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THE NAVIER-STOKES-ω/NAVIER-STOKES-ω MODEL FOR FLUID-FLUID INTERACTION USING AN UNCONDITIONALLY STABLE FINITE ELEMENT SCHEME

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Abstract. In this article, for solving fluid-fluid interaction problem, we consider a Navier-Stokes-ω/Navier-Stokes-ω model, which includes two Navier-Stokes-ω equations coupled by some nonlinear interface conditions. Based on an auxiliary variable, we propose a fully discrete, decouple finite element scheme. We adopt the backward Euler scheme and mixed finite element approximation for temporal-spatial discretization, and explicit treatment for the interface terms and nonlinear terms. Moreover, the proposed scheme is shown to be unconditionally stable. Then, we establish error estimate of the numerical solution. Finally, with a series of numerical experiments we illustrate the stability and effectiveness of the proposed scheme and its ability to capture basic phenomenological features of the fluid-fluid interaction.

Key words. Navier-Stokes-ω model, fluid-fluid interaction, auxiliary variable, unconditional stability.

1. Introduction

In many scientific fields and practical applications, numerical simulation is an important aspect for multi-physics and multi-domain of one immiscible fluid with another fluid, such as simulations of atmosphere-ocean interaction problem [1, 2, 3, 4] in environmental engineering and so on. Herein, we will consider a fluid-fluid interaction problem with some nonlinear interface conditions, which are known as the rigid-lid condition (e.g. approximate the atmosphere-ocean surface as flat). It allows for the energy to transfer back and forth across the interface, while the global energy of the system remains conserved [5].

At the time of writing, numerous works are devoted to fluid-fluid interaction models with nonlinear interface condition [6, 7]. Bernardi et al. [8, 9, 10] have studied two immiscible turbulent fluids on adjacent subdomains, but it limits the effectiveness of calculations because of a large and decoupled system. Connors et al. [11] have proposed two decoupled time stepping methods based on the partitioned time stepping methods. One of them is the geometric averaging method, whose key benefit is the unconditional stability. This method has further developed in [12, 13, 14, 15, 16, 17]. Recently, Aggul et al. [18] have proposed a large eddy simulation with correction model, which used the defect correction to control efficiently the model error.

When at least one of the flow enters the turbulent state in subdomain, we need to choose a turbulence model. Recently, the approximate deconvolution model of turbulence is considered for fluid-fluid interaction [19]. In this paper, we will consider the Navier-Stokes-ω model, which has non-filtered velocity on the interface. In fact, Layton et al. [20] showed that the discrete Navier-Stokes-ω simulation had greater accuracy at less cost and required significantly fewer degrees of freedom than a comparable Navier-Stokes-α simulation. In [21], the authors proved existence and

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uniqueness of strong solutions of Navier-Stokes-ω model and gained convergence to a weak solution of the Navier-Stokes equations as the averaging radius decreased to zero. Furthermore, Manica et al. [22] used van Cittert approximate deconvolution to improve accuracy and Scott-Vogelius elements to provide pointwise mass conservative solutions, and removed the dependence of the Bernoulli pressure error on the velocity error for Navier-Stokes-ω model. Recently, with help of the geometric averaging approach, Aggul et al. [23] considered the Navier-Stokes-ω model for fluid-fluid interaction problem, which was showed to be unconditionally stable. Further, based on large eddy simulation with correction, the authors designed a second-order scheme in time [18].

In this article, we will design a fully explicit treatment to decouple nonlinear interface conditions and also obtain the unconditional stability. From the point of view of numerical discretization for partial differential equations, it is natural to hope that nonlinear terms can be treated explicitly. In the meantime, the unconditional stability will be kept. The scalar auxiliary variable has been initially applied to the gradient flow problem in [24, 25], which can keep unconditional energy stability. Inspired by this idea, Li et al. [26] have solved the magnetohydrodynamic by an exponential scalar auxiliary variable method. Jiang and Yang [27] have developed two decoupled ensemble schemes for the Stokes-Darcy system, by combining the scalar auxiliary variable idea with the ensemble time step method. Additionally, Li et al. [28] have further established an unconditionally energy stable finite element scheme for a nonlinear fluid-fluid interaction model. A generalized scalar auxiliary variable approach has been considered for the Navier-Stokes-ω/Navier-Stokes-ω equations based on the grad-div stabilization [29].

In current work, we introduce an auxiliary variable in exponential function to deal with the Navier-Stokes-ω/Navier-Stokes-ω model, where nonlinear terms and nonlinear interface terms are treated explicitly. The proposed scheme enjoys the following features: (i) it is unconditionally stable; (ii) unlike the geometric averaging scheme, explicit treatment is considered for the interface terms. The structure of this paper is arranged as follows: In Section 2, we will introduce some notations and function spaces. In Section 3, the Navier-Stokes-ω/Navier-Stokes-ω model is showed. Moreover, a fully discrete finite element scheme based on an auxiliary variable is designed, and unconditional stability is obtained. Section 4 develops the theory for the scheme, showing analysis of convergence. In Section 5, we use several numerical experiments to test the stability and convergence of the proposed scheme, and to show its ability to capture basic phenomenological features of the fluid-fluid interaction. We close with some concluding remarks in Section 6.

2. Preliminary

This section summarizes some necessary notations and inequalities.

Consider the spatial domain $\varOmega \subset \mathbb{R}^d$ $(d = 2, 3)$ that consists of two subdomains $\varOmega_1$ and $\varOmega_2$ coupled across their shared interface $I \subsetneqq \partial\varOmega$. Next, $\|\cdot\|_0$ and $(\cdot, \cdot)$ are represented as $L^2(\varOmega_i)$ $(i = 1, 2)$ norm and its inner product. Additionally, $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$ are expressed as the Lebesgue space $L^p(\varOmega_i)$ norms and the Sobolev space $W^{m,p}(\varOmega_i)$ norms for $m \in \mathbb{N}^+$, $1 \leqslant p \leqslant \infty$. Besides, for X_i being a normed function space in $\varOmega_i$, $L^p(0, T, X_i)$ is the space of all functions defined on $[0, T] \times \varOmega_i$ and it's norm represents

$$\|\mathbf{u}\|_{L^p(0,T,\mathbf{X}_i)} = \left(\int_0^T \|\mathbf{u}\|_{X_i}^p dt\right)^{\frac{1}{p}}, \ p \in [1, \infty).$$

Moreover, we introduce the velocity and pressure spaces

$$\mathbf{X}_i = \left\{ \mathbf{v}_i \in W^{1,2}(\Omega_i)^d ;\ \mathbf{v}_i|_{\partial\Omega_i \setminus I} = 0;\ \mathbf{v}_i \cdot \mathbf{n}_i = 0 \text{ on } I \right\},$$
$$M_i = \left\{ q_i \in L^2(\Omega_i);\ (q_i, 1) = 0 \right\}.$$

For space $\mathbf{X}_i$, its dual space is defined by $\mathbf{X}_i'$ and the norm is denoted as

$$\|\mathbf{f}_i\|_{-1} = \sup_{\mathbf{v}_i \in \mathbf{X}_i} \frac{|(\mathbf{f}_i, \mathbf{v}_i)|}{\|\nabla \mathbf{v}_i\|_0},$$

where $\mathbf{f}_i$ is an element in the dual space of $\mathbf{X}_i$.

For each subdomain Ω_i it is regular partitioned into a mesh π_i^h and $\pi^h = \pi_1^h \cup \pi_2^h$, which comprises of K that is triangle for $d = 2$ and tetrahedral element for $d = 3$. We assume that h is the largest diameter of the element in π^h. According, we define the finite element spaces on π_i^h by $\mathbf{X}_i^h \subset \mathbf{X}_i$ and $M_i^h \subset M_i$. Suppose that the finite element spaces $\mathbf{X}_i^h$ and M_i^h satisfy the discrete LBB condition

$$\beta_i \|q_{i,h}\|_0 \leqslant \sup_{\mathbf{v}_{i,h} \in \mathbf{X}_i^h} \frac{|(\nabla \cdot \mathbf{v}_{i,h}, q_{i,h})|}{\|\nabla \mathbf{v}_{i,h}\|_0},\ \forall q_{i,h} \in M_i^h,$$

where $\beta_i > 0$ is only dependent on Ω_i. In this paper, we apply the MINI element for the velocity and pressure.

We introduce the following trilinear form on $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i \in \mathbf{X}_i$ by

$$b_\omega(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i) = ((\nabla \times \mathbf{u}_i) \times \mathbf{v}_i, \mathbf{w}_i).$$

This rotation version of the trilinear term satisfies some properties listed in the following lemma.

Lemma 2.1. [20, 23] *For $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i \in \mathbf{X}_i$, $i = 1, 2$, $b_\omega(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i)$ satisfies*

$$b_\omega(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i) \leqslant c \|\nabla \times \mathbf{u}_i\|_0 \|\nabla \mathbf{v}_i\|_0 \|\nabla \mathbf{w}_i\|_0,$$
$$b_\omega(\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i) \leqslant c \|\mathbf{u}_i\|_0^{\frac{1}{2}} \|\nabla \mathbf{u}_i\|_0^{\frac{1}{2}} \|\nabla \mathbf{v}_i\|_0 \|\nabla \mathbf{w}_i\|_0,$$

where c are some positive constants and only related to Ω.

In this paper we use c (with or without a subscript) to denote a generic positive constant, which is possibly different at different occurrences but always independent of mesh size and time step.

Next, some commonly useful inequalities are introduced as follows.

Lemma 2.2. [30] *There exist some constants $c = c(\Omega_i) > 0$, satisfying*

$$\|\mathbf{v}_i\|_{L^2(I)} \leqslant c \|\mathbf{v}_i\|_0^{\frac{1}{2}} \|\nabla \mathbf{v}_i\|_0^{\frac{1}{2}}, \quad \|\mathbf{v}_i\|_{L^4(I)} \leqslant c \|\nabla \mathbf{v}_i\|_0.$$

Lemma 2.3. [31] *For $\mathbf{v}_{i,h} \in \mathbf{X}_i^h$, then there hold*

$$\|\nabla \mathbf{v}_{i,h}\|_0 \leqslant c_1 h^{-1} \|\mathbf{v}_{i,h}\|_0.$$

Lemma 2.4. [32] *Let c, k and a_n, b_n, d_n, for integers $n_1 \leqslant n \leqslant m$, be nonnegative numbers such that*

$$a_m + k \sum_{n=n_1}^{m} b_n \leqslant k \sum_{n=n_1}^{m-1} a_n d_n + c,\ \forall m \geqslant n_1.$$

Then, one has

$$a_m + k \sum_{n=n_1}^{m} b_n \leqslant \exp\left(k \sum_{n=n_1}^{m-1} d_n \right) c,\ \forall m \geqslant n_1.$$

3. A Navier-Stokes-ω/Navier-Stokes-ω model

In this section, we show the Navier-Stokes-ω/Navier-Stokes-ω model for resolution of fluid-fluid interaction problem, then propose a fully discrete finite element scheme for the equivalent system of the considered model, and finally prove unconditional stability of the scheme.

Firstly, for $i, j = 1, 2,\ i \neq j$, we recall the governing equations of the fluid-fluid interaction problem, called the Navier-Stokes/Navier-Stokes model, as follows [11]:

$$\begin{aligned}
&\mathbf{u}_{i,t} - \nu_i \Delta \mathbf{u}_i = \mathbf{f}_i - (\mathbf{u}_i \cdot \nabla)\,\mathbf{u}_i - \nabla p_i,\ \nabla \cdot \mathbf{u}_i = 0, && \text{in } \Omega_i,\\
&-\nu_i \mathbf{n}_i \cdot \nabla \mathbf{u}_i \cdot \boldsymbol{\tau} = \kappa\,|\mathbf{u}_i - \mathbf{u}_j|\,(\mathbf{u}_i - \mathbf{u}_j) \cdot \boldsymbol{\tau},\ \mathbf{u}_i \cdot \mathbf{n}_i = 0, && \text{on } I,\\
&\mathbf{u}_i\,(0, \mathbf{x}) = \mathbf{u}_{i,0}\,(\mathbf{x}), && \text{in } \Omega_i,\\
&\mathbf{u}_i = \mathbf{0}, && \text{on } \Gamma_i := \partial\Omega_i \backslash I,
\end{aligned} \tag{1}$$

where the vector field $\mathbf{u}_i$ are the velocities of the fluid, and p_i stand for the pressures in Ω_i. The positive parameters ν_i represent the kinematic viscosity, and κ is the friction coefficient. Besides, $\mathbf{f}_i$ are the given body forces. $|\cdot|$ is the Euclidean norm, the vectors $\mathbf{n}_i$ are the unit normals on $\partial\Omega_i$, and $\boldsymbol{\tau}$ represents any vector on I such that $\boldsymbol{\tau} \cdot \mathbf{n}_i = 0$.

Secondly, based on the Navier-Stokes-ω model proposed and studied in [20, 21], for $i, j = 1, 2,\ i \neq j$, we consider the following Navier-Stokes-ω/Navier-Stokes-ω model for the fluid-fluid interaction problem

$$\begin{aligned}
&\mathbf{u}_{i,t} - \nu_i \Delta \mathbf{u}_i = \mathbf{f}_i - (\nabla \times \bar{\mathbf{u}}_i) \times \mathbf{u}_i - \nabla p_i,\ \nabla \cdot \mathbf{u}_i = 0, && \text{in } \Omega_i,\\
&\bar{\mathbf{u}}_i - \delta_i^2 \Delta \bar{\mathbf{u}}_i = \mathbf{u}_i, && \text{in } \Omega_i,\\
&-\nu_i \mathbf{n}_i \cdot \nabla \mathbf{u}_i \cdot \boldsymbol{\tau} = \kappa\,|\mathbf{u}_i - \mathbf{u}_j|\,(\mathbf{u}_i - \mathbf{u}_j) \cdot \boldsymbol{\tau},\ \mathbf{u}_i \cdot \mathbf{n}_i = 0, && \text{on } I,\\
&\mathbf{u}_i\,(0, \mathbf{x}) = \mathbf{u}_{i,0}\,(\mathbf{x}), && \text{in } \Omega_i,\\
&\mathbf{u}_i = \mathbf{0}, && \text{on } \Gamma_i,
\end{aligned} \tag{2}$$

where the vector fields $\bar{\mathbf{u}}_i$ are the filtered velocities, and δ_i are given filtering radii.

