INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 22, Number 2, Pages 268–306

doi: 10.4208/ijnam2025-1013

FOURIER CONVERGENCE ANALYSIS FOR FOKKER-PLANCK EQUATION OF TEMPERED FRACTIONAL LANGEVIN-BROWNIAN MOTION AND NONLINEAR TIME FRACTIONAL DIFFUSION EQUATION

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Abstract. Fourier analysis works well for the finite difference schemes of the linear partial differential equations. However, the presence of nonlinear terms leads to the fact that the method cannot be applied directly to deal with nonlinear problems. In the current work, we introduce an effective approach to enable Fourier methods to effectively deal with nonlinear problems and elaborate on it in detail by rigorously proving that the difference scheme for two-dimensional nonlinear problem considered in this paper is strictly unconditionally stable and convergent. Further, some numerical experiments are performed to confirm the rates of convergence and the robustness of the numerical scheme.

Key words. Time-fractional Fokker-Planck model, $L1$ scheme, Nonlinearity, Fourier stability-convergence analysis.

1. Introduction

Anomalous dynamics are ubiquitous in the nature world, especially in the complex system, the applications of which have a broad range, including physics [2], chemistry [14], and biology [32], etc. Unlike the classic mathematical and physical model for describing the diffusion, the anomalous diffusion processes no longer obey Fourier's or Fick's law [20, 27, 28]. We usually distinguish between normal and anomalous diffusive processes according to the mean squared displacement (MSD) (see [34] and the references therein), the MSD of the anomalous diffusing species $\langle x^2(t)\rangle$ scales as the following nonlinear power law, i.e.,

$$\langle x^2(t)\rangle \sim \kappa_\beta t^\beta,$$

where β is the anomalous diffusion index and κ_β the diffusion coefficient. According to β the anomalous diffusions are distinguished into subdiffusion if $0 < \beta < 1$, normal diffusion if $\beta = 1$, and superdiffusion if $\beta > 1$. Especially, we call it underballistic hyperdiffusion, ballistic diffusion, and hyperballistic diffusion as $1 < \beta < 2$, $\beta = 2$, and $\beta > 2$, respectively; see, e.g., [21]. Several effective methods are restored to describe the anomalous subdiffusive transport processes, including continuous time random walk model, fractal diffusion equation, fractional Klein-Kramers equation, and fractional Brownian and Langevin motion [6, 8, 9, 16, 22], etc.

In this paper, we aim to give an efficient proof idea to extend the Fourier method to deal with the nonlinear problems with nonlinear term $f(u)$. For this purpose, we consider the following two-dimensional nonlinear time fractional Fokker-Planck

Received by the editors on November 22, 2023 and, accepted November 4, 2024.
2000 *Mathematics Subject Classification.* 65M12, 65M06.
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equation
(1)

$$
\begin{aligned}
{}_0^C D_t^\beta u(x,y,t) =& \frac{\overline{A}}{\beta}\Big[\, t\ {}_0 D_t^{1-\beta} \triangle\ u(x,y,t)\Big] - (A,A)\cdot \nabla u(x,y,t) + f(u(x,y,t),x,y,t),\\
& 0 \le x,\ y \le L,\ 0 \le t \le T,
\end{aligned}
$$

where $0 < \beta < 1$, $\overline{A}$ and A are positive constants, and (A,A) is a two-dimensional vector. The time fractional Caputo derivative operator ${}_0^C D_t^\beta$ is defined by [24]

$$
{}_0^C D_t^\beta u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial u(s)}{\partial s} ds,
$$

and the time fractional Riemann-Liouville derivative operator ${}_0 D_t^{1-\beta}$ is defined as [24]

$$
{}_0 D_t^{1-\beta} u(t) = \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t (t-s)^{\beta-1} u(s) ds,
$$

where $\Gamma(z) := \int_0^\infty s^{z-1} e^{-s} ds$ (for $\Re(z) > 0$) denotes the Gamma function. Here, the solution $u(x,y,t)$ of model (1) represents the probability density function of particle position (see, e.g., [6, 31]).

For the problem (1), the theoretical analysis will be challenged as the right-hand side of the equation contains the term $(t\ {}_0D_t^{1-\beta} \triangle u)$. Thus, in this paper, we first consider the case of $f(u) = -\kappa u + h(x,y,t)$ $(\kappa > 0)$ in model (1), i.e.,

$$
\begin{aligned}
{}_0^C D_t^\beta u(x,y,t) =& \frac{\overline{A}}{\beta}\Big[\, t\ {}_0 D_t^{1-\beta} \triangle\ u(x,y,t)\Big] - (A,A)\cdot \nabla u(x,y,t)\\
& - \kappa u(x,y,t) + h(x,y,t),\ 0 \le x,\ y \le L,\ 0 \le t \le T,
\end{aligned} \tag{2}
$$

with the boundary conditions

$$
\begin{aligned}
u(0,y,t) &= \varphi_1(y,t),\ u(L,y,t) = \varphi_2(y,t),\ 0 \le\ y \le\ L,\ 0 \le\ t \le T,\\
u(x,0,t) &= \psi_1(x,t),\ u(x,L,t) = \psi_2(x,t),\ 0 \le\ x \le\ L,\ 0 \le\ t \le T,
\end{aligned} \tag{3}
$$

and the initial condition

$$
u(x,y,0) = \phi(x,y), \quad 0 \le\ x,y \le\ L. \tag{4}
$$

Then in Section 4, we consider the general two-dimensional nonlinear time fractional sub-diffusion problem

$$
{}_0^C D_t^\beta u = p \triangle u + (q,q)\cdot \nabla u + f(u,x,y,t), \tag{5}
$$

where $p > 0$, $q \in R$.

Due to their wide applications, fractional partial differential equations (FPDEs) have generated much interest in developing stable and accurate numerical methods as well as rigorous mathematical and numerical analysis; see [12, 15] and the references therein. We know that although some analytical solutions of FPDEs are expressed in terms of some special functions, these special functions are always difficult to evaluate numerically. This has naturally led to the rapid development of various effective numerical methods, including finite difference methods [5], finite element methods [7, 13], finite volume methods [11], spectral method [10], and collocation methods [17], etc. Among the existing approaches, the finite difference approximation to the fractional derivative seems to be the most studied one. And the $L1$ method [18] and Grünwald Letnikov formula [29] are effective discretization methods and are widely used in discretization of fractional differential operators.

As we all know, the Fourier stability analysis works well and is popular for the finite difference schemes of the linear partial differential equations with constant coefficients or variable coefficients independent of spatial variables; see for example, [3, 4, 25] and the references therein. Recently, for such linear problems as mentioned above, we have given a strict Fourier convergence proof in [30, 31] to remove the unreasonable assumptions required by the original literatures to prove convergence. Since the Fourier method is effective in proving the stability and convergence of the difference scheme for linear problems, there are several works also directly use the Fourier method to deal with nonlinear problems; one can see [1, 23, 33]. However, due to the presence of nonlinear terms, we find that Fourier method cannot be directly extended to deal with nonlinear problems as it does to linear cases. For example, in terms of stability analysis, in [23], for the perturbation error equation $\left(1+(\frac{1}{12}-\mu_1-\mu_2)\delta_x^2\right)\rho_j^k = \left(1+(\frac{1}{12}+\mu_1\lambda_1+\mu_1\upsilon_1)\delta_x^2\right)\rho_j^{k-1} + \sum_{l=0}^{k-2}(\mu_1\lambda_{k-l}+\mu_2\upsilon_{k-l})\delta_x^2\rho_j^l + \tau\left(1+\frac{1}{12}\delta_x^2\right)\left(f_j^{k-1}-\widetilde{f}_j^{k-1}\right)$ for $k=1,2,\ldots,N,\ j=1,2,\ldots,M-1$. The function $\rho^k(x)$ can be expanded into a Fourier series $\rho^k(x)=\sum_{l=-\infty}^{\infty}d_k(l)e^{i2\pi lx/L}$ when it is defined by piecewise constant functions. Due to the orthogonality of the basis function $e^{i2\pi lx/L}$ with respect to l, when f is linear, the results obtained by incorporating the result $\rho_j^k=\rho^k(x_j)=\sum_{l=-\infty}^{\infty}d_k(l)e^{i2\pi lx_j/L}$ in the entire frequency domain or the result $d_k(l)e^{i2\pi lx_j/L}$ in one frequency domain into the above error equation are the same. However, for nonlinear problems with term $f(u)$, obviously, bringing $\rho_j^k=\sum_{l=-\infty}^{\infty}d_k(l)e^{i2\pi lx_j/L}$ or $d_k(l)e^{i2\pi lx_j/L}$ into the error equation will yield a completely different result, i.e.,

$$
\begin{aligned}
&\sum_{l=-\infty}^{\infty}\Big[1+(\frac{1}{12}-\mu_1-\mu_2)(e^{i\sigma h}-2+e^{-i\sigma h})d_k(l)\\
&-\Big(1+(\frac{1}{12}+\mu_1\lambda_1+\mu_1\upsilon_1)(e^{i\sigma h}-2+e^{-i\sigma h})\Big)d_{k-1}(l)\\
&-\sum_{l=0}^{k-2}(\mu_1\lambda_{k-l}+\mu_2\upsilon_{k-l})(e^{i\sigma h}-2+e^{-i\sigma h})d_l(l)\Big]e^{i\sigma jh}\\
&-\tau(1+\frac{1}{12}\delta_x^2)\Big(f_j^{k-1}-\widetilde{f}_j^{k-1}\Big)=0
\end{aligned}
\tag{6}
$$

is not equivalent to

$$
\begin{aligned}
&\Big[1+(\frac{1}{12}-\mu_1-\mu_2)(e^{i\sigma h}-2+e^{-i\sigma h})d_k(l)\\
&-\Big(1+(\frac{1}{12}+\mu_1\lambda_1+\mu_1\upsilon_1)(e^{i\sigma h}-2+e^{-i\sigma h})\Big)d_{k-1}(l)\\
&-\sum_{l=0}^{k-2}(\mu_1\lambda_{k-l}+\mu_2\upsilon_{k-l})(e^{i\sigma h}-2+e^{-i\sigma h})d_l(l)\Big]e^{i\sigma jh}\\
&-\tau(1+\frac{1}{12}\delta_x^2)\Big(f_j^{k-1}-\widetilde{f}_j^{k-1}\Big)=0,
\end{aligned}
\tag{7}
$$

where $\sigma=2\pi l/L$. In other words, the stability analysis should be analyzed based on Eq. (6) instead of Eq. (7). However, the stability analysis in [23] is based on Eq. (7), which is incorrect. Similar issues arise in the convergence analysis.

Inspired by the issues, in this paper, we aim to provide an effective technique to extend the Fourier analysis method to deal with nonlinear problems with $f(u)$ by drawing on the ideas of [31]. To this end, we provide the fully discrete finite difference schemes for problems (2)-(4) and (50), and strictly prove that the difference schemes are unconditionally stable and convergent with new proof techniques. For nonlinear problem (50), the core idea of the proof is to perform a Fourier series expansion on nonlinear terms as well, first analyzing in one frequency domain, then continuing the analysis back to the whole frequency domain, and finally using the discrete fractional Grönwall inequality to obtain the final desired results. To our best knowledge, this proof technique is given for the first time; and without an extra assumption on the truncated error function, there is no other available literatures directly dealing with the nonlinear problems based on the idea of Fourier analysis except [1, 23, 33], where the given theoretical analysis is not appropriate, as we mentioned above.

The rest of this paper is organized as follows. In Section 2, we construct the fully discrete scheme for the two-dimensional time fractional Fokker-Planck problem (2)-(4). In Section 3, we prove that the fully discrete scheme (16) is unconditionally stable and strictly convergent. In Section 4, we construct the nonuniform numerical scheme (54) for the general two-dimensional nonlinear time fractional sub-diffusion problem (50) and provide a new proof technique to show that it is strictly unconditionally stable and convergent. In Section 5, several numerical examples are then provided to validate the theoretical results and to demonstrate the efficiency of the proposed method. Finally, we conclude the paper with a brief discussion.

Notation: Throughout this paper, C is a generic positive constant independent of h, h_x, h_y, and N.

2. Derivation of the numerical scheme

In this section, we will provide a fully discrete finite difference scheme for the two-dimensional time fractional Fokker-Planck problem (2)-(4), where we assume that $u(x,y,t) \in C^{4,4,2}_{x,y,t}([0,L]\times[0,L]\times[0,T])$.

We adopt a uniform grid of mesh points (x_j, y_m, t_n) with $x_j = jh_x$, $j = 0,1,\cdots,M_1$, $y_m = mh_y$, $m = 0,1,\cdots,M_2$, and $t_n = n\tau$, $n = 0,1,\cdots,N$, where M_1, M_2, and N are given positive integers, $h_x = \frac{L}{M_1}$, $h_y = \frac{L}{M_2}$, and $\tau = \frac{T}{N}$ are the uniform spatial and temporal mesh sizes, respectively. The exact and numerical solutions at the mesh point (x_j, y_m, t_n) are denoted by $u^n_{j,m}$ and $U^n_{j,m}$, respectively.

Next we use the following difference operators to discretize the first- and second-order spatial derivatives, respectively, namely,

$$\frac{u^n_{j+1,m} - u^n_{j-1,m}}{2h_x} = \frac{\partial u(x,y,t)}{\partial x}\Big|_{(x_j,y_m,t_n)} + O(h_x^2), \tag{8}$$

$$\frac{u^n_{j,m+1} - u^n_{j,m-1}}{2h_y} = \frac{\partial u(x,y,t)}{\partial y}\Big|_{(x_j,y_m,t_n)} + O(h_y^2), \tag{9}$$

$$\frac{u^n_{j+1,m} - 2u^n_{j,m} + u^n_{j-1,m}}{h_x^2} = \frac{\partial^2 u(x,y,t)}{\partial x^2}\Big|_{(x_j,y_m,t_n)} + O(h_x^2), \tag{10}$$

$$\frac{u^n_{j,m+1} - 2u^n_{j,m} + u^n_{j,m-1}}{h_y^2} = \frac{\partial^2 u(x,y,t)}{\partial y^2}\Big|_{(x_j,y_m,t_n)} + O(h_y^2). \tag{11}$$

For the term of time fractional Caputo derivative operator ${}_0^C D_t^\beta u(t)$, it is approximated by the standard $L1$ scheme [35]

(12)
$$
\begin{aligned}
{}_0^C D_t^\beta u(t)\Big|_{t_n} &= \frac{1}{\Gamma(1-\beta)}\int_0^{t_n}(t_n-s)^{-\beta}\frac{\partial u(s)}{\partial s}ds\\
&= \frac{1}{\Gamma(1-\beta)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(t_n-s)^{-\beta}\frac{\partial u(s)}{\partial s}ds\\
&= \frac{1}{\Gamma(1-\beta)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(t_n-s)^{-\beta}\frac{u^k-u^{k-1}}{\tau}ds+R_1\\
&= \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Big[a_0^{(1-\beta)}u^n-\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})u^k-a_{n-1}^{(1-\beta)}u^0\Big]+R_1,
\end{aligned}
$$

where the coefficients $a_k^{(1-\beta)} := (k+1)^{1-\beta}-k^{1-\beta}$, $k=0,1,\cdots,n-1$; and the truncation error $|R_1| = O(\tau^{2-\beta})$ being rigorously proved in [19] under the assumption that $u(t)\in C^2[0,T]$. For the time fractional Riemann-Liouville operator ${}_0D_t^{1-\beta}u(t)$, we adopt the Grünwald-Letnikov approximation given in [18]

$$
{}_0D_t^{1-\beta}u(t)\Big|_{t_n} = \tau^{\beta-1}\sum_{k=0}^{n}g_k^{(1-\beta)}u^{n-k}+O(\tau), \tag{13}
$$

where $g_k^{(1-\beta)} := \frac{\Gamma(k+\beta-1)}{\Gamma(\beta-1)\Gamma(k+1)} = (-1)^k\binom{1-\beta}{k}$ are the normalized Grünwald weights. In addition, the coefficients $g_k^{(1-\beta)}$ satisfy the recursive relationship as follows

$$
g_0^{(1-\beta)}=1,\ g_k^{(1-\beta)}=\left(1-\frac{2-\beta}{k}\right)g_{k-1}^{(1-\beta)}.
$$

