INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 22, Number 2, Pages 157–177

doi: 10.4208/ijnam2025-1008

WEAK GALERKIN FINITE ELEMENT METHOD BASED ON POD FOR NONLINEAR PARABOLIC EQUATIONS

JIANGHONG ZHANG AND FUZHENG GAO* AND JINTAO CUI*

Abstract. In this paper, we establish a novel reduced-order weak Galerkin (ROWG) finite element method for solving parabolic equation with nonlinear compression coefficient. We first present the classical weak Galerkin finite element discretization scheme and derive the optimal error estimates. Then we apply a proper orthogonal decomposition (POD) technique to develop the ROWG method, which can effectively reduce degrees of freedom and CPU time. The optimal order error estimates are also derived, and the algorithm flow is provided. Finally, some numerical experiments illustrate the performance of the ROWG method. The numerical results show that the proposed ROWG method is efficient for solving nonlinear parabolic equations.

Key words. Weak Galerkin finite element method, nonlinear parabolic equations, proper orthogonal decomposition.

1. Introduction

In this paper, we consider the following parabolic equations with nonlinear compression coefficient:

$$g(u)u_t - \nabla \cdot (D\nabla u) = f, \quad (x,t) \in \Omega \times J, \tag{1a}$$

$$u = u_0, \quad (x,t) \in \Omega \times \{t = 0\}, \tag{1b}$$

$$u = \phi, \quad (x,t) \in \partial\Omega \times J, \tag{1c}$$

where Ω is a polygonal region in $\mathbb{R}^2$ with Lipschitz continuous boundary. Here D is a symmetric positive definite matrix, $g(u)$ is a sufficiently smooth function with bounded derivatives up to the second-order, and there exist two constants g_*, g^* such that

$$0 \le g_* \le g(u) \le g^*, \quad \|u\|_\infty < \infty.$$

And the assumptions that the solution of (1) satisfies can be found in the literature [4].

The weak formulation of (1) is to find $u \in H^1(\Omega)$ such that

$$(g(u)u_t, v) + (D\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{2}$$

The weak Galerkin (WG) finite element method is first proposed in [14, 15, 12]. It can be viewed as an extension of the standard finite element method. The key to WG method is the introduction of weak functions and weak gradients. In comparison with conventional finite element method (FEM), the WG method has higher robustness in boundary processing and is more suitable for grids with hanging points. In recent years, the WG method has been widely used to solve the Darcy-Stokes equation [3, 8], quasi-linear elliptic problems [18, 2], etc.

The proper orthogonal decomposition (POD) technique has been combined with the finite element method since 2001 and successfully applied to solve parabolic equations [7]. This method uses several layers of images to perform a low-dimensional

Received by the editors on June 7, 2024 and, accepted on September 21, 2024.
2000 *Mathematics Subject Classification.* 78M10.
*Corresponding authors.

approximation of the piecewise polynomial function space. In this way, a new finite element function space is constructed. Within the allowed error range, POD can effectively transform the original high-dimensional model into a low-dimensional one, which significantly improves the computational efficiency. In recent years, researchers have been actively exploring the combination of POD and different types of numerical methods, including the finite difference method, the finite volume method, and the hybrid finite element method [1, 5, 6, 9], etc. Very recently, Zhang el at. in [16] considers the formulation and theoretical analysis of a reduced-order numerical method constructed by POD for nonlocal diffusion problems. Zhao et al. [17] first linked POD with weak Galerkin finite element method, but only gave the algorithm flow without the corresponding theoretical analysis.

In this paper, we apply the POD technique to develop a novel reduced-order weak Galerkin (ROWG) finite element method [17, 10, 13] for solving the nonlinear parabolic problem (1). We construct a new correlation matrix and provide convergence analysis under the L^2 and the discrete H^1 norms. The rest of the paper is organized as follows: In Section 2, we first introduce the concepts of discrete weak functions and weak derivatives for WG method. Then we establish the fully discrete WG scheme for problem (1), and derive the optimal error estimates. In Section 3, we construct the POD basis and build the fully discrete ROWG scheme. The optimal error estimates for the ROWG scheme are presented, and the algorithm process is shown. In Section 4, we give some numerical examples and compare the CPU time of the ROWG scheme and the WG scheme for all examples. Conclusions are given in Section 5.

2. Classical WG method

In this section, we consider the following discrete weak Galerkin finite element space $WG(P_r, P_{r-1}; P_{r-1}^2)$. Let $\mathcal{T}_h$ be a partition of Ω that satisfies the conditions in [11]. We denote

$$V_h = \Big\{ v = \{v_0, v_b\} : v_0|_K \in P_r(K), v_b|_e \in P_{r-1}(e), e \subset \partial K, K \in \mathcal{T}_h \Big\}, \tag{3}$$

and its subspace V_h^0 as

$$V_h^0 = \Big\{ v \in V_h : v_b|_{\partial\Omega} = 0 \Big\}. \tag{4}$$

For any $v \in V_h$, its weak gradient $\nabla_\omega v$ satisfes that

$$(\nabla_\omega v, \phi)_K = -(v_0, \nabla \cdot \phi)_K + \langle v_b, \phi \cdot \mathbf{n} \rangle_{\partial K} \quad \forall \phi \in P_{r-1}^2(K), \tag{5}$$

Let $t_n = n\Delta t \, (n = 1, 2, \cdots, N)$, $\Delta t = T/N$, and denote $u^n = u(t_n)$. The fully discrete WG finite element scheme for (1) is to find $U^n = \{U_0^n, U_b^n\} \in V_h$ such that

$$\left(g(U^{n-1}) \frac{U^n - U^{n-1}}{\Delta t}, v \right) + a_s(U^n, v) = (f^n, v), \quad \forall v = \{v_0, v_b\} \in V_h^0, \tag{6}$$

with the initial value $U^0 = Q_h u^0$. Here the bilinear form

$$a_s(u, v) = \sum_{K \in \mathcal{T}_h} (D \nabla_\omega u, \nabla_\omega v)_K + h_K^{-1} \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{\partial K}.$$

We define $|||v||| = \sqrt{a_s(v, v)}$ and $\|v\|_h = \left(\sum_{K \in \mathcal{T}_h} \|\nabla v_0\|_T^2 + h_T^{-1} \|Q_b v_0 - v_b\|_{\partial K}^2 \right)^{\frac{1}{2}}$ on V_h, and $|||\cdot|||$ is equivalent to $\|\cdot\|_h$ (cf. [12]).

We give the estimate of the errors between WG solution and the analytical solution. Denote $\theta_h^n = E_h u^n - U^n$, $\eta_h^n = u^n - Q_h u^n$, $\tau_h^n = Q_h u^n - E_h u^n$, where local L_2 projection operator $Q_h = \{Q_0, Q_b\}$ and elliptic projection operator E_h are defined as follows:

$$(u,\phi)_K = (Q_0 u,\phi)_K, \quad \forall \phi \in P_r(K),$$
$$\langle u,\varphi\rangle_e, = \langle Q_b u,\varphi\rangle_e, \quad \forall \varphi \in P_{r-1}(e), \tag{7}$$
$$a_s(E_h u, v) = (-\nabla\cdot(D\nabla u), v), \quad \forall v \in V_h^0.$$

From [14], we have the following estimates of η_h^n and τ_h^n .

Lemma 2.1. *Suppose that $\mathcal{T}_h$ satisfies the conditions in* [11]. *If $u(t) \in H^{r+1}(\Omega)$, we have*

$$\Sigma_{K\in\mathcal{T}_h}\|\eta_h^n\|^2 + \Sigma_{K\in\mathcal{T}_h} h_K^2\|\nabla\eta_h^n\|^2 \le Ch^{2(r+1)}\|u^n\|_{r+1}^2,$$
$$\Sigma_{K\in\mathcal{T}_h}\|\tau_h^n\|^2 + \Sigma_{K\in\mathcal{T}_h} h_K^2\|\nabla\tau_h^n\|^2 \le Ch^{2(r+1)}\|u^n\|_{r+1}^2. \tag{8}$$

Lemma 2.2. [11] *Assume that $\mathcal{T}_h$ satisfies some shape regularity, we have*

$$\|v\| \le C|\!|\!|v|\!|\!|, \quad \forall\, v \in V_h. \tag{9}$$

Lemma 2.3. *Take $U^0 = Q_h u^0$. Let u^m and U^m $(m = 1, 2, \cdots, N)$ be the analytical solution of* (1) *and the numerical solution of* (6)*, respectively. Let Δt and h satisfy $\Delta t \le Ch$. Then there exists a constant C independent of h and Δt such that*

$$\|\theta_h^m\|^2 \le C\left((\Delta t)^2 + h^{2r+2}\|u\|^2_{L^\infty(J;H^{r+1}(\Omega))} + h^{2r+2}\|u_t\|^2_{L^2(J;H^{r+1}(\Omega))}\right). \tag{10}$$

