$\lvert Y_0-Y^0\rvert$	4.748E-02	3.924E-02	3.344E-02	2.913E-02	2.581E-02	1.036
	$\lvert Z_0^1-Z_1^0\rvert$	1.306E-02	1.084E-02	9.281E-03	8.112E-03	7.206E-03	1.011
	$\lvert Z_0^2-Z_2^0\rvert$	2.558E-02	2.140E-02	1.840E-02	1.616E-02	1.440E-02	0.977
	RT	1.030s	2.038s	3.387s	5.508s	8.107s	
$\alpha^1=0.5$, $\alpha^2=0.5$	$\lvert Y_0-Y^0\rvert$	7.960E-03	6.729E-03	5.814E-03	5.104E-03	4.558E-03	0.950
	$\lvert Z_0^1-Z_1^0\rvert$	1.490E-02	1.233E-02	1.043E-03	9.038E-03	7.980E-03	1.066
	$\lvert Z_0^2-Z_2^0\rvert$	2.892E-02	2.405E-02	2.048E-02	1.787E-02	1.583E-02	1.026
	RT	1.011s	1.938s	3.112s	5.112s	7.991s	
$\alpha^1=0.25$, $\alpha^2=0.75$	$\lvert Y_0-Y^0\rvert$	2.335E-02	1.942E-02	1.662E-02	1.451E-02	1.288E-03	1.011
	$\lvert Z_0^1-Z_1^0\rvert$	1.362E-02	1.111E-02	9.495E-03	8.249E-03	7.301E-03	1.057
	$\lvert Z_0^2-Z_2^0\rvert$	2.662E-02	2.181E-02	1.870E-02	1.636E-02	1.453E-02	1.025
	RT	1.043s	2.062s	3.179s	5.006s	7.560s	
$\alpha^1=0.75$, $\alpha^2=0.25$	$\lvert Y_0-Y^0\rvert$	1.659E-02	1.379E-02	1.179E-02	1.029E-02	9.130E-03	1.016
	$\lvert Z_0^1-Z_1^0\rvert$	1.502E-02	1.228E-02	1.040E-02	9.026E-03	7.973E-03	1.077
	$\lvert Z_0^2-Z_2^0\rvert$	2.908E-02	2.403E-02	2.048E-02	1.787E-02	1.584E-02	1.032
	RT	1.059s	1.898s	3.085s	4.965s	7.607s	

Example 2. We use our schemes for solving the following two-dimensional BSDE

$$Y_t = \varphi(X_T) + \int_t^T Y_s Z_s \tilde{\boldsymbol{\sigma}} ds - \int_t^T Z_s dW_s, \tag{64}$$

where $X_t = X_0 + \sigma W_t$ with $X_t = (X_t^1, X_t^2)$, $Z_t = (Z_t^1, Z_t^2)$, $\tilde{\boldsymbol{\sigma}} = (-\frac{1}{\sigma}, -\frac{1}{\sigma})^\top$, W_t is a standard two-dimensional Brownian motion, and the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is given. According to the nonlinear Feynman-Kac formula, the solutuon (Y_t, Z_t) of (64) can be represented as

$$Y_t = u(T-t, X_t), \quad Z_t = \sigma \nabla_x u(T-t, X_t), \quad t \in [0, T], \tag{65}$$

where $u : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is the classical solution of the following two-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + u \nabla u - \frac{\sigma^2}{2} \Delta u = 0 \quad \text{in} \quad (0, T] \times \mathbb{R}^2 \tag{66}$$

with initial condition $u(0, \boldsymbol{x}) = \varphi(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^2$.

In this example, the function φ and $\sigma = \sqrt{0.3}$ are chosen such that the exact solution u is given by $u = \frac{1}{1+\exp(\frac{X_t^1+X_t^2-t}{\sigma^2})}$. The errors $|Y_0 - Y^0|$, $|Z_0^1 - Z_1^0|$ and $|Z_0^2 - Z_2^0|$, CR and RT by Schemes 1, 2 and 3 for different time partitions with different parameters of α^i $(i = 1, 2)$ are listed in Tables 4-6. The same conclusions as shown in Example 1 are observed for solving the BSDE (64).

TABLE 4. Numerical results of Schemes 1, 2 and 3 for Example 2 with $\alpha^1 = 0, \alpha^2 = 1$.