Denote

$$
\begin{aligned}
R_{j,m}^n &= \frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Big[a_0^{(1-\beta)}u_{j,m}^n-\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})u_{j,m}^k-a_{n-1}^{(1-\beta)}u_{j,m}^0\Big]\\
&\quad-\frac{\overline{A}}{\beta}\Big[\ t_n\ \tau^{\beta-1}\sum_{k=0}^{n}g_k^{(1-\beta)}\Big(\frac{\delta_x^2u_{j,m}^{n-k}}{h_x^2}+\frac{\delta_y^2u_{j,m}^{n-k}}{h_y^2}\Big)\Big]+\kappa u_{j,m}^n\\
&\quad+A\Big(\frac{u_{j+1,m}^n-u_{j-1,m}^n}{2h_x}+\frac{u_{j,m+1}^n-u_{j,m-1}^n}{2h_y}\Big)-h_{j,m}^n,\\
&\quad j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N,
\end{aligned} \tag{14}
$$

where $\delta_x^2u_{j,m}^k = u_{j-1,m}^k-2u_{j,m}^k+u_{j+1,m}^k$ and $\delta_y^2u_{j,m}^k = u_{j,m-1}^k-2u_{j,m}^k+u_{j,m+1}^k$. According to (9)-(13), one can easily obtain that there is a positive constant C such that

$$
\begin{gathered}
|R_{j,m}^n|\le C(\tau+h_x^2+h_y^2),\\
j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N.
\end{gathered} \tag{15}
$$

Collecting the above approximations (8)-(13), one can get the finite difference scheme of problem (2)-(4) with the truncation error (15), i.e.,

$$\begin{aligned}
&\frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Big[a_0^{(1-\beta)}U_{j,m}^n-\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})U_{j,m}^k-a_{n-1}^{(1-\beta)}U_{j,m}^0\Big]\\
&=\frac{\overline{A}}{\beta}\Big[\ t_n\ \tau^{\beta-1}\sum_{k=0}^{n}g_k^{(1-\beta)}\Big(\frac{\delta_x^2U_{j,m}^{n-k}}{h_x^2}+\frac{\delta_y^2U_{j,m}^{n-k}}{h_y^2}\Big)\Big]\\
&-A\Big(\frac{U_{j+1,m}^n-U_{j-1,m}^n}{2h_x}+\frac{U_{j,m+1}^n-U_{j,m-1}^n}{2h_y}\Big)-\kappa U_{j,m}^n+h_{j,m}^n, \qquad (16)\\
&j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N,\\
&U_{0,m}^n=\varphi_1(y_m,t_n),\ U_{L,m}^n=\varphi_2(y_m,t_n),\ m=0,1,\ldots,M_2,\ n=1,2,\ldots,N,\\
&U_{j,0}^n=\psi_1(x_j,t_n),\ U_{j,L}^n=\psi_2(x_j,t_n),\ j=0,1,\ldots,M_1,\ n=1,2,\ldots,N,\\
&U_{j,m}^0=\phi(jh_x,mh_y),\ j=0,1,\ldots,M_1,\ m=0,1,\ldots,M_2,
\end{aligned}$$

where $\delta_x^2U_{j,m}^n=U_{j-1,m}^n-2U_{j,m}^n+U_{j+1,m}^n$, $\delta_y^2U_{j,m}^n=U_{j,m-1}^n-2U_{j,m}^n+U_{j,m+1}^n$.

3. Stability and convergence analysis

In this section, we will combine the ideas of Fourier analysis to present an efficient technique that enables the Fourier analysis method to handle the case $f(u)=-\kappa u+h(x,y,t)$, and rigorously prove that the finite difference scheme (16) is unconditionally stable and strictly convergent. Before carrying out the stability analysis of the difference scheme (16), we first give the following lemma for the subsequent theoretical analysis.

Lemma 3.1 ([18]). *Let the coefficients* $a_k^{(1-\beta)}=(k+1)^{1-\beta}-k^{1-\beta}$ $(k=0,1,\cdots,n-1)$ *and* $g_k^{(1-\beta)}=(-1)^k\binom{1-\beta}{k}$ $(k=0,1,\cdots,n)$ *be given in (12) and (13), respectively. Then*

(1) $1=a_0^{(1-\beta)}>a_1^{(1-\beta)}>\cdots>a_n^{(1-\beta)}\to 0$ *as* $n\to\infty$;

(2) $\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})+a_{n-1}^{(1-\beta)}=1$;

(3) $g_0^{(1-\beta)}=1,\ g_1^{(1-\beta)}=\beta-1,\ g_k^{(1-\beta)}<0\ (k=1,2,\cdots)$;

(4) $\sum_{k=0}^{\infty}g_k^{(1-\beta)}=0,\ -\sum_{k=1}^{n}g_k^{(1-\beta)}<1.$

3.1. Stability analysis. In this subsection, we will rigorously prove that the difference scheme (16) is unconditionally stable. Let $\widetilde{U}_{j,m}^n$ be the approximate solutions of the difference scheme (16) and denote the error

$$\rho_{j,m}^n=U_{j,m}^n-\widetilde{U}_{j,m}^n,$$

$$j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=0,1,\cdots,N.$$

Then we obtain the error equation

$$
\begin{aligned}
&\frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Big[a_0^{(1-\beta)}\rho_{j,m}^n-\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})\rho_{j,m}^k-a_{n-1}^{(1-\beta)}\rho_{j,m}^0\Big]\\
&=\frac{\overline{A}}{\beta}\Big[\ t_n\ \tau^{\beta-1}\sum_{k=0}^{n}g_k^{(1-\beta)}\Big(\frac{\delta_x^2\rho_{j,m}^{n-k}}{h_x^2}+\frac{\delta_y^2\rho_{j,m}^{n-k}}{h_y^2}\Big)\Big]\\
&-A\Big(\frac{\rho_{j+1,m}^n-\rho_{j-1,m}^n}{2h_x}+\frac{\rho_{j,m+1}^n-\rho_{j,m-1}^n}{2h_y}\Big)-\kappa\rho_{j,m}^n,\\
&j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N,
\end{aligned}
\tag{17}
$$

where $\delta_x^2\rho_{j,m}^k=\rho_{j+1,m}^k-2\rho_{j,m}^k+\rho_{j-1,m}^k$ and $\delta_y^2\rho_{j,m}^k=\rho_{j,m+1}^k-2\rho_{j,m}^k+\rho_{j,m-1}^k$. To facilitate discussion and understanding, we briefly review some knowledge about Fourier analysis. For $n=0,1,\dots,N$, we define the grid function

$$
\rho^n(x,y)=\begin{cases}\rho_{j,m}^n,\ \text{when } x_j-\dfrac{h_x}{2}<x\le x_j+\dfrac{h_x}{2},\ y_m-\dfrac{h_y}{2}<y\le y_m+\dfrac{h_y}{2},\\ \ (j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1),\\ 0,\ \text{when } 0\le x\le \frac{h_x}{2} \text{ or } L-\frac{h_x}{2}<x\le L,\\ \ \text{or } 0\le y\le\frac{h_y}{2} \text{ or } L-\frac{h_y}{2}<y\le L.\end{cases}
$$

Then $\rho^n(x,y)$ can be expanded by Fourier series, namely,

$$
\rho^n(x,y)=\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}d_n(l_1,l_2)e^{2\pi i(l_1x/L+l_2y/L)},\ n=0,1,\cdots,N,
$$

where

$$
d_n(l_1,l_2)=\frac{1}{L^2}\int_0^L\int_0^L\rho^n(x,y)e^{-2\pi i(l_1x/L+l_2y/L)}dxdy,
$$

Letting

$$
\rho^n=\Big[\rho_{1,1}^n,\rho_{1,2}^n,\cdots,\rho_{1,M_2-1}^n,\cdots,\rho_{M_1-1,1}^n,\rho_{M_1-1,2}^n,\cdots,\rho_{M_1-1,M_2-1}^n\Big]^T,
$$

and applying the Parseval equality

$$
\int_0^L\int_0^L|\rho^n(x,y)|^2dxdy=L^2\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|d_n(l_1,l_2)|^2,
$$

and

$$
\|\rho^n\|_2^2=\sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1}h_xh_y|\rho_{j,m}^n|^2=\int_0^L\int_0^L|\rho^n(x,y)|^2dxdy,
\tag{18}
$$

we obtain

$$
\|\rho^n\|_2^2=L^2\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|d_n(l_1,l_2)|^2,\ n=0,1,\cdots,N.
\tag{19}
$$

Based on the above knowledge, we next prove the numerical stability. Putting $\rho^n(x_j,y_m)=\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}d_n(l_1,l_2)e^{2\pi i(l_1jh_x/L+l_2mh_y/L)}$ into the error equation (17) and using Euler's formula lead to

(20)

$$
\begin{aligned}
&\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\Big\{\frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Big[a_0^{(1-\beta)}d_n(l_1,l_2)-\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})d_k(l_1,l_2)\\
&-a_{n-1}^{(1-\beta)}d_0(l_1,l_2)\Big]\Big\}e^{2\pi i(l_1jh_x/L+l_2mh_y/L)}\\
&=-\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\Big\{\frac{\overline{A}}{\beta}\Big[\ t_n\ \tau^{\beta-1}\sum_{k=0}^{n}g_k^{(1-\beta)}\Big(\frac{4\sin^2(\frac{\sigma_1h_x}{2})}{h_x^2}+\frac{4\sin^2(\frac{\sigma_2h_y}{2})}{h_y^2}\Big)d_{n-k}(l_1,l_2)\Big]\\
&-iA\Big(\frac{\sin(\sigma_1h_x)}{h_x}+\frac{\sin(\sigma_2h_y)}{h_y}\Big)d_n(l_1,l_2)-\kappa d_n(l_1,l_2)\Big\}e^{2\pi i(l_1jh_x/L+l_2mh_y/L)},\\
&j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N,
\end{aligned}
$$

Further, we obtain

(21)

$$
\begin{aligned}
&\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\Big\{\frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Big[a_0^{(1-\beta)}d_n(l_1,l_2)\\
&-\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})d_k(l_1,l_2)-a_{n-1}^{(1-\beta)}d_0(l_1,l_2)\Big]\\
&+\frac{\overline{A}}{\beta}\Big[\ t_n\ \tau^{\beta-1}\sum_{k=0}^{n}g_k^{(1-\beta)}\Big(\frac{4\sin^2(\frac{\sigma_1h_x}{2})}{h_x^2}+\frac{4\sin^2(\frac{\sigma_2h_y}{2})}{h_y^2}\Big)d_{n-k}(l_1,l_2)\Big]\\
&+iA\Big(\frac{\sin(\sigma_1h_x)}{h_x}+\frac{\sin(\sigma_2h_y)}{h_y}\Big)d_n(l_1,l_2)+\kappa d_n(l_1,l_2)\Big\}e^{2\pi i(l_1jh_x/L+l_2mh_y/L)}=0,\\
&j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N.
\end{aligned}
$$

Due to the orthogonality of $\{e^{2\pi i(l_1jh_x/L+l_2mh_y/L)}\}$ with respect to l_1 and l_2, it further follows from Eq. (21) that

(22)

$$
\begin{aligned}
&\frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Big[a_0^{(1-\beta)}d_n(l_1,l_2)-\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})d_k(l_1,l_2)-a_{n-1}^{(1-\beta)}d_0(l_1,l_2)\Big]\\
&+\frac{\overline{A}}{\beta}\Big[\ t_n\ \tau^{\beta-1}\sum_{k=0}^{n}g_k^{(1-\beta)}\Big(\frac{4\sin^2(\frac{\sigma_1h_x}{2})}{h_x^2}+\frac{4\sin^2(\frac{\sigma_2h_y}{2})}{h_y^2}\Big)d_{n-k}(l_1,l_2)\\
&+iA\Big(\frac{\sin(\sigma_1h_x)}{h_x}+\frac{\sin(\sigma_2h_y)}{h_y}\Big)d_n(l_1,l_2)+\kappa d_n(l_1,l_2)\Big\}=0,\\
&j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N.
\end{aligned}
$$

In fact, we can only analyze them directly in one frequency domain (l_1,l_2), namely, we check the evolution of one particular frequency (l_1,l_2) of the solution ρ^n of Eq. (17), i.e., inserting $\rho_{j,m}^n=d_n(l_1,l_2)e^{i(\sigma_1jh_x+\sigma_2mh_y)}$ with $\sigma_1:=2\pi l_1/L$ and

$\sigma_2 := 2\pi l_2/L$ into the error equation (17) and using Euler's formula lead to

(23)

$$\begin{aligned}
&d_n(l_1,l_2)+\frac{\Gamma(2-\beta)\overline{A}\ t_n\tau^{2\beta-1}}{\beta}\Big(\frac{4\sin^2(\frac{\sigma_1 h_x}{2})}{h_x^2}+\frac{4\sin^2(\frac{\sigma_2 h_y}{2})}{h_y^2}\Big)d_n(l_1,l_2)\\
&\quad+i\Gamma(2-\beta)A\tau^\beta\Big(\frac{\sin(\sigma_1 h_x)}{h_x}+\frac{\sin(\sigma_2 h_y)}{h_y}\Big)d_n(l_1,l_2)+\kappa\Gamma(2-\beta)\tau^\beta d_n(l_1,l_2)\\
&\quad=-\frac{\Gamma(2-\beta)\overline{A}\ t_n\tau^{2\beta-1}}{\beta}\sum_{k=1}^{n-1}g_k^{(1-\beta)}\Big(\frac{4\sin^2(\frac{\sigma_1 h_x}{2})}{h_x^2}+\frac{4\sin^2(\frac{\sigma_2 h_y}{2})}{h_y^2}\Big)d_{n-k}(l_1,l_2)\\
&\quad+\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})d_k(l_1,l_2)+a_{n-1}^{(1-\beta)}d_0(l_1,l_2)\\
&\quad-\frac{\Gamma(2-\beta)\overline{A}\ t_n\tau^{2\beta-1}g_n^{(1-\beta)}}{\beta}\Big(\frac{4\sin^2(\frac{\sigma_1 h_x}{2})}{h_x^2}+\frac{4\sin^2(\frac{\sigma_2 h_y}{2})}{h_y^2}\Big)d_0(l_1,l_2)\\
&j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N.
\end{aligned}$$

Obviously, Eq. (22) and Eq. (23) are the same. So here and later we always consider the evolution on a single frequency domain (l_1,l_2). For the sake of simplicity, we reformulate Eq. (23) by introducing the following notations

(24)

$$C_{1.n}:=\frac{4\Gamma(2-\beta)\overline{A}\ t_n\tau^{2\beta-1}\sin^2(\frac{\sigma_1 h_x}{2})}{\beta h_x^2};\quad C_{2,n}:=\frac{4\Gamma(2-\beta)\overline{A}\ t_n\tau^{2\beta-1}\sin^2(\frac{\sigma_2 h_y}{2})}{\beta h_y^2};$$

$$b1:=\frac{\Gamma(2-\beta)A\tau^\beta\sin(\sigma_1 h_x)}{h_x};\ b2:=\frac{\Gamma(2-\beta)A\tau^\beta\sin(\sigma_2 h_y)}{h_y};\ b3:=\kappa\Gamma(2-\beta)\tau^\beta.$$

Thus Eq. (23) can be rewritten as

$$\begin{aligned}
&\Big(1+C_{1,n}+C_{2,n}+b_3+i(b_1+b_2)\Big)d_n(l_1,l_2)\\
&=\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})d_k(l_1,l_2)-(C_{1,n}+C_{2,n})\sum_{k=1}^{n-1}g_k^{(1-\beta)}d_{n-k}(l_1,l_2)\\
&\quad+\Big(a_{n-1}^{(1-\beta)}-(C_{1,n}+C_{2,n})g_n^{(1-\beta)}\Big)d_0(l_1,l_2)\\
&n=1,2,\cdots,N.
\end{aligned}\tag{25}$$

Lemma 3.2. *Assume that $d_n(l_1,l_2)$ $(n=1,2,\cdots,N)$ are the solutions of Eq. (25). Then we have*

$$|d_n(l_1,l_2)|^2\le|d_0(l_1,l_2)|^2,\ n=1,2,\cdots,N.$$

Proof. Notice that $0 < \beta < 1$, $C_{1,n} > 0$, and $C_{2,n} > 0$. By virtue of Lemma 3.1, from Eq. (25), we have