Proof. First, we make the induction hypothesis

$$\|U^n\|_\infty \le C, \quad n = 0, 1, \cdots, N. \tag{11}$$

By Lemma 2.1 and the inverse inequality, we have

$$\|U^0\|_\infty \le \|u^0\|_\infty + \|u^0 - Q_h u^0\|_\infty \le \|u^0\|_\infty + Ch^{-1}\cdot h^{r+1} \le C, \tag{12}$$

and (11) hold for $n = 0$. Subtracting (6) from (2), observing (7) and choosing test function $v = \theta_h^n$, we have

$$\begin{aligned}
&\left(g\left(U^{n-1}\right)\partial_t\theta_h^n, \theta_h^n\right) + a_s\left(\theta_h^n, \theta_h^n\right)\\
&\quad = -\left(g\left(u^n\right)\left(u_t^n - \partial_t u^n\right), \theta_h^n\right) - \left(\left(g\left(u^n\right) - g\left(u^{n-1}\right)\right)\partial_t u^n, \theta_h^n\right)\\
&\qquad - \left(\left(g\left(u^{n-1}\right) - g\left(U^{n-1}\right)\right)\partial_t u^n, \theta_h^n\right) - \left(g\left(U^{n-1}\right)\partial_t(\eta_h^n + \tau_h^n), \theta_h^n\right),
\end{aligned} \tag{13}$$

where $\partial_t u^n = \frac{u^n - u^{n-1}}{\Delta t}$. Using the equality $a\,(a-b) = \frac{(a-b)^2}{2} + \frac{a^2-b^2}{2}$, we have

$$\begin{aligned}
&\left(g\left(U^{n-1}\right)\partial_t\theta_h^n, \theta_h^n\right)\\
&= \frac{1}{2\Delta t}\left(g\left(U^n\right)\theta_h^n, \theta_h^n\right) - \frac{1}{2\Delta t}\left(g\left(U^{n-1}\right)\theta_h^{n-1}, \theta_h^{n-1}\right)\\
&\quad + \frac{\Delta t}{2}\left(g\left(U^{n-1}\right)\partial_t\theta_h^n, \partial_t\theta_h^n\right) - \frac{1}{2\Delta t}\left(\left(g\left(U^n\right) - g\left(U^{n-1}\right)\right)\theta_h^n, \theta_h^n\right).
\end{aligned} \tag{14}$$

Multiply by $2\Delta t$ and sum over time, we can rewrite (13) as

$$\begin{aligned}
&\left(g\left(U^{m-1}\right)\theta_h^m,\theta_h^m\right)+(\Delta t)^2\sum_{n=1}^{m}\left(g\left(U^{n-1}\right)\partial_t\theta_h^n,\partial_t\theta_h^n\right)+2\Delta t\sum_{n=1}^{m}a_s(\theta_h^n,\theta_h^n)\\
&=\left(g\left(U^0\right)\theta_h^0,\theta_h^0\right)+\sum_{n=1}^{m}\left(\left(g\left(U^n\right)-g\left(U^{n-1}\right)\right)\theta_h^n,\theta_h^n\right)\\
&\quad-\sum_{n=1}^{m}\left(g\left(u^n\right)\left(u_t^n-\partial_t u^n\right),\theta_h^n\right)-\sum_{n=1}^{m}\left(\left(g\left(u^n\right)-g\left(u^{n-1}\right)\right)\partial_t u^n,\theta_h^n\right)\\
&\quad-\sum_{n=1}^{m}\left(\left(g\left(u^{n-1}\right)-g\left(U^{n-1}\right)\right)\partial_t u^n,\theta_h^n\right)-\sum_{n=1}^{m}\left(g\left(U^{n-1}\right)\partial_t(\eta_h^n+\tau_h^n),\theta_h^n\right)\\
&=\left(g\left(U^0\right)\theta_h^0,\theta_h^0\right)+T_1+T_2+T_3+T_4+T_5,
\end{aligned}\tag{15}$$

where T_1, T_2, T_3, T_4, T_5 represent the last five terms on the right-hand side of equation (15).

Denote the left-hand side terms of (15) by $H_i(i=1,2,3)$, we have the estimate

$$\sum_{i=1}^{3}H_i\geq C(\|\theta_h^m\|^2+\sum_{n=1}^{m}\|\partial_t\theta_h^n\|^2(\Delta t)^2+\sum_{n=1}^{m}\|\theta_h^n\|_h^2\Delta t).\tag{16}$$

For T_1, we have

$$\begin{aligned}
|T_1|&=2\left|\sum_{n=1}^{m}\left(\int_{t_{n-1}}^{t_n}g'\left(\frac{t-t_{n-1}}{\Delta t}U^n-\frac{t-t_n}{\Delta t}U^{n-1}\right)dt\cdot\theta_h^n,\theta_h^n\right)\right|\\
&\leq 2\sum_{n=1}^{m}\left(\int_{t_{n-1}}^{t_n}\left|g'\left(\frac{t-t_{n-1}}{\Delta t}U^n-\frac{t-t_n}{\Delta t}U^{n-1}\right)\right|dt\cdot\theta_h^n,\theta_h^n\right)\\
&\leq C\sum_{n=1}^{m}\|\theta_h^n\|^2\Delta t,
\end{aligned}\tag{17}$$

where $g'(\cdot)=\frac{\partial g}{\partial t}$. The term of T_2 can be estimated directly as

$$\begin{aligned}
|T_2|&\leq\left(\sum_{n=1}^{m}\|u_t^n-\partial_t u^n\|^2\Delta t+\sum_{n=1}^{m}\|\theta_h^n\|^2\Delta t\right)\\
&\leq\left(\sum_{n=1}^{m}\|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(t-t_{n-1})u_{tt}dt\|^2\Delta t+C\sum_{n=1}^{m}\|\theta_h^n\|^2\Delta t\right)\\
&\leq C\left((\Delta t)^2\|u_{tt}\|_{L^2(J;L^2(\Omega))}+\sum_{n=1}^{m}\|\theta_h^n\|^2\Delta t\right).
\end{aligned}\tag{18}$$

For T_3 and T_4, by applying Cauchy mean value theorem, we obtain

$$\begin{aligned}
|T_3|&=2\left|\sum_{n=1}^{m}(g'(\phi^n)(u^n-u^{n-1})u_t(\lambda^n),\theta_h^n)\Delta t\right|\\
&\leq C\sum_{n=1}^{m}\left|(g'(\phi^n)u_t(\beta^n)u_t(\lambda^n),\theta_h^n)\right|(\Delta t)^2\\
&\leq C\left((\Delta t)^2+\sum_{n=1}^{m}\|\theta_h^n\|^2\Delta t\right),
\end{aligned}\tag{19}$$

$$
\begin{aligned}
|T_4| &\le 2\sum_{n=1}^{m} |(g'(\xi^{n-1})(\theta_h^{n-1}+\eta_h^{n-1}+\tau_h^{n-1})u_t(\gamma^n),\theta_h^n)|\Delta t \\
&\le C\left(\sum_{n=1}^{m}\|\theta_h^n\|^2\Delta t+\|\eta_h\|^2_{L^\infty(J;L^2(\Omega))}+\|\tau_h\|^2_{L^\infty(J;L^2(\Omega)}+\|\theta_h^0\|^2\Delta t\right),
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
&\phi^n = w_1^n u^n + (1-w_1^n)u^{n-1}, && \beta^n = w_2^n t^n + (1-w_2^n)t^{n-1},\\
&\lambda^n = w_3^n t^n + (1-w_3^n)t^{n-1}, && \xi^n = w_4^n u^n + (1-w_4^n)U^n,\\
&\gamma^n = w_5^n t^n + (1-w_5^n)t^{n-1},
\end{aligned}
$$

and $w_i^n \in [0,1](i=1,2,3,4,5)$. For T_5, we have

$$
|T_5| \le C\left(\|\eta_{h,t}\|^2_{L^2(J;L^2(\Omega))}+\|\tau_{h,t}\|^2_{L^2(J;L^2(\Omega))}+\sum_{n=1}^{m}\|\theta^n\|^2\Delta t\right). \tag{21}
$$

The combination of (15)–(21) leads to

$$
\begin{aligned}
&\|\theta_h^m\|^2+(\Delta t)^2\sum_{n=1}^{m}\|\partial_t\theta_h^n\|^2+\Delta t\sum_{n=1}^{m}\|\theta_h^n\|_h^2\\
\le & C\Bigg(\|\theta_h^0\|^2+(\Delta t)^2\|u_{tt}\|_{L^2(J;L^2(\Omega))}+\sum_{n=1}^{m}\|\theta_h^n\|^2\Delta t\\
&+\|\eta_h\|^2_{L^\infty(J;L^2(\Omega))}+\|\tau_h\|^2_{L^\infty(J;L^2(\Omega))}\\
&+\|\eta_{h,t}\|^2_{L^2(J;L^2(\Omega))}+\|\tau_{h,t}\|^2_{L^2(J;L^2(\Omega))}\Bigg).
\end{aligned} \tag{22}
$$