From (2) and (1), we notice that the nonlinear term in (1) is convective form while the one in (2) is rotational form. In fact, if the divergence constraint $\nabla \cdot \mathbf{u}_i = 0$ holds pointwise, then these formulations are equivalent by modifying pressure (the pressure here is the dynamic pressure, rather than the usual pressure in (1), although we use the same notation p_i). Besides, if $\delta_i = 0$, then (2) and (1) are the same. In fact, according to [33] one has $\bar{\mathbf{u}}_i = \mathbf{u}_i + O(\delta_i^2)$.

Note that from [20, 23], for $\mathbf{v}_i \in \mathbf{X}_i$, we have the following inequalities

$$\|\bar{\mathbf{v}}_i\|_0 \leqslant \|\mathbf{v}_i\|_0, \quad \|\nabla \bar{\mathbf{v}}_i\|_0 \leqslant \|\nabla \mathbf{v}_i\|_0, \quad \|\nabla \times \bar{\mathbf{v}}_i\|_0 \leqslant \sqrt{2}\,\|\nabla \mathbf{v}_i\|_0. \tag{3}$$

Next, based on exponential function of t, we introduce an auxiliary variable $Q\,(t) = \exp\left(-\frac{t}{T}\right)$. Then, we obtain equivalent equations of (2), i.e., for $i, j = 1, 2,$ $i \neq j$,

$$\begin{aligned}
&\mathbf{u}_{i,t} - \nu_i \Delta \mathbf{u}_i = \mathbf{f}_i - \exp\left(\frac{t}{T}\right) Q\,(\nabla \times \bar{\mathbf{u}}_i) \times \mathbf{u}_i - \nabla p_i,\ \nabla \cdot \mathbf{u}_i = 0, && \text{in } \Omega_i,\\
&\bar{\mathbf{u}}_i - \delta_i^2 \Delta \bar{\mathbf{u}}_i = \mathbf{u}_i, && \text{in } \Omega_i,\\
&\nu_i \mathbf{n}_i \cdot \nabla \mathbf{u}_i \cdot \boldsymbol{\tau} = \exp\left(\frac{t}{T}\right) Q \kappa\,|\mathbf{u}_j - \mathbf{u}_i|\,(\mathbf{u}_i - \mathbf{u}_j) \cdot \boldsymbol{\tau},\ \mathbf{u}_i \cdot \mathbf{n}_i = 0, && \text{on } I,\\
&\mathbf{u}_i\,(0, \mathbf{x}) = \mathbf{u}_{i,0}\,(\mathbf{x}), && \text{in } \Omega_i,\\
&\mathbf{u}_i = \mathbf{0}, && \text{on } \Gamma_i.
\end{aligned} \tag{4}$$

In addition, we need the following ordinary differential equation, which is equivalent to the derivative of the auxiliary variable Q with respect to t.

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} =& -\frac{1}{T}Q + \exp\left(\frac{t}{T}\right)\sum_{i=1}^{2} b_\omega\left(\bar{\mathbf{u}}_i, \mathbf{u}_i, \mathbf{u}_i\right)\\ &+ \exp\left(\frac{t}{T}\right)\sum_{i=1, i\neq j}^{2}\int_I \kappa\left|\mathbf{u}_i - \mathbf{u}_j\right|\left(\mathbf{u}_i - \mathbf{u}_j\right)\cdot\mathbf{u}_i \mathrm{d}s\\ &+ \exp\left(\frac{2t}{T}\right)Q\sum_{i=1, i\neq j}^{2}\int_I \kappa\left|\mathbf{u}_i - \mathbf{u}_j\right|\left(\mathbf{u}_j - \mathbf{u}_i\right)\cdot\mathbf{u}_i \mathrm{d}s. \end{aligned} \tag{5}$$

In fact, it is easy to verify that $b_\omega\left(\bar{\mathbf{u}}_i, \mathbf{u}_i, \mathbf{u}_i\right) = 0$ and

$$\begin{aligned}&\exp\left(\frac{t}{T}\right)\sum_{i=1, i\neq j}^{2}\int_I \kappa\left|\mathbf{u}_i - \mathbf{u}_j\right|\left(\mathbf{u}_i - \mathbf{u}_j\right)\cdot\mathbf{u}_i \mathrm{d}s\\ &\quad+ \exp\left(\frac{2t}{T}\right)Q\sum_{i=1, i\neq j}^{2}\int_I \kappa\left|\mathbf{u}_i - \mathbf{u}_j\right|\left(\mathbf{u}_j - \mathbf{u}_i\right)\cdot\mathbf{u}_i \mathrm{d}s = 0.\end{aligned}$$

Then, the corresponding variational formulation of (4) is given as follows: find $\left(\mathbf{u}_i, p_i, \bar{\mathbf{u}}_i\right) \in L^2\left(0, T, \mathbf{X}_i\right) \times L^2\left(0, T, M_i\right) \times L^2\left(0, T, \mathbf{X}_i\right)$ for all $\left(\mathbf{v}_i, q_i, \bar{\mathbf{v}}_i\right) \in \mathbf{X}_i \times M_i \times \mathbf{X}_i$, $i, j = 1, 2$ and $i \neq j$ such that

$$\begin{aligned}&\left(\mathbf{u}_{i,t}, \mathbf{v}_i\right) + \nu_i\left(\nabla\mathbf{u}_i, \nabla\mathbf{v}_i\right) - \left(\nabla\cdot\mathbf{v}_i, p_i\right) + \exp\left(\frac{t}{T}\right)Q b_\omega\left(\bar{\mathbf{u}}_i, \mathbf{u}_i, \mathbf{v}_i\right)\\ &\quad+ \left(\nabla\cdot\mathbf{u}_i, q_i\right) + \exp\left(\frac{t}{T}\right)Q\int_I \kappa\left|\mathbf{u}_i - \mathbf{u}_j\right|\left(\mathbf{u}_i - \mathbf{u}_j\right)\cdot\mathbf{v}_i \mathrm{d}s = \left(\mathbf{f}_i, \mathbf{v}_i\right),\\ &\delta_i^2\left(\nabla\bar{\mathbf{u}}_i, \nabla\bar{\mathbf{v}}_i\right) + \left(\bar{\mathbf{u}}_i, \bar{\mathbf{v}}_i\right) = \left(\mathbf{u}_i, \bar{\mathbf{v}}_i\right).\end{aligned} \tag{6}$$

Now, let $\varDelta t > 0$ and $N := \frac{T}{\varDelta t}$ for N an integer, and $t_n = n\varDelta t$ with $n = 0, 1, \ldots, N$. Denote $(\mathbf{u}_{i,h}^{n+1}, p_{i,h}^{n+1})$ the fully discrete approximation to the solution $(\mathbf{u}_i(t_{n+1}), p_i(t_{n+1}))$ of the problem (1) at $t = t_n$. Next, involving the backward Euler scheme, a fully discrete finite element scheme for the equivalent system (6) of the considered model (2).

Given $\mathbf{u}_{i,h}^n \in \mathbf{X}_i^h$ and $Q^n \in \mathbb{R}$, find $\left(\mathbf{u}_{i,h}^{n+1}, \bar{\mathbf{u}}_{i,h}^n, p_{i,h}^{n+1}, Q^{n+1}\right) \in \mathbf{X}_i^h \times \mathbf{X}_i^h \times M_i^h \times \mathbb{R}$, satisfying that for all $(\mathbf{v}_{i,h}, \bar{\mathbf{v}}_{i,h}, q_{i,h}) \in \mathbf{X}_i^h \times \mathbf{X}_i^h \times M_i^h$, with $i, j = 1, 2$, $i \neq j$,

$$\begin{aligned}&\left(\frac{\mathbf{u}_{i,h}^{n+1} - \mathbf{u}_{i,h}^n}{\varDelta t}, \mathbf{v}_{i,h}\right) + \exp\left(\frac{t_{n+1}}{T}\right)Q^{n+1} b_\omega\left(\bar{\mathbf{u}}_{i,h}^n, \mathbf{u}_{i,h}^n, \mathbf{v}_{i,h}\right)\\ &\quad+ \exp\left(\frac{t_{n+1}}{T}\right)Q^{n+1}\int_I \kappa\left|\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n\right|\left(\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n\right)\cdot\mathbf{v}_{i,h}\mathrm{d}s\\ &\quad+ \nu_i\left(\nabla\mathbf{u}_{i,h}^{n+1}, \nabla\mathbf{v}_{i,h}\right) - \left(\nabla\cdot\mathbf{v}_{i,h}, p_{i,h}^{n+1}\right) + \left(\nabla\cdot\mathbf{u}_{i,h}^{n+1}, q_{i,h}\right)\\ &= \left(\mathbf{f}_i(t_{n+1}), \mathbf{v}_{i,h}\right),\\ &\delta_i^2\left(\nabla\bar{\mathbf{u}}_{i,h}^n, \nabla\bar{\mathbf{v}}_{i,h}\right) + \left(\bar{\mathbf{u}}_{i,h}^n, \bar{\mathbf{v}}_{i,h}\right) = \left(\mathbf{u}_{i,h}^n, \bar{\mathbf{v}}_{i,h}\right),\end{aligned} \tag{7}$$

and

$$
\begin{aligned}
\frac{Q^{n+1}-Q^n}{\Delta t} &= -\frac{1}{T}Q^{n+1} + \exp\left(\frac{t_{n+1}}{T}\right)\sum_{i=1}^{2} b_\omega\left(\bar{\mathbf{u}}_{i,h}^n, \mathbf{u}_{i,h}^n, \mathbf{u}_{i,h}^{n+1}\right) \\
&\quad + \exp\left(\frac{t_{n+1}}{T}\right)\sum_{i=1}^{2}\int_I \kappa\left|\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n\right|\left(\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n\right)\cdot \mathbf{u}_{i,h}^{n+1}\mathrm{d}s \\
&\quad + \exp\left(\frac{2t_{n+1}}{T}\right)Q^{n+1}\sum_{i=1}^{2}\int_I \kappa\left|\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n\right|\left(\mathbf{u}_{j,h}^n - \mathbf{u}_{i,h}^n\right)\cdot \mathbf{u}_{i,h}^{n}\mathrm{d}s,
\end{aligned} \tag{8}
$$

where we pick $Q^0 = \exp(0) = 1$.

In last part of this section, the stability of the fully discrete finite element scheme (7)-(8) is explored.

Theorem 3.1. *Let $\mathbf{u}_{i,h}^{n+1}$ and Q^{n+1} be the solution of the scheme (7)-(8). If we assume that the initial values and body forces have the following bound $\|\mathbf{u}_{i,h}^0\|_0 + \|\mathbf{f}_i^{n+1}\|_{-1} \leqslant c$, then the fully discrete scheme (7)-(8) is unconditionally stable, and the solution has the following bound*

$$
\begin{aligned}
&\sum_{i=1}^{2}\left(\left\|\mathbf{u}_{i,h}^N\right\|_0^2 + \sum_{n=0}^{N-1}\left\|\mathbf{u}_{i,h}^{n+1} - \mathbf{u}_{i,h}^n\right\|_0^2 + \nu_i \Delta t\sum_{n=0}^{N-1}\left\|\nabla \mathbf{u}_{i,h}^{n+1}\right\|_0^2\right) \\
&\quad + \left|Q^N\right|^2 + \sum_{n=0}^{N-1}\left|Q^{n+1}-Q^n\right|^2 + \frac{2\Delta t}{T}\sum_{n=0}^{N-1}\left|Q^{n+1}\right|^2 \\
&\quad + 2\Delta t\sum_{n=0}^{N-1}\exp\left(\frac{2t_{n+1}}{T}\right)\left|Q^{n+1}\right|^2\int_I \kappa\left|\mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^n\right|^3 ds \\
&\leqslant \sum_{i=1}^{2}\left\|\mathbf{u}_{i,h}^0\right\|_0^2 + \left|Q^0\right|^2 + \sum_{i=1}^{2}\Delta t \nu_i^{-1}\sum_{n=0}^{N-1}\left\|\mathbf{f}_i^{n+1}\right\|_{-1}^2 =: c_2.
\end{aligned}
$$

Proof. Choosing $(\mathbf{v}_{i,h}, q_{i,h}) = 2\Delta t\left(\mathbf{u}_{i,h}^{n+1}, p_{i,h}^{n+1}\right)$ in (7), then using the polarization identity and finally adding up the ensuing equation from $i=1$ to 2, we arrive at

$$
\begin{aligned}
&\sum_{i=1}^{2}\left(\left\|\mathbf{u}_{i,h}^{n+1}\right\|_0^2 - \left\|\mathbf{u}_{i,h}^n\right\|_0^2 + \left\|\mathbf{u}_{i,h}^{n+1} - \mathbf{u}_{i,h}^n\right\|_0^2 + 2\nu_i\Delta t\left\|\nabla \mathbf{u}_{i,h}^{n+1}\right\|_0^2\right) \\
&\quad + 2\Delta t\exp\left(\frac{t_{n+1}}{T}\right)Q^{n+1}\sum_{i=1}^{2} b_\omega\left(\bar{\mathbf{u}}_{i,h}^n, \mathbf{u}_{i,h}^n, \mathbf{u}_{i,h}^{n+1}\right) \\
&\quad + 2\Delta t\exp\left(\frac{t_{n+1}}{T}\right)Q^{n+1}\sum_{i=1}^{2}\int_I \kappa\left|\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n\right|\left(\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n\right)\cdot \mathbf{u}_{i,h}^{n+1}\mathrm{d}s \\
&= 2\Delta t\sum_{i=1}^{2}\left(\mathbf{f}_i^{n+1}, \mathbf{u}_{i,h}^{n+1}\right).
\end{aligned} \tag{9}
$$