(26)

$$
\begin{aligned}
|d_n(l_1,l_2)| \le{}& |1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|^{-1} \Big[\sum_{k=1}^{n-1} (a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)}) |d_k(l_1,l_2)| \\
& - (C_{1,n} + C_{2,n}) \sum_{k=1}^{n-1} g_k^{(1-\beta)} |d_{n-k}(l_1,l_2)| \\
& + \Big(a_{n-1}^{(1-\beta)} - (C_{1,n} + C_{2,n}) g_n^{(1-\beta)} \Big) |d_0(l_1,l_2)| \Big] \\
={}& |1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|^{-1} \Big[\sum_{k=1}^{n-1} (a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)}) |d_k(l_1,l_2)| \\
& - (C_{1,n} + C_{2,n}) \sum_{k=1}^{n-1} g_{n-k}^{(1-\beta)} |d_k(l_1,l_2)| \\
& + \Big(a_{n-1}^{(1-\beta)} - (C_{1,n} + C_{2,n}) g_n^{(1-\beta)} \Big) |d_0(l_1,l_2)| \Big], \\
& n = 1, 2, \cdots, N.
\end{aligned}
$$

For simplicity of presentation, we denote $w_{n,k} := a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)} - (C_{1,n} + C_{2,n}) g_{n-k}^{(1-\beta)}$ for $k = 1, 2, \ldots, n-1$, and $w_{n,0} = a_{n-1}^{(1-\beta)} - (C_{1,n} + C_{2,n}) g_n^{(1-\beta)}$. Obviously, the coefficients $w_{n,k}$ $(k = 0, 1, \ldots, n-1)$ above are all positive according to the definition of $w_{n,k}$ and Lemma 3.1. Then Eq. (26) can be treated as follows

$$
\begin{aligned}
|d_n(l_1,l_2)| \le{}& |1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|^{-1} \\
& \cdot \Big(\sum_{k=1}^{n-1} w_{n,k} |d_k(l_1,l_2)| + w_{n,0} |d_0(l_1,l_2)| \Big).
\end{aligned} \tag{27}
$$

Multiplying $|d_n(l_1,l_2)|$ on both sides of the inequality (27) yields

$$
\begin{aligned}
|d_n(l_1,l_2)|^2 \le{}& |1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|^{-1} \\
& \cdot \Big(\sum_{k=1}^{n-1} w_{n,k} |d_k(l_1,l_2)| |d_n(l_1,l_2)| + w_{n,0} |d_0(l_1,l_2)| |d_n(l_1,l_2)| \Big).
\end{aligned} \tag{28}
$$

For the two terms on the right hand side of (28), it follows from the Cauchy inequality with ε that

$$
|d_k(l_1,l_2)| |d_n(l_1,l_2)| \le \varepsilon |d_n(l_1,l_2)|^2 + \frac{1}{4\varepsilon} |d_k(l_1,l_2)|^2 \tag{29}
$$

and

$$
|d_0(l_1,l_2)| |d_n(l_1,l_2)| \le \varepsilon |d_n(l_1,l_2)|^2 + \frac{1}{4\varepsilon} |d_0(l_1,l_2)|^2. \tag{30}
$$

Bringing (29) and (30) into (28) yields

$$
\begin{aligned}
|d_n(l_1,l_2)|^2 \le{}& \frac{\varepsilon (\sum_{k=1}^{n-1} w_{n,k} + w_{n,0})}{|1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|} |d_n(l_1,l_2)|^2 \\
& + \frac{1}{4\varepsilon |1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|} \\
& \cdot \Big(\sum_{k=1}^{n-1} w_{n,k} |d_k(l_1,l_2)|^2 + w_{n,0} |d_0(l_1,l_2)|^2 \Big).
\end{aligned} \tag{31}
$$

By virtue of Lemma 3.1, a straightforward calculation shows

$$
\begin{aligned}
&\frac{\sum_{k=1}^{n-1} w_{n,k} + w_{n,0}}{|1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|}\\
=&\frac{\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)} - (C_{1,n} + C_{2,n})g_{n-k}^{(1-\beta)}\right) + a_{n-1}^{(1-\beta)} - (C_{1,n} + C_{2,n})g_n^{(1-\beta)}}{|1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|}\\
=&\frac{1 - (C_{1,n} + C_{2,n})\sum_{k=0}^{n-1} g_{n-k}^{(1-\beta)}}{|1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|}\\
<&\frac{1 + C_{1,n} + C_{2,n}}{|1 + C_{1,n} + C_{2,n}|}\\
<&1.
\end{aligned}
$$

Taking $\varepsilon = 1/2$, according to the above result, we deduce from (31) that

$$
\begin{aligned}
|d_n(l_1, l_2)|^2 \leq& |1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_3 + b_2)|^{-1}\\
&\cdot\Big[\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)} - (C_{1,n} + C_{2,n})g_{n-k}^{(1-\beta)}\right)|d_k(l_1, l_2)|^2\\
&+\left(a_{n-1}^{(1-\beta)} - (C_{1,n} + C_{2,n})g_n^{(1-\beta)}\right)|d_0(l_1, l_2)|^2\Big]\\
\leq&(1 + C_{1,n} + C_{2,n})^{-1}\Big[\sum_{k=1}^{n-1}\Big(a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)}\\
&-(C_{1,n} + C_{2,n})g_{n-k}^{(1-\beta)}\Big)|d_k(l_1, l_2)|^2\\
&+\left(a_{n-1}^{(1-\beta)} - (C_{1,n} + C_{2,n})g_n^{(1-\beta)}\right)|d_0(l_1, l_2)|^2\Big].
\end{aligned}
\tag{32}
$$

Next, we will prove $|d_n(l_1, l_2)|^2 \leq |d_0(l_1, l_2)|^2$ using the mathematical induction. When $n = 1$, applying Lemma 3.1, one has

$$
|d_1(l_1, l_2)|^2 \leq \frac{1 - g_1^{(1-\beta)}(C_{1,1} + C_{2,1})}{1 + C_{1,1} + C_{2,1}}|d_0(l_1, l_2)|^2 \leq |d_0(l_1, l_2)|^2.
$$

Suppose that

$$
|d_k(l_1, l_2)|^2 \leq |d_0(l_1, l_2)|^2,\ for\ k = 1, 2, \cdots, n-1.
$$

Then for $k = n$, applying Lemma 3.1, from Eq. (32), one has

$$\begin{aligned}|d_n(l_1,l_2)|^2 \leq&(1+C_{1,n}+C_{2,n})^{-1}\Big[\sum_{k=1}^{n-1}\Big(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)}\\&-(C_{1,n}+C_{2,n})g_{n-k}^{(1-\beta)}\Big)|d_k(l_1,l_2)|^2\\&+\Big(a_{n-1}^{(1-\beta)}-(C_{1,n}+C_{2,n})g_n^{(1-\beta)}\Big)|d_0(l_1,l_2)|^2\Big]\\ \leq&(1+C_{1,n}+C_{2,n})^{-1}\Big[\sum_{k=1}^{n-1}\Big(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)}-(C_{1,n}+C_{2,n})g_{n-k}^{(1-\beta)}\Big)\\&+\Big(a_{n-1}^{(1-\beta)}-(C_{1,n}+C_{2,n})g_n^{(1-\beta)}\Big)\Big]|d_0(l_1,l_2)|^2\\=&\frac{1+(C_{1,n}+C_{2,n})\sum_{k=0}^{n-1}|g_{n-k}^{(1-\beta)}|}{1+C_{1,n}+C_{2,n}}|d_0(l_1,l_2)|^2\\ \leq&|d_0(l_1,l_2)|^2,\end{aligned}$$

i.e.,

$$|d_n(l_1,l_2)|^2 \leq |d_0(l_1,l_2)|^2.$$

The proof is then completed. □

From Lemma 3.2, we immediately obtain the following stability result.

Theorem 3.3. *The finite difference scheme (16) is unconditionally stable.*

Proof. From Lemma 3.2 and (19), one has

$$\|\rho^n\|_2^2 = L^2\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|d_n(l_1,l_2)|^2 \leq L^2\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|d_0(l_1,l_2)|^2 = \|\rho^0\|_2^2.$$

This proof is then completed. □

3.2. Convergence analysis. In this subsection, similar to the trick of stability analysis, without extra assumptions on the truncated error function, we will present the strict Fourier convergence proof for the finite difference scheme (16).

According to Eq. (14), one has

$$\begin{aligned}&\frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Big[a_0^{(1-\beta)}u_{j,m}^n-\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)})u_{j,m}^k-a_{n-1}^{(1-\beta)}u_{j,m}^0\Big]\\&=\frac{\overline{A}}{\beta}\Big[\ t_n\ \tau^{\beta-1}\sum_{k=0}^{n}g_k^{(1-\beta)}\Big(\frac{\delta_x^2u_{j,m}^{n-k}}{h_x^2}+\frac{\delta_y^2u_{j,m}^{n-k}}{h_y^2}\Big)\Big]\\&-A\Big(\frac{u_{j+1,m}^n-u_{j-1,m}^n}{2h_x}+\frac{u_{j,m+1}^n-u_{j,m-1}^n}{2h_y}\Big)-\kappa u_{j,m}^n+R_{j,m}^n,\\&j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N.\end{aligned}\tag{33}$$

Subtracting the first equality of Eq. (16) from Eq. (33), there exists the error equation

$$
\begin{aligned}
&\frac{\tau^{-\beta}}{\Gamma(2-\beta)}\Big[e^n_{j,m}-\sum_{k=1}^{n-1}(a^{(1-\beta)}_{n-k-1}-a^{(1-\beta)}_{n-k})e^k_{j,m}\Big]\\
&=\frac{\overline{A}}{\beta}\Big[\ t_n\ \tau^{\beta-1}\sum_{k=0}^{n-1}g^{(1-\beta)}_k\Big(\frac{\delta_x^2 e^{n-k}_{j,m}}{h_x^2}+\frac{\delta_y^2 e^{n-k}_{j,m}}{h_y^2}\Big)\Big]\\
&\quad-A\Big(\frac{e^n_{j+1,m}-e^n_{j-1,m}}{2h_x}+\frac{e^n_{j,m+1}-e^n_{j,m-1}}{2h_y}\Big)-\kappa e^n_{j,m}+R^n_{j,m},\\
&j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N,
\end{aligned}
\tag{34}
$$

where $e^n_{j,m}=u^n_{j,m}-U^n_{j,m}$. Notice that the error equation satisfies the boundary conditions

$$
\begin{aligned}
e^n_{0,m}=e^n_{M_1,m}=0,\ m=0,1,\cdots,M_2,\ n=1,2,\cdots,N,\\
e^n_{j,0}=e^n_{j,M_2}=0,\ j=0,1,\cdots,M_1,\ n=1,2,\cdots,N,
\end{aligned}
$$

and the initial condition

$$
e^0_{j,m}=0,\ j=0,1,\cdots,M_1,\ m=0,1,\cdots,M_2.
$$

Similar to the stability analyses, for $n=0,1,\cdots,N$, we also define the grid function

$$
e^n(x,y)=\begin{cases}
e^n_{j,m},\ \text{when}\ x_j-\dfrac{h_x}{2}<x\le x_j+\dfrac{h_x}{2},\ y_m-\dfrac{h_y}{2}<y\le y_m+\dfrac{h_y}{2},\\
(j=1,2,\cdot\cdot\cdot,M_1-1,\ m=1,2,\cdot\cdot\cdot,M_2-1)\\
0,\ \text{when}\ 0\le x\le\frac{h_x}{2}\ \text{or}\ L-\frac{h_x}{2}<x\le L,\\
\ \text{or}\ 0\le y\le\frac{h_y}{2}\ \text{or}\ L-\frac{h_y}{2}<y\le L,
\end{cases}
$$

and for $n=1,2,\ldots,N$, define the grid function

$$
R^n(x,y)=\begin{cases}
R^n_{j,m},\ \text{when}\ x_j-\dfrac{h_x}{2}<x\le x_j+\dfrac{h_x}{2},\ y_m-\dfrac{h_y}{2}<y\le y_m+\dfrac{h_y}{2},\\
(j=1,2,\cdot\cdot\cdot,M_1-1,\ m=1,2,\cdot\cdot\cdot,M_2-1)\\
0,\ \text{when}\ 0\le x\le\frac{h_x}{2}\ \text{or}\ L-\frac{h_x}{2}<x\le L,\\
\ \text{or}\ 0\le y\le\frac{h_y}{2}\ \text{or}\ L-\frac{h_y}{2}<y\le L.
\end{cases}
$$

Then, the functions $e^n(x,y)$ and $R^n(x,y)$ can also be expanded by Fourier series, respectively, namely,

$$
e^n(x,y)=\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\xi_n(l_1,l_2)e^{2\pi i(l_1x/L+l_2y/L)},\ n=0,1,\cdot\cdot\cdot,N
$$

and

$$
R^n(x,y)=\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\eta_n(l_1,l_2)e^{2\pi i(l_1x/L+l_2y/L)},\ n=1,2,\cdot\cdot\cdot,N,
$$

where

$$
\begin{aligned}
\xi_n(l_1,l_2)&=\frac{1}{L^2}\int_0^L\int_0^L e^n(x,y)e^{-2\pi i(l_1x/L+l_2y/L)}dxdy,\\
\eta_n(l_1,l_2)&=\frac{1}{L^2}\int_0^L\int_0^L R^n(x,y)e^{-2\pi i(l_1x/L+l_2y/L)}dxdy.
\end{aligned}
$$

Letting

$$e^n = \Big[e^n_{1,1}, e^n_{1,2}, \cdots, e^n_{1,M_2-1}, \cdots, e^n_{M_1-1,1}, e^n_{M_1-1,2}, \cdots, e^n_{M_1-1,M_2-1}\Big]^T,$$

$$R^n = \Big[R^n_{1,1}, R^n_{1,2}, \cdots, R^n_{1,M_2-1}, \cdots, R^n_{M_1-1,1}, R^n_{M_1-1,2}, \cdots, R^n_{M_1-1,M_2-1}\Big]^T,$$

and applying the Parseval equalities

$$\int_0^L \int_0^L |e^n(x,y)|^2 dxdy = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |\xi_n(l_1,l_2)|^2, \ n = 0, 1, \cdots, N,$$

$$\int_0^L \int_0^L |R^n(x,y)|^2 dxdy = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |\eta_n(l_1,l_2)|^2, \ n = 1, 2, \cdots, N,$$

and

$$\|e^n\|_2^2 = \sum_{j=1}^{M_1-1} \sum_{m=1}^{M_2-1} h_x h_y |e^n_{j,m}|^2 = \int_0^L \int_0^L |e^n(x,y)|^2 dxdy, \ n = 0, 1, \cdots, N,$$

(35)

$$\|R^n\|_2^2 = \sum_{j=1}^{M_1-1} \sum_{m=1}^{M_2-1} h_x h_y |R^n_{j,m}|^2 = \int_0^L \int_0^L |R^n(x,y)|^2 dxdy, \ n = 1, 2, \cdots, N,$$

one can obtain

$$\|e^n\|_2^2 = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |\xi_n(l_1,l_2)|^2, \ n = 0, 1, \cdots, N, \tag{36}$$

and

$$\|R^n\|_2^2 = L^3 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |\eta_n(l_1,l_2)|^2, \ n = 1, 2, \cdots, N. \tag{37}$$

We directly consider the error in one particular frequency (l_1, l_2), i.e., take

$$e^n_{j,m} = \xi_n(l_1,l_2) e^{i(\sigma_1 j h_x + \sigma_2 m h_y)} \tag{38}$$

and

$$R^n_{j,m} = \eta_n(l_1,l_2) e^{i(\sigma_1 j h_x + \sigma_2 m h_y)} \tag{39}$$

respectively, where $\sigma_1 := 2\pi l_1/L$ and $\sigma_2 := 2\pi l_2/L$. Substituting (38) and (39) into Eq. (34) and using Euler's formula result in

(40)

$$\begin{aligned}
&\xi_n(l_1,l_2) + \frac{\Gamma(2-\beta)\overline{A}\ t_n\ \tau^{2\beta-1}}{\beta}\Big(\frac{4\sin^2(\frac{\sigma_1 h_x}{2})}{h_x^2} + \frac{4\sin^2(\frac{\sigma_2 h_y}{2})}{h_y^2}\Big)\xi_n(l_1,l_2) \\
&\quad + \kappa\Gamma(2-\beta)\tau^\beta \xi_n(l_1,l_2) + i\Gamma(2-\beta)A\tau^\beta\Big(\frac{\sin(\sigma_1 h_x)}{h_x} + \frac{\sin(\sigma_2 h_y)}{h_y}\Big)\xi_n(l_1,l_2) \\
&= -\frac{\Gamma(2-\beta)\overline{A}\ t_n\ \tau^{2\beta-1}}{\beta}\sum_{k=1}^{n-1} g_k^{(1-\beta)}\Big(\frac{4\sin^2(\frac{\sigma_1 h_x}{2})}{h_x^2} + \frac{4\sin^2(\frac{\sigma_2 h_y}{2})}{h_y^2}\Big)\xi_{n-k}(l_1,l_2) \\
&\quad + \sum_{k=1}^{n-1}(a^{(1-\beta)}_{n-k-1} - a^{(1-\beta)}_{n-k})\xi_k(l_1,l_2) + \Gamma(2-\beta)\tau^\beta \eta_n(l_1,l_2), \\
&\quad j = 1, 2, \cdots, M_1-1, \ m = 1, 2, \cdots, M_2-1, \ n = 1, 2, \cdots, N.
\end{aligned}$$

Using the same notations as (24), then Eq. (40) can be rewritten as follows

$$
\begin{aligned}
&\Big(1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)\Big)\xi_n(l_1, l_2) \\
&= \sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)})\xi_k(l_1, l_2) - (C_{1,n} + C_{2,n})\sum_{k=1}^{n-1} g_k^{(1-\beta)}\xi_{n-k}(l_1, l_2) \\
&\quad + \Gamma(2-\beta)\tau^\beta\eta_n(l_1, l_2), \\
&\quad n = 1, 2, \cdots, N.
\end{aligned} \tag{41}
$$

Next, without extra assumptions on the truncated error function, we will finish the proof of Lemma 3.4 below.