		$N = 15$	$N = 18$	$N = 21$	$N = 24$	$N = 27$	CR
Scheme 1	$\lvert Y_0 - Y^0\rvert$	7.166E-02	5.767E-02	4.835E-02	4.165E-02	3.660E-02	1.142
	$\lvert Z_0^1 - Z_1^0\rvert$	1.224E-02	1.071E-02	9.134E-03	7.939E-03	6.924E-03	0.977
	$\lvert Z_0^2 - Z_2^0\rvert$	1.253E-02	1.100E-02	9.568E-03	8.362E-03	7.445E-03	0.894
	RT	4.494s	9.501s	17.961s	31.191s	49.978s	
Scheme 2 Scheme 3	$\lvert Y_0 - Y^0\rvert$	6.145E-02	5.027E-02	4.256E-02	3.691E-02	3.259E-02	1.079
	$\lvert Z_0^1 - Z_1^0\rvert$	2.124E-02	1.620E-02	1.305E-02	1.091E-02	9.369E-03	1.392
	$\lvert Z_0^2 - Z_2^0\rvert$	2.128E-02	1.628E-02	1.318E-02	1.106E-02	9.579E-03	1.359
	RT(Scheme 2)	3.412s	7.140s	14.139s	23.893s	38.941s	
	RT(Scheme 3)	2.322s	4.821s	9.398s	16.430s	26.228s	

TABLE 5. Numerical results of Scheme 1 for Example 2 with different α^i $(i = 1, 2)$.

		$N = 15$	$N = 18$	$N = 21$	$N = 24$	$N = 27$	CR
$\alpha^1 = 1$ $\alpha^2 = 0$	$\lvert Y_0 - Y^0\rvert$	5.498E-02	4.306E-02	3.532E-02	2.992E-02	2.594E-02	1.277
	$\lvert Z_0^1 - Z_1^0\rvert$	7.995E-02	6.718E-02	5.756E-02	5.021E-02	4.445E-02	1.000
	$\lvert Z_0^2 - Z_2^0\rvert$	7.981E-02	6.699E-02	5.723E-02	4.983E-02	4.393E-02	1.016
	RT	4.367s	8.909s	17.965s	32.009s	50.096s	
$\alpha^1 = 0.5$ $\alpha^2 = 0.5$	$\lvert Y_0 - Y^0\rvert$	8.295E-03	6.664E-03	5.573E-03	4.784E-03	4.194E-03	1.159
	$\lvert Z_0^1 - Z_1^0\rvert$	3.541E-02	2.809E-02	2.331E-02	1.990E-02	1.738E-02	1.210
	$\lvert Z_0^2 - Z_2^0\rvert$	3.515E-02	2.782E-02	2.290E-02	1.949E-02	1.687E-02	1.247
	RT	4.488s	9.566s	18.722s	32.302s	52.083s	
$\alpha^1 = 0.25$ $\alpha^2 = 0.75$	$\lvert Y_0 - Y^0\rvert$	3.384E-02	2.715E-02	2.269E-02	1.949E-02	1.708E-02	1.161
	$\lvert Z_0^1 - Z_1^0\rvert$	1.903E-02	1.460E-02	1.192E-02	1.005E-02	8.739E-03	1.323
	$\lvert Z_0^2 - Z_2^0\rvert$	1.876E-02	1.432E-02	1.150E-02	9.641E-03	8.226E-03	1.400
	RT	4.754s	9.155s	18.662s	31.806s	51.899s	
$\alpha^1 = 0.75$ $\alpha^2 = 0.25$	$\lvert Y_0 - Y^0\rvert$	3.785E-03	2.793E-03	2.203E-03	1.826E-03	1.554E-03	1.512
	$\lvert Z_0^1 - Z_1^0\rvert$	4.427E-02	3.466E-02	2.844E-02	2.409E-02	2.089E-02	1.277
	$\lvert Z_0^2 - Z_2^0\rvert$	4.401E-02	3.438E-02	2.803E-02	2.368E-02	2.038E-02	1.308
	RT	4.805s	9.632s	18.528s	32.356s	52.055s	

TABLE 6. Numerical results of Scheme 2 for Example 2 with different α^i $(i = 1, 2)$.