It follows from Lemma 2.1 and the discrete Gronwall's inequality that

$$
\|\theta_h^m\|^2 \le C\left((\Delta t)^2+h^{2r+2}\|u\|^2_{L^\infty(J;H^{r+1}(\Omega))}+h^{2r+2}\|u_t\|^2_{L^2(J;H^{r+1}(\Omega))}\right). \tag{23}
$$

When $\Delta t = O(h), r \ge 0$, we have

$$
\|U^m\|_\infty \le \|u^m\|_\infty + Ch^{-1}((\Delta t)+h^{r+1}) \le C. \tag{24}
$$

This complete the proof of the induction hypothesis. □

Theorem 2.4. *Take $U^0 = Q_h u^0$. Let u^m and U^m $(m = 1, 2, \cdots, N)$ be the analytical solution of* (1) *and the numerical solution of* (6)*, respectively. Let Δt and h satisfy $\Delta t \le Ch$. Then there exists a constant C independent of h and Δt such that*

$$
\|u^m - U^m\|^2 \le C\left((\Delta t)^2 + h^{2r+2}\right). \tag{25}
$$

Proof. By triangle inequality, Lemma 2.1 and Lemma 2.3, (25) is easy to obtain. □

Lemma 2.5. *Take $U^0 = Q_h u^0$. Let u^m and U^m $(m = 1, 2, \cdots, N)$ be the analytical solution of* (1) *and the numerical solution of* (6)*, respectively. Let Δt and h satisfy $\Delta t \le Ch$. Then there exists a constant C independent of h and Δt such that*

$$
\begin{aligned}
\|\theta_h^m\|_h^2 \le & C\Big(h^{2r}\|u\|^2_{L^\infty(J;H^{r+1}(\Omega))}+(\Delta t)^2\|u_{tt}\|^2_{L^2(J;L^2(\Omega))}\\
&+h^{2r+2}\|u\|^2_{L^\infty(J;H^{r+1}(\Omega))}+h^{2r+2}\|u_t\|^2_{L^2(J;H^{r+1}(\Omega))}\Big).
\end{aligned} \tag{26}
$$

Proof. Subtracting (6) from (2), observing (7) and choosing the test function $v = \partial_t \theta_h^n$, we have

$$
\begin{aligned}
&2\Delta t \sum_{n=1}^{m} \left(g\left(U^{n-1}\right) \partial_t \theta_h^n, \partial_t \theta_h^n\right) + a_s\left(\theta_h^m, \theta_h^m\right) + (\Delta t)^2 \sum_{n=1}^{m} a_s\left(\partial_t \theta_h^n, \partial_t \theta_h^n\right) \\
=&a_s\left(\theta_h^0, \theta_h^0\right) - 2\sum_{n=1}^{m}\left(g\left(u^n\right)\left(u_t^n - \partial_t u^n\right), \theta_h^n\right)\Delta t \\
&- 2\sum_{n=1}^{m}\left(\left(g\left(u^n\right) - g\left(u^{n-1}\right)\right)\partial_t u^n, \partial_t \theta_h^n\right)\Delta t \\
&- 2\sum_{n=1}^{m}\left(\left(g\left(u^{n-1}\right) - g\left(U^{n-1}\right)\right)\partial_t u^n, \partial_t \theta_h^n\right)\Delta t \\
&- 2\sum_{n=1}^{m}\left(g\left(U^{n-1}\right)\partial_t(\eta_h^n + \tau_h^n), \partial_t \theta_h^n\right)\Delta t \\
=&a_s\left(\theta_h^0, \theta_h^0\right) + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4,
\end{aligned} \tag{27}
$$

where $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$ represent the last four terms on the right-hand side of equation (27).

Denote the left-hand side terms of (27) by $\mathcal{H}_i\,(i = 1, 2, 3)$, we have the estimate

$$
\sum_{i=1}^{3} \mathcal{H}_i \geq C(\|\theta_h^m\|_h^2 + \sum_{n=1}^{m} \|\partial_t \theta_h^n\|^2 \Delta t + \sum_{n=1}^{m} \|\partial_t \theta_h^n\|_h^2 (\Delta t)^2). \tag{28}
$$

By using (18) and the ε-Cauchy inequality, we have

$$
|\mathcal{T}_1| \leq C\left(\Delta t\right)^2 \|u_{tt}\|_{L^2(J;L^2(\Omega))} + \varepsilon \sum_{n=1}^{m} \|\partial_t \theta_h^n\|^2 \Delta t. \tag{29}
$$

Applying Cauchy mean value theorem and the ε-Cauchy inequality, we have

$$
|\mathcal{T}_2| \leq C(\Delta t)^2 + \varepsilon \sum_{n=1}^{m} \|\partial_t \theta_h^n\|^2 \Delta t. \tag{30}
$$

It follows from Lemma 2.2 and Cauchy mean value theorem that

$$
\begin{aligned}
|\mathcal{T}_3| &\leq C\left(\sum_{n=0}^{m} \|\theta_h^n\|^2 \Delta t + \|\eta_h\|_{L^\infty(J;L^2(\Omega))}^2 + \|\tau_h\|_{L^\infty(J;L^2(\Omega))}^2\right) + \varepsilon \sum_{n=1}^{m} \|\partial_t \theta_h^n\|^2 \Delta t \\
&\leq C\left(\|\eta_h\|_{L^\infty(J;L^2(\Omega))}^2 + \|\tau_h\|_{L^\infty(J;L^2(\Omega))}^2 + \sum_{n=0}^{m} |||\theta_h^n|||^2 \Delta t\right) + \varepsilon \sum_{n=1}^{m} \|\partial_t \theta_h^n\|^2 \Delta t.
\end{aligned} \tag{31}
$$

Moreover, it is easy to see that

$$
|\mathcal{T}_4| \leq C(\|\eta_{h,t}\|_{L^2(J;L^2(\Omega))}^2 + \|\tau_{h,t}\|_{L^2(J;L^2(\Omega))}^2) + \varepsilon \sum_{n=1}^{m} \|\partial_t \theta_h^n\|^2 \Delta t. \tag{32}
$$

When ε is sufficiently small, the combination of (27) – (32) leads to
(33)

$$\|\theta_h^m\|_h^2 + \sum_{n=1}^{m} \|\partial_t \theta_h^n\|^2 \Delta t + \sum_{n=1}^{m} \|\partial_t \theta_h^n\|_h^2 (\Delta t)^2 \leq C\Big(\|\theta_h^0\|_h^2 + (\Delta t)^2 \|u_{tt}\|^2_{L^2(J;L^2(\Omega))} + \|\eta_{h,t}\|^2_{L^2(J;L^2(\Omega))} + \|\tau_{h,t}\|^2_{L^2(J;L^2(\Omega))} + \|\eta_h\|^2_{L^\infty(J;L^2(\Omega))} + \|\tau_h\|^2_{L^\infty(J;L^2(\Omega))} + \sum_{n=1}^{m} \|\theta_h^n\|_h^2 \Delta t \Big).$$

Finally, by applying Lemma 2.1 and the discrete Gronwall's inequality, we get
(34)

$$\|\theta_h^m\|_h^2 \leq C(h^{2r}\|u\|^2_{L^\infty(J;H^{r+1}(\Omega))} + (\Delta t)^2 \|u_{tt}\|^2_{L^2(J;L^2(\Omega))} + h^{2r+2}\|u\|^2_{L^\infty(J;H^{r+1}(\Omega))} + h^{2r+2}\|u_t\|^2_{L^2(J;H^{r+1}(\Omega))}).$$

□

By triangle inequality, Lemma 2.1 and Lemma 2.5, we arrive at the following theorem.

Theorem 2.6. *Take $U^0 = Q_h u^0$. Let u^m and U^m $(m = 1, 2, \cdots, N)$ be the analytical solution of* (1) *and the numerical solution of* (6)*, respectively. Let Δt and h satisfy $\Delta t \leq Ch$. Then there exists a constant C independent of h and Δt such that*

$$\|Q_h u^m - U^m\|_h^2 \leq C\left((\Delta t)^2 + h^{2r}\right). \tag{35}$$

3. Reduced-order WG scheme based on POD

In this section, we apply the ROWG scheme (cf. [17, 10, 13]) to solve problem (1), and give the corresponding theoretical analysis. A complete ROWG algorithm flow is given in **Algorithm 1** in Section 4.