Lemma 3.4. *Assume that* $\xi_n(l_1, l_2)$ $(n = 1, 2, \cdots, N)$ *are the solutions of Eq. (41). Then*

$$
\|e^n\|_2 \le \widetilde{C}\Big(\tau^{(1+\beta)/2} + \tau^{(\beta-1)/2}(h_x^2 + h_y^2)\Big), \quad n = 1, 2, \cdots, N,
$$

where $\widetilde{C} = \sqrt{\Gamma(1-\beta)\Gamma(2-\beta)T^\beta}\hat{C}$.

Proof. Noticing that $e^0 = 0$, we have $\xi_0(l_1, l_2) = 0$. According to (15) and the first equality (35), we have

$$
\begin{aligned}
\|R^n\|_2 &\le C\sqrt{M_1 h_x}\sqrt{M_2 h_y}(\tau + h_x^2 + h_y^2) \\
&\le CL(\tau + h_x^2 + h_y^2) \\
&= \hat{C}(\tau + h_x^2 + h_y^2), \ n = 1, 2, \cdots, N,
\end{aligned} \tag{42}
$$

where $\hat{C} = CL$. Next, from Eq. (41), one has

$$
\begin{aligned}
|\xi_n(l_1, l_2)| &\le |1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|^{-1}\Big[\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)})|\xi_k(l_1, l_2)| \\
&\quad - (C_{1,n} + C_{2,n})\sum_{k=1}^{n-1} g_k^{(1-\beta)}|\xi_{n-k}(l_1, l_2)| + \Gamma(2-\beta)\tau^\beta|\eta_n(l_1, l_2)|\Big] \\
&= |1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|^{-1}\Big[\sum_{k=1}^{n-1}(a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)})|\xi_k(l_1, l_2)| \\
&\quad - (C_{1,n} + C_{2,n})\sum_{k=1}^{n-1} g_{n-k}^{(1-\beta)}|\xi_k(l_1, l_2)| + \Gamma(2-\beta)\tau^\beta|\eta_n(l_1, l_2)|\Big], \\
&\quad n = 1, 2, \cdots, N.
\end{aligned} \tag{43}
$$

We also denote $w_{n,k} := a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)} - (C_{1,n} + C_{2,n})g_{n-k}^{(1-\beta)}$. Then Eq. (43) can be simplified as

$$
\begin{aligned}
|\xi_n(l_1, l_2)| \le &|1 + C_{1,n} + C_{2,n} + b_3 + i(b_1 + b_2)|^{-1} \\
&\cdot\Big(\sum_{k=1}^{n-1} w_{n,k}|\xi_k(l_1, l_2)| + \Gamma(2-\beta)\tau^\beta|\eta_n(l_1, l_2)|\Big).
\end{aligned} \tag{44}
$$

Multiplying $|\xi_n(l_1,l_2)|$ on both sides of the inequality (44) yields

(45)
$$\begin{aligned}|\xi_n(l_1,l_2)|^2 \leq & \Big|1+C_{1,n}+C_{2,n}+b_3+i(b_1+b_2)\Big|^{-1}\Big(\sum_{k=1}^{n-1} w_{n,k}|\xi_k(l_1,l_2)||\xi_n(l_1,l_2)| \\ & + \Gamma(2-\beta)\tau^\beta|\eta_n(l_1,l_2)||\xi_n(l_1,l_2)|\Big).\end{aligned}$$

For the two terms on the right side of (45), it follows from the Cauchy inequality with ε that

$$|\xi_k(l_1,l_2)||\xi_n(l_1,l_2)| \leq \varepsilon|\xi_n(l_1,l_2)|^2 + \frac{1}{4\varepsilon}|\xi_k(l_1,l_2)|^2 \tag{46}$$

and

(47)
$$\Gamma(2-\beta)\tau^\beta|\eta_n(l_1,l_2)||\xi_n(l_1,l_2)| \leq \varepsilon a_{n-1}^{(1-\beta)}|\xi_n(l_1,l_2)|^2 + \frac{\Gamma(2-\beta)^2\tau^{2\beta}}{4\varepsilon a_{n-1}^{(1-\beta)}}|\eta_n(l_1,l_2)|^2.$$

Bringing (46) and (47) into (45) yields

$$\begin{aligned}|\xi_n(l_1,l_2)|^2 \leq & \frac{\varepsilon(\sum_{k=1}^{n-1} w_{n,k} + a_{n-1}^{(1-\beta)})}{|1+C_{1,n}+C_{2,n}+b_3+i(b_1+b_2)|}|\xi_n(l_1,l_2)|^2 \\ & + \frac{1}{4\varepsilon|1+C_{1,n}+C_{2,n}+b_3+i(b_1+b_2)|} \\ & \cdot\Big(\sum_{k=1}^{n-1} w_{n,k}|\xi_k(l_1,l_2)|^2 + \frac{\Gamma(2-\beta)^2\tau^{2\beta}}{a_{n-1}^{(1-\beta)}}|\eta_n(l_1,l_2)|^2\Big).\end{aligned} \tag{48}$$

By virtue of Lemma 3.1, a straightforward calculation shows

$$\begin{aligned}&\frac{\sum_{k=1}^{n-1} w_{n,k} + a_{n-1}^{(1-\beta)}}{|1+C_{1,n}+C_{2,n}+b_3+i(b_1+b_2)|} \\ =&\frac{\sum_{k=1}^{n-1}\Big(a_{n-k-1}^{(1-\beta)} - a_{n-k}^{(1-\beta)} - (C_{1,n}+C_{2,n})g_{n-k}^{(1-\beta)}\Big) + a_{n-1}^{(1-\beta)}}{|1+C_{1,n}+C_{2,n}+b_3+i(b_1+b_2)|} \\ =&\frac{1-(C_{1,n}+C_{2,n})\sum_{k=1}^{n-1} g_{n-k}^{(1-\beta)}}{|1+C_{1,n}+C_{2,n}+b_3+i(b_1+b_2)|} \\ <&\frac{1+C_{1,n}+C_{2,n}}{|1+C_{1,n}+C_{2,n}+i(b_1+b_2)|} \\ <&1.\end{aligned}$$

Taking $\varepsilon = 1/2$ and using the result $\frac{1}{a_{n-1}^{(1-\beta)}} \leq \frac{n^\beta}{1-\beta}$, according to the above result, we deduce from (48) that

$$\begin{aligned}|\xi_n(l_1,l_2)|^2 \leq & \Big|1+C_{1,n}+C_{2,n}+b_3+i(b_1+b_2)\Big|^{-1}\Big(\sum_{k=1}^{n-1} w_{n,k}|\xi_k(l_1,l_2)|^2 \\ & + \Gamma(1-\beta)\Gamma(2-\beta)(n\tau)^\beta\tau^\beta|\eta_n(l_1,l_2)|^2\Big).\end{aligned}$$

Further, using $n\tau \le T$, we obtain

$$
\begin{aligned}
|\xi_n(l_1,l_2)|^2 \le &(1+C_{1,n}+C_{2,n})^{-1}\Big[\sum_{k=1}^{n-1}\Big(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)} \\
&-(C_{1,n}+C_{2,n})g_{n-k}^{(1-\beta)}\Big)|\xi_k(l_1,l_2)|^2 \\
&+\Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta|\eta_n(l_1,l_2)|^2\Big].
\end{aligned} \tag{49}
$$

Next, using the recursion relation (49) above, for $n=1$, one has

$$
|\xi_1(l_1,l_2)|^2 \le \Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta|\eta_1(l_1,l_2)|^2.
$$

For $n=2$, it holds

$$
\begin{aligned}
|\xi_2(l_1,l_2)|^2 \le& \frac{1-a_1^{(1-\beta)}-(C_{1,2}+C_{2,2})g_1^{(1-\beta)}}{1+C_{1,2}+C_{2,2}}|\xi_1(l_1,l_2)|^2 \\
&+\Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta|\eta_2(l_1,l_2)|^2 \\
&\le |\xi_1(l_1,l_2)|^2+\Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta|\eta_2(l_1,l_2)|^2 \\
&\le \Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta\Big(|\eta_1(l_1,l_2)|^2+|\eta_2(l_1,l_2)|^2\Big).
\end{aligned}
$$

Similarly, for $n\ge 3$, there exists

$$
\begin{aligned}
|\xi_n(l_1,l_2)|^2 \le& (1+C_{1,n}+C_{2,n})^{-1}\Big[\sum_{k=1}^{n-1}\Big(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)}-(C_{1,n}+C_{2,n})g_{n-k}^{(1-\beta)}\Big) \\
&\cdot|\xi_k(l_1,l_2)|^2+\Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta|\eta_n(l_1,l_2)|^2\Big] \\
&\le \frac{\Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta}{1+C_{1,n}+C_{2,n}}\sum_{k=1}^{n-1}\Big(a_{n-k-1}^{(1-\beta)}-a_{n-k}^{(1-\beta)}-(C_{1,n}+C_{2,n})g_{n-k}^{(1-\beta)}\Big) \\
&\cdot\sum_{j=1}^{n-1}|\eta_j(l_1,l_2)|^2+\Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta|\eta_n(l_1,l_2)|^2 \\
&\le \frac{\Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta\Big(1-a_{n-1}^{(1-\beta)}+(C_{1,n}+C_{2,n})\Big)}{1+C_{1,n}+C_{2,n}} \\
&\cdot\sum_{j=1}^{n-1}|\eta_j(l_1,l_2)|^2+\Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta|\eta_n(l_1,l_2)|^2 \\
&\le \Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta\sum_{j=1}^{n-1}|\eta_j(l_1,l_2)|^2 \\
&+\Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta|\eta_n(l_1,l_2)|^2 \\
&= \Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta\sum_{j=1}^{n}|\eta_j(l_1,l_2)|^2,
\end{aligned}
$$

i.e.,

$$
|\xi_n(l_1,l_2)|^2 \le \Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta\sum_{j=1}^{n}|\eta_j(l_1,l_2)|^2.
$$

Summing the above inequality for l_1 from $-\infty$ to ∞ and l_2 from $-\infty$ to ∞, respectively, and exchanging the order of summation yield

$$\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|\xi_n(l_1,l_2)|^2 \le \Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta\sum_{j=1}^{n}\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|\eta_j(l_1,l_2)|^2.$$

This, together with (36), (37), and (42), leads to

$$\begin{aligned}\|e^n\|_2^2 &\le \Gamma(1-\beta)\Gamma(2-\beta)T^\beta\tau^\beta\sum_{j=1}^{n}\|R^j\|_2^2\\ &\le \Gamma(1-\beta)\Gamma(2-\beta)T^\beta n\tau^\beta\hat{C}^2(\tau+h_x^2+h_y^2)^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|e^n\|_2 &\le \sqrt{\Gamma(1-\beta)\Gamma(2-\beta)T^\beta}\,\hat{C}(n\tau^\beta)^{\frac{1}{2}}(\tau+h_x^2+h_y^2)\\ &\le \sqrt{\Gamma(1-\beta)\Gamma(2-\beta)T^\beta}\,\hat{C}\Big(\tau^{(1+\beta)/2}+\tau^{(\beta-1)/2}(h_x^2+h_y^2)\Big)\\ &= \widetilde{C}\Big(\tau^{(1+\beta)/2}+\tau^{(\beta-1)/2}(h_x^2+h_y^2)\Big),\end{aligned}$$

where $\widetilde{C}=\sqrt{\Gamma(1-\beta)\Gamma(2-\beta)T^\beta}\,\hat{C}$. The proof is then completed. □

Obviously, according to Lemma 3.4 above, we naturally obtain the following convergence result.

Theorem 3.5. *The finite difference scheme (16) is convergent with the order* $O(\tau^{(1+\beta)/2}+\tau^{(\beta-1)/2}(h_x^2+h_y^2))$.

Remark 3.6. *For the problem (2), the presence of the term* $(t\ {}_0D_t^{1-\beta}\triangle u)$ *causes the convergence analysis to be difficult. Thus, in the above proof of Lemma 3.4, our analysis only yields the convergence order* $(\tau^{\frac{1+\beta}{2}}+\tau^{\frac{\beta-1}{2}}(h_x^2+h_y^2))$ *assuming that the solution is sufficiently smooth. However, based on the idea of Lemma 3.4 above, we will present an effective technique to enable the Fourier method to effectively handle such nonlinear problems* ${}_0^CD_t^\beta u = p\triangle u + q\nabla u + f(u)$ *and can obtain the optimal convergence order, as shown in the next section.*

4. Application of the methodology to general nonlinear problem (5)

In this section, we apply the analysis method described in the previous section to enable the Fourier method to handle nonlinear problems with the term $f(u)$ under the help of the discrete fractional Grönwall inequality. For this purpose, we consider the general time fractional nonlinear equation (5), i.e.,

$$\begin{aligned}&{}_0^CD_t^\beta u = p\triangle u + (q,q)\cdot\nabla u + f(u,x,y,t),\ 0\le x,\ y\le L,\ 0\le t\le T,\\ &u(0,y,t)=\varphi_3(y,t),\ u(L,y,t)=\varphi_4(y,t),\ 0\le\ y\le\ L,\ 0\le\ t\le T,\\ &u(x,0,t)=\psi_3(x,t),\ u(x,L,t)=\psi_4(x,t),\ 0\le\ x\le\ L,\ 0\le\ t\le T,\\ &u(x,y,0)=\phi_0(x,y),\ \ 0\le\ x,y\le\ L.\end{aligned}\tag{50}$$

In addition, we assume that the nonlinear source term $f(u,x,y,t)$ has the first order continuous partial derivative $\frac{\partial f(u,x,y,t)}{\partial t}$, and satisfies the Lipschitz condition with respect to u, that is,

$$|f(\bar{u},x,y,t)-f(\tilde{u},x,y,t)|\le \mathbf{L}|\bar{u}-\tilde{u}|,\ \forall\ \bar{u},\tilde{u},\tag{51}$$

where $\mathbf{L}$ is a Lipschitz constant.