In (9), we remark that $j=2$ for $i=1$ and $j=1$ for $i=2$. Next, multiply (8) by $2\Delta t Q^{n+1}$ to get

$$\begin{aligned}
&\left|Q^{n+1}\right|^2-\left|Q^n\right|^2+\left|Q^{n+1}-Q^n\right|^2+\frac{2\Delta t}{T}\left|Q^{n+1}\right|^2\\
=&\,2\Delta t\exp\left(\frac{t_{n+1}}{T}\right)Q^{n+1}\sum_{i=1}^{2}b_\omega\left(\bar{\mathbf{u}}_{i,h}^n,\mathbf{u}_{i,h}^n,\mathbf{u}_{i,h}^{n+1}\right)\\
&+2\Delta t\exp\left(\frac{t_{n+1}}{T}\right)Q^{n+1}\sum_{i=1}^{2}\int_I\kappa\left|\mathbf{u}_{i,h}^n-\mathbf{u}_{j,h}^n\right|\left(\mathbf{u}_{i,h}^n-\mathbf{u}_{j,h}^n\right)\cdot\mathbf{u}_{i,h}^{n+1}\mathrm{d}s\\
&+2\Delta t\exp\left(\frac{2t_{n+1}}{T}\right)\left|Q^{n+1}\right|^2\sum_{i=1}^{2}\int_I\kappa\left|\mathbf{u}_{i,h}^n-\mathbf{u}_{j,h}^n\right|\left(\mathbf{u}_{j,h}^n-\mathbf{u}_{i,h}^n\right)\cdot\mathbf{u}_{i,h}^{n}\mathrm{d}s.
\end{aligned}\tag{10}$$

Now, combining (9) with (10) and utilizing the Young inequality lead to

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\left\|\mathbf{u}_{i,h}^{n+1}\right\|_0^2-\left\|\mathbf{u}_{i,h}^{n}\right\|_0^2+\left\|\mathbf{u}_{i,h}^{n+1}-\mathbf{u}_{i,h}^{n}\right\|_0^2+\nu_i\Delta t\left\|\nabla\mathbf{u}_{i,h}^{n+1}\right\|_0^2\right)\\
&\quad+\left|Q^{n+1}\right|^2-\left|Q^n\right|^2+\left|Q^{n+1}-Q^n\right|^2+\frac{2\Delta t}{T}\left|Q^{n+1}\right|^2\\
&\quad+2\Delta t\exp\left(\frac{2t_{n+1}}{T}\right)\left|Q^{n+1}\right|^2\int_I\kappa\left|\mathbf{u}_{1,h}^n-\mathbf{u}_{2,h}^n\right|^3ds\\
&\leqslant\sum_{i=1}^{2}\Delta t\nu_i^{-1}\left\|\mathbf{f}_i^{n+1}\right\|_{-1}^2.
\end{aligned}\tag{11}$$

Summing (11) with respect to n from 0 to $N-1$ and applying the assumptions on the body forces and initial values, we arrive at the desired result. □

4. Error analysis

In this section, we mainly analyze the fully discrete error between the numerical solution of the proposed scheme (7)-(8) and the true solution of the original problem (1), i.e., the error coming from discretization of the Navier-Stokes-ω/Navier-Stokes-ω equations (2) for approximating the fluid-fluid interaction problem (1).

Here, for $i=1,2$, we denote $\mathbf{e}_i^n=\mathbf{u}_i(t_n)-\mathbf{u}_{i,h}^n$, $e_{i,p}^n=p_i(t_n)-p_{i,h}^n$ and $e_Q^n=Q(t_n)-Q^n$ as the velocity, pressure and auxiliary variable errors, respectively. Then, the velocity and pressure errors are decomposed as follows:

$$\mathbf{e}_i^n=\boldsymbol{\eta}_i^n+\boldsymbol{\phi}_i^n,\quad e_{i,p}^n=\theta_i^n+\zeta_i^n,$$

where $\boldsymbol{\eta}_i^n=\mathbf{u}_i(t_n)-I_h\mathbf{u}_i$, $\boldsymbol{\phi}_i^n=I_h\mathbf{u}_i-\mathbf{u}_{i,h}^n$ and $\theta_i^n=p_i(t_n)-\rho_hp_i$, $\zeta_i^n=\rho_hp_i-p_{i,h}^n$ with the Stokes-Stokes projection [34]: find $(I_h\mathbf{u}_i,\rho_hp_i)\in\mathbf{X}_i^h\times M_i^h$ such that

$$\begin{aligned}
\nu_i\left(\nabla\left(\mathbf{u}_i-I_h\mathbf{u}_i\right),\nabla\mathbf{v}_i\right)-\left(\nabla\cdot\mathbf{v}_i,p_i-\rho_hp_i\right)&=0,\ \forall\mathbf{v}_i\in\mathbf{X}_i^h,\\
\left(\nabla\cdot\left(\mathbf{u}_i-I_h\mathbf{u}_i\right),q_i\right)&=0,\ \forall q_i\in M_i^h,
\end{aligned}\tag{12}$$

whose approximation properties are listed as follows:

$$\left\|\mathbf{u}_i-I_h\mathbf{u}_i\right\|_0+h\left(\left\|\nabla\left(\mathbf{u}_i-I_h\mathbf{u}_i\right)\right\|_0+\left\|p_i-\rho_hp_i\right\|_0\right)\leqslant ch^2\left(\left\|\mathbf{u}_i\right\|_{W^{2,2}}+\left\|p_i\right\|_{W^{1,2}}\right).\tag{13}$$

Moreover, in order to obtain the error equation, setting $(\mathbf{v}_i,q_i)=(\mathbf{v}_{i,h},q_{i,h})$ in (6) with $t=t_{n+1}$ and replacing the trilinear term $b_\omega\left(\bar{\mathbf{u}}_i(t_{n+1}),\mathbf{u}_i(t_{n+1}),\mathbf{v}_{i,h}\right)$ by

$b_\omega(\mathbf{u}_i(t_{n+1}), \mathbf{u}_i(t_{n+1}), \mathbf{v}_{i,h})$, we get

(14)
$$\begin{aligned}&\left(\frac{\mathbf{u}_i(t_{n+1})-\mathbf{u}_i(t_n)}{\Delta t}, \mathbf{v}_{i,h}\right)+\nu_i(\nabla \mathbf{u}_i(t_{n+1}), \nabla \mathbf{v}_{i,h})-(\nabla\cdot \mathbf{v}_{i,h}, p_i(t_{n+1}))\\
&+(\nabla\cdot \mathbf{u}_i(t_{n+1}), q_{i,h})+\exp\left(\frac{t_{n+1}}{T}\right)Q(t_{n+1})\, b_\omega(\mathbf{u}_i(t_{n+1}), \mathbf{u}_i(t_{n+1}), \mathbf{v}_{i,h})\\
&+\exp\left(\frac{t_{n+1}}{T}\right)Q(t_{n+1})\int_I \kappa\left|\mathbf{u}_i(t_{n+1})-\mathbf{u}_j(t_{n+1})\right|(\mathbf{u}_i(t_{n+1})-\mathbf{u}_j(t_{n+1}))\cdot \mathbf{v}_{i,h}\mathrm{d}s\\
&=\left(\mathbf{f}_i^{n+1}, \mathbf{v}_{i,h}\right)+\left(\frac{\mathbf{u}_i(t_{n+1})-\mathbf{u}_i(t_n)}{\Delta t}-\mathbf{u}_{i,t}(t_{n+1}), \mathbf{v}_{i,h}\right).\end{aligned}$$

Note that the pressure in the proposed scheme (7) is the dynamic pressure, rather than the usual pressure in the original problem (1). Hence, in order to gain the reasonable error equation, we modify the usual pressure in the original problem (1) as the dynamic pressure by rewriting the nonlinearity in the rotational form, and obtain (14) as the continuous equation.

Next, subtracting (7) from (14), we give the error equation

(15)
$$\begin{aligned}&\left(\frac{\mathbf{e}_i^{n+1}-\mathbf{e}_i^n}{\Delta t}, \mathbf{v}_{i,h}\right)+\nu_i\left(\nabla \mathbf{e}_i^{n+1}, \nabla \mathbf{v}_{i,h}\right)\\
&+\exp\left(\frac{t_{n+1}}{T}\right)Q(t_{n+1})\, b_\omega(\mathbf{u}_i(t_{n+1}), \mathbf{u}_i(t_{n+1}), \mathbf{v}_{i,h})\\
&-\exp\left(\frac{t_{n+1}}{T}\right)Q^{n+1} b_\omega\left(\bar{\mathbf{u}}_{i,h}^n, \mathbf{u}_{i,h}^n, \mathbf{v}_{i,h}\right)-\left(\nabla\cdot \mathbf{v}_{i,h}, e_{i,p}^{n+1}\right)+\left(\nabla\cdot \mathbf{e}_i^{n+1}, q_{i,h}\right)\\
&+\exp\left(\frac{t_{n+1}}{T}\right)Q(t_{n+1})\int_I \kappa\left|\mathbf{u}_i(t_{n+1})-\mathbf{u}_j(t_{n+1})\right|(\mathbf{u}_i(t_{n+1})-\mathbf{u}_j(t_{n+1}))\cdot \mathbf{v}_{i,h}\mathrm{d}s\\
&-\exp\left(\frac{t_{n+1}}{T}\right)Q^{n+1}\int_I \kappa\left|\mathbf{u}_{i,h}^n-\mathbf{u}_{j,h}^n\right|\left(\mathbf{u}_{i,h}^n-\mathbf{u}_{j,h}^n\right)\cdot \mathbf{v}_{i,h}\mathrm{d}s=\left(\mathbf{T}_i^{n+1}, \mathbf{v}_{i,h}\right),\end{aligned}$$

where

$$\mathbf{T}_i^{n+1}=\frac{\mathbf{u}_i(t_{n+1})-\mathbf{u}_i(t_n)}{\Delta t}-\mathbf{u}_{i,t}(t_{n+1})=\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(t_n-t)\mathbf{u}_{i,tt}\mathrm{d}t.$$

Now, we derive the main error estimate result in following theorem.

Theorem 4.1. *Under the assumptions of Theorem 3.1, if Δt, h and δ_i satisfy the following bound*

(16)
$$\begin{aligned}&c_5\left(\nu_i^{-1}\left(1+\kappa^2\right)+c_4\kappa^2\right)c_3\exp(TN_1)\left(\Delta t+h+\delta_i^2\right)\left(\Delta t^{\frac{1}{2}}\left(\nu_i^{-\frac{1}{2}}+\nu_j^{-\frac{1}{2}}\right)+c_1\right)\\
&\leqslant \min\left\{\frac{\nu_i}{6}, \frac{\nu_j}{6}\right\},\end{aligned}$$

where N_1 is defined in (36) and $c_3=c_3(\nu_1,\nu_2,\kappa,c_2,T)$, $c_4=\left(\nu_i^{-1}+\nu_j^{-1}\right)c_2$ and $c_5=c_5(c_2)$, and the true solution of (1) is smooth, then we have

$$\begin{aligned}&\sum_{i=1}^2\left\|\mathbf{u}_i(t_N)-\mathbf{u}_{i,h}^N\right\|_0^2+\Delta t\sum_{i=1}^2\sum_{n=0}^{N-1}\nu_i\left\|\nabla\left(\mathbf{u}_i(t_{n+1})-\mathbf{u}_{i,h}^{n+1}\right)\right\|_0^2\\
&\quad+\left|Q(t_N)-Q^N\right|^2+\sum_{n=0}^{N-1}\frac{\Delta t}{T}\left|Q(t_{n+1})-Q^{n+1}\right|^2\leqslant c\left(\Delta t^2+h^2+\delta_i^4\right).\end{aligned}$$

Proof. Selecting $(\mathbf{v}_{i,h}, q_{i,h}) = 2\Delta t\left(\boldsymbol{\phi}_i^{n+1}, \zeta_i^{n+1}\right)$ in (15), we get

$$\begin{aligned}
&\left\|\boldsymbol{\phi}_i^{n+1}\right\|_0^2 - \left\|\boldsymbol{\phi}_i^{n}\right\|_0^2 + \left\|\boldsymbol{\phi}_i^{n+1} - \boldsymbol{\phi}_i^{n}\right\|_0^2 + 2\left(\boldsymbol{\eta}_i^{n+1} - \boldsymbol{\eta}_i^{n}, \boldsymbol{\phi}_i^{n+1}\right) \\
&\quad + 2\Delta t b_\omega\left(\mathbf{u}_i(t_{n+1}), \mathbf{u}_i(t_{n+1}), \boldsymbol{\phi}_i^{n+1}\right) + 2\Delta t \nu_i \left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2 \\
&\quad - 2\exp\left(\frac{t_{n+1}}{T}\right) Q^{n+1} \Delta t \int_I \kappa \left|\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n\right| \left(\mathbf{u}_{i,h}^n - \mathbf{u}_{j,h}^n\right) \cdot \boldsymbol{\phi}_i^{n+1} \mathrm{d}s \\
&\quad + 2\Delta t \int_I \kappa \left|\mathbf{u}_i(t_{n+1}) - \mathbf{u}_j(t_{n+1})\right| \left(\mathbf{u}_i(t_{n+1}) - \mathbf{u}_j(t_{n+1})\right) \cdot \boldsymbol{\phi}_i^{n+1} \mathrm{d}s \\
&\quad - 2\exp\left(\frac{t_{n+1}}{T}\right) Q^{n+1} \Delta t b_\omega\left(\bar{\mathbf{u}}_{i,h}^n, \mathbf{u}_{i,h}^n, \boldsymbol{\phi}_i^{n+1}\right) = 2\Delta t\left(\mathbf{T}_i^{n+1}, \boldsymbol{\phi}_i^{n+1}\right).
\end{aligned} \tag{17}$$