		$N = 15$	$N = 18$	$N = 21$	$N = 24$	$N = 27$	CR
$\alpha^1 = 1$ $\alpha^2 = 0$	$\lvert Y_0 - Y^0\rvert$	5.498E-02	4.306E-02	3.532E-02	2.992E-02	2.594E-02	1.277
	$\lvert Z_0^1 - Z_1^0\rvert$	7.995E-02	6.718E-02	5.756E-02	5.021E-02	4.445E-02	1.000
	$\lvert Z_0^2 - Z_2^0\rvert$	7.981E-02	6.699E-02	5.723E-02	4.983E-02	4.393E-02	1.016
	RT	3.413s	7.117s	14.155s	24.083s	39.303s	
$\alpha^1 = 0.5$ $\alpha^2 = 0.5$	$\lvert Y_0 - Y^0\rvert$	1.674E-02	1.339E-02	1.116E-02	9.571E-03	8.377E-03	1.177
	$\lvert Z_0^1 - Z_1^0\rvert$	3.032E-02	2.346E-02	1.914E-02	1.617E-02	1.400E-02	1.313
	$\lvert Z_0^2 - Z_2^0\rvert$	3.016E-02	2.328E-02	1.885E-02	1.588E-02	1.363E-02	1.348
	RT	3.535s	7.319s	14.303s	24.514s	39.739s	
$\alpha^1 = 0.25$ $\alpha^2 = 0.75$	$\lvert Y_0 - Y^0\rvert$	3.709E-02	2.979E-02	2.492E-02	2.143E-02	1.881E-02	1.154
	$\lvert Z_0^1 - Z_1^0\rvert$	1.021E-02	7.873E-03	6.451E-03	5.484E-03	4.777E-03	1.290
	$\lvert Z_0^2 - Z_2^0\rvert$	1.010E-02	7.736E-03	6.236E-03	5.261E-03	4.484E-03	1.376
	RT	3.571s	7.822s	14.631s	25.032s	39.400s	
$\alpha^1 = 0.75$ $\alpha^2 = 0.25$	$\lvert Y_0 - Y^0\rvert$	3.701E-03	3.184E-03	2.754E-03	2.403E-03	2.129E-03	0.944
	$\lvert Z_0^1 - Z_1^0\rvert$	4.229E-02	3.233E-02	2.610E-02	2.187E-02	1.882E-02	1.377
	$\lvert Z_0^2 - Z_2^0\rvert$	4.207E-02	3.210E-02	2.575E-02	2.152E-02	1.838E-02	1.407
	RT	3.342s	7.816s	14.758s	24.607s	39.633s	

Moreover, to clearly see the efficiency of Scheme 3, we list the errors $|Y_0 - Y^0|$, $|Z_0^1 - Z_1^0|$ and $|Z_0^2 - Z_2^0|$, CR and RT by Schemes 3 and the Euler scheme [35, 36, 37, 38] in Table 7 for the different time partition number N. The numerical results in Table 7 show that Scheme 3 is more efficient than the Euler scheme.

TABLE 7. Numerical results of Schemes 3 and the Euler scheme for Example 2.

		$N=20$	$N=24$	$N=28$	$N=32$	$N=36$	CR
Scheme 3	$\|Y_0 - Y^0\|$	4.486E-02	3.691E-02	3.136E-02	2.726E-02	2.411E-02	1.055
	$\|Z_0^1 - Z_1^0\|$	1.395E-02	1.091E-02	8.947E-03	7.578E-03	6.571E-03	1.281
	$\|Z_0^2 - Z_2^0\|$	1.407E-02	1.106E-02	9.171E-03	7.859E-03	6.887E-03	1.213
	RT	6.574s	13.726s	26.858s	45.930s	67.241s	
Euler scheme	$\|Y_0 - Y^0\|$	2.793E-02	2.259E-02	1.895E-02	1.632E-02	1.432E-02	1.136
	$\|Z_0^1 - Z_1^0\|$	4.287E-02	3.336E-02	2.717E-02	2.287E-02	1.972E-02	1.322
	$\|Z_0^2 - Z_2^0\|$	4.287E-02	3.336E-02	2.717E-02	2.287E-02	1.972E-02	1.322
	RT	8.256s	18.013s	33.551s	57.885s	83.846s	

In conclusion, all our numerical tests above show that the splitting Schemes 1, 2 and 3 are effective and first-order accurate for solving BSDEs, which are consistent with our theoretical results.

5. Conclusions

This paper focuses on splitting methods for solving BSDEs. By splitting the BSDE (1) into d BSDEs on each time subinterval $[t_n, t_{n+1}]$ and approximating these split BSDEs, we proposed splitting schemes for solving the BSDE (1). The key feature of the splitting schemes is that only one-dimensional approximations are needed to calculate the conditional expectations, which may reduce computational cost. We rigorously analyzed the splitting schemes, and derived the first-order convergence rate for the schemes, which are validated by our numerical tests.

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