The POD method consists in finding a set of standard orthogonal bases $\{\psi_i\}_{i=1}^d$ that satisfies a minimization problem. With the initial L layers of the discrete WG scheme (6) as snapshots $\{U^n\}_{n=1}^L$, for a positive integer $1 \leq d \leq l$, we can establish the corresponding minimization problem and the POD basis.

Definition 3.1. *Define the following minimization problem*

$$\min_{\{\psi_j\}_{j=1}^d} \frac{1}{L} \sum_{n=1}^{L} \left|\left|\left| U^n - \sum_{j=1}^{d} a_s(U^n, \psi_j)\psi_j \right|\right|\right| \tag{36a}$$

$$s.t. \quad a_s(\psi_i, \psi_j) = \delta_{ij}, \ 1 \leq i \leq j \leq d. \tag{36b}$$

leq For U^n $(1 \leq n \leq L)$, we have $U^n = \sum_{i=1}^{l} a_s(U^n, \psi_i)\psi_i$.

Definition 3.2. *Define the correlation matrix $\boldsymbol{A} \in R^{L\times L}$ such that*

$$A_{ij} = \frac{1}{L} \sum_{K\in\mathcal{T}_h} ((\nabla_\omega U^i, \nabla_\omega U^j)_K + h_K^{-1}\langle Q_b U_0^i - U_b^i, Q_b U_0^j - U_b^j\rangle_{\partial K}). \tag{37}$$

It is easy to see from definition that $\boldsymbol{A}$ is a symmetric positive definite matrix, and there exist positive eigenvalues and corresponding standard orthogonal eigenvectors $\{\lambda_i, \boldsymbol{v_i}\}_{i=1}^L$.

Definition 3.3. [10, 17] *Assuming $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \lambda_L > 0$, the POD basis is formed by*

$$\psi_0^i = \frac{1}{\sqrt{L\lambda_i}} \sum_{j=1}^{L} (\boldsymbol{v}_i)_j U_0^j, \quad \psi_b^i = \frac{1}{\sqrt{L\lambda_i}} \sum_{j=1}^{L} (\boldsymbol{v}_i)_j U_b^j, \quad i = 1, 2, \ldots, d, \tag{38}$$

where $(\boldsymbol{v}_i)_j (1 \leq j \leq L)$ represents the j-th component of the standard orthogonal eigenvector $\boldsymbol{v}_i$.

Denote by $V_d = V_{d,0} \oplus V_{d,b}$, where $V_{d,0} = \text{span}\{\psi_0^1, \psi_0^2, \cdots, \psi_0^d\}$, $V_{d,b} = \text{span}\{\psi_b^1, \psi_b^2, \cdots, \psi_b^d\}$. One can observe that $V_d \subseteq V_h$. The fully discrete ROWG scheme for (1) is: Find $U_d^n = \{U_{d0}^n, U_{db}^n\} \in V_d$ such that

$$(g(U_d^{n-1})\partial_t U_d^n, v_d) + a_s(U_d^n, v_d) = (f^n, v_d), \quad \forall\, v_d \in V_d^0, n = L+1, L+2, \cdots, N. \tag{39}$$

with $U_d^L = \sum\limits_{j=1}^{d} a_s(U^L, \psi_j)\psi_j$.

Denote the L_2 projection operator $Q_d = \{Q_{d0}, Q_{db}\} : V_h \to V_d$ and the elliptic projection operator $E_d : V_h \to V_d$ such that for any $v_h = \{v_0, v_b\} \in V_h$, $K \in \mathcal{T}_h$, $e \in \partial K$, there holds

$$\begin{aligned} a_s\left(v_h, \phi\right) &= a_s\left(E_d v_h, \phi\right) \quad \forall \phi \in V_d, \\ \left(v_0, \phi\right)_K &= \left(Q_{d0} v_0, \phi\right)_K \quad \forall \phi \in V_d, \\ \left\langle v_b, \phi\right\rangle_e &= \left\langle Q_{db} v_b, \phi\right\rangle_e \quad \forall \phi \in V_d. \end{aligned} \tag{40}$$

In fact, we can define $E_h\phi = E_d\phi$ and $Q_h\phi = Q_d\phi$ when $\phi \in V_h$.

Then, we give the estimates of the errors between the fully discrete WG scheme (6) and the fully discrete ROWG scheme (39). Denote $\eta_d^n = U^n - E_d U^n$, and $\theta_d^n = E_d U^n - U_d^n$. In order to estimate η_d^n, we need the following lemmas.

Lemma 3.4. *For every $n\,(1 \leq n \leq L)$, if $r \geq 0$, η_d^n satisfies*

$$\|\eta_d^n\| \leq Ch\|\eta_d^n\|_h, \tag{41a}$$

and if $r \geq 1$, η_d^n satisfies

$$\|\eta_d^n\|_{-h} \leq Ch^2\|\eta_d^n\|_h, \tag{42a}$$

where $\|u\|_{-h} = \sup_{v \in V_h} \frac{(u_0, v_0)}{\|v\|_h}$ for any $u \in V_h$.

Proof. Consider the dual problem that seeks $w \in H_0^1(\Omega)$ satisfying

$$-\nabla \cdot (D\nabla w) = \eta_d^n. \tag{43}$$

If the dual problem has the usual H^{r+2}-regularity, then there exists a constant C such that $\|w\|_{r+2} \leq C\|\eta_d^n\|_r$. With the definition of the elliptic projection E_h, we have

$$a_s(E_h w, v) = (\eta_d^n, v), \quad \forall v \in V_h^0. \tag{44}$$

Taking $v = \eta_d^n$, it then follows from (40) that

$$\begin{aligned} \|\eta_d^n\|^2 &= a_s(E_h w, \eta_d^n) \\ &= a_s(E_h w - Q_d(E_h w), \eta_d^n) \\ &\leq C\|E_h w - Q_d(E_h w)\|_h \|\eta_d^n\|_h. \end{aligned} \tag{45}$$

By applying the triangle inequality and Lemma 2.1, we have

$$\begin{aligned}\|E_h w - Q_d(E_h w)\|_h \leq & \|E_h w - Q_h w\|_h + \|Q_h(w - E_h w)\|_h \\ & + \|Q_h(E_h w) - Q_d(E_h w)\|_h \\ \leq & \|E_h w - Q_h w\|_h + C\|w - E_h w\|_h \\ \leq & C(\|E_h w - Q_h w\|_h + \|w - Q_h w\|_h) \\ \leq & Ch^{r+1}\|w\|_{r+2}.\end{aligned} \tag{46}$$

When $r \geq 0$, in view of the regularity assumption, (41a) holds.

For any $\phi \in V_h$, consider the dual problem that seeks $w \in H_0^1(\Omega)$ satisfying

$$-\nabla \cdot (D\nabla w) = \phi. \tag{47}$$

Similarly, there exists a constant C such that $\|w\|_{r+2} \leq C\|\phi\|_r$ when it has the usual H^{r+2} - regularity. Moreover, we have

$$a_s(E_h w, v) = (\phi, v), \quad \forall\, v \in V_h^0. \tag{48}$$

Taking $v = \eta_d^n$ and applying (40), we have

$$\begin{aligned}(\phi, \eta_d^n) &= a_s(E_h w, \eta_d^n) \\ &= a_s(E_h w - Q_d(E_h w), \eta_d^n) \\ &\leq Ch^{r+1}\|w\|_{r+2}\|\eta_d^n\|_h \\ &\leq Ch^{r+1}\|\phi\|_r\|\eta_d^n\|_h.\end{aligned} \tag{49}$$

When $r \geq 1$, we have

$$\|\eta_d^n\|_{-h} = \sup_{\phi \in V_h} \frac{(\phi, \eta_d^n)}{\|\phi\|_h} \leq \frac{Ch^2\|\phi\|_1\|\eta_d^n\|_h}{\|\phi\|_h} \leq Ch^2\|\eta_d^n\|_h, \tag{50}$$

and then (42a) holds. □

Lemma 3.5. *For every $d\,(1 \leq d \leq l)$ and $r \geq 0$, there exists a constant C independent of h such that*

$$\frac{1}{L}\sum_{n=1}^{L}(\|\eta_d^n\|^2 + h^2\|\eta_d^n\|_h^2) \leq Ch^2 \sum_{i=d+1}^{L} \lambda_i. \tag{51}$$