Then, set $[\mathbf{u}(t_{n+1})] = \mathbf{u}_i(t_{n+1}) - \mathbf{u}_j(t_{n+1})$, $\left[\mathbf{u}_h^{n+1}\right] = \mathbf{u}_{i,h}^{n+1} - \mathbf{u}_{j,h}^{n+1}$ and $[I_h\mathbf{u}(t_{n+1})] = I_h\mathbf{u}_i(t_{n+1}) - I_h\mathbf{u}_j(t_{n+1})$. The interface terms in (17) are rewritten as follows:

$$\begin{aligned}
\text{interface term} = {} & 2\Delta t \int_I \kappa \left(\left|[\mathbf{u}(t_{n+1})]\right| [\mathbf{u}(t_{n+1})] - \left|[\mathbf{u}(t_n)]\right| [\mathbf{u}(t_n)]\right) \cdot \boldsymbol{\phi}_i^{n+1} \mathrm{d}s \\
& + 2\Delta t \int_I \kappa \left(\left|[\mathbf{u}(t_n)]\right| [\mathbf{u}(t_n)] - \left|[\mathbf{u}_h^n]\right| [\mathbf{u}_h^n]\right) \cdot \boldsymbol{\phi}_i^{n+1} \mathrm{d}s \\
& + 2\Delta t \exp\left(\frac{t_{n+1}}{T}\right) e_Q^{n+1} \int_I \kappa \left|[\mathbf{u}_h^n]\right| [\mathbf{u}_h^n] \cdot \boldsymbol{\phi}_i^{n+1} \mathrm{d}s =: \sum_{k=1}^{3} A_k.
\end{aligned} \tag{18}$$

Using the Hölder inequality, Young inequality and Lemma 2.2, we arrive at

$$\begin{aligned}
& A_1 + A_2 \\
\leqslant & \varepsilon_1 \nu_i \Delta t \left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2 + \varepsilon_2 \nu_i \Delta t \left\|\nabla \boldsymbol{\phi}_{i,u}^{n}\right\|_0^2 + \varepsilon_3 \nu_j \Delta t \left\|\nabla \boldsymbol{\phi}_{j,u}^{n}\right\|_0^2 \\
& + c\nu_i^{-1} \kappa^2 \Delta t^2 \left\|[\mathbf{u}_t]\right\|_{L^2\left(t_n, t_{n+1}, W^{1,2}(\Omega_i)^d\right)}^2 \left(\left\|\nabla [\mathbf{u}(t_{n+1})]\right\|_0^2 + \left\|\nabla [\mathbf{u}(t_n)]\right\|_0^2\right) \\
& + c\nu_i^{-1} \kappa^2 \Delta t \left\|\nabla [\boldsymbol{\eta}_u^n]\right\|_0^2 \left(\left\|\nabla [\mathbf{u}(t_n)]\right\|_0^2 + \left\|\nabla [\mathbf{u}_h^n]\right\|_0^2\right) \\
& + c\kappa^4 \left(\nu_i^{-3} + \nu_i^{-2} \nu_j^{-1}\right) \Delta t \left\|[\boldsymbol{\phi}_u^n]\right\|_0^2 \left(\left\|\nabla [\mathbf{u}(t_n)]\right\|_0^4 + \left\|\nabla [I_h\mathbf{u}(t_n)]\right\|_0^4\right) \\
& + c\nu_i^{-1} \kappa^2 \Delta t \left\|[\boldsymbol{\phi}_u^n]\right\|_0 \left\|\nabla [\boldsymbol{\phi}_u^n]\right\|_0 \left\|\nabla [\boldsymbol{\phi}_u^n]\right\|_0^2.
\end{aligned} \tag{19}$$

Moreover, we rewrite the nonlinear terms in (17).

$$\begin{aligned}
& \text{nonlinear term} \\
= {} & 2\Delta t b_\omega\left(\mathbf{u}_i(t_{n+1}), \mathbf{u}_i(t_{n+1}) - \mathbf{u}_i(t_n), \boldsymbol{\phi}_i^{n+1}\right) \\
& + 2\Delta t b_\omega\left(\mathbf{u}_i(t_{n+1}), \boldsymbol{\eta}_i^n + \boldsymbol{\phi}_i^n, \boldsymbol{\phi}_i^{n+1}\right) + 2\Delta t b_\omega\left(\mathbf{u}_i(t_{n+1}) - \mathbf{u}_i(t_n), \mathbf{u}_{i,h}^n, \boldsymbol{\phi}_i^{n+1}\right) \\
& + 2\Delta t b_\omega\left(\mathbf{u}_i(t_n) - \bar{\mathbf{u}}_i(t_n), \mathbf{u}_{i,h}^n, \boldsymbol{\phi}_i^{n+1}\right) + 2\Delta t b_\omega\left(\bar{\mathbf{u}}_i(t_n) - \bar{\mathbf{u}}_{i,h}^n, \mathbf{u}_{i,h}^n, \boldsymbol{\phi}_i^{n+1}\right) \\
& + 2\Delta t \exp\left(\frac{t_{n+1}}{T}\right) e_Q^{n+1} b_\omega\left(\bar{\mathbf{u}}_{i,h}^n, \mathbf{u}_{i,h}^n, \boldsymbol{\phi}_i^{n+1}\right) =: \sum_{k=1}^{6} B_k.
\end{aligned} \tag{20}$$

Then, based on Lemma 2.1 and the Cauchy-Schwarz inequality, the Young inequality, each B_k except B_6 is estimated

(21)

$$
\begin{aligned}
B_1 \leqslant & \varepsilon_4 \nu_i \Delta t \left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2+c \nu_i^{-1} \Delta t^2\left\|\nabla \mathbf{u}_{i, t}\right\|_{L^2\left(t_n, t_{n+1}, W^{1,2}\left(\Omega_i\right)^d\right)}^2\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2, \\
B_2 \leqslant & \varepsilon_5 \nu_i \Delta t\left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2+\varepsilon_6 \nu_i \Delta t\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2+c \nu_i^{-1} \Delta t\left\|\nabla \boldsymbol{\eta}_i^n\right\|_0^2\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2 \\
& +c \nu_i^{-3} \Delta t\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^4\left\|\boldsymbol{\phi}_i^n\right\|_0^2, \\
B_3 \leqslant & \varepsilon_7 \nu_i \Delta t\left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2+\varepsilon_8 \nu_i \Delta t\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2 \\
& +c \nu_i^{-1} \Delta t^2\left\|\nabla \mathbf{u}_{i, t}\right\|_{L^2\left(t_n, t_{n+1}, W^{1,2}\left(\Omega_i\right)^d\right)}^2\left\|\nabla I_h \mathbf{u}_i\left(t_n\right)\right\|_0^2 \\
& +c \nu_i^{-3} \Delta t\left(\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^4+\left\|\nabla \mathbf{u}_i\left(t_n\right)\right\|_0^4\right)\left\|\boldsymbol{\phi}_i^n\right\|_0^2, \\
B_4 \leqslant & \varepsilon_9 \nu_i \Delta t\left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2+c \nu_i^{-1} \Delta t \delta_i^4\left\|\nabla \Delta \overline{\mathbf{u}}_i\left(t_n\right)\right\|_0^2\left\|\nabla \mathbf{u}_{i, h}^n\right\|_0^2, \\
B_5 \leqslant & \varepsilon_{10} \nu_i \Delta t\left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2+\varepsilon_{11} \nu_i \Delta t\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2+c \nu_i^{-1} \Delta t\left\|\nabla \boldsymbol{\eta}_i^n\right\|_0^2\left\|\nabla \mathbf{u}_{i, h}^n\right\|_0^2 \\
& +c \nu_i^{-3} \Delta t\left\|\nabla I_h \mathbf{u}_i\left(t_n\right)\right\|_0^4\left\|\boldsymbol{\phi}_i^n\right\|_0^2+c \nu_i^{-1} \Delta t\left\|\boldsymbol{\phi}_i^n\right\|_0\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2,
\end{aligned}
$$

where for B_4 we have noticed the fact that $\mathbf{u}_i(t_n)-\overline{\mathbf{u}}_i(t_n)=-\delta_i^2 \Delta \overline{\mathbf{u}}_i(t_n)$ from (2). Next, applying the Young inequality for the remaining terms of (17) yields

$$
\begin{aligned}
& 2\left(\boldsymbol{\eta}_i^{n+1}-\boldsymbol{\eta}_i^n, \boldsymbol{\phi}_i^{n+1}\right) \leqslant \varepsilon_{12} \nu_i \Delta t\left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2+c \nu_i^{-1}\left\|\boldsymbol{\eta}_{i, t}\right\|_{L^2\left(t_n, t_{n+1}, \mathbf{X}_i^{\prime}\right)}^2, \\
& 2 \Delta t\left(\mathbf{T}_i^{n+1}, \boldsymbol{\phi}_i^{n+1}\right) \leqslant \varepsilon_{13} \nu_i \Delta t\left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2+c \nu_i^{-1} \Delta t^2\left\|\mathbf{u}_{i, t t}\right\|_{L^2\left(t_n, t_{n+1}, \mathbf{X}_i^{\prime}\right)}^2 .
\end{aligned} \tag{22}
$$

Now, picking $\varepsilon_1+\varepsilon_4+\varepsilon_5+\varepsilon_7+\varepsilon_9+\varepsilon_{10}+\varepsilon_{12}+\varepsilon_{13}=\frac{2}{9}$ and $\varepsilon_2+\varepsilon_6+\varepsilon_8+\varepsilon_{11}=\frac{1}{9}$, we get

(23)

$$
\begin{aligned}
& \left\|\boldsymbol{\phi}_i^{n+1}\right\|_0^2-\left\|\boldsymbol{\phi}_i^n\right\|_0^2+\left\|\boldsymbol{\phi}_i^{n+1}-\boldsymbol{\phi}_i^n\right\|_0^2+\frac{5}{3} \Delta t \nu_i\left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2 \\
& \quad+\frac{1}{9} \Delta t \nu_i\left(\left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2-\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2\right)+A_3+B_6 \\
& \leqslant \varepsilon_3 \nu_j \Delta t\left\|\nabla \boldsymbol{\phi}_{j, u}^n\right\|_0^2+c \nu_i^{-1} \Delta t \delta_i^4\left\|\nabla \Delta \overline{\mathbf{u}}_i\left(t_n\right)\right\|_0^2\left\|\nabla \mathbf{u}_{i, h}^n\right\|_0^2+c \nu_i^{-1}\left\|\boldsymbol{\eta}_{i, t}\right\|_{L^2\left(t_n, t_{n+1}, \mathbf{X}_i^{\prime}\right)}^2 \\
& \quad+c \nu_i^{-1} \Delta t^2\left\|\mathbf{u}_{i, t t}\right\|_{L^2\left(t_n, t_{n+1}, \mathbf{X}_i^{\prime}\right)}^2+c \nu_i^{-1} \Delta t^2\left\|\nabla \mathbf{u}_{i, t}\right\|_{L^2\left(t_n, t_{n+1}, W^{1,2}\left(\Omega_i\right)^d\right)}^2 \\
& \quad \times\left(\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2+\left\|\nabla I_h \mathbf{u}_i\left(t_n\right)\right\|_0^2\right)+c \nu_i^{-1} \Delta t\left\|\nabla \boldsymbol{\eta}_i^n\right\|_0^2\left(\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\right. \\
& \quad\left.+\left\|\nabla \mathbf{u}_{i, h}^n\right\|_0^2\right)+c \nu_i^{-1} \kappa^2 \Delta t\left\|\nabla\left[\boldsymbol{\eta}_u^n\right]\right\|_0^2\left(\left\|\nabla\left[\mathbf{u}\left(t_n\right)\right]\right\|_0^2+\left\|\nabla\left[\mathbf{u}_h^n\right]\right\|_0^2\right) \\
& \quad+c \nu_i^{-1} \kappa^2 \Delta t^2\left\|\left[\mathbf{u}_t\right]\right\|_{L^2\left(t_n, t_{n+1}, W^{1,2}(\Omega)^d\right)}^2\left(\left\|\nabla\left[\mathbf{u}\left(t_{n+1}\right)\right]\right\|_0^2+\left\|\nabla\left[\mathbf{u}\left(t_n\right)\right]\right\|_0^2\right) \\
& \quad+c \kappa^4\left(\nu_i^{-3}+\nu_i^{-2} \nu_j^{-1}\right) \Delta t\left\|\left[\boldsymbol{\phi}_u^n\right]\right\|_0^2\left(\left\|\nabla\left[\mathbf{u}\left(t_n\right)\right]\right\|_0^4+\left\|\nabla\left[I_h \mathbf{u}\left(t_n\right)\right]\right\|_0^4\right) \\
& \quad+c \nu_i^{-1} \kappa^2 \Delta t\left\|\left[\boldsymbol{\phi}_u^n\right]\right\|_0\left\|\nabla\left[\boldsymbol{\phi}_u^n\right]\right\|_0\left\|\nabla\left[\boldsymbol{\phi}_u^n\right]\right\|_0^2+c \nu_i^{-1} \Delta t\left\|\boldsymbol{\phi}_i^n\right\|_0\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2 \\
& \quad+c \nu_i^{-3} \Delta t\left(2\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^4+\left\|\nabla I_h \mathbf{u}_i\left(t_n\right)\right\|_0^4+\left\|\nabla \mathbf{u}_i\left(t_n\right)\right\|_0^4\right)\left\|\boldsymbol{\phi}_i^n\right\|_0^2 .
\end{aligned}
$$