4.1. The discrete problem. In this subsection, we adopt L1 scheme on graded meshes to discrete ${}_0^C D_t^\beta g$. Let N be a positive integer and $r \geq 1$. We set $t_n = T(\frac{n}{N})^r$, $n = 0, 1, \ldots, N$, and $\tau_n = t_n - t_{n-1}$. The nonuniform L1 approximation of the Caputo derivative ${}_0^C D_t^\beta g$ is given by [26]

$$
\begin{aligned}
{}_0^C D_t^\beta g^n &\approx \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} w_{1-\beta}(t_n - s) \nabla_\tau g^n / \tau_k ds \\
&= \sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau g^k = a_0^{(n)} g^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(n)} - a_{n-k}^{(n)}) g^k - a_{n-1}^{(n)} g^0,
\end{aligned} \tag{52}
$$

where $w_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, the difference operator $\nabla_\tau g^n := g^n - g^{n-1}$, and

$$
a_{n-k}^{(n)} := \int_{t_{k-1}}^{t_k} \frac{w_{1-\beta}(t_n - s)}{\tau_k} ds = \frac{(t_n - t_{k-1})^{(1-\beta)} - (t_n - t_k)^{(1-\beta)}}{\Gamma(2-\beta)\tau_k}, \quad 1 \leq k \leq n.
$$

Using the mean value theorem one can easily prove that

$$
a_{n-k+1}^{(n)} < w_{1-\beta}(t_n - t_{k-1}) < a_{n-k}^{(n)}, \qquad 1 \leq k \leq n.
$$

Since $f(u, x, y, t)$ has the first order continuous derivative $\frac{\partial f(u,x,y,t)}{\partial t}$, it follows that

$$
f(u(x_j, y_m, t_n), x_j, y_m, t_n) = f(u(x_j, y_m, t_{n-1}), x_j, y_m, t_{n-1}) + O(\tau). \tag{53}
$$

Applying (8)-(11), (52) and (53), one can get the following nonuniform finite difference scheme for (50) with the truncation error $|\tilde{R}_{j,m}^n| = O(N^{-\min\{1, r\beta\}} + h_x^2 + h_y^2)$, i.e.,

$$
\begin{aligned}
&\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau U_{j,m}^k = p\Big(\frac{\delta_x^2 U_{j,m}^n}{h_x^2} + \frac{\delta_y^2 U_{j,m}^n}{h_y^2}\Big) \\
&+ q\Big(\frac{U_{j+1,m}^n - U_{j-1,m}^n}{2h_x} + \frac{U_{j,m+1}^n - U_{j,m-1}^n}{2h_y}\Big) + f_{j,m}^{n-1}, \\
&j = 1, 2, \cdots, M_1 - 1,\ m = 1, 2, \cdots, M_2 - 1,\ n = 1, 2, \cdots, N, \\
&U_{0,m}^n = \varphi_1(y_m, t_n),\ U_{L,m}^n = \varphi_2(y_m, t_n),\ m = 0, 1, \cdots, M_2,\ n = 1, 2, \ldots, N, \\
&U_{j,0}^n = \psi_1(x_j, t_n),\ U_{j,L}^n = \psi_2(x_j, t_n),\ j = 0, 1, \cdots, M_1,\ n = 1, 2, \ldots, N, \\
&U_{j,m}^0 = \phi(jh_x, mh_y),\ j = 0, 1, \cdots, M_1,\ m = 0, 1, \cdots, M_2,
\end{aligned} \tag{54}
$$

where $f_{j,m}^{n-1} = f(U_{j,m}^{n-1}, x_j, y_m, t_{n-1})$.

Before carrying out the theoretical analyses of the difference scheme (54), we first give the following lemmas that will be used later.

Lemma 4.1 ([26]). *If $0 < a_k^{(n)} \leq a_{k-1}^{(n)}$, for $1 \leq k \leq n \leq N$, the discrete Caputo formula (52) satisfies*

$$
v^n\Big(\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau v^k\Big) \geq \frac{1}{2} \sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau \Big(|v^k|^2\Big) \quad for\ 1 \leq n \leq N.
$$

Lemma 4.2 ([26]). *Assume that the discrete convolution kernels $a_{n-k}^{(n)}$ hold the following three properties:*
A1. The discrete kernel is monotone, that is,

$$
0 < a_{k-1}^{(n)} \leq a_{k-2}^{(n)} \qquad 2 \leq k \leq n \leq N.
$$

A2. There exists a constant $\pi_A > 0$, holding that

$$a_{n-k}^{(n)} \geq \frac{1}{\pi_A} \int_{t_{k-1}}^{t_k} \frac{w_{1-\beta}(t_n - s)}{\tau_k} ds, \quad 1 \leq k \leq n \leq N.$$

A3. There exists a positive constant ρ such that the step size ratios $\rho_k := \frac{\tau_k}{\tau_{k+1}}$ satisfy

$$\rho_k \leq \rho, \quad 1 \leq k \leq N-1.$$

Let $(\eta^n)_{n=1}^N$ and $(\lambda_l)_{l=0}^{N-1}$ be given non-negative sequence. Assume that there exists a constant Λ (independent of the step sizes) such that $\Lambda \geq \sum_{l=0}^{N-1} \lambda_l$, and that the maximum step size satisfies $\tau \leq \frac{1}{\sqrt[\beta]{2\pi_A \Gamma(2-\beta)\Lambda}}$. For any non-negative sequence $(v^k)_{k=0}^N$ such that

$$\sum_{k=1}^n a_{n-k}^{(n)} \nabla_\tau (v^k)^2 \leq \sum_{k=1}^n \lambda_{n-k} (v^k)^2 + (\eta^n)^2 \ for \ \ 1 \leq n \leq N.$$

Then it holds that, for $1 \leq n \leq N$,

$$v^n \leq 2E_\beta(2\max\{1,\rho\}\pi_A \Lambda t_n^\beta)\Big(v^0 + \sqrt{\pi_A \Gamma(1-\beta) \max_{1\leq k\leq N}\{t_k^{\beta/2}\eta^k\}}\Big),$$

where $E_\beta(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+k\beta)}$ is the Mittag-Leffler function.

4.2. Stability analysis. In this subsection, we will rigorously prove that the difference scheme (54) is unconditionally stable, which is different from the previous works [1, 23, 33].

Let $\widetilde{U}_{j,m}^n$ be the approximate solutions of the difference scheme (54) and denote the error

$$\rho_{j,m}^n = U_{j,m}^n - \widetilde{U}_{j,m}^n,$$
$$j = 1, 2, \cdots, M_1 - 1, \ m = 1, 2, \cdots, M_2 - 1, \ n = 0, 1, \cdots, N.$$

Then we obtain the error equation

(55)
$$\begin{aligned}\sum_{k=1}^n a_{n-k}^{(n)} \nabla_\tau (\rho_{j,m}^k) =& p\Big(\frac{\delta_x^2 \rho_{j,m}^n}{h_x^2} + \frac{\delta_y^2 \rho_{j,m}^n}{h_y^2}\Big) + q\Big(\frac{\rho_{j+1,m}^n - \rho_{j-1,m}^n}{2h_x} + \frac{\rho_{j,m+1}^n - \rho_{j,m-1}^n}{2h_y}\Big)\\ &+ f_{j,m}^{n-1} - \tilde{f}_{j,m}^{n-1},\\ & j = 1, 2, \cdots, M_1 - 1, \ m = 1, 2, \cdots, M_2 - 1, \ n = 1, 2, \cdots, N,\end{aligned}$$

where $\delta_x^2 \rho_{j,m}^k = \rho_{j+1,m}^k - 2\rho_{j,m}^k + \rho_{j-1,m}^k$, $\delta_y^2 \rho_{j,m}^k = \rho_{j,m+1}^k - 2\rho_{j,m}^k + \rho_{j,m-1}^k$, and $\tilde{f}_{j,m}^{n-1} = f(\tilde{U}_{j,m}^{n-1}, x_j, y_m, t_{n-1})$.

Similar to subsection 3.1, for $n = 0, 1, \ldots, N$, we define the grid function

$$\rho^n(x,y) = \begin{cases} \rho_{j,m}^n, \text{ when } x_j - \dfrac{h_x}{2} < x \leq x_j + \dfrac{h_x}{2}, \ y_m - \dfrac{h_y}{2} < y \leq y_m + \dfrac{h_y}{2}, \\ \ (j = 1, 2, \cdots, M_1 - 1, \ m = 1, 2, \cdots, M_2 - 1), \\ 0, \text{ when } 0 \leq x \leq \frac{h_x}{2} \text{ or } L - \frac{h_x}{2} < x \leq L, \\ \ \text{or } 0 \leq y \leq \frac{h_y}{2} \text{ or } L - \frac{h_y}{2} < y \leq L, \end{cases}$$

and for $n = 1, 2, \ldots, N$, define the grid function

$$g^{n-1}(x,y) = \begin{cases} g_{j,m}^{n-1}, \text{ when } x_j - \dfrac{h_x}{2} < x \leq x_j + \dfrac{h_x}{2}, \ y_m - \dfrac{h_y}{2} < y \leq y_m + \dfrac{h_y}{2}, \\ \ (j = 1, 2, \cdots, M_1 - 1, \ m = 1, 2, \cdots, M_2 - 1), \\ 0, \text{ when } 0 \leq x \leq \frac{h_x}{2} \text{ or } L - \frac{h_x}{2} < x \leq L, \\ \ \text{or } 0 \leq y \leq \frac{h_y}{2} \text{ or } L - \frac{h_y}{2} < y \leq L, \end{cases}$$

where $g_{j,m}^{n-1} := f_{j,m}^{n-1} - \tilde{f}_{j,m}^{n-1}$. Then $\rho^n(x,y)$ and $g^{n-1}(x,y)$ can be expanded by Fourier series, respectively, namely,

$$\rho^n(x,y) = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} d_n(l_1,l_2) e^{2\pi i(l_1 x/L + l_2 y/L)}, \ n = 0, 1, \cdots, N,$$

and

$$g^{n-1}(x,y) = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \alpha_{n-1}(l_1,l_2) e^{2\pi i(l_1 x/L + l_2 y/L)}, \ n = 1, \cdots, N,$$

where

$$d_n(l_1,l_2) = \frac{1}{L^2} \int_0^L \int_0^L \rho^n(x,y) e^{-2\pi i(l_1 x/L + l_2 y/L)} dxdy,$$

$$\alpha_{n-1}(l_1,l_2) = \frac{1}{L^2} \int_0^L \int_0^L g^{n-1}(x,y) e^{-2\pi i(l_1 x/L + l_2 y/L)} dxdy.$$

Letting

$$\rho^n = \left[\rho_{1,1}^n, \rho_{1,2}^n, \cdots, \rho_{1,M_2-1}^n, \cdots, \rho_{M_1-1,1}^n, \rho_{M_1-1,2}^n, \cdots, \rho_{M_1-1,M_2-1}^n\right]^T,$$

$$g^{n-1} = \left[g_{1,1}^{n-1}, g_{1,2}^{n-1}, \cdots, g_{1,M_2-1}^{n-1}, \cdots, g_{M_1-1,1}^{n-1}, \rho_{M_1-1,2}^{n-1}, \cdots, g_{M_1-1,M_2-1}^{n-1}\right]^T,$$

and applying the Parseval equality

$$\int_0^L \int_0^L |\rho^n(x,y)|^2 dxdy = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |d_n(l_1,l_2)|^2,$$

$$\int_0^L \int_0^L |g^{n-1}(x,y)|^2 dxdy = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |\alpha_{n-1}(l_1,l_2)|^2,$$

and

$$\|\rho^n\|_2^2 = \sum_{j=1}^{M_1-1} \sum_{m=1}^{M_2-1} h_x h_y |\rho_{j,m}^n|^2 = \int_0^L \int_0^L |\rho^n(x,y)|^2 dxdy, \tag{56}$$

$$\|g^{n-1}\|_2^2 = \sum_{j=1}^{M_1-1} \sum_{m=1}^{M_2-1} h_x h_y |g_{j,m}^{n-1}|^2 = \int_0^L \int_0^L |g^{n-1}(x,y)|^2 dxdy, \tag{57}$$

we obtain

$$\|\rho^n\|_2^2 = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |d_n(l_1,l_2)|^2, \ n = 0, 1, \cdots, N, \tag{58}$$

and

$$\|g^{n-1}\|_2^2 = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |\alpha_{n-1}(l_1,l_2)|^2, \ n = 1, 2, \cdots, N. \tag{59}$$

Since the nonlinear term g^{n-1} is also expanded into Fourier series, we can analyze them directly in one frequency domain (l_1, l_2), namely, we check the evolution of one particular frequency (l_1, l_2) of the solution ρ^n of Eq. (55), i.e., inserting $\rho_{j,m}^n =$

$d_n(l_1,l_2)e^{i(\sigma_1 jh_x+\sigma_2 mh_y)}$ and $g_{j,m}^{n-1} = \alpha_{n-1}(l_1,l_2)e^{i(\sigma_1 jh_x+\sigma_2 mh_y)}$ with $\sigma_1 := 2\pi l_1/L$ and $\sigma_2 := 2\pi l_2/L$ into the error equation (55) and using Euler's formula lead to

$$\begin{aligned}
&\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau \Big(d_k(l_1,l_2)\Big)\\
&= -p\Big(\frac{4\sin^2(\frac{\sigma_1 h_x}{2})}{h_x^2} + \frac{4\sin^2(\frac{\sigma_2 h_y}{2})}{h_y^2}\Big)d_n(l_1,l_2)\\
&\quad + iq\Big(\frac{\sin(\sigma_1 h_x)}{h_x} + \frac{\sin(\sigma_2 h_y)}{h_y}\Big)d_n(l_1,l_2) + \alpha_{n-1}(l_1,l_2),\\
&\quad j = 1,2,\cdots,M_1-1,\ m = 1,2,\cdots,M_2-1,\ n = 1,2,\cdots,N.
\end{aligned} \tag{60}$$

For the sake of simplicity, we denote

$$\lambda_1 := \frac{4p\sin^2(\frac{\sigma_1 h_x}{2})}{h_x^2},\ \lambda_2 := \frac{4p\sin^2(\frac{\sigma_2 h_y}{2})}{h_y^2},\ \lambda_3 := \frac{q\sin(\sigma_1 h_x)}{h_x},\ \lambda_4 := \frac{q\sin(\sigma_2 h_y)}{h_y}. \tag{61}$$

Thus Eq. (60) can be rewritten as

$$\begin{aligned}
&a_0^{(n)} d_n(l_1,l_2) - \sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})d_k(l_1,l_2) - a_{n-1}^{(n)} d_0(l_1,l_2)\\
&= -(\lambda_1+\lambda_2)d_n(l_1,l_2) + i(\lambda_3+\lambda_4)d_n(l_1,l_2) + \alpha_{n-1}(l_1,l_2),\\
&\qquad n = 1,2,\cdots,N.
\end{aligned} \tag{62}$$

Remark 4.3. *Obviously, when the nonlinear term is also expanded by Fourier series, Eq. (60) can be easily obtained from Eq. (55), being different from the works in* [1, 23, 33]*, which do not treat the nonlinear term. Next, we will complete the proof of Lemma 4.4 below by applying the discrete fractional Grönwall inequality.*

Lemma 4.4. *Assume that $d_n(l_1,l_2)$ $(n = 1,2,\cdots,N)$ are the solutions of Eq. (62). Then we have*

$$\|\rho^n\|_2 \le 2E_\beta(2\max\{1,\rho\}\Lambda t_n^\beta)\|\rho^0\|_2,$$

where $\Lambda = 1 + \mathbf{L}^2$.