Proof. In fact, $U^n = \sum_{i=1}^{L} a_s(U^n, \psi_i)\psi_i\ (n = 1, 2, \cdots, L)$ holds. So we have

$$\begin{aligned}\frac{1}{L}\sum_{n=1}^{L}\left|\left|\left| U^n - \sum_{i=1}^{d} a_s(U^n, \psi_i)\psi_i \right|\right|\right|^2 &= \frac{1}{L}\sum_{n=1}^{L}\left|\left|\left| \sum_{i=d+1}^{L} a_s(U^n, \psi_i)\psi_i \right|\right|\right|^2 \\ &= \frac{1}{L}\sum_{i=d+1}^{L}\sum_{n=1}^{L} a_s(U^n, \psi_i)^2 \\ &= \sum_{i=d+1}^{L} \frac{1}{\lambda_i}(\boldsymbol{A}\boldsymbol{v}_i)^T(\boldsymbol{A}\boldsymbol{v}_i) \\ &= \sum_{i=d+1}^{L} \lambda_i.\end{aligned} \tag{52}$$

From (40), for any $v_h \in V_h$ and $v_d \in V_d$, it satisfies

$$\begin{aligned} \|v_h - E_d v_h\|_h^2 &\le C a_s(v_h - E_d v_h, v_h - E_d v_h) \\ &\le C a_s(v_h - E_d v_h, v_h - v_d) \\ &\le C\|v_h - E_d v_h\|_h \|v_h - v_d\|_h. \end{aligned} \tag{53}$$

Taking $v_h = U^n$ and $v_d = \sum_{i=1}^{d}(u_h^n, \psi_i)_h \psi_i$, we have

$$\begin{aligned} \frac{1}{L}\sum_{n=1}^{L}\|\eta_d^n\|_h^2 &\le \frac{C}{L}\sum_{n=1}^{L}\|U^n - \sum_{i=1}^{d}(U^n, \psi_i)_h\psi_i\|_h^2 \\ &\le C\sum_{i=d+1}^{L}\lambda_i, \end{aligned} \tag{54}$$

which together with Lemma 3.4 implies (51). □

Lemma 3.6. *For every $n\,(L+1 \le n \le N)$, and $r \ge 0$, there exists a constant C independent of h such that*

$$\|\eta_d^n\|^2 + h^2\|\eta_d^n\|_h^2 \le C\left((\Delta t)^2 + h^{2r+2}\right). \tag{55}$$

Proof. Note that $E_h U^n = E_d U^n$. It follows from Lemma 2.1 and Lemma 2.6 that

$$\begin{aligned} \|\eta_d^n\|_h^2 &\le \|U^n - u^n\|_h^2 + \|u^n - E_h u^n\|_h^2 + \|E_h(u^n - U^n)\|_h^2 + \|E_h U^n - E_d U^n\|_h^2 \\ &\le C(\|u^n - Q_h u^n\|_h^2 + \|Q_h u^n - E_h u^n\|_h^2 + \|E_h u^n - U^n\|_h^2) \\ &\le C\left((\Delta t)^2 + h^{2r}\right). \end{aligned} \tag{56}$$

Combining (56) and Lemma 3.4, one can derive (55). □

Lemma 3.7. *If $r \ge 1$ and $\Delta t \le Ch$. . Let $U^n \in V_h\,(n = 1, 2, \cdots, N)$ be the numerical solution of (6), then there exist constants K_1 and K_2 such that*

$$\sum_{n=1}^{N}\|\partial_t U^n\|_\infty \Delta t \le K_1, \tag{57a}$$

$$\sum_{n=1}^{N}\|\partial_t U^n\|_\infty^2 \Delta t \le K_2. \tag{57b}$$

Proof. Using the triangle inequality and inverse inequality, we have

$$\begin{aligned} \sum_{n=1}^{N}\|\partial_t U^n\|_\infty^2\Delta t \le& \sum_{n=1}^{N}(\|\partial_t\theta_h^n\|_\infty^2 + \|\partial_t\tau_h^n\|_\infty^2 + \|\partial_t\eta_h^n\|_\infty^2 + \|\partial_t u^n\|_\infty^2)\Delta t \\ \le& C\Big(h^{-2}\sum_{n=1}^{N}\|\partial_t\theta_h^n\|^2\Delta t + h^{-2}\|\tau_{h,t}\|_{L^2(J;L^2(\Omega))}^2 \\ &+ \|\eta_{h,t}\|_{L^2(J;L^\infty(\Omega))}^2 + \|u_t\|_{\infty,\infty}^2\Big). \end{aligned} \tag{58}$$

If $r \ge 1$ and $\Delta t = O(h)$, it follows from (8) and (33) that

$$\sum_{n=1}^{N}\|\partial_t U^n\|_\infty^2\Delta t \le C\left(h^{-2}((\Delta t)^2 + h^{2r}) + \|u_t\|_{\infty,\infty}^2\right) \le K_2. \tag{59}$$

Moreover, (57a) can be estimated directly as

$$\sum_{n=1}^{N} \|\partial_t U^n\|_\infty \Delta t \le \frac{1}{2}(\sum_{n=1}^{N} \|\partial_t U^n\|_\infty^2 \Delta t + N\Delta t) \le K_1. \tag{60}$$

So (57a) and (57b) hold. □

Theorem 3.8. *If $\Delta t \le Ch$, and let $U^m \in V_h$ and $U_d^m \in V_d (m = 1, 2, \cdots, N)$ be the solutions of* (6) *and* (39), *respectively. Then there exists a constant C independent of h and Δt such that*

$$\|U^m - U_d^m\|^2 \le C\left(K(m) + h^2 \sum_{i=d+1}^{L} \lambda_i\right), \quad 1 \le m \le N, \tag{61}$$

where $K(m) = 0$ for $1 \le m \le L$ and $K(m) = (1 + (m - L)\Delta t)\left((\Delta t)^2 + h^{2r+2}\right)$ for $L + 1 \le m \le N$.

Proof. We make an induction hypothesis

$$\|U_d^n\|_\infty \le C, \quad n = 0, 1, \cdots, N. \tag{62}$$

In fact, for every $1 \le n \le L$, $\theta_d^n = 0$. Using the triangle inequality and Lemma 3.5, we have

$$\|U^n - U_d^n\|^2 \le \|\eta_d^n\|^2 + \|\theta_d^n\|^2 \le Ch^2 \sum_{i=d+1}^{L} \lambda_i. \tag{63}$$

By (11), we have

$$\|U_d^n\|_\infty \le \|U^n\|_\infty + \|\eta_d^n\|_\infty + \|\theta_d^n\|_\infty \le \|U^n\|_\infty + Ch^{-1} \cdot h^2 \sum_{i=d+1}^{L} \lambda_i \le C. \tag{64}$$

We can choose d that is sufficiently large and satisfies $\sum_{i=d+1}^{L} \lambda_i \le (\Delta t)^2 + h^{2r+2}$.

Subtracting (39) from (6), observing (40) and choosing test function $v = \theta_d^n$, we get

$$\begin{aligned} &(g(U_d^{n-1})\partial_t \theta_d^n, \theta_d^n) + a_s(\theta_d^n, \theta_d^n) \\ =& -((g(U^{n-1}) - g(U_d^{n-1}))\partial_t U^n, \theta_d^n) - (g(U_d^{n-1})\partial_t \eta_d^n, \theta_d^n). \end{aligned} \tag{65}$$

By using $a(a + b) = \frac{a^2-b^2}{2} + \frac{(a-b)^2}{2}$, we have

$$\begin{aligned} (g(U_d^{n-1})\partial_t \theta_d^n, \theta_d^n) =& \frac{1}{2\Delta t}(g(U_d^n)\theta_d^n, \theta_d^n) - \frac{1}{2\Delta t}(g(U_d^{n-1})\theta_d^{n-1}, \theta_d^{n-1}) \\ &+ \frac{\Delta t}{2}(g(U_d^{n-1})\partial_t \theta_d^n, \partial_t \theta_d^n) - \frac{1}{2\Delta t}\left(\left(g(U_d^n) - g(U_d^{n-1})\right)\theta_d^n, \theta_d^n\right). \end{aligned} \tag{66}$$

Multiplying by $2\Delta t$ and summing over time (from $L+1$ to m, $m \geq L+1$), we can rewrite (65) as

(67)
$$\begin{aligned}
&(g(U_d^m)\theta_d^n, \theta_d^n) + \Delta t \sum_{n=L+1}^{m} [2a_s(\theta_d^n, \theta_d^n) + \Delta t(g(U_d^{n-1})\partial_t\theta_d^n, \partial_t\theta_d^n)] \\
&= (g(U_d^0)\theta_d^0, \theta_d^0) + \sum_{n=L+1}^{m} \left(\left(g(U_d^n) - g(U_d^{n-1})\right)\theta_d^n, \theta_d^n\right) \\
&- \sum_{n=L+1}^{m} 2((g(U^{n-1}) - g(U_d^{n-1}))\partial_t U^n, \theta_d^n)\Delta t - 2\sum_{n=L+1}^{m} (g(U_d^{n-1})\partial_t\eta_d^n, \theta_d^n)\Delta t \\
&= (g(u_d^0)\theta_d^0, \theta_d^0) + P_1 + P_2 + P_3,
\end{aligned}$$

where P_1, P_2, P_3 represent the last three terms on the right-hand side of equation (67).