Setting $\varepsilon_3=\frac{1}{9}$ and adding up (23) from $i=1,2, i \neq j$ and $n=0,1,2, \cdots, m$ $(0 \leqslant m \leqslant N-1)$, and then using the projection properties (13), Theorem 3.1, we arrive at

(24)
$$
\begin{aligned}
&\sum_{i=1}^{2}\left\|\boldsymbol{\phi}_i^{m+1}\right\|_0^2+\sum_{i=1}^{2}\sum_{n=0}^{m}\left\|\boldsymbol{\phi}_i^{n+1}-\boldsymbol{\phi}_i^{n}\right\|_0^2+\frac{2}{9}\Delta t\sum_{i=1}^{2}\nu_i\left\|\nabla\boldsymbol{\phi}_i^{m+1}\right\|_0^2\\
&\quad+\frac{14}{9}\Delta t\sum_{i=1}^{2}\sum_{n=0}^{m}\nu_i\left\|\nabla\boldsymbol{\phi}_i^{n+1}\right\|_0^2+\sum_{i=1}^{2}\sum_{n=0}^{m}(A_3+B_6)\\
&\leqslant c_3\left(\Delta t^2+h^2+\delta_i^4\right)+c\Delta t\sum_{i=1}^{2}\sum_{n=0}^{m}\Big(\nu_i^{-3}\left\|\boldsymbol{\phi}_i^n\right\|_0^2+\kappa^4\left(\nu_i^{-3}+\nu_i^{-2}\nu_j^{-1}\right)\left\|[\boldsymbol{\phi}_u^n]\right\|_0^2\\
&\quad+\nu_i^{-1}\left\|\boldsymbol{\phi}_i^n\right\|_0\left\|\nabla\boldsymbol{\phi}_i^n\right\|_0\left\|\nabla\boldsymbol{\phi}_i^n\right\|_0^2+\nu_i^{-1}\kappa^2\left\|[\boldsymbol{\phi}_u^n]\right\|_0\left\|\nabla\left[\boldsymbol{\phi}_u^n\right]\right\|_0\left\|\nabla\left[\boldsymbol{\phi}_u^n\right]\right\|_0^2\Big).
\end{aligned}
$$

On the other hand, subtracting (8) from (5) at $t=t_{n+1}$, and then multiplying the ensuing equation by $2\Delta t e_Q^{n+1}$, we have

(25)
$$
\begin{aligned}
&\left|e_Q^{n+1}\right|^2-\left|e_Q^{n}\right|^2+\left|e_Q^{n+1}-e_Q^{n}\right|^2+2\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2\\
&=2\exp\left(\frac{t_{n+1}}{T}\right)\Delta t e_Q^{n+1}\sum_{i=1}^{2}\Big(b_\omega\left(\bar{\mathbf{u}}_i\left(t_{n+1}\right),\mathbf{u}_i\left(t_{n+1}\right),\mathbf{u}_i\left(t_{n+1}\right)\right)\\
&\quad-b_\omega(\bar{\mathbf{u}}_{i,h}^n,\mathbf{u}_{i,h}^n,\mathbf{u}_{i,h}^{n+1})\\
&\quad+\int_I\kappa\left|\left[\mathbf{u}\left(t_{n+1}\right)\right]\right|\left[\mathbf{u}\left(t_{n+1}\right)\right]\cdot\mathbf{u}_i\left(t_{n+1}\right)\mathrm{d}s-\int_I\kappa\left|[\mathbf{u}_h^n]\right|\left[\mathbf{u}_h^n\right]\cdot\mathbf{u}_{i,h}^{n+1}\mathrm{d}s\Big)\\
&\quad+2\exp\left(\frac{2t_{n+1}}{T}\right)\Delta t e_Q^{n+1}\sum_{i=1}^{2}\Big(Q^{n+1}\int_I\kappa\left|[\mathbf{u}_h^n]\right|\left[\mathbf{u}_h^n\right]\cdot\mathbf{u}_{i,h}^n\mathrm{d}s\\
&\quad-Q\left(t_{n+1}\right)\int_I\kappa\left|\left[\mathbf{u}\left(t_{n+1}\right)\right]\right|\left[\mathbf{u}\left(t_{n+1}\right)\right]\cdot\mathbf{u}_i\left(t_{n+1}\right)\mathrm{d}s\Big)+2\Delta t e_Q^{n+1}\mathbf{T}_Q^{n+1},
\end{aligned}
$$

where

$$
\mathbf{T}_Q^{n+1}=\frac{Q\left(t_{n+1}\right)-Q\left(t_n\right)}{\Delta t}-Q_t\left(t_{n+1}\right)=\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\left(t_n-t\right)Q_{tt}\mathrm{d}t.
$$

Note that the rotation version of the trilinear term holds

(26)
$$
\begin{aligned}
&2\exp\left(\frac{t_{n+1}}{T}\right)\Delta t e_Q^{n+1}\Big(b_\omega\left(\bar{\mathbf{u}}_i\left(t_{n+1}\right),\mathbf{u}_i\left(t_{n+1}\right),\mathbf{u}_i\left(t_{n+1}\right)\right)-b_\omega(\bar{\mathbf{u}}_{i,h}^n,\mathbf{u}_{i,h}^n,\mathbf{u}_{i,h}^{n+1})\Big)\\
=&2\exp\left(\frac{t_{n+1}}{T}\right)\Delta t e_Q^{n+1}\Big(b_\omega\left(\bar{\mathbf{u}}_i\left(t_{n+1}\right),\mathbf{u}_i\left(t_{n+1}\right),\boldsymbol{\eta}_i^{n+1}\right)\\
&+b_\omega\left(\bar{\mathbf{u}}_i\left(t_{n+1}\right),\mathbf{u}_i\left(t_{n+1}\right)-\mathbf{u}_{i,h}^n,I_h\mathbf{u}_i\left(t_{n+1}\right)\right)\\
&+b_\omega\left(\bar{\mathbf{u}}_i\left(t_{n+1}\right)-\bar{\mathbf{u}}_{i,h}^n,\mathbf{u}_{i,h}^n,I_h\mathbf{u}_i\left(t_{n+1}\right)\right)+b_\omega\left(\bar{\mathbf{u}}_{i,h}^n,\mathbf{u}_{i,h}^n,\boldsymbol{\phi}_i^{n+1}\right)\Big)=:\sum_{k=1}^{4}D_k.
\end{aligned}
$$

Then, based on Lemma 2.1, Theorem 3.1, (3) and the Young inequality and Poincaré inequality, we arrive at

(27)

$$\begin{aligned}
D_1 \leqslant& \varepsilon_{14}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2 + c\Delta t\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\left\|\nabla \boldsymbol{\eta}_i^{n+1}\right\|_0^2,\\
D_2 \leqslant& \varepsilon_{15}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2 + \varepsilon_{16}\nu_i\Delta t\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2\\
&+ c\Delta t^2\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\left\|\nabla \mathbf{u}_{i,t}\right\|_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\\
&+ c\Delta t\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\left\|\nabla \boldsymbol{\eta}_i^n\right\|_0^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\\
&+ c\nu_i^{-1}\Delta t\left\|\boldsymbol{\phi}_i^n\right\|_0^2\left\|\nabla \mathbf{u}_i\left(t_{n+1}\right)\right\|_0^4\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^4,\\
D_3 \leqslant& \frac{\Delta t}{16c_4}\left|e_Q^{n+1}\right|^2\left\|\nabla \mathbf{u}_{i,h}^n\right\|_0^2 + cc_4\Delta t^2\left\|\nabla \bar{\mathbf{u}}_{i,t}\right\|_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\\
&+ cc_4\Delta t\left\|\nabla \boldsymbol{\eta}_i^n\right\|_0^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2 + \varepsilon_{17}\nu_i\Delta t\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2\\
&+ cc_4^2\nu_i^{-1}\Delta t\left\|\boldsymbol{\phi}_i^n\right\|_0^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^4.
\end{aligned}$$

Additionally, we consider the interface terms in (25).

(28)

$$\begin{aligned}
&2\exp\left(\frac{t_{n+1}}{T}\right)\Delta t e_Q^{n+1}\int_I \kappa\left|\left[\mathbf{u}\left(t_{n+1}\right)\right]\right|\left[\mathbf{u}\left(t_{n+1}\right)\right]\cdot \mathbf{u}_i\left(t_{n+1}\right)\mathrm{d}s\\
&\quad- 2\exp\left(\frac{t_{n+1}}{T}\right)\Delta t e_Q^{n+1}\int_I \kappa\left|\left[\mathbf{u}_h^n\right]\right|\left[\mathbf{u}_h^n\right]\cdot \mathbf{u}_{i,h}^{n+1}\mathrm{d}s\\
&\leqslant 2\exp\left(\frac{t_{n+1}}{T}\right)\Delta t e_Q^{n+1}\left(\int_I \kappa\left|\left[\mathbf{u}\left(t_{n+1}\right)\right]\right|\left[\mathbf{u}\left(t_{n+1}\right)\right]\cdot \boldsymbol{\eta}_i^{n+1}\mathrm{d}s\right.\\
&\quad+ \int_I \kappa\left(\left|\left[\mathbf{u}\left(t_{n+1}\right)\right]\right| - \left|\left[\mathbf{u}_h^n\right]\right|\right)\left[\mathbf{u}\left(t_{n+1}\right)\right]\cdot I_h\mathbf{u}_i\left(t_{n+1}\right)\mathrm{d}s\\
&\quad\left.+ \int_I \kappa\left|\left[\mathbf{u}_h^n\right]\right|\left(\left[\mathbf{u}\left(t_{n+1}\right)\right] - \left[\mathbf{u}_h^n\right]\right)\cdot I_h\mathbf{u}_i\left(t_{n+1}\right)ds + \int_I \kappa\left|\left[\mathbf{u}_h^n\right]\right|\left[\mathbf{u}_h^n\right]\cdot \boldsymbol{\phi}_i^{n+1}ds\right)\\
&=: \sum_{k=1}^4 E_k.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and Young inequality, we get

(29)

$$\begin{aligned}
E_1 + E_2 \leqslant& \varepsilon_{18}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2 + \varepsilon_{19}\nu_i\Delta t\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2 + \varepsilon_{20}\nu_j\Delta t\left\|\nabla \boldsymbol{\phi}_j^n\right\|_0^2\\
&+ c\kappa^2\Delta t\left\|\nabla\left[\mathbf{u}\left(t_{n+1}\right)\right]\right\|_0^4\left\|\nabla \boldsymbol{\eta}_i^{n+1}\right\|_0^2\\
&+ c\kappa^2\Delta t\left\|\nabla\left[\boldsymbol{\eta}^n\right]\right\|_0^2\left\|\nabla\left[\mathbf{u}\left(t_{n+1}\right)\right]\right\|_0^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\\
&+ c\kappa^2\Delta t^2\left\|\left[\mathbf{u}_t\right]\right\|_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}^2\left\|\nabla\left[\mathbf{u}\left(t_{n+1}\right)\right]\right\|_0^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\\
&+ c\kappa^4\left(\nu_i^{-1} + \nu_j^{-1}\right)\Delta t\left\|\left[\boldsymbol{\phi}^n\right]\right\|_0^2\left\|\nabla\left[\mathbf{u}\left(t_{n+1}\right)\right]\right\|_0^4\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2,\\
E_3 \leqslant& \frac{\Delta t}{32c_4}\left|e_Q^{n+1}\right|^2\left\|\nabla\left[\mathbf{u}_h^n\right]\right\|_0^2 + \varepsilon_{21}\nu_i\Delta t\left\|\nabla \boldsymbol{\phi}_i^n\right\|_0^2 + \varepsilon_{22}\nu_j\Delta t\left\|\nabla \boldsymbol{\phi}_j^n\right\|_0^2\\
&+ cc_4\kappa^2\Delta t\left\|\nabla\left[\boldsymbol{\eta}^n\right]\right\|_0^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\\
&+ cc_4\kappa^2\Delta t^2\left\|\left[\mathbf{u}_t\right]\right\|_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^2\\
&+ cc_4^2\kappa^4\left(\nu_i^{-1} + \nu_j^{-1}\right)\Delta t\left\|\left[\boldsymbol{\phi}^n\right]\right\|_0^2\left\|\nabla I_h\mathbf{u}_i\left(t_{n+1}\right)\right\|_0^4.
\end{aligned}$$

Besides, rewrite the last two interface terms in (25).