Proof. Notice that $0 < \beta < 1$, $a_0^{(n)} = \frac{\tau_n^{-\beta}}{\Gamma(2-\beta)} > 0$, and $\lambda_i > 0$ $(i = 1,2,3,4)$. From Eq. (62), we have

$$\begin{aligned}
\Big(1 + Q_n(\lambda_1+\lambda_2) - iQ_n(\lambda_3+\lambda_4)\Big)d_n(l_1,l_2) =& Q_n\Big(\sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})d_k(l_1,l_2)\\
&+ a_{n-1}^{(n)} d_0(l_1,l_2) + \alpha_{n-1}(l_1,l_2)\Big),\\
&n = 1,2,\cdots,N,
\end{aligned}$$

where $Q_n = \frac{1}{a_0^{(n)}}$. Further, there exists

$$\begin{aligned}
|d_n(l_1,l_2)| =& \frac{Q_n}{|1 + Q_n(\lambda_1+\lambda_2) - iQ_n(\lambda_3+\lambda_4)|}\Big(\sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})|d_k(l_1,l_2)|\\
&+ a_{n-1}^{(n)}|d_0(l_1,l_2)| + |\alpha_{n-1}(l_1,l_2)|\Big),\\
\le& Q_n\Big(\sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})|d_k(l_1,l_2)| + a_{n-1}^{(n)}|d_0(l_1,l_2)| + |\alpha_{n-1}(l_1,l_2)|\Big),\\
&n = 1,2,\cdots,N.
\end{aligned}$$

Then, one has

$$
\begin{aligned}
&a_0^{(n)}|d_n(l_1,l_2)| - \sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})|d_k(l_1,l_2)| - a_{n-1}^{(n)}|d_0(l_1,l_2)| \\
&\leq |\alpha_{n-1}(l_1,l_2)|, \\
&\qquad n = 1,2,\cdots,N,
\end{aligned}
$$

i.e.,

$$
\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau \Big(|d_k(l_1,l_2)|\Big) \leq \alpha_{n-1}(l_1,l_2)|, \; n = 1,2,\cdots,N. \tag{63}
$$

Multiplying $|d_n(l_1,l_2)|$ on both sides of the inequality (63) yields

$$
|d_n(l_1,l_2)|\Big(\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau \Big(|d_k(l_1,l_2)|\Big)\Big) \leq \alpha_{n-1}(l_1,l_2)||d_n(l_1,l_2)|. \tag{64}
$$

From Lemma 4.1 and with the help of the Cauchy inequality, it follows that

$$
\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau \Big(|d_k(l_1,l_2)|^2\Big) \leq |d_n(l_1,l_2)|^2 + |\alpha_{n-1}(l_1,l_2)|^2.
$$

Summing the above inequality for l_1 from $-\infty$ to ∞ and l_2 from $-\infty$ to ∞, respectively, and exchanging the order of summation yield

$$
\begin{aligned}
&\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau \Big(\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} |d_k(l_1,l_2)|^2\Big) \\
&\leq \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} |d_n(l_1,l_2)|^2 + \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} |\alpha_{n-1}(l_1,l_2)|^2.
\end{aligned}
$$

According to (58) and (59), we have

$$
\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau \Big(\|\rho^k\|_2^2\Big) \leq \|\rho^n\|_2^2 + \|g^{n-1}\|_2^2. \tag{65}
$$

By (51), $g_{j,m}^{n-1} := f_{j,m}^{n-1} - \tilde{f}_{j,m}^{n-1}$, and the first equality of (56) and (57), we obtain

$$
\begin{aligned}
\|g^{n-1}\|_2^2 &= \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |g_{j,m}^{n-1}|^2 = \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |f_{j,m}^{n-1} - \tilde{f}_{j,m}^{n-1}|^2 \quad (by\ (57)) \\
&\leq \mathbf{L}^2 \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |U_{j,m}^{n-1} - \tilde{U}_{j,m}^{n-1}|^2 \quad (by\ (51)) \\
&= \mathbf{L}^2 \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |\rho_{j,m}^{n-1}|^2 \quad (by\ (56)) \\
&= \mathbf{L}^2 \|\rho^{n-1}\|_2^2.
\end{aligned}
$$

Thus, from (65) we can get

$$
\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau (\|\rho^k\|_2^2) \leq \|\rho^n\|_2^2 + \mathbf{L}^2 \|\rho^{n-1}\|_2^2.
$$

Applying the discrete fractional Grönwall inequality Lemma 4.2 with the substitutions

$$v^k := \|\rho^k\|_2,\quad \eta^n := 0\ (1 \le n \le N),\quad \lambda_0 := 1,\quad \lambda_1 := \mathbf{L}^2,\quad \lambda_l := 0\ (l \ge 2)$$

to the above inequality, one has

$$\|\rho^n\|_2 \le 2E_\beta(2\max\{1,\rho\}\Lambda t_n^\beta)\|\rho^0\|_2,$$

where $\Lambda = 1 + \mathbf{L}^2$. The proof is then completed. □

As a direct consequence of Lemma 4.4, we immediately obtain the following stability result.

Theorem 4.5. *The finite difference scheme (54) is unconditionally stable.*

4.3. Convergence analysis. In this subsection, similar to stability analysis, without extra assumptions on the truncated error function, we will present the strict Fourier convergence proof of the finite difference scheme (54).

According to Eq. (54), one has

(66)
$$\begin{aligned}\sum_{k=1}^{n} a_{n-k}^{(n)}\nabla_\tau u_{j,m}^k &= p\Big(\frac{\delta_x^2 u_{j,m}^n}{h_x^2} + \frac{\delta_y^2 u_{j,m}^n}{h_y^2}\Big) + q\Big(\frac{u_{j+1,m}^n - u_{j-1,m}^n}{2h_x} + \frac{u_{j,m+1}^n - u_{j,m-1}^n}{2h_y}\Big)\\ &\quad + f(u_{j,m}^{n-1}, x_j, y_m, t_{n-1}) + \tilde{R}_{j,m}^n,\\ &\quad j = 1,2,\cdots,M_1-1,\ m = 1,2,\cdots,M_2-1,\ n = 1,2,\cdots,N.\end{aligned}$$

Subtracting the first equality of Eq. (54) from Eq. (66), there exists the error equation

(67)
$$\begin{aligned}\sum_{k=1}^{n} a_{n-k}^{(n)}\nabla_\tau e_{j,m}^k &= p\Big(\frac{\delta_x^2 e_{j,m}^n}{h_x^2} + \frac{\delta_y^2 e_{j,m}^n}{h_y^2}\Big) + q\Big(\frac{e_{j+1,m}^n - e_{j-1,m}^n}{2h_x} + \frac{e_{j,m+1}^n - e_{j,m-1}^n}{2h_y}\Big)\\ &\quad + f(u_{j,m}^{n-1}, x_j, y_m, t_{n-1}) - f_{j,m}^{n-1} + \tilde{R}_{j,m}^n,\\ &\quad j = 1,2,\cdots,M_1-1,\ m = 1,2,\cdots,M_2-1,\ n = 1,2,\cdots,N,\end{aligned}$$

where $e_{j,m}^n = u_{j,m}^n - U_{j,m}^n$. Notice that the error equation satisfies the boundary conditions

$$\begin{aligned} e_{0,m}^n = e_{M_1,m}^n = 0,\ m = 0,1,\cdots,M_2,\ n = 1,2,\cdots,N,\\ e_{j,0}^n = e_{j,M_2}^n = 0,\ j = 0,1,\cdots,M_1,\ n = 1,2,\cdots,N,\end{aligned}$$

and the initial condition

$$e_{j,m}^0 = 0,\ j = 0,1,\cdots,M_1,\ m = 0,1,\cdots,M_2.$$

Denote $\tilde{g}_{j,m}^{n-1} := f(u_{j,m}^{n-1}, x_j, y_m, t_{n-1}) - f_{j,m}^{n-1}$. Similar to the stability analyses, for $n = 0,1,\ldots,N$, we also define the grid function

$$e^n(x,y) = \begin{cases} e_{j,m}^n,\ \text{when } x_j - \dfrac{h_x}{2} < x \le x_j + \dfrac{h_x}{2},\ y_m - \dfrac{h_y}{2} < y \le y_m + \dfrac{h_y}{2},\\ \ (j = 1,2,\cdots,M_1-1,\ m = 1,2,\cdots,M_2-1)\\ 0,\ \text{when } 0 \le x \le \frac{h_x}{2} \text{ or } L - \frac{h_x}{2} < x \le L,\\ \ \text{or } 0 \le y \le \frac{h_y}{2} \text{ or } L - \frac{h_y}{2} < y \le L,\end{cases}$$

and for $n = 1, 2, \cdots, N$, define the grid functions

$$\tilde{R}^n(x,y) = \begin{cases} \tilde{R}^n_{j,m}, \text{ when } x_j - \dfrac{h_x}{2} < x \le x_j + \dfrac{h_x}{2}, \ y_m - \dfrac{h_y}{2} < y \le y_m + \dfrac{h_y}{2}, \\ (j = 1, 2, \cdots, M_1 - 1, \ m = 1, 2, \cdots, M_2 - 1) \\ 0, \text{ when } 0 \le x \le \frac{h_x}{2} \text{ or } L - \frac{h_x}{2} < x \le L, \\ \text{or } 0 \le y \le \frac{h_y}{2} \text{ or } L - \frac{h_y}{2} < y \le L, \end{cases}$$

and

$$\tilde{g}^{n-1}(x,y) = \begin{cases} \tilde{g}^{n-1}_{j,m}, \text{ when } x_j - \dfrac{h_x}{2} < x \le x_j + \dfrac{h_x}{2}, \ y_m - \dfrac{h_y}{2} < y \le y_m + \dfrac{h_y}{2}, \\ (j = 1, 2, \cdots, M_1 - 1, \ m = 1, 2, \cdots, M_2 - 1) \\ 0, \text{ when } 0 \le x \le \frac{h_x}{2} \text{ or } L - \frac{h_x}{2} < x \le L, \\ \text{or } 0 \le y \le \frac{h_y}{2} \text{ or } L - \frac{h_y}{2} < y \le L, \end{cases}$$

respectively. Then, the functions $e^n(x,y)$, $R^n(x,y)$, and $\tilde{g}^{n-1}(x,y)$ can also be expanded by Fourier series, respectively, namely,

$$e^n(x,y) = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \xi_n(l_1,l_2) e^{2\pi i(l_1 x/L + l_2 y/L)}, \ n = 0, 1, \cdots, N,$$

$$\tilde{R}^n(x,y) = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \tilde{\eta}_n(l_1,l_2) e^{2\pi i(l_1 x/L + l_2 y/L)}, \ n = 1, 2, \cdots, N,$$

and

$$\tilde{g}^{n-1}(x,y) = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \beta_{n-1}(l_1,l_2) e^{2\pi i(l_1 x/L + l_2 y/L)}, \ n = 1, 2, \cdots, N,$$

where

$$\xi_n(l_1,l_2) = \frac{1}{L^2} \int_0^L \int_0^L e^n(x,y) e^{-2\pi i(l_1 x/L + l_2 y/L)} dxdy,$$

$$\tilde{\eta}_n(l_1,l_2) = \frac{1}{L^2} \int_0^L \int_0^L R^n(x,y) e^{-2\pi i(l_1 x/L + l_2 y/L)} dxdy,$$

and

$$\beta_{n-1}(l_1,l_2) = \frac{1}{L^2} \int_0^L \int_0^L \tilde{g}^{n-1}(x,y) e^{-2\pi i(l_1 x/L + l_2 y/L)} dxdy.$$

Letting

$$e^n = \left[e^n_{1,1}, e^n_{1,2}, \cdots, e^n_{1,M_2-1}, \cdots, e^n_{M_1-1,1}, e^n_{M_1-1,2}, \cdots, e^n_{M_1-1,M_2-1}\right]^T,$$

$$\tilde{R}^n = \left[\tilde{R}^n_{1,1}, \tilde{R}^n_{1,2}, \cdots, \tilde{R}^n_{1,M_2-1}, \cdots, \tilde{R}^n_{M_1-1,1}, \tilde{R}^n_{M_1-1,2}, \cdots, \tilde{R}^n_{M_1-1,M_2-1}\right]^T,$$

$$\tilde{g}^{n-1} = \left[\tilde{g}^{n-1}_{1,1}, \tilde{g}^{n-1}_{1,2}, \cdots, \tilde{g}^{n-1}_{1,M_2-1}, \cdots, \tilde{g}^{n-1}_{M_1-1,1}, \tilde{g}^{n-1}_{M_1-1,2}, \cdots, \tilde{g}^{n-1}_{M_1-1,M_2-1}\right]^T,$$

and applying the Parseval equalities

$$\int_0^L \int_0^L |e^n(x,y)|^2 dxdy = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |\xi_n(l_1,l_2)|^2, \ n = 0, 1, \cdots, N,$$

$$\int_0^L \int_0^L |\tilde{R}^n(x,y)|^2 dxdy = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |\tilde{\eta}_n(l_1,l_2)|^2, \ n = 1, 2, \cdots, N,$$

$$\int_0^L \int_0^L |\tilde{g}^{n-1}(x,y)|^2 dxdy = L^2 \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} |\beta_{n-1}(l_1,l_2)|^2, \ n = 1, 2, \cdots, N,$$

and

$$\|e^n\|_2^2 = \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |e_{j,m}^n|^2 = \int_0^L \int_0^L |e^n(x,y)|^2 dxdy,\ n=0,1,\cdots,N, \tag{68}$$

$$\|\tilde{R}^n\|_2^2 = \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |\tilde{R}_{j,m}^n|^2 = \int_0^L \int_0^L |\tilde{R}^n(x,y)|^2 dxdy,\ n=1,2,\cdots,N, \tag{69}$$

$$\|\tilde{g}^{n-1}\|_2^2 = \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |\tilde{g}_{j,m}^{n-1}|^2 = \int_0^L \int_0^L |\tilde{g}^{n-1}(x,y)|^2 dxdy,\ n=1,2,\cdots,N, \tag{70}$$

one can obtain

$$\|e^n\|_2^2 = L^2 \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} |\xi_n(l_1,l_2)|^2,\ n=0,1,\cdots,N, \tag{71}$$

$$\|\tilde{R}^n\|_2^2 = L^2 \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} |\tilde{\eta}_n(l_1,l_2)|^2,\ n=1,2,\cdots,N, \tag{72}$$

and

$$\|\tilde{g}^{n-1}\|_2^2 = L^2 \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} |\beta_{n-1}(l_1,l_2)|^2,\ n=1,2,\cdots,N. \tag{73}$$

Due to the fact that the nonlinear term $\tilde{g}^{n-1}$ is expanded in Fourier series, we directly consider the error in one particular frequency (l_1,l_2), i.e., take

$$e_{j,m}^n = \xi_n(l_1,l_2)e^{i(\sigma_1 j h_x + \sigma_2 m h_y)} \tag{74}$$

$$\tilde{R}_{j,m}^n = \tilde{\eta}_n(l_1,l_2)e^{i(\sigma_1 j h_x + \sigma_2 m h_y)} \tag{75}$$

and

$$\tilde{g}_{j,m}^{n-1} = \beta_{n-1}(l_1,l_2)e^{i(\sigma_1 j h_x + \sigma_2 m h_y)}, \tag{76}$$

respectively, where $\sigma_1 := 2\pi l_1/L$ and $\sigma_2 := 2\pi l_2/L$. Substituting (74), (75), and (76) into Eq. (67) and using Euler's formula result in

$$\begin{aligned}
&\sum_{k=1}^{n} a_{n-k}^{(n)} \nabla_\tau \Big(\xi_k(l_1,l_2)\Big) \\
&= -p\Big(\frac{4\sin^2(\frac{\sigma_1 h_x}{2})}{h_x^2} + \frac{4\sin^2(\frac{\sigma_2 h_y}{2})}{h_y^2}\Big)\xi_n(l_1,l_2) \\
&\quad + iq\Big(\frac{\sin(\sigma_1 h_x)}{h_x} + \frac{\sin(\sigma_2 h_y)}{h_y}\Big)\xi_n(l_1,l_2) + \beta_{n-1}(l_1,l_2) + \tilde{\eta}_n(l_1,l_2), \\
&j=1,2,\cdots,M_1-1,\ m=1,2,\cdots,M_2-1,\ n=1,2,\cdots,N.
\end{aligned} \tag{77}$$

Using the same notations as (61), then Eq. (77) can be rewritten as

$$\begin{aligned} &a_0^{(n)}\xi_n(l_1,l_2) - \sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})\xi_k(l_1,l_2) - a_{n-1}^{(n)}\xi_0(l_1,l_2) \\ &= -(\lambda_1+\lambda_2)\xi_n(l_1,l_2) + i(\lambda_3+\lambda_4)\xi_n(l_1,l_2) + \beta_{n-1}(l_1,l_2) + \tilde{\eta}_n(l_1,l_2), \\ &\quad n = 1,2,\cdots,N. \end{aligned} \tag{78}$$

Next, without extra assumption on the truncated error function, we apply a similar proof technique to Lemma 4.4 to finish the proof of Lemma 4.6 below.

Lemma 4.6. *Assume that $\xi_n(l_1,l_2)$ $(n = 1,2,\cdots,N)$ are the solutions of Eq. (78). Then we have*

$$\begin{aligned} \|e^n\|_2 \le &2E_\beta(2\max\{1,\rho\}\Lambda t_n^\beta)\sqrt{\Gamma(1-\beta)}t_n^{\beta/2}\hat{C} \\ &\cdot (N^{-\min\{1,r\beta\}} + h_x^2 + h_y^2),\ n = 1,2,\cdots,N, \end{aligned}$$

where $\Lambda = 2 + \mathbf{L}^2$ and $\hat{C} = CL$.