Denote the left-hand side terms of (67) by L_i $(i = 1, 2, 3)$, we have the estimate

$$\sum_{i=1}^{3} L_i \geq C\left(\|\theta_d^m\|^2 + \sum_{n=L+1}^{m} \|\theta_d^n\|_h^2\Delta t + \sum_{n=L+1}^{m} \|\partial_t\theta_d^n\|^2(\Delta t)^2\right), \tag{68}$$

Following (17), we have

$$\begin{aligned}
|P_1| =& |\sum_{n=L+1}^{m} \left(\int_{t_{n-1}}^{t_n} g'\left(\frac{t-t_{n-1}}{\Delta t}U_d^n - \frac{t-t_n}{\Delta t}U_d^{n-1}\right) dt \cdot \theta_d^n, \theta_d^n\right)| \\
&\leq C\sum_{n=L+1}^{m} \|\theta_d^n\|^2\Delta t.
\end{aligned} \tag{69}$$

We can rewrite P_2 as follows:

$$\begin{aligned}
P_2 =& -2\Delta t\sum_{n=L+1}^{m} ((g(U^{n-1}) - g(E_dU^{n-1}))\partial_t U^n, \theta_d^n) \\
&\quad - 2\Delta t\sum_{n=L+1}^{m} ((g(E_dU^{n-1}) - g(U_d^{n-1}))\partial_t U^n, \theta_d^n) \\
=& P_{2,1} + P_{2,2}.
\end{aligned} \tag{70}$$

If $r \geq 1$ and $\Delta t = O(h)$, it follows from (11), (62), Lemma 3.4 and Lemma 3.7 that

$$\begin{aligned}
|P_{2,1}| &\leq C\sum_{n=L+1}^{m} \|\partial_t U^n\|_\infty^2\|\eta_d^n\|_{-h}^2\Delta t + \varepsilon\sum_{n=L+1}^{m} \|\theta_d^n\|_h^2\Delta t \\
&\leq C\sum_{n=L+1}^{m} \|\eta_d^n\|_{-h}^2(\Delta t)^{-1} + \varepsilon\sum_{n=L+1}^{m} \|\theta_d^n\|_h^2\Delta t \\
&\leq Ch^2\sum_{n=L+1}^{m} \|\eta_d^n\|_h^2\Delta t + \varepsilon\sum_{n=L+1}^{m} \|\theta_d^n\|_h^2\Delta t,
\end{aligned} \tag{71}$$

$$\begin{aligned}
|P_3| &\leq C\sum_{n=L+1}^{m} \|\partial_t\eta_d^n\|_{-h}^2\Delta t + \epsilon\sum_{L+1}^{m} \|\theta_d^n\|_h^2\Delta t \\
&\leq Ch^2\sum_{n=L}^{m} \|\eta_d^n\|_h^2\Delta t + \varepsilon\sum_{L+1}^{m} \|\theta_d^n\|_h^2\Delta t.
\end{aligned}$$

Suppose there exist sequences $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m(a_i, b_i \geq 0, i = 1, 2, \cdots, m)$ such that

$$\sum_{n=L+1}^{m} a_n b_n \leq max_n a_n \sum_{n=1}^{m} b_n \leq \frac{C}{m}(\sum_{n=1}^{m} a_n)(\sum_{n=1}^{m} b_n) \tag{72}$$

and $\frac{1}{m} = O(\Delta t)$. Combining Lemma 3.7 and $\theta_d^n = 0$ for $n = 1, 2, \cdots, L$, we can obtain

(73)

$$\begin{aligned} |P_{2,2}| \leq C \sum_{n=L+1}^{m} \|\partial_t U^n\|_\infty \|\theta_d^n\|^2 \Delta t &\leq C(\sum_{n=L+1}^{m} \|\partial_t U^n\|_\infty \Delta t)(\sum_{n=L+1}^{m} \|\theta_d^n\|^2 \Delta t) \\ &\leq C \sum_{n=L+1}^{m} \|\theta_d^n\|^2 \Delta t. \end{aligned}$$

When ε is sufficiently small, the combination of (67) - (73) leads to

$$\begin{aligned} &\|\theta_d^m\|^2 + \sum_{n=L+1}^{m} \|\theta_d^n\|_h \Delta t + \sum_{n=L+1}^{m} \|\partial_t \theta_d^n\|^2 (\Delta t)^2 \\ \leq &C \left(h^2 \|\eta_d^L\|_h^2 \Delta t + \sum_{n=L+1}^{m} \|\theta_d^n\|^2 \Delta t + h^2 \sum_{n=L+1}^{m} \|\eta_d^n\|_h^2 \Delta t \right). \end{aligned} \tag{74}$$

It then follows from Lemma 3.5, Lemma 3.6 and the discrete Gronwalls inequality that

$$\begin{aligned} \|\theta_d^m\|^2 &\leq Ch^2 \left(\|\eta_d^L\|_h^2 \Delta t + \sum_{n=L+1}^{m} \|\eta_d^n\|_h^2 \Delta t \right) \\ &\leq C \left((m-L)\Delta t((\Delta t)^2 + h^{2r+2}) + \Delta t h^2 \sum_{i=d+1}^{L} \lambda_i \right). \end{aligned} \tag{75}$$

In other words, for every $L + 1 \leq m \leq N$, we have

$$\begin{aligned} \|U^m - U_d^m\|^2 &\leq \|\eta_d^m\|^2 + \|\theta_d^m\|^2 \\ &\leq C \left((1 + (m-L)\Delta t)((\Delta t)^2 + h^{2r+2}) + \Delta t h^2 \sum_{i=d+1}^{L} \lambda_i \right). \end{aligned} \tag{76}$$

Hence (61) can be obtained from (63) and (76). Finally, we need to complete the induction hypothesis (62). If $r \geq 0$ and $\Delta t = O(h)$, from (11), (76), we have

$$\|U_d^m\|_\infty \leq \|U^m\|_\infty + \|U^m - U_d^m\|_\infty \leq \|U^m\|_\infty + Ch^{-1} \cdot \left(\Delta t + h^{r+1}\right) \leq C, \tag{77}$$

which completes the proof. □

Theorem 3.9. *If* $\Delta t \leq Ch$. *Let* $U^m \in V_h$ *and* $U_d^m \in V_d$ *be the solutions of* (6) *and* (39)*, respectively. There exists a constant* C *independent of* h *and* Δt *such that*

$$\|U^m - U_d^m\|_h^2 \leq C \left(I(m) \sum_{i=d+1}^{L} \lambda_i + F(m) \right), \quad 1 \leq m \leq N, \tag{78}$$

where $I(m) = 1$ *for* $1 \leq m \leq L$ *and* $I(m) = \Delta t$ *for* $L + 1 \leq m \leq N$, $F(m) = 0$ *for* $1 \leq m \leq L$ *and* $F(m) = (m - L - 1)((\Delta t)^2 + h^{2r+2})\Delta t + (\Delta t)^2 + h^{2r}$ *for* $L + 1 \leq m \leq N$.

Proof. For every $1 \leq n \leq L$, we have

$$\|U^n - U_d^n\|_h^2 \leq \|\eta_d^n\|_h^2 + \|\theta_d^n\|_h^2 \leq C \sum_{i=d+1}^{L} \lambda_i. \tag{79}$$

Subtracting (39) from (6), observing(40) and choosing test function $v = \partial_t \theta_d^n$, we have

$$\begin{aligned}
&a_s(\theta_d^m, \theta_d^m) + 2\Delta t \sum_{n=L+1}^{m} (g(U_d^{n-1})\partial_t\theta_d^n, \partial_t\theta_d^n) + (\Delta t)^2 \sum_{n=L+1}^{m} a_s(\partial_t\theta_d^n, \partial_t\theta_d^n)\\
=&a_s(\theta_d^L, \theta_d^L) - 2\Delta t \sum_{n=L+1}^{m} ((g(U^{n-1}) - g(U_d^{n-1}))\partial_t U^n, \partial_t\theta_d^n)\\
&- 2\Delta t \sum_{n=L+1}^{m} (g(U_d^{n-1})\partial_t\eta_d^n, \partial_t\theta_d^n)\\
=&a_s(\theta_d^L, \theta_d^L) + \mathcal{P}_1 + \mathcal{P}_2.
\end{aligned} \tag{80}$$

where $\mathcal{P}_1$, $\mathcal{P}_2$ represent the last two terms on the right-hand side of equation (80).