(30)

$$
\begin{aligned}
&2\exp\left(\frac{2t_{n+1}}{T}\right)\Delta t e_Q^{n+1}\left(Q^{n+1}\int_I \kappa\left|[\mathbf{u}_h^n]\right|[\mathbf{u}_h^n]\cdot\mathbf{u}_{i,h}^n\mathrm{d}s\right.\\
&\quad\left.-Q(t_{n+1})\int_I\kappa\left|[\mathbf{u}(t_{n+1})]\right|[\mathbf{u}(t_{n+1})]\cdot\mathbf{u}_i(t_{n+1})\mathrm{d}s\right)\\
=&2\exp\left(\frac{2t_{n+1}}{T}\right)Q^{n+1}\Delta t e_Q^{n+1}\left(\int_I\kappa\left|[\mathbf{u}_h^n]\right|[\mathbf{u}_h^n]\cdot\left(\mathbf{u}_{i,h}^n-\mathbf{u}_i(t_{n+1})\right)\mathrm{d}s\right.\\
&\quad+\int_I\kappa\left|[\mathbf{u}_h^n]\right|\left([\mathbf{u}_h^n]-[\mathbf{u}(t_{n+1})]\right)\cdot\mathbf{u}_i(t_{n+1})\mathrm{d}s\\
&\quad\left.+\int_I\kappa\left(\left|[\mathbf{u}_h^n]\right|-\left|[\mathbf{u}(t_{n+1})]\right|\right)[\mathbf{u}(t_{n+1})]\cdot\mathbf{u}_i(t_{n+1})\mathrm{d}s\right)\\
&\quad-2\exp\left(\frac{2t_{n+1}}{T}\right)\Delta t\left|e_Q^{n+1}\right|^2\int_I\kappa\left|[\mathbf{u}(t_{n+1})]\right|[\mathbf{u}(t_{n+1})]\cdot\mathbf{u}_i(t_{n+1})\mathrm{d}s=:\sum_{k=1}^4F_k.
\end{aligned}
$$

Then, using the Young inequality and Lemma 2.3, we arrive at

(31)

$$
\begin{aligned}
F_1+F_2\leqslant&\frac{\Delta t}{16c_4}\left|e_Q^{n+1}\right|^2\|\nabla[\mathbf{u}_h^n]\|_0^2+\varepsilon_{23}\nu_i\Delta t\|\nabla\boldsymbol{\phi}_i^n\|_0^2+\varepsilon_{24}\nu_j\Delta t\left\|\nabla\boldsymbol{\phi}_j^n\right\|_0^2\\
&+cc_4\left|Q^{n+1}\right|^2\kappa^2\Delta t\Big(\|\nabla[\boldsymbol{\eta}^n]\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2+\|\nabla\boldsymbol{\eta}_i^n\|_0^2\|\nabla[\mathbf{u}_h^n]\|_0^2\\
&+\Delta t\|\mathbf{u}_{i,t}\|^2_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}\|\nabla[I_h\mathbf{u}(t_n)]\|_0^2\\
&+\Delta t\|[\mathbf{u}_t]\|^2_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2+\|\boldsymbol{\phi}_i^n\|_0\|\nabla\boldsymbol{\phi}_i^n\|_0\|\nabla[\boldsymbol{\phi}^n]\|_0^2\Big)\\
&+cc_4^2\left|Q^{n+1}\right|^4\kappa^4\left(\nu_i^{-1}+\nu_j^{-1}\right)\Delta t\Big(\|\boldsymbol{\phi}_i^n\|_0^2\|\nabla[I_h\mathbf{u}(t_n)]\|_0^4\\
&+\Delta t^2\|[\boldsymbol{\phi}^n]\|_0^2\|\mathbf{u}_{i,t}\|^4_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}+\|[\boldsymbol{\phi}^n]\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^4\Big),\\
F_3\leqslant&\varepsilon_{25}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2+\varepsilon_{26}\nu_i\Delta t\|\nabla\boldsymbol{\phi}_i^n\|_0^2+\varepsilon_{27}\nu_j\Delta t\left\|\nabla\boldsymbol{\phi}_j^n\right\|_0^2\\
&+c\left|Q^{n+1}\right|^2\kappa^2\Delta t\Big(\|\nabla[\boldsymbol{\eta}^n]\|_0^2\|\nabla[\mathbf{u}(t_{n+1})]\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2\\
&+\Delta t\|[\mathbf{u}_t]\|^2_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}\|\nabla[\mathbf{u}(t_{n+1})]\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2\\
&+c\left|Q^{n+1}\right|^4\kappa^4\left(\nu_i^{-1}+\nu_j^{-1}\right)\Delta t\|[\boldsymbol{\phi}^n]\|_0^2\|\nabla[\mathbf{u}(t_{n+1})]\|_0^4\|\nabla\mathbf{u}_i(t_{n+1})\|_0^4.
\end{aligned}
$$

Finally, to estimate the residual interaction term of (25), we apply the Cauchy-Schwarz inequality

$$
2\Delta t e_Q^{n+1}\mathbf{T}_Q^{n+1}\leqslant\varepsilon_{28}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2+c\Delta t^2\int_{t_n}^{t_{n+1}}|Q_{tt}|^2dt. \tag{32}
$$

Now, setting $\varepsilon_{14}+\varepsilon_{15}+\varepsilon_{18}+\varepsilon_{25}+\varepsilon_{28}=1$, $\varepsilon_{16}+\varepsilon_{17}+\varepsilon_{19}+\varepsilon_{21}+\varepsilon_{23}+\varepsilon_{26}=\frac{1}{9}$, $\varepsilon_{20}+\varepsilon_{22}+\varepsilon_{24}+\varepsilon_{27}=\frac{1}{9}$ and combining (25) with (26)-(32), we get

(33)

$$
\begin{aligned}
&\left|e_Q^{n+1}\right|^2-\left|e_Q^n\right|^2+\left|e_Q^{n+1}-e_Q^n\right|^2+\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2-F_4\\
\leqslant&\sum_{i=1}^2\left(\frac{\Delta t}{4c_4}\left|e_Q^{n+1}\right|^2\left(\left\|\nabla\mathbf{u}_{i,h}^n\right\|_0^2+\left\|\nabla\mathbf{u}_{j,h}^n\right\|_0^2\right)+c\Delta t\left(c_4\|\nabla\boldsymbol{\eta}_i^n\|_0^2\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^2\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\nabla\boldsymbol{\eta}_i^{n+1}\right\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2\\
&+\|\nabla\boldsymbol{\eta}_i^{n}\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^2\\
&+c\Delta t^2\|\nabla\mathbf{u}_{i,t}\|^2_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^2\\
&+c\kappa^2\Delta t\left(\|\nabla[\mathbf{u}(t_{n+1})]\|_0^4\left\|\nabla\boldsymbol{\eta}_i^{n+1}\right\|_0^2+\|\nabla[\boldsymbol{\eta}^n]\|_0^2\|\nabla[\mathbf{u}(t_{n+1})]\|_0^2\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^2\right)\\
&+c\kappa^2\Delta t^2\|[\mathbf{u}_t]\|^2_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}\left(\|\nabla[\mathbf{u}(t_{n+1})]\|_0^2+c_4\right)\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^2\\
&+cc_4\left|Q^{n+1}\right|^2\kappa^2\Delta t\left(\|\nabla[\boldsymbol{\eta}^n]\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2+\|\nabla\boldsymbol{\eta}_i^n\|_0^2\|\nabla[\mathbf{u}_h^n]\|_0^2\right)\\
&+cc_4\left|Q^{n+1}\right|^2\kappa^2\Delta t^2\|\mathbf{u}_{i,t}\|^2_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}\|\nabla[I_h\mathbf{u}(t_n)]\|_0^2\\
&+cc_4\Delta t^2\|\nabla\bar{\mathbf{u}}_{i,t}\|^2_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^2+D_4+E_4\\
&+c\left|Q^{n+1}\right|^2\kappa^2\Delta t\|\nabla[\boldsymbol{\eta}^n]\|_0^2\|\nabla[\mathbf{u}(t_{n+1})]\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2\\
&+c\left|Q^{n+1}\right|^2\kappa^2\Delta t^2\|[\mathbf{u}_t]\|^2_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}\left(\|\nabla[\mathbf{u}(t_{n+1})]\|_0^2+c_4\right)\|\nabla\mathbf{u}_i(t_{n+1})\|_0^2\\
&+cc_4\kappa^2\Delta t\|\nabla[\boldsymbol{\eta}^n]\|_0^2\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^2\Big)\Big)+c\Delta t^2\int_{t_n}^{t_{n+1}}|Q_{tt}|^2\,\mathrm{d}t\\
&+c\Delta t\sum_{i=1}^{2}\Big(\nu_i^{-1}\|\boldsymbol{\phi}_i^n\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^4\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^4\\
&+c_4^2\nu_i^{-1}\|\boldsymbol{\phi}_i^n\|_0^2\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^4\\
&+\kappa^4\left(\nu_i^{-1}+\nu_j^{-1}\right)\|[\boldsymbol{\phi}^n]\|_0^2\|\nabla I_h\mathbf{u}_i(t_{n+1})\|_0^4\left(c_4^2+\|[\mathbf{u}(t_{n+1})]\|_0^4\right)\\
&+c_4^2\left|Q^{n+1}\right|^4\kappa^4\left(\nu_i^{-1}+\nu_j^{-1}\right)\Big(\Delta t^2\|[\boldsymbol{\phi}^n]\|_0^2\|\mathbf{u}_{i,t}\|^4_{L^2\left(t_n,t_{n+1},W^{1,2}(\Omega_i)^d\right)}\\
&+\|[\boldsymbol{\phi}^n]\|_0^2\|\nabla\mathbf{u}_i(t_{n+1})\|_0^4\left(1+\|\nabla[\mathbf{u}(t_{n+1})]\|_0^4\right)+\|\boldsymbol{\phi}_i^n\|_0^2\|\nabla[I_h\mathbf{u}(t_n)]\|_0^4\Big)\Big)\\
&+\Delta t\sum_{i=1}^{2}\left(cc_4\left|Q^{n+1}\right|^2\kappa^2\|\boldsymbol{\phi}_i^n\|_0\|\nabla\boldsymbol{\phi}_i^n\|_0\|\nabla[\boldsymbol{\phi}^n]\|_0^2+\frac{1}{9}\nu_i\|\nabla\boldsymbol{\phi}_i^n\|_0^2+\frac{1}{9}\nu_j\left\|\nabla\boldsymbol{\phi}_j^n\right\|_0^2\right).
\end{aligned}
$$

Summing (33) from $n=0,1,\cdots,n^*$, applying the projection properties (13) and Theorem 3.1, and noticing that

$$
\frac{\Delta t}{4c_4}\sum_{i=1}^{2}\sum_{n=0}^{n^*+1}\left(\left|e_Q^{n+1}\right|^2\left(\left\|\nabla\mathbf{u}_{i,h}^n\right\|_0^2+\left\|\nabla\mathbf{u}_{j,h}^n\right\|_0^2\right)\right)\leqslant\frac{1}{2}\left|e_Q^{n^*+1}\right|^2,
$$

where we assume that $\left|e_Q^{n^*+1}\right|^2$ is the maximum among all $\left|e_Q^n\right|^2$, we discover that

(34)

$$
\begin{aligned}
&\frac{1}{2}\left|e_Q^{n^*+1}\right|^2+\sum_{n=0}^{n^*}\left|e_Q^{n+1}-e_Q^n\right|^2+\sum_{n=0}^{n^*}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2\\
\leqslant{}& c\Delta t\sum_{i=1}^{2}\sum_{n=0}^{n^*}\Big(\left(\nu_i^{-1}+c_4^2\nu_i^{-1}\right)\|\boldsymbol{\phi}_i^n\|_0^2+\kappa^4\left(\nu_i^{-1}+\nu_j^{-1}\right)\left(c_4^2+1\right)\|[\boldsymbol{\phi}^n]\|_0^2\\
&+c_4^2c_2^2\kappa^4\left(\nu_i^{-1}+\nu_j^{-1}\right)\left(\Delta t^2\|[\boldsymbol{\phi}^n]\|_0^2+\|[\boldsymbol{\phi}^n]\|_0^2+\|\boldsymbol{\phi}_i^n\|_0^2\right)\Big)+c_3\left(h^2+\Delta t^2\right)
\end{aligned}
$$

$$+\Delta t\sum_{i=1}^{2}\sum_{n=0}^{n^*}\left(cc_2c_4\kappa^2\left\|\boldsymbol{\phi}_i^n\right\|_0\left\|\nabla\boldsymbol{\phi}_i^n\right\|_0\left\|\nabla\left[\boldsymbol{\phi}^n\right]\right\|_0^2+\frac{1}{9}\nu_i\left\|\nabla\boldsymbol{\phi}_i^n\right\|_0^2+\frac{1}{9}\nu_j\left\|\nabla\boldsymbol{\phi}_j^n\right\|_0^2\right)$$
$$+\sum_{n=0}^{n^*}\sum_{i=1}^{2}(D_4+E_4).$$

Combine (24) with $m=n^*$ and (34) to get

(35)
$$\sum_{i=1}^{2}\left\|\boldsymbol{\phi}_i^{n^*+1}\right\|_0^2+\sum_{i=1}^{2}\sum_{n=0}^{n^*}\left\|\boldsymbol{\phi}_i^{n+1}-\boldsymbol{\phi}_i^n\right\|_0^2+\frac{4\Delta t}{3}\sum_{i=1}^{2}\sum_{n=0}^{n^*}\nu_i\left\|\nabla\boldsymbol{\phi}_i^{n+1}\right\|_0^2$$
$$+\frac{1}{2}\left|e_Q^{n^*+1}\right|^2+\sum_{n=0}^{n^*}\left|e_Q^{n+1}-e_Q^n\right|^2+\sum_{n=0}^{n^*}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2$$
$$\leqslant\Delta t\sum_{i=1}^{2}\sum_{n=0}^{n^*}N_1\left\|\boldsymbol{\phi}_i^n\right\|_0^2+\Delta t\sum_{i=1}^{2}\sum_{n=0}^{n^*}N_2^n\left(\left\|\nabla\boldsymbol{\phi}_i^n\right\|_0^2+\left\|\nabla\boldsymbol{\phi}_j^n\right\|_0^2\right)+c_3\left(h^2+\Delta t^2+\delta_i^4\right),$$

where

(36)
$$N_1=c\left(\nu_i^{-3}+\nu_i^{-1}+c_4^2\nu_i^{-1}+c_2^2c_4^2\left(\nu_i^{-1}+\nu_j^{-1}\right)\right)$$
$$+c\kappa^4\sum_{i=1,i\neq j}^{2}\left(\nu_i^{-1}\nu_j^{-2}+\nu_i^{-3}+\left(\nu_i^{-1}+\nu_j^{-1}\right)\left(1+c_4^2\left(c_2^2\left(\Delta t^2+1\right)+1\right)\right)\right),$$

and

$$N_2^n=c\left(\nu_i^{-1}+c_2c_4\kappa^2\right)\left\|\boldsymbol{\phi}_i^n\right\|_0\left\|\nabla\boldsymbol{\phi}_i^n\right\|_0+c\kappa^2\nu_i^{-1}\left\|\left[\boldsymbol{\phi}_u^n\right]\right\|_0\left\|\nabla\left[\boldsymbol{\phi}_u^n\right]\right\|_0.$$