Proof. Noticing that $e^0 = 0$, we have $\xi_0(l_1,l_2) = 0$. According to $|\tilde{R}_{j,m}^n| \le C(N^{-\min\{1,r\beta\}} + h_x^2 + h_y^2)$ and the first equality (69), we have

$$\begin{aligned} \|\tilde{R}^n\|_2 &\le C\sqrt{M_1h_x}\sqrt{M_2h_y}(N^{-\min\{2-\beta,r\beta\}} + h_x^2 + h_y^2) \\ &\le CL(N^{-\min\{1,r\beta\}} + h_x^2 + h_y^2) \\ &= \hat{C}(N^{-\min\{1,r\beta\}} + h_x^2 + h_y^2),\ n = 1,2,\cdots,N, \end{aligned} \tag{79}$$

where $\hat{C} = CL$. From Eq. (78) and denoting $Q_n := \frac{1}{a_0^{(n)}}$, there exists

$$\begin{aligned} \Big(1 + Q_n(\lambda_1+\lambda_2) - iQ_n(\lambda_3+\lambda_4)\Big)\xi_n(l_1,l_2) =& Q_n\Big(\sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})\xi_k(l_1,l_2) \\ &+ \beta_{n-1}(l_1,l_2) + \tilde{\eta}_n(l_1,l_2)\Big), \\ &n = 1,2,\cdots,N. \end{aligned}$$

Further, one has

$$\begin{aligned} |\xi_n(l_1,l_2)| =& \frac{Q_n}{|1 + Q_n(\lambda_1+\lambda_2) - iQ_n(\lambda_3+\lambda_4)|}\Big(\sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})|\xi_k(l_1,l_2)| \\ &+ |\beta_{n-1}(l_1,l_2)| + |\tilde{\eta}_n(l_1,l_2)|\Big), \\ \le& Q_n\Big(\sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})|\xi_k(l_1,l_2)| + |\beta_{n-1}(l_1,l_2)| + |\tilde{\eta}_n(l_1,l_2)|\Big), \\ &n = 1,2,\cdots,N. \end{aligned}$$

Then, we obtain

$$\begin{aligned} &a_0^{(n)}|\xi_n(l_1,l_2)| - \sum_{k=1}^{n-1}(a_{n-k-1}^{(n)} - a_{n-k}^{(n)})|\xi_k(l_1,l_2)| \le |\beta_{n-1}(l_1,l_2)| + |\tilde{\eta}_n(l_1,l_2)|, \\ &\quad n = 1,2,\cdots,N, \end{aligned}$$

i.e.,

$$\sum_{k=1}^{n} a_{n-k}^{(n)}\nabla_\tau\Big(|\xi_k(l_1,l_2)|\Big) \le |\beta_{n-1}(l_1,l_2)| + |\tilde{\eta}_n(l_1,l_2)|,\ n = 1,2,\cdots,N. \tag{80}$$

Multiplying $|\xi_n(l_1,l_2)|$ on both sides of the inequality (80) yields

(81)
$$|\xi_n(l_1,l_2)|\Big(\sum_{k=1}^{n} a_{n-k}^{(n)}\nabla_\tau\Big(|\xi_k(l_1,l_2)|\Big)\Big) \le |\beta_{n-1}(l_1,l_2)||\xi_n(l_1,l_2)| + |\tilde{\eta}_n(l_1,l_2)||\xi_n(l_1,l_2)|.$$

For the two terms on the right hand side of (81), it follows from the Cauchy inequality that

$$|\beta_{n-1}(l_1,l_2)||\xi_n(l_1,l_2)| \le \frac{1}{2}|\beta_{n-1}(l_1,l_2)|^2 + \frac{1}{2}|\xi_n(l_1,l_2)|^2,$$
$$|\tilde{\eta}_n(l_1,l_2)||\xi_n(l_1,l_2)| \le \frac{1}{2}|\tilde{\eta}_n(l_1,l_2)|^2 + \frac{1}{2}|\xi_n(l_1,l_2)|^2.$$

With the help of Lemma 4.1, from (81) one has

$$\sum_{k=1}^{n} a_{n-k}^{(n)}\nabla_\tau\Big(|\xi_k(l_1,l_2)|^2\Big) \le 2|\xi_n(l_1,l_2)|^2 + |\beta_{n-1}(l_1,l_2)|^2 + |\tilde{\eta}_n(l_1,l_2)|^2.$$

Summing the above inequality for l_1 from $-\infty$ to ∞ and l_2 from $-\infty$ to ∞, respectively, and exchanging the order of summation yield

$$\begin{aligned}
&\sum_{k=1}^{n} a_{n-k}^{(n)}\nabla_\tau\Big(\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|\xi_k(l_1,l_2)|^2\Big)\\
\le& 2\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|\xi_n(l_1,l_2)|^2 + \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|\beta_{n-1}(l_1,l_2)|^2\\
&+\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}|\tilde{\eta}_n(l_1,l_2)|^2.
\end{aligned}$$

According to (71)-(73), we have

$$\sum_{k=1}^{n} a_{n-k}^{(n)}\nabla_\tau\Big(\|e^k\|_2^2\Big) \le 2\|e^n\|_2^2 + \|\tilde{g}^{n-1}\|_2^2 + \|\tilde{R}^n\|_2^2. \tag{82}$$

By using (51), $\tilde{g}_{j,m}^{n-1} := f(u_{j,m}^{n-1}, x_j, y_m, t_{n-1}) - f_{j,m}^{n-1}$, and the first equality of (68) and (70), we obtain

$$\begin{aligned}
\|\tilde{g}^{n-1}\|_2^2 &= \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |\tilde{g}_{j,m}^{n-1}|^2\\
&= \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |f(u_{j,m}^{n-1}, x_j, y_m, t_{n-1}) - f_{j,m}^{n-1}|^2 \ (by\ (70))\\
&\le \mathbf{L}^2 \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |u_{j,m}^{n-1} - U_{j,m}^{n-1}|^2 \quad (by\ (51))\\
&= \mathbf{L}^2 \sum_{j=1}^{M_1-1}\sum_{m=1}^{M_2-1} h_x h_y |e_{j,m}^{n-1}|^2 \quad (by\ (68))\\
&= \mathbf{L}^2 \|e^{n-1}\|_2^2.
\end{aligned}$$

Thus, from (82) we can get

$$\sum_{k=1}^{n} a_{n-k}^{(n)}\nabla_\tau\Big(\|e^k\|_2^2\Big) \le 2\|e^n\|_2^2 + \mathbf{L}^2\|e^{n-1}\|_2^2 + \|\tilde{R}^n\|_2^2.$$

Applying the discrete fractional Grönwall inequality Lemma 4.2 with the substitutions

$$v^k := \|e^k\|_2,\quad \eta^n := \|\tilde{R}^n\|,\quad \lambda_0 := 2,\quad \lambda_1 := \mathbf{L}^2,\quad \lambda_l := 0\ (l \geq 2)$$

to the above inequality, and according to (79) one has

$$\begin{aligned}\|e^n\|_2 &\leq 2E_\beta(2\max\{1,\rho\}\Lambda t_n^\beta)\Big(\sqrt{\Gamma(1-\beta)}\max_{1\leq k\leq n}\{t_k^{\beta/2}\|\tilde{R}^k\|_2\}\Big)\\ &\leq 2E_\beta(2\max\{1,\rho\}\Lambda t_n^\beta)\sqrt{\Gamma(1-\beta)}t_n^{\beta/2}\hat{C}(N^{-\min\{1,r\beta\}}+h_x^2+h_y^2),\end{aligned}$$

where $\Lambda = 2 + \mathbf{L}^2$. This proof is then completed. □

Obviously, according to Lemma 4.6 above, we naturally obtain the following convergence result.

Theorem 4.7. *The finite difference scheme (54) is convergent with the order* $O(N^{-\min\{1,r\beta\}} + h_x^2 + h_y^2)$.

5. Numerical experiments

To demonstrate the effectiveness of the numerical scheme and verify the error estimates obtained in Theorem 3.5 and Theorem 4.7, we now present some numerical simulations for the discrete problem (16) and (54) in one and two dimensional cases. In our computations, we always take $T = 1$, $L = \pi$. The errors are measured in the sense of L_2 norm and L_∞ norm, respectively.

5.1. Numerical results for difference scheme (16). Here, the orders of convergence are calculated by the standard formulas as follows

$$\textit{Spatial convergence rate} = \log_2\Big(\frac{E_p(M,N)}{E_p(2M,N)}\Big)$$

and

$$\textit{Time convergence rate} = \log_2\Big(\frac{E_p(M,N)}{E_p(M,2N)}\Big),$$

where the positive integer N is the number of time intervals and M is the number of uniform space partition in one direction (both directions have the same number of partitions for the two dimensional cases). The error $E_p(M,N)$ is calculated by (e.g., in two dimensional case)

$$E_p(M,N) := \|u(x_j, y_m, t_N) - U_{j,m}^N\|_p, \tag{83}$$

where $p = 2$ or ∞.

Example 5.1. *We consider the one-dimensional initial-boundary value problem of Fokker-Planck equation with a source term:*

$$\begin{aligned}{}_0^C D_t^\beta u(x,t) &= \frac{\overline{A}}{\beta}\Big[\ t\ {}_0D_t^{1-\beta}\frac{\partial^2 u(x,t)}{\partial x^2}\Big] - A\frac{\partial u(x,t)}{\partial x}\\ &\qquad - \kappa u + h(x,t),\ (x,t)\in(0,\pi)\times(0,1],\\ u(0,t) &= u(\pi,t) = 0,\ t\in(0,1],\\ u(x,0) &= 0,\ x\in[0,\pi],\end{aligned}\tag{84}$$

where the source term

$$h(x,t) = \Big(\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + \frac{2\overline{A}t^{1+\beta}}{\beta\Gamma(2+\beta)} + \kappa t^2\Big)\sin(x) + At^2\cos(x)$$

is specially chosen such that the problem (84) admits an exact solution in the form of $u(x,t) = t^2\sin(x)$.

Tables 1-2 show the L_2 norm errors, the L_∞ norm errors, and time-space convergence orders of the finite difference scheme (16) for the problem (84). To investigate the convergence in time and eliminate the influence from spatial discretization, we take $M = 600$ large enough such that the spatial discretization does not pollute the temporal error in Table 1. From Table 1, it is observed that the proposed scheme has first-order convergence with respect to the time direction in the sense of two different norms, which is consistent with the theoretical estimate. Table 2 shows the numerical L_2 norm errors, L_∞ norm errors, and the order of convergence in the spatial direction with respect to different values of β when $N = M^2$. From Table 2, it can be clearly seen that the rate of convergence is of order 2 in space direction with respect to both norms. From Tables 1-2, it is confirmed that the convergence rates are strictly $O(\tau + h^2)$. This suggests that the excellent agreement of the numerical results with the theoretical predictions is experimentally supported.

Example 5.2. *We consider the one-dimensional initial-boundary value problem of Fokker-Planck equation:*

(85)

$$\begin{aligned}
&{}_0^CD_t^\beta u(x,t) = \frac{\overline{A}}{\beta}\Big[\ t\ {}_0D_t^{1-\beta}\frac{\partial^2 u(x,t)}{\partial x^2}\Big] - A\frac{\partial u(x,t)}{\partial x} - \kappa u(x,t),\ (x,t)\in(0,\pi)\times(0,1],\\
&u(0,t) = u(\pi,t) = 0,\ t\in(0,1],\\
&u(x,0) = \sin(x),\ x\in[0,\pi].
\end{aligned}$$

Since the analytical solution is unknown for problem (85), the orders of the convergence of the numerical results are computed by the two-mesh principle

$$rate_\tau = \log_2\Big(\frac{e(M,N)}{e(2M,2N)}\Big),$$

where $e(M,N) := \max\limits_{1\le n\le N}\|U_{m/2}^{n/2} - U_m^n\|_p,\ p = 2,\infty$. With $u(x,0) = \sin(x)$, the solution of Example 5.2 will have a weak singularity at $t = 0$. Since the spatial error $O(h^2)$ is standard, here we only investigate the temporal error on uniform meshes. Table 3 shows maximum of the errors and convergence orders of the finite difference scheme (16) for the problem (85), where we take $M = N$ such that $e(M,N) \approx e(N)$ to verify the convergence orders in the temporal directions. From Table 3, the numerical data indicate that the order of temporal convergence is about $O(\tau^\beta)$ when the solution is not smooth.

Example 5.3. *We consider the two-dimensional initial-boundary value problem of the Fokker-Planck equation with a source term:*

$$\begin{aligned}
{}_0^CD_t^\beta u(x,y,t) = \frac{\overline{A}}{\beta}\ t\ {}_0D_t^{1-\beta}\Big(&\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\Big)\\
&- A\Big(\frac{\partial u(x,y,t)}{\partial x} + \frac{\partial u(x,y,t)}{\partial y}\Big) - u(x,y,t) + h(x,y,t),\\
&(x,y,t)\in(0,\pi)\times(0,\pi)\times(0,1],\\
u(0,y,t) = u(\pi,y,t) &= u(x,0,t) = u(x,\pi,t) = 0,\ t\in(0,1],\\
u(x,y,0) = 0,\ &(x,y)\in[0,\pi]\times[0,\pi],
\end{aligned} \tag{86}$$

where the source term

$$\begin{aligned}
h(x,y,t) =&\Big(\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + \frac{4\overline{A}t^{2+\beta}}{\beta\Gamma(2+\beta)} + t^2\Big)\sin(x)\sin(y)\\
&+ At^2\Big(\cos(x)\sin(y) + \sin(x)\cos(y)\Big)
\end{aligned}$$

TABLE 1. The L_2 errors, L_∞ errors, and orders of temporal convergence of the scheme (16) with $\overline{A} = 2$, $A = 2$, and $\kappa = 1$ for Example 5.1.

β	$M = 600$					
		$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
$\beta = 0.4$	L_2 error	1.6000e-02	7.9000e-03(1.0181)	4.000e-03(0.9819)	2.0000e-03(1.0000)	9.8255e-04(1.0254)
	L_∞ error	1.2800e-02	6.3000e-03(1.0227)	3.2000e-03(0.9773)	1.6000e-03(1.0000)	7.8495e-04(1.0274)
$\beta = 0.6$	L_2 error	1.0100e-02	4.9000e-03(1.0435)	2.4000e-03(1.0297)	1.2000e-03(1.0000)	5.7717e-04(1.0560)
	L_∞ error	8.1000e-03	3.9000e-03(1.0544)	1.9000e-03(1.0375)	9.3637e-04(1.0208)	4.6159e-04(1.0205)
$\beta = 0.8$	L_2 error	7.8000e-03	3.7000e-03(1.0759)	1.7000e-02(1.1220)	8.0653e-04(1.0757)	3.8152e-04(1.0800)
	L_∞ error	6.3000e-03	2.9000e-03(1.1193)	1.4000e-03(1.0506)	6.4694e-04(1.1137)	3.0593e-04(1.0804)

TABLE 2. The L_2 errors, L_∞ errors, and orders of spatial convergence of the scheme (16) with $\overline{A} = 2$, $A = 2$, and $\kappa = 1$ for Example 5.1.

β	$N = M^2$					
		$M = 8, N = 64$	$M = 16, N = 256$	$M = 32, N = 1024$	$M = 64, N = 4096$	$M = 128, N = 16384$
$\beta = 0.4$	L_2 error	4.320e-02	1.100e-02(1.9735)	2.800e-03(1.9740)	6.963e-04(2.0076)	1.7417e-04(1.9992)
	L_∞ error	3.460e-02	8.800e-03(1.9752)	2.200e-03(2.0000)	5.5921e-04(1.9760)	1.3989e-04(1.9991)
$\beta = 0.6$	L_2 error	4.750e-02	1.230e-02(1.9493)	3.100e-03(1.9883)	7.8129e-04(1.9883)	1.9571e-04(1.9971)
	L_∞ error	3.810e-02	9.900e-03(1.9443)	2.500e-03(1.9855)	6.3162e-04(1.9848)	1.5826e-04(1.9968)
$\beta = 0.8$	L_2 error	4.890e-02	1.280e-02(1.9337)	3.300e-03(1.9556)	8.2844e-04(1.9940)	2.0868e-04(1.9891)
	L_∞ error	3.930e-02	1.050e-02(1.9041)	2.700e-03(1.9594)	6.7635e-04(1.9971)	1.7044e-04(1.9885)

TABLE 3. The L_2 errors, L_∞ errors, and orders of temporal convergence of the scheme (16) with $\overline{A} = 2$, $A = 2$, and $\kappa = 1$ for Example 5.2.