Denote the left-hand side terms of (80) by $\mathcal{L}_i (i = 1, 2, 3)$, and we have

$$\sum_{i=1}^{3} \mathcal{L}_i \geq C \left(\|\theta_d^m\|_h^2 + \sum_{n=L+1}^{m} \|\partial_t\theta_d^n\|^2 \Delta t + \sum_{n=L+1}^{m} \|\partial_t\theta_d^n\|_h^2 (\Delta t)^2 \right). \tag{81}$$

The term $\mathcal{P}_1$ on the right-hand side of (80) can be rewritten as

$$\begin{aligned}
\mathcal{P}_1 =& -2\Delta t \sum_{n=L+1}^{m} ((g(U^{n-1}) - g(E_d U^{n-1}))\partial_t U^n, \partial_t\theta_d^n)\\
&- 2\Delta t \sum_{n=L+1}^{m} ((g(E_d U^{n-1}) - g(U_d^{n-1}))\partial_t U^n, \partial_t\theta_d^n)\\
=& \mathcal{P}_{1,1} + \mathcal{P}_{1,2}.
\end{aligned} \tag{82}$$

From (73), we have

$$\begin{aligned}
|\mathcal{P}_{1,1}| &\leq C \sum_{n=L+1}^{m} \|\partial_t U^n\|_\infty^2 \|\eta_d^{n-1}\|^2 \Delta t + \varepsilon \sum_{n=L+1}^{m} \|\partial_t\theta_d^n\|^2 \Delta t\\
&\leq C(\sum_{n=L+1}^{m} \|\partial_t U^n\|_\infty^2 \Delta t)(\sum_{n=L+1}^{m} \|\eta_d^{n-1}\|^2 \Delta t) + \varepsilon \sum_{n=L+1}^{m} \|\partial_t\theta_d^n\|^2 \Delta t\\
&\leq C \sum_{n=L+1}^{m} \|\eta_d^{n-1}\|^2 \Delta t + \varepsilon \sum_{n=L+1}^{m} \|\partial_t\theta_d^n\|^2 \Delta t.
\end{aligned} \tag{83}$$

By Lemma 2.2 and the fact that $\theta_d^L = 0$, we have

$$\begin{aligned}
|\mathcal{P}_{1,2}| &\leq C(\sum_{n=L+1}^{m} \|\partial_t U^n\|_\infty^2 \Delta t)(\sum_{n=L+1}^{m} \|\theta_d^{n-1}\|_h^2 \Delta t) + \varepsilon \sum_{n=L+1}^{m} \|\partial_t\theta_d^n\|^2 \Delta t\\
&\leq C \sum_{n=L+1}^{m} \|\theta_d^n\|_h^2 \Delta t + \varepsilon \sum_{n=L+1}^{m} \|\partial_t\theta_d^n\|^2 \Delta t.
\end{aligned} \tag{84}$$

From (62), we have

$$|\mathcal{P}_2| \le C \sum_{n=L+1}^{m} \|\partial_t \eta_d^n\|^2 \Delta t + \varepsilon \sum_{n=L+1}^{m} \|\partial_t \theta_d^n\|^2 \Delta t. \tag{85}$$

When ε is sufficiently small, the combination of (80) - (85) leads to

$$\begin{aligned} &\|\theta_d^m\|_h^2 + \sum_{n=L+1}^{m} \|\partial_t \theta_d^n\|^2 \Delta t + \sum_{n=L+1}^{m} \|\partial_t \theta_d^n\|_h^2 (\Delta t)^2 \\ \le& C \left(\|\theta_d^L\|_h + \sum_{n=L+1}^{m} \|\eta_d^{n-1}\|^2 \Delta t + \sum_{n=L+1}^{m} \|\partial_t \eta_d^n\|^2 \Delta t + \sum_{n=L+1}^{m} \|\theta_d^n\|_h^2 \Delta t \right). \end{aligned} \tag{86}$$

Note that

$$\begin{aligned} \|\partial_t \eta_d^n\| \le& \|\partial_t (U^n - u^n)\| + \|\partial_t (u^n - E_h u^n)\| \\ &+ \|\partial_t (E_h(u^n - U^n))\| + \|\partial_t (E_h U^n - E_d U^n)\| \\ \le& C(\|\partial_t \theta_h^n\| + \|\partial_t \eta_h^n\| + \|\partial_t \tau_h^n\|), \end{aligned} \tag{87}$$

which together with (8), (33) implies

$$\begin{aligned} \sum_{n=L+1}^{m} \|\partial_t \eta_d^n\|^2 \Delta t \le& C(\sum_{n=1}^{m} \|\partial_t \theta_h^n\|^2 \Delta t + \sum_{n=1}^{m} \|\partial_t \eta_h^n\|^2 + \sum_{n=1}^{m} \|\partial_t \tau_h^n\|^2) \\ \le& C((\Delta t)^2 + h^{2r}). \end{aligned} \tag{88}$$

Then by Lemma 3.6 and the discrete Gronwall's inequality, we obtain

$$\begin{aligned} &\|\theta_d^m\|_h^2 + \Delta t \sum_{n=L+1}^{m} \|\partial_t \theta_d^n\|^2 + (\Delta t)^2 \sum_{n=L+1}^{m} \|\partial_t \theta_d^n\|_h \\ \le& C\left((m-L)((\Delta t)^2 + h^{2r+2})\Delta t + (\Delta t)^2 + h^{2r}\right), \end{aligned} \tag{89}$$

In other words, for every $L+1 \le m \le N$, we have

$$\begin{aligned} \|U^m - U_d^m\|_h^2 \le& \|\theta_d^m\|_h^2 + \|\eta_d^m\|_h^2 \\ \le& C\left(\Delta t \sum_{i=d+1}^{L} \lambda_i + (m-L-1)((\Delta t)^2 + h^{2r+2})\Delta t + (\Delta t)^2 + h^{2r}\right), \end{aligned} \tag{90}$$

Finally, (78) can be obtained from (79) and (90). □

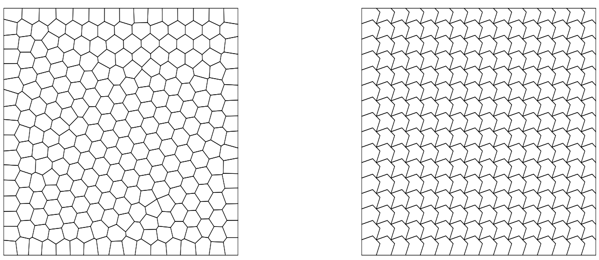

FIGURE 1. 16×16 meshes for all examples, left: Voronoi mesh, right: Nonconv mesh.

Algorithm 1: ROWG algorithm

Step 1 : Generate the snapshots $U^m = \{U_0^m, U_b^m\}, m = 1, 2, \cdots, L$ by solving the WG scheme (6).
Step 2 : Calculate the correlation matrix $\mathbf{A} = (A_{ij})_{L\times L}$ according to (37).
Step 3 : Find the positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \lambda_L > 0$ of $\mathbf{A}$ and the corresponding standard orthonormal eigenvectors
$\mathbf{v}_i = (v_1^i, v_2^i, \cdots, v_L^i)^T, 1 \leq i \leq L$.
Step 4 : Determine the number of POD bases d such that
$\sum_{i=d+1}^{L} \lambda_i \leq (\Delta t)^2 + h^{2r+2}$.
Step 5 : Construct the POD basis $\boldsymbol{\psi_i} = \{\psi_{i0}, \psi_{ib}\}$ by (38) and denote the POD space by $V_d = span\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \cdots \boldsymbol{\psi}_d\}$.
Step 6 : Solve the ROWG scheme (39), and then the numerical solution $\boldsymbol{U}_d^m \in \boldsymbol{V}_d$ for each time layer is obtained.
Step 7 : If $\|\boldsymbol{U}_{d,0}^{m+1} - \boldsymbol{U}_{d,0}^m\| \leq \|\boldsymbol{U}_{d,0}^m - \boldsymbol{U}_{d,0}^{m-1}\|, m = L, L+1, \cdots N-1$, then $\boldsymbol{U}_d^m (m = L+1, \cdots, N)$ are the ROWG solutions satisfying the desirable accuracy. Else extract the new snapshots $U^{m+j-L}, j = 1, 2 \cdots L$ and return to **Step 2**.

4. Numerical experiments

In this section, we show three numerical examples to verify the accuracy and efficiency of the POD weak finite element method. We perform numerical simulations on domain $\Omega = [0,1] \times [0,1]$, $T = 1$. We take $r = 1$ and $r = 2$ for the discrete weak finite element function space $WG(P_r, P_{r-1}; P_{r-1}^2)$. When $r = 1$, we take $\Delta t = 500$ and $L = 20$; when $r = 2$, we take $\Delta t = 1000$ and $L = 30$.