Next, we will derive $\left|e_Q^{n^*+1}\right|\leqslant c\left(h+\Delta t+\delta_i^2\right)$. To do that, we need to prove the following bound by applying the inductive method

$$\sum_{i=1}^{2}\left\|\boldsymbol{\phi}_i^{n^*+1}\right\|_0^2+\sum_{i=1}^{2}\sum_{n=0}^{n^*}\left\|\boldsymbol{\phi}_i^{n+1}-\boldsymbol{\phi}_i^n\right\|_0^2+\Delta t\sum_{i=1}^{2}\sum_{n=0}^{n^*}\nu_i\left\|\nabla\boldsymbol{\phi}_i^{n+1}\right\|_0^2$$
$$+\frac{1}{2}\left|e_Q^{n^*+1}\right|^2+\sum_{n=0}^{n^*}\left|e_Q^{n+1}-e_Q^n\right|^2+\sum_{n=0}^{n^*}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2 \tag{37}$$
$$\leqslant c_3\exp\left(\Delta t\sum_{n=0}^{n^*}N_1\right)\left(\Delta t^2+h^2+\delta_i^4\right).$$

First, when $n^*=0$ in (35), we get

$$\sum_{i=1}^{2}\left\|\boldsymbol{\phi}_i^1\right\|_0^2+\sum_{i=1}^{2}\left\|\boldsymbol{\phi}_i^1-\boldsymbol{\phi}_i^0\right\|_0^2+\Delta t\sum_{i=1}^{2}\nu_i\left\|\nabla\boldsymbol{\phi}_i^1\right\|_0^2+\frac{1}{2}\left|e_Q^1\right|^2 \tag{38}$$
$$+\left|e_Q^1-e_Q^0\right|^2+\frac{\Delta t}{T}\left|e_Q^1\right|^2\leqslant c_3\left(\Delta t^2+h^2+\delta_i^4\right).$$

Second, we assume that (37) holds at the case of $k \leqslant n^* - 1$

$$\begin{aligned} &\sum_{i=1}^{2} \left\|\boldsymbol{\phi}_i^{k+1}\right\|_0^2 + \sum_{i=1}^{2}\sum_{n=0}^{k} \left\|\boldsymbol{\phi}_i^{n+1} - \boldsymbol{\phi}_i^{n}\right\|_0^2 + \Delta t \sum_{i=1}^{2}\sum_{n=0}^{k} \nu_i \left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2 \\ &\quad + \frac{1}{2}\left|e_Q^{k+1}\right|^2 + \sum_{n=0}^{k}\left|e_Q^{n+1} - e_Q^{n}\right|^2 + \sum_{n=0}^{k}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2 \\ &\leqslant c_3 \exp\left(\Delta t \sum_{n=0}^{k} N_1\right)\left(\Delta t^2 + h^2 + \delta_i^4\right), \end{aligned} \tag{39}$$

which, combines (16) to gain

$$\begin{aligned} N_2^{k+1} &= c\left(\nu_i^{-1} + c_2 c_4 \kappa^2\right)\left\|\boldsymbol{\phi}_i^{k+1}\right\|_0 \left\|\nabla \boldsymbol{\phi}_i^{k+1}\right\|_0 + c\kappa^2 \nu_i^{-1}\left\|\left[\boldsymbol{\phi}_u^{k+1}\right]\right\|_0 \left\|\nabla\left[\boldsymbol{\phi}_u^{k+1}\right]\right\|_0 \\ &\leqslant c_5\left(\nu_i^{-1}\left(1+\kappa^2\right) + c_4\kappa^2\right)\left(\left\|\boldsymbol{\phi}_i^{k+1}\right\|_0 + \left\|\boldsymbol{\phi}_j^{k+1}\right\|_0\right)\left(\left\|\nabla\boldsymbol{\phi}_i^{k+1}\right\|_0 + \left\|\nabla\boldsymbol{\phi}_j^{k+1}\right\|_0\right) \\ &\leqslant c_5\left(\nu_i^{-1}\left(1+\kappa^2\right) + c_4\kappa^2\right) c_3 \exp\left(T N_1\right) \\ &\cdot \left(\Delta t + h + \delta_i^2\right)\left(\Delta t^{\frac{1}{2}}\left(\nu_i^{-\frac{1}{2}} + \nu_j^{-\frac{1}{2}}\right) + c_1\right) \\ &\leqslant \min\left\{\frac{\nu_i}{6}, \frac{\nu_j}{6}\right\}, \end{aligned} \tag{40}$$

$c_3 = c_3\left(\nu_1, \nu_2, \kappa, c_2, T\right)$, $c_4 = \left(\nu_i^{-1} + \nu_j^{-1}\right) c_2$ and $c_5 = c_5\left(c_2\right)$.

Hence, when $n = n^*$, the following inequality holds with help of (35) and (40)

$$\begin{aligned} &\sum_{i=1}^{2} \left\|\boldsymbol{\phi}_i^{n^*+1}\right\|_0^2 + \sum_{i=1}^{2}\sum_{n=0}^{n^*} \left\|\boldsymbol{\phi}_i^{n+1} - \boldsymbol{\phi}_i^{n}\right\|_0^2 + \Delta t \sum_{i=1}^{2}\sum_{n=0}^{n^*} \nu_i \left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2 \\ &\quad + \frac{1}{2}\left|e_Q^{n^*+1}\right|^2 + \sum_{n=0}^{n^*}\left|e_Q^{n+1} - e_Q^{n}\right|^2 + \sum_{n=0}^{n^*}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2 \\ &\leqslant \Delta t \sum_{i=1}^{2}\sum_{n=0}^{n^*} N_1 \left\|\boldsymbol{\phi}_i^{n}\right\|_0^2 + c_3\left(\Delta t^2 + h^2 + \delta_i^4\right). \end{aligned} \tag{41}$$

Applying Lemma 2.4 to (41), we complete the induction.

Besides, adding up (33) from $n = 0, 1, \cdots, N-1$, combining (24) with (33) at the case of $m = N - 1$ and applying Theorem 3.1 and the projection properties and $\left|e_Q^{n^*+1}\right| \leqslant c\left(h + \Delta t + \delta_i^2\right)$, we get

$$\begin{aligned} &\sum_{i=1}^{2} \left\|\boldsymbol{\phi}_i^{N}\right\|_0^2 + \frac{4\Delta t}{3}\sum_{i=1}^{2}\sum_{n=0}^{N-1} \nu_i \left\|\nabla \boldsymbol{\phi}_i^{n+1}\right\|_0^2 + \left|e_Q^{N}\right|^2 + \sum_{n=0}^{N-1}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2 \\ &\leqslant \Delta t \sum_{i=1}^{2}\sum_{n=0}^{N-1} N_1 \left\|\boldsymbol{\phi}_i^{n}\right\|_0^2 + \Delta t \sum_{i=1}^{2}\sum_{n=0}^{N-1} N_2^n \left(\left\|\nabla\boldsymbol{\phi}_i^{n}\right\|_0^2 + \left\|\nabla\boldsymbol{\phi}_j^{n}\right\|_0^2\right) \\ &\quad + c_3\left(h^2 + \Delta t^2 + \delta_i^4\right). \end{aligned} \tag{42}$$

Now, we use the induction method to prove

$$\begin{aligned}&\sum_{i=1}^{2}\left\|\phi_i^N\right\|_0^2+\Delta t\sum_{i=1}^{2}\sum_{n=0}^{N-1}\nu_i\left\|\nabla\phi_i^{n+1}\right\|_0^2+\left|e_Q^N\right|^2+\sum_{n=0}^{N-1}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2\\&\leqslant c_3\exp\left(\Delta t\sum_{n=0}^{N-1}N_1\right)\left(\Delta t^2+h^2+\delta_i^4\right).\end{aligned}\tag{43}$$

In fact, when $N=1$, it follows (38) that (43) holds. Then we assume that (43) holds for the case of $k\leqslant N-1$:

$$\begin{aligned}&\sum_{i=1}^{2}\left\|\phi_i^k\right\|_0^2+\Delta t\sum_{i=1}^{2}\sum_{n=0}^{k-1}\nu_i\left\|\nabla\phi_i^{n+1}\right\|_0^2+\left|e_Q^k\right|^2+\sum_{n=0}^{k-1}\frac{\Delta t}{T}\left|e_Q^{n+1}\right|^2\\&\leqslant c_3\exp\left(\Delta t\sum_{n=0}^{k-1}N_1\right)\left(\Delta t^2+h^2+\delta_i^4\right).\end{aligned}\tag{44}$$

Finally, by a similar argument as (40) we get $N_2^k\leqslant\min\left\{\frac{\nu_i}{6},\frac{\nu_j}{6}\right\}$. Applying Lemma 2.4 to (42) finishes the induction, and then utilizing the triangle inequality completes the proof. □

5. Numerical experiments

In this section, several numerical experiments are showed to test the scheme presented herein. These examples will demonstrate its stability, convergence, and numerical performance for some benchmark problems compared with the numerical results of the Navier-Stokes/Navier-Stokes equations.

5.1. Stability. In this subsection, we will illustrate the stability by some numerical examples. We consider the problem (1) on the domain $\Omega=\Omega_1\cup\Omega_2$ with $\Omega_1=[0,1]\times[0,1]$ and $\Omega_2=[0,1]\times[-1,0]$.

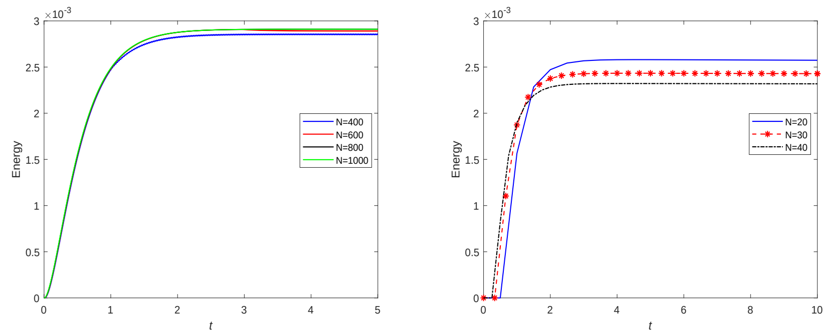
First, set $f_{1,1}=f_{1,2}=\cos(x)\sin(y)$ and $f_{2,1}=f_{2,2}=\cos(y)\sin(x)$ in (1). Then, we choose the parameters $\kappa=100$, $\nu_1=0.05$, $\nu_2=0.5$ and the initial velocity $\mathbf{u}_{1,0}=\mathbf{u}_{2,0}=\mathbf{0}$. Next, we denote the energy by $\|\mathbf{u}_1^n\|_0^2+\|\mathbf{u}_2^n\|_0^2$. Here, we pick $T=5$, $h=\frac{1}{32}$ and $N=400, 600, 800, 1000$. In addition, we set $f_{1,1}=f_{1,2}=f_{1,3}=\cos(x)\sin(y)\cos(z)$, $f_{2,1}=f_{2,2}=f_{2,3}=\sin(x)\cos(y)\sin(z)$ and pick $\kappa=1$, $\nu_1=0.05$, $\nu_2=0.5$, $T=10$, $h=\frac{1}{10}$ with $N=20,30,40$. In Figure 1, we can see that the energy keeps bounded by a constant with different N.

Second, we consider numerical value of Q under different time step Δt. Choose the homogeneous Dirichlet boundary conditions and set zero forcing. Besides, the initial value is presented as follows:

$$u_{i,1}=\sin(2\pi y)\sin^2(\pi x),u_{i,2}=-\sin(2\pi x)\sin^2(\pi y),i=1,2.$$

Moreover, set $\nu_1=0.05$, $\nu_2=0.1$, $\kappa=1$, $T=1$ and $h=\frac{1}{32}$. Figure 2 shows some numerical results of the auxiliary variable Q for four different time steps $\Delta t=0.1$, 0.05, 0.01 and 0.005. We find that the numerical value is approximate to the reference value. The large time step has little effect on the current scheme.

Finally, we will test the considered scheme with the problem at smaller viscosity. In addition, a fully coupled, monolithic approximation will be taken as "truth". According to [35], we plot the ratio of Taylor microscale to mesh size with different values of the viscosity in Figure 3. It is found that the result of the current scheme is closer to the result of the coupled scheme at ν_1 =5.0e-2, but there is a big error between two schemes when ν_1 =1.0e-6.



(a) (b)

FIGURE 1. Stability of the presented scheme for the 2D model (a) and 3D model (b) with different values of N.

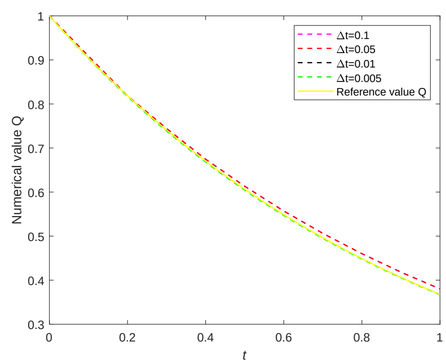
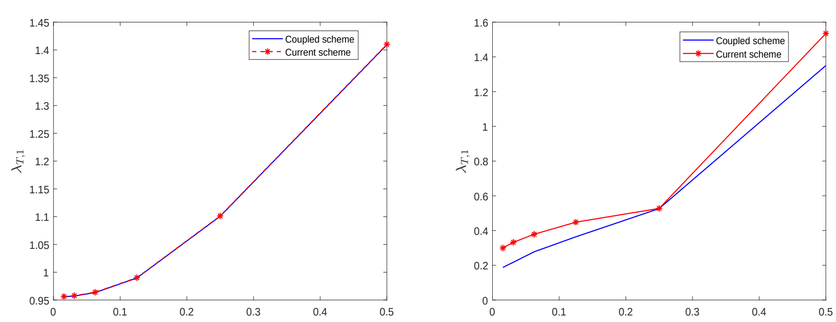


FIGURE 2. Numerical value of Q.