β	$M = N$					
		$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
$\beta = 0.4$	L_2 error	4.1700e-02	3.2400e-02(0.3641)	2.5000e-02(0.3741)	1.9200e-02(0.3808)	1.4700e-02(0.3853)
	L_∞ error	4.9700e-02	3.8700e-02(0.3609)	2.9700e-02(0.3819)	2.2600e-02(0.3941)	1.7100e-02(0.4023)
$\beta = 0.6$	L_2 error	2.2500e-02	1.4900e-02(0.5946)	9.9000e-03(0.5898)	6.5000e-03(0.6070)	4.3000e-03(0.5961)
	L_∞ error	2.7200e-02	1.7800e-02(0.6117)	1.1500e-02(0.6302)	7.4000e-03(0.6360)	4.9000e-03(0.5947)
$\beta = 0.8$	L_2 error	1.1600e-02	6.5000e-03(0.8356)	3.6000e-03(0.8524)	2.0000e-03(0.8480)	1.1000e-03(0.8625)
	L_∞ error	1.5000e-02	8.4000e-03(0.8365)	4.6000e-03(0.8688)	2.5000e-03(0.8797)	1.4000e-03(0.8365)

TABLE 4. The L_2 errors, L_∞ errors, and orders of temporal convergence of the scheme (16) with $M_1 = M_2 = M$, $\overline{A} = 2$ and $A = 2$ for Example 5.3.

β	$M_1 = M_2 = 64$					
		$N = 2^4$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$
$\beta = 0.4$	L_2 error	2.541e-01	1.332e-01(0.9318)	6.820e-02(0.9658)	3.440e-02(0.9874)	1.720e-02(1.0000)
	L_∞ error	1.625e-01	8.530e-02(0.9298)	4.370e-02(0.9649)	2.210e-02(0.9836)	1.100e-02(1.0065)
$\beta = 0.6$	L_2 error	2.581e-01	1.361e-01(0.9233)	7.000e-02(0.9592)	3.550e-02(1.0632)	1.770e-02(1.0041)
	L_∞ error	1.659e-01	8.780e-02(0.9180)	4.520e-02(0.9579)	2.290e-02(0.9810)	1.150e-02(0.9937)
$\beta = 0.8$	L_2 error	2.588e-01	1.377e-02(0.9103)	7.130e-02(0.9496)	3.630e-02(0.9739)	1.820e-02(0.9960)
	L_∞ error	1.676e-01	8.970e-02(0.9018)	4.660e-02(0.9448)	2.370e-02(0.9754)	1.190e-02(0.9939)

is specially chosen such that the problem (86) admits an exact solution in the form of $u(x, y, t) = t^2 \sin(x) \sin(y)$.

TABLE 5. The L_2 errors, L_∞ errors, and orders of spatial convergence of the scheme (16) with $M_1 = M_2 = M$, $\overline{A} = 2$ and $A = 2$ for Example 5.3.

β	$N = M^2$	$M = 8, N = 64$	$M = 12, N = 144$	$M = 18, N = 324$	$M = 27, N = 729$
$\beta = 0.4$	L_2 error	3.130e-02	1.380e-02(2.0198)	6.100e-03(2.0134)	2.700e-03(2.0101)
	L_∞ error	3.200e-02	1.440e-02(1.9694)	6.400e-03(2.0000)	2.900e-03(1.9523)
$\beta = 0.6$	L_2 error	2.550e-02	1.120e-02(2.0292)	5.000e-03(1.9890)	2.200e-03(2.0248)
	L_∞ error	3.370e-02	1.520e-02(1.9637)	6.800e-03(1.9838)	3.100e-03(1.9373)
$\beta = 0.8$	L_2 error	2.130e-02	9.400e-03(2.00174)	4.100e-03(2.0463)	1.800e-03(2.0303)
	L_∞ error	3.540e-02	1.610e-02(1.9432)	7.300e-03(1.9507)	3.300e-03(1.9581)

Table 4 displays the L_2 norm errors, L_∞ norm errors, and the corresponding convergence order with respect to the time step size with $M_1 = M_2 = 64$ large enough so that the error with respect to spatial discretization can be omitted. From Table 4, it is observed that in the time direction the convergence is first-order for different values of N and β. Table 5 shows the L_2 norm errors, L_∞ norm errors, and the order of convergence in the spatial direction with respect to different values of β when $N = M^2 = M_1^2 = M_2^2$, and the order of convergence is 2 in space direction. From Tables 4-5, it can be seen that the rate of convergence is of order $O(\tau + h_x^2 + h_y^2)$.

5.2. Numerical results for nonuniform difference scheme (54).

Example 5.4. *We consider the following one-dimensional nonlinear subdiffusion problem:*

$$
\begin{aligned}
&{}_0^C D_t^\beta u = \frac{\partial^2 u}{\partial x^2} + q\frac{\partial u}{\partial x} + u(1-u) + g_1(x,t),\ (x,t) \in (0,\pi) \times (0,1], \\
&u(0,t) = u(\pi,t) = 0,\ t \in (0,1], \\
&u(x,0) = 0,\ x \in [0,\pi],
\end{aligned} \tag{87}
$$

where $g(x, t)$ *is specially chosen such that the problem (87) admits an exact solution in the form of* $u(x, t) = (t^2 + t^\beta) \sin(x)$.

Here, the temporal convergence order is calculated by the formula $\log_2\left(\frac{e(M,N)}{e(2M,2N)}\right)$, where $e(M, N) := \max_{1\le n\le N} \|u_m^n - U_m^n\|_p$, $p = 2, \infty$. In Tables 6 and 7, we take $M = N$ to verify the temporal convergence order for $r = 1$ and $r = (2 - \beta)/\beta$, respectively. One can see that the temporal convergence rates are $O(N^{-\min\{1,r\beta\}})$. In addition, as shown in Table 8, we set $N = M^2$ to verify the order of spatial convergence, and it is obvious that the spatial convergence order is 2. In short, the numerical results in Tables 6-8 confirm that the convergence order is $O(N^{-\min\{1,r\beta\}} + h^2)$.

TABLE 6. The L_2 errors, L_∞ errors, and orders of temporal convergence of the scheme (54) for Example 5.4 with $q = 1$ and $r = 1$.

β	$M = N$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
$\beta = 0.4$	L_2 error	3.146e-01	2.378e-01(0.4038)	1.800e-01(0.4018)	1.346e-01(0.4002)	1.034e-01(0.3996)
	L_∞ error	2.510e-01	1.897e-01(0.4040)	1.436e-01(0.4017)	1.088e-01(0.4004)	8.250e-02(0.3992)
$\beta = 0.6$	L_2 error	1.579e-01	1.037e-01(0.6066)	6.830e-02(0.6025)	4.500e-02(0.6020)	2.970e-02(0.5995)
	L_∞ error	1.260e-01	8.270e-02(0.6075)	5.450e-02(0.6016)	3.590e-02(0.6023)	2.370e-04(0.5991)
$\beta = 0.8$	L_2 error	8.170e-02	4.530e-02(0.8508)	2.590e-02(0.8066)	1.490e-02(0.7976)	8.500e-03(0.8098)
	L_∞ error	6.380e-02	3.610e-02(0.8216)	2.070e-02(0.8024)	1.190e-02(0.7987)	6.800e-03(0.8074)

TABLE 7. The L_2 errors, L_∞ errors, and orders of temporal convergence of the scheme (54) for Example 5.4 with $q = 1$ and $r = (2 - \beta)/\beta$.

β	$M = N$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
$\beta = 0.4$	L_2 error	1.831e-01	8.870e-02(1.0456)	4.370e-02(1.0213)	2.170e-02(1.0099)	1.080e-02(1.0067)
	L_∞ error	1.442e-01	6.990e-02(1.0447)	3.440e-02(1.0229)	1.710e-02(1.0084)	8.500e-03(1.0085)
$\beta = 0.6$	L_2 error	1.128e-01	5.550e-02(1.0232)	2.750e-02(1.0131)	1.370e-02(1.0053)	6.900e-03(0.9895)
	L_∞ error	8.890e-01	4.370e-02(1.0246)	2.170e-02(1.0099)	1.080e-02(1.0067)	5.400e-04(1.0000)
$\beta = 0.8$	L_2 error	8.000e-02	3.960e-02(1.0145)	1.970e-02(1.0073)	9.900e-03(0.9927)	5.000e-03(0.9855)
	L_∞ error	6.270e-02	3.110e-02(1.0116)	1.550e-02(1.0046)	7.800e-03(0.9907)	3.900e-03(1.0000)

TABLE 8. The L_2 errors, L_∞ errors, and orders of spatial convergence of the scheme (54) for Example 5.4 with $q = 1$.

β	$N = M^2$					
		$M = 8, N = 64$	$M = 16, N = 256$	$M = 32, N = 1024$	$M = 64, N = 4096$	$M = 128, N = 16384$
$\beta = 0.4$	L_2 error	7.860e-02	1.920e-02(2.0334)	4.800e-03(2.0000)	1.200e-03(2.0000)	2.9885e-04(2.0054)
	L_∞ error	5.960e-02	1.480e-02(2.0097)	3.700e-03(2.0000)	9.1999e-04(2.0078)	2.3004e-04(1.9997)
$\beta = 0.6$	L_2 error	4.570e-02	1.130e-02(2.0159)	2.800e-03(2.0128)	7.1130e-04(1.9769)	1.7817e-04(1.9972)
	L_∞ error	3.460e-02	8.600e-03(2.0086)	2.200e-03(1.9668)	5.4046e-04(2.0252)	1.3537e-04(2.0237)
$\beta = 0.8$	L_2 error	3.100e-02	7.700e-03(2.0093)	1.900e-03(2.0189)	4.8757e-04(1.9623)	1.2266e-04(1.9909)
	L_∞ error	2.350e-02	5.900e-03(1.9939)	1.500e-03(1.9758)	3.7108e-04(2.0152)	9.3344e-04(1.9911)

TABLE 9. The L_2 errors, L_∞ errors, and orders of temporal convergence of the scheme (54) for Example 5.5 with $M = M_1 = M_2$, $q = 1$ and $r = 1$.

β	$M = 64$					
		$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
$\beta = 0.4$	L_2 error	3.942e-01	2.980e-01(0.4036)	2.256e-01(0.4015)	1.717e-01(0.3998)	1.295e-01(0.4010)
	L_∞ error	2.510e-01	1.897e-01(0.4040)	1.436e-01(0.4017)	1.088e-01(0.4004)	8.250e-02(0.3992)
$\beta = 0.6$	L_2 error	1.979e-01	1.299e-01(0.6074)	8.560e-02(0.6017)	5.640e-02(0.6019)	3.720e-02(0.6004)
	L_∞ error	1.260e-01	8.270e-02(0.6075)	5.450e-02(0.6016)	3.590e-02(0.6023)	2.370e-04(0.5991)
$\beta = 0.8$	L_2 error	1.002e-01	5.680e-02(0.8474)	3.250e-02(0.8055)	1.860e-02(0.8051)	1.070e-02(0.7977)
	L_∞ error	6.350e-02	3.610e-02(0.8148)	2.070e-02(0.8024)	1.190e-02(0.7987)	6.800e-03(0.8074)

TABLE 10. The L_2 errors, L_∞ errors, and orders of temporal convergence of the scheme (54) for Example 5.5 with $M = M_1 = M_2$, $q = 1$ and $r = (2-\beta)/\beta$.

β	$M = 64$					
		$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
$\beta = 0.4$	L_2 error	2.979e-01	1.468e-01(1.0210)	7.260e-02(1.0158)	3.600e-02(1.0120)	1.780e-02(1.0161)
	L_∞ error	1.805e-01	8.830e-02(1.0315)	4.360e-02(1.0181)	2.160e-02(1.0133)	1.060e-03(1.0270)
$\beta = 0.6$	L_2 error	1.867e-01	9.220e-02(1.0179)	4.570e-02(1.0126)	2.270e-02(1.0095)	1.120e-02(1.0192)
	L_∞ error	1.125e-01	5.530e-02(1.0246)	2.740e-02(1.0131)	1.350e-02(1.0212)	6.700e-03(1.0107)
$\beta = 0.8$	L_2 error	1.317e-01	6.520e-02(1.0143)	3.240e-02(1.0089)	1.600e-03(1.0179)	7.900e-03(1.0181)
	L_∞ error	7.900e-02	3.900e-02(1.0184)	1.930e-02(1.0149)	9.600e-03(1.0075)	4.700e-03(1.0304)

TABLE 11. The L_2 errors, L_∞ errors, and orders of spatial convergence of the scheme (54) with $M = M_1 = M_2$ and $q = 1$ for Example 5.5.

β	$N = M^2$				
		$M = 8, N = 64$	$M = 12, N = 144$	$M = 18, N = 324$	$M = 27, N = 729$
$\beta = 0.4$	L_2 error	1.265e-01	5.580e-02(2.0186)	2.470e-02(2.0100)	1.100e-02(1.9950)
	L_∞ error	7.410e-02	3.240e-02(2.0251)	1.440e-02(2.0152)	6.400e-03(2.0000)
$\beta = 0.6$	L_2 error	7.250e-01	3.210e-02(2.0094)	1.420e-02(2.0116)	6.300e-03(2.0043)
	L_∞ error	4.220e-02	1.870e-02(2.0073)	8.300e-03(2.0033)	3.700e-03(1.9926)
$\beta = 0.8$	L_2 error	4.630e-02	2.050e-02(2.0093)	9.100e-03(2.0030)	4.100e-03(1.9663)
	L_∞ error	2.660e-02	1.180e-02(2.0046)	5.300e-03(1.9740)	2.400e-03(1.9539)

Example 5.5. *We consider the following two-dimensional nonlinear subdiffusion problem:*

(88)

$$
\begin{aligned}
&{}_0^C D_t^\beta u = \triangle u + (q,q)\cdot \nabla u + u(1-u) + g_2(x,y,t),\ (x,y,t) \in (0,\pi)\times(0,\pi)\times(0,1],\\
&u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0,\ t\in(0,1],\\
&u(x,y,0) = 0,\ (x,y)\in[0,\pi]\times[0,\pi],
\end{aligned}
$$

where $g_2(x,y,t)$ is specially chosen such that the problem (88) admits an exact solution in the form of $u(x,y,t) = (t^2 + t^\beta)\sin(x)\sin(y)$.

In Tables 9 and 10, we take $M = M_1 = M_2 = 64$, and take different N to verify the temporal convergence order for $r = 1$ and $r = (2-\beta)/\beta$, respectively. One can also see that the temporal convergence rates are $O(N^{-\min\{1,r\beta\}})$. In addition, as shown in Table 11, we set $N = M^2 = M_1^2 = M_2^2$ to verify the order of spatial convergence, and it is obvious that the spatial convergence order is 2. Therefore, the numerical results in Tables 9-11 also verify that the convergence order of the difference scheme (54) is $O(N^{-\min\{1,r\beta\}} + h_x^2 + h_y^2)$, which is consistent with our theoretical analysis.

6. Conclusion

In this paper, we mainly propose an effective proof technique that combines the ideas of Fourier analysis, enabling the Fourier method to be extended to handle nonlinear problems. This paper provides the finite difference schemes for two types of two-dimensional problems. We first consider the time fractional Fokker-Planck Eq. (2)-(4), where L1 method and Grünwald-Letnikov formula are used to discrete the time fractional Caputo and Riemann-Liouville operators on uniform meshes, respectively. Then we consider the general time fractional Eq. (50), where L1 method is used to discrete the time fractional Caputo operator on graded meshes. Additionally, the strict unconditional stability and convergence of the two fully discrete schemes are rigorously proved. Finally, a series of numerical examples in one and two dimensional cases are provided to confirm the theoretical results and to demonstrate the efficiency of the proposed method.

Declarations

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest The authors declared that they have no conflict of interest to this work.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant Nos. 12071195 and 12225107, the Innovative Groups of Basic Research in Gansu Province under Grant No. 22JR5RA391, the Major Science and Technology Projects in Gansu Province - Leading Talents in Science and Technology under Grant No. 23ZDKA0005, and Lanzhou Talent Work Special Fund.

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