(a) ν_1 =5.0e-2 (b) ν_1 =1.0e-6

FIGURE 3. Taylor microscale with different mesh sizes.

5.2. Convergence. We consider a manufactured true solution in $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = [0,1] \times [0,1]$ and $\Omega_2 = [0,1] \times [-1,0]$.

$$\begin{aligned}
u_{1,1}(t,x,y) =& x^2 \exp(-t)(x-1)^2(1-y),\\
u_{1,2}(t,x,y) =& xy\exp(-t)\left(6x+y-3xy+2x^2y-4x^2-2\right),\\
u_{2,1}(t,x,y) =& x\exp(-t)(1-x)\Big(x(x-1)y^2\left(\nu_1\nu_2^{-1}+1\right)-\nu_1^{1/2}\kappa^{-1/2}y^2\exp(t/2)\\
&-x(x-1)+\nu_1^{1/2}\kappa^{-1/2}\exp(t/2)+\nu_1\nu_2^{-1}xy(x-1)\Big),\\
u_{2,2}(t,x,y) =& \frac{1}{3}\nu_2^{-1}\kappa^{-\frac{1}{2}}y(1-2x)\exp(-t)\Big(6\nu_2x^2\kappa^{1/2}-6\nu_2x\kappa^{1/2}-3\nu_1^{1/2}\nu_2\exp(t/2)\\
&-2\nu_1x^2y^2\kappa^{1/2}-2\nu_2x^2y^2\kappa^{1/2}+3\nu_1xy\kappa^{1/2}+2\nu_1xy^2\kappa^{1/2}\\
&-3\nu_1x^2y\kappa^{1/2}+2\nu_2xy^2\kappa^{1/2}+\nu_1^{1/2}\nu_2y^2\exp(t/2)\Big),\\
p_1(t,x,y) =& p_2(t,x,y) = \exp(-t)\cos(\pi x)\sin(\pi y).
\end{aligned}$$

Additionally, we denote the errors

$$Err(\mathbf{u}_i) = \left(\Delta t \sum_{n=1}^{N} \|\nabla(\mathbf{u}_i(t_n) - \mathbf{u}_i^n)\|_0^2\right)^{\frac{1}{2}}, \; Err(p_i) = \left(\Delta t \sum_{n=1}^{N} \|p_i(t_n) - p_i^n\|_0^2\right)^{\frac{1}{2}}.$$

In the meantime, pick $256\delta^2 = h$, $\nu_1 = 0.05$, $\nu_2 = 0.5$ and the coupling coefficient $\kappa = 100$.

TABLE 1. Spatial errors and convergence rates of the velocities and pressures with $T = 0.01$.

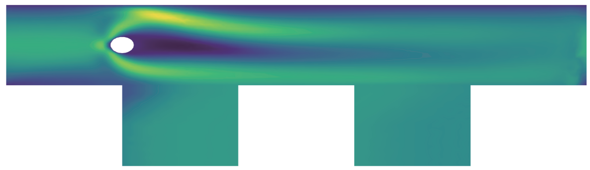
h^{-1}	$Err(\mathbf{u}_1)$	Rate	$Err(\mathbf{u}_2)$	*Rate*	$err(p_1)$	Rate	$err(p_2)$	*Rate*
4	2.72e-2	–	2.90e-2	–	6.00e-2	–	8.41e-2	–
8	1.28e-2	1.09	1.45e-2	1.00	1.50e-2	2.00	2.09e-2	2.01
16	5.93e-3	1.11	7.23e-3	1.00	3.70e-3	2.03	5.14e-3	2.02
32	2.83e-3	1.07	3.61e-3	1.00	9.18e-4	2.01	1.30e-3	1.99
64	1.39e-3	1.02	1.80e-3	1.00	2.31e-4	1.99	4.04e-4	1.68

TABLE 2. Temporal errors and convergence rates of the velocities and pressures with $T = 1.0$.

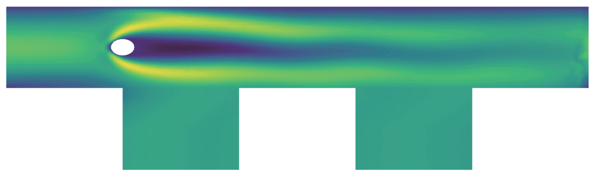
Δt^{-1}	$Err(\mathbf{u}_1)$	Rate	$Err(\mathbf{u}_2)$	*Rate*	$err(p_1)$	Rate	$err(p_2)$	*Rate*
4	2.57e-1	–	2.56e-1	–	1.42e-2	–	3.05e-2	–
8	9.91e-2	1.37	1.12e-1	1.20	3.92e-3	1.85	1.28e-2	1.25
16	4.03e-2	1.30	4.93e-2	1.18	1.19e-3	1.72	5.23e-3	1.29
24	2.47e-2	1.21	3.10e-2	1.14	6.02e-4	1.69	3.14e-3	1.26
32	1.77e-2	1.15	2.25e-2	1.11	3.76e-4	1.64	2.30e-3	1.08

On the one hand, considering the convergence order with respect to h, we choose a small time step $\Delta t = 0.001$. Then pick five mesh sizes $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ and $\frac{1}{64}$. In Table 1, we list the convergence orders with respect to h in L^2 norm for the pressure and $W^{1,2}$ norm for the velocity, respectively. From this figure, we can see that the current scheme works well and gain the right convergence order.

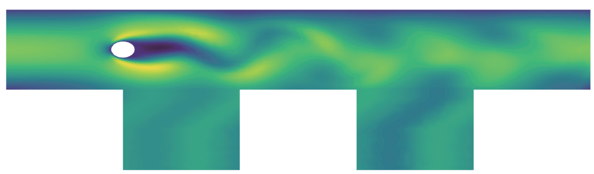
On the other hand, considering the convergence order with respect to Δt, we choose $\Delta t = h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{24}$ and $\frac{1}{32}$. Numerical results are collected in Table 2. The first order temporal accuracy is obtained.



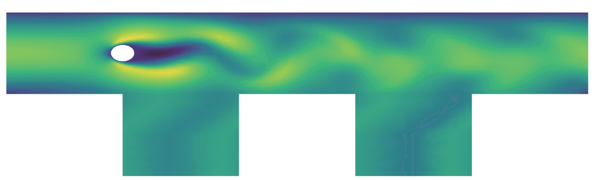
(a) Result of the Navier-Stokes-ω/Navier-Stokes-ω model with $\delta = 0.1$



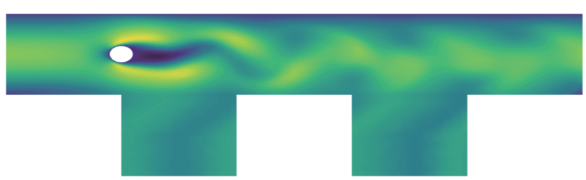
(b) Result of the Navier-Stokes-ω/Navier-Stokes-ω model with $\delta = 0.05$



(c) Result of the Navier-Stokes-ω/Navier-Stokes-ω model with $\delta = 0.01$



(d) Result of the Navier-Stokes-ω/Navier-Stokes-ω model with $\delta = 0.005$



(e) Result of the Navier-Stokes/Navier-Stokes model

FIGURE 4. Contour plots of the velocity magnitudes.

5.3. Flow past a cylinder problem. In this experiment, we test the presented scheme for the equivalent form (4) of the Navier-Stokes-ω/Navier-Stokes-ω equations (2) by the flow past a cylinder problem. The physical background of this problem describes a parabolic inflow in the top fluid domain that passes a cylindrical obstacle before it meets the fluid domain below.

A parabolic inflow drives the flow in the upper rectangular domain 5×1 a circular obstacle of radius 0.1 and centered at $(1, 0.5)$. Meanwhile, the lower domain includes two unit square region. Next, the parabolic flow's inlet/outlet are

$$u_{1,1}(0, y, t) = u_{1,1}(5.0, y, t) = 2.0y(1-y),$$
$$u_{1,2}(0, y, t) = u_{1,2}(5.0, y, t) = 0.$$

Besides, the homogeneous Dirichlet boundary conditions are enforced on the rest of the boundaries of the upper domain. Additionally, the no-slip boundary is enforced on the walls of the lower domain.

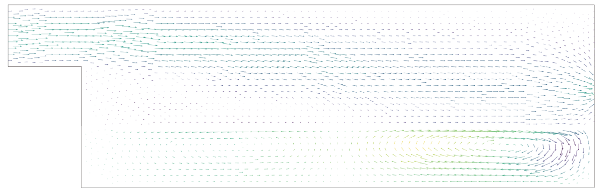
The problem parameters are given as follows: $\nu_1 = 0.001$, $\nu_2 = 1.0$, $\kappa = 0.01$ and $T = 20.0$. Also, $\Delta t = 0.01$ and the mesh of the domain Ω_1 and Ω_2 consists of 8930 nodes and 2912 nodes, respectively. Figure 4 shows contour plots of the velocity magnitudes with different δ. From this figure, we can find that decreasing δ value makes the solution approach that of the Navier-Stokes/Navier-Stokes equations step by step. It is not surprising because when $\delta_i = 0$ the Navier-Stokes-ω/Navier-Stokes-ω equations and Navier-Stokes/Navier-Stokes equations are equivalent. Besides, for the bigger value of δ, the Navier-Stokes-ω/Navier-Stokes-ω equations for simulation this problem is too dissipative to yield reliable result.

5.4. Flow pass a backward-facing step. In the subsection, a flow pass a backward-facing step is simulated by the current scheme for the Navier-Stokes-ω/Navier-Stokes-ω model and the Navier-Stokes/Navier-Stokes model. The setup has been proposed in the literature [19]. Most notably, it is pointed out that the real physical phenomenon of this setup is a description of the coast mountain, cliff and so on.

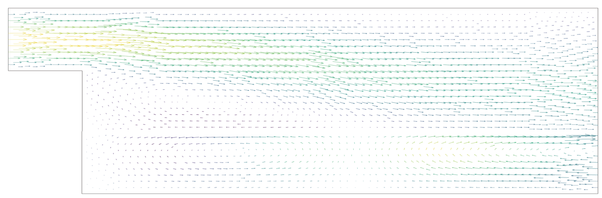
From above subsection, we can see that the Navier-Stokes-ω/Navier-Stokes-ω model with decreasing δ value gradually makes the solution approach that of the Navier-Stokes/Navier-Stokes equations. Hence, it is natural to point out why we still consider here the Navier-Stokes-ω/Navier-Stokes-ω model. In fact, the Navier-Stokes-ω/Navier-Stokes-ω model can be considered as a sequel and a complement of the Navier-Stokes/Navier-Stokes model for the simulation of turbulence. However, in some extreme cases, the disadvantages of the Navier-Stokes/Navier-Stokes model emerge.

The main goal of this test is to show that when the viscosity coefficient ν_1 is moderate, both governing equations can capture the key features of physical solution, but when ν_1 is small, only the Navier-Stokes-ω/Navier-Stokes-ω model simulates well.

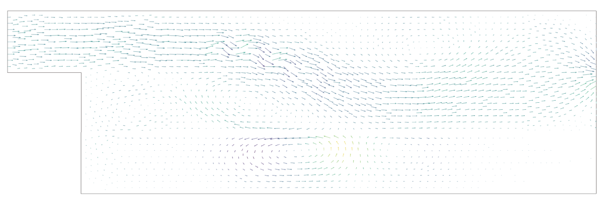
The no-slip condition is imposed on the step, the oceanic wall and the top wall of the atmosphere. Besides, the parabolic inflow with maximum inlet 10 drives the flow in the atmosphere, and the "do-nothing" condition is applied to the outflow. We consider $\nu_2 = 0.05$, $\kappa = 0.1$, $T = 15$, $\Delta t = 0.01$ and $h = \frac{1}{10}$ with varying ν_1. The Figure 5 illustrates that both governing equations produce similar results with $\nu_1 = 0.025$. However, decreasing ν_1 value results in a failure for the Navier-Stokes/Navier-Stokes model. On the other hand, the Navier-Stokes-ω/Navier-Stokes-ω model can still capture the key physical phenomenon.



(a) Result of the Navier-Stokes-ω/Navier-Stokes-ω model with $\nu_1 = 0.025$



(b) Result of the Navier-Stokes/Navier-Stokes model with $\nu_1 = 0.025$



(c) Result of the Navier-Stokes-ω/Navier-Stokes-ω model with $\nu_1 = 0.00025$

FIGURE 5. Velocity fields (the Navier-Stokes/Navier-Stokes model with $\nu_1 = 0.00025$ fails).

6. Conclusions

In this work, we consider the Navier-Stokes-ω/Navier-Stokes-ω model for the fluid-fluid interaction problem. Based on the auxiliary variable, we design the fully discrete and decoupled scheme, which is unconditionally energy stable, and is explicit treatment for the nonlinear terms and interaction terms. The proved stability and accuracy and the ability to capture basic phenomenological features of the proposed scheme are demonstrated by a series of numerical tests.

In [19], the authors find a model that avoids filtering in the interface but keeps the unconditional stability. Then, in this work, we design another model which also avoids filtering in the interface and obtains the unconditional stability. Compared with the numerical scheme in [19], the presented scheme applies explicit treatment for the nonlinear coupling conditions and nonlinear terms, and yields a linear system with a constant coefficient matrix which is easy to solve.

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