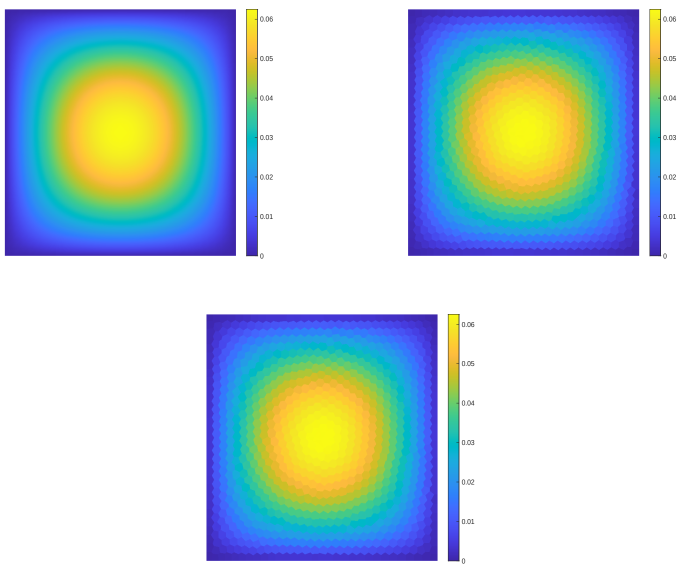

FIGURE 2. Comparison of analytic solution, numerical solution of WG scheme, and numerical solution of the ROWG scheme for Example 4.2 at $t = 1$ on Voronoi mesh.

TABLE 1. Error rate and CPU time of two methods for Example 4.1 on Voronoi mesh. Top half: $r = 1$, Bottom half: $r = 2$.

	ROWG method					WG method				
$\mathcal{T}_h$	$\|u-U\|$	rate	$\|Q_h u-U\|$	rate	CPU time	$\|u-U\|$	rate	$\|Q_h u-U\|$	rate	CPU time
4×4	8.9410E-02	-	7.0792E-01	-	6.49s	8.9417E-02	-	7.0783E-01	-	6.08s
8×8	2.2418E-02	2.00	3.3156E-01	1.09	23.61s	2.2421E-02	2.00	3.3161E-01	1.09	24.03s
16×16	5.5185E-03	2.02	1.6225E-01	1.03	101.64s	5.5191E-03	2.02	1.6226E-01	1.03	166.04s
32×32	1.3536E-03	2.03	8.0335E-02	1.01	471.54s	1.3537E-03	2.03	8.0334E-02	1.01	1849.68s
64×64	3.5407E-04	1.93	4.0829E-02	0.98	4043.95s	3.5411E-04	1.93	4.0829E-02	0.98	55560.67s
4×4	9.1622E-03	-	1.9029E-01	-	13.53s	9.1665E-03	-	1.9033E-01	-	13.89s
8×8	1.2073E-03	2.92	5.0245E-02	1.92	57.20s	1.2073E-03	2.92	5.0249E-02	1.92	69.11s
16×16	1.4154E-04	3.09	1.1839E-02	2.09	310.13s	1.4154E-04	3.09	1.1839E-02	2.09	794.52s
32×32	1.6760E-05	3.08	2.7970E-03	2.08	1422.25s	1.6800E-05	3.07	2.7979E-03	2.08	16018.09s

TABLE 2. Error rate and CPU time of two methods for Example 4.2 on Voronoi mesh. Top half: $r = 1$, Bottom half: $r = 2$.

	ROWG method					WG method				
$\mathcal{T}_h$	$\|u-U\|$	rate	$\|Q_h u-U\|$	rate	CPU time	$\|u-U\|$	rate	$\|Q_h u-U\|$	rate	CPU time
4×4	1.5190E-02	-	1.3023E-01	-	5.16s	1.7641E-02	-	1.4016E-01	-	5.06s
8×8	4.1366E-03	1.88	6.6843E-02	0.96	20.80s	4.6423E-03	1.93	6.9855E-02	1.00	23.58s
16×16	1.0868E-03	1.93	3.4483E-02	0.95	99.29s	1.1347E-03	2.03	3.4374E-02	1.02	162.36s
32×32	2.7601E-04	1.98	1.7481E-02	0.98	477.60s	2.7149E-04	2.06	1.6914E-02	1.02	1873.10s
64×64	6.9250E-05	1.99	8.7831E-03	0.99	4022.36s	6.9146E-05	1.97	8.5624E-03	0.98	56105.72s
4×4	9.1622E-03	-	1.9029E-01	-	13.53s	9.1665E-03	-	1.9033E-01	-	13.89s
8×8	1.2073E-03	2.92	5.0245E-02	1.92	57.20s	1.2073E-03	2.92	5.0249E-02	1.92	69.11s
16×16	1.4154E-04	3.09	1.1839E-02	2.09	310.13s	1.4154E-04	3.09	1.1839E-02	2.09	794.52s
32×32	1.6760E-05	3.08	2.7970E-03	2.08	1422.25s	1.6800E-05	3.07	2.7979E-03	2.08	16018.09s

TABLE 3. Error rate and CPU time of two methods for Example 4.3 on Voronoi mesh. Top half: $r = 1$, Bottom half: $r = 2$.

	ROWG method					WG method				
$\mathcal{T}_h$	$\|u-U\|$	rate	$\|Q_hu-U\|$	rate	CPU time	$\|u-U\|$	rate	$\|Q_hu-U\|$	rate	CPU time
4×4	5.7476E-02	-	5.2630E-01	-	5.16s	6.8925E-02	-	5.6308E-01	-	5.56 s
8×8	1.4515E-02	1.99	2.4516E-01	1.10	21.64s	1.6254E-02	2.08	2.5328E-01	1.15	25.68 s
16×16	3.7986E-03	1.93	1.2351E-01	0.99	103.82s	3.9469E-03	2.04	1.2348E-01	1.04	175.31 s
32×32	9.6352E-04	1.98	6.1225E-02	1.01	516.13s	9.9284E-04	1.99	6.1830E-02	1.00	2047.37 s
64×64	2.4787E-04	1.96	3.0611E-02	1.00	4571.39s	2.5916E-04	1.94	3.1204E-02	0.99	56165.14 s
4×4	7.5079E-03	-	1.5627E-01	-	11.76s	7.8418E-03	-	1.6343E-01	-	13.10s
8×8	1.1140E-03	2.75	4.1925E-02	1.90	54.04s	1.0100E-03	2.96	4.2176E-02	1.95	71.72s
16×16	1.3273E-04	3.07	1.0074E-02	2.06	315.46s	1.2132E-04	3.06	1.0098E-02	2.06	799.10s
32×32	1.7975E-05	2.88	2.4078E-03	2.06	1574.28s	1.4974E-05	3.02	2.4098E-03	2.07	16034.55s

TABLE 4. Error rate and CPU time of two methods for Example 4.1 on Nonconv mesh. Top half: $r = 1$, Bottom half: $r = 2$.

	ROWG method					WG method				
$\mathcal{T}_h$	$\|u-U\|$	rate	$\|Q_hu-U\|$	rate	CPU time	$\|u-U\|$	rate	$\|Q_hu-U\|$	rate	CPU time
4×4	7.7582E-02	-	6.5292E-01	-	12.77 s	7.7607E-02	-	6.5342E-01	-	12.23 s
8×8	2.0534E-02	1.92	3.2223E-01	1.02	45.96 s	2.0536E-02	1.92	3.2228E-01	1.02	47.76 s
16×16	5.3462E-03	1.94	1.6392E-01	0.98	196.48 s	5.3466E-03	1.94	1.6392E-01	0.98	283.40 s
32×32	1.3612E-03	1.97	8.2724E-02	0.99	861.57 s	1.3613E-03	1.97	8.2723E-02	0.99	2849.33 s
64×64	3.4715E-04	1.97	4.1522E-02	0.99	7415.64 s	3.4717E-04	1.97	4.1522E-02	0.99	93801.36 s
4×4	9.4349E-03	-	1.9942E-01	-	24.20 s	9.4509E-03	-	1.9925E-01	-	26.89 s
8×8	1.0081E-03	3.23	4.8025E-02	2.05	99.45 s	1.0083E-03	3.23	4.8019E-02	2.05	126.98 s
16×16	1.1601E-04	3.12	1.1620E-02	2.05	451.10 s	1.1601E-04	3.12	1.1620E-02	2.05	1259.54 s
32×32	1.4597E-05	2.99	2.8598E-03	2.02	2618.36 s	1.4607E-05	2.99	2.8597E-03	2.02	28240.32 s

TABLE 5. Error rate and CPU time of two methods for Example 4.2 on Nonconv mesh. Top half: $r = 1$, Bottom half: $r = 2$.

	ROWG method					WG method				
$\mathcal{T}_h$	$\|u-U\|$	rate	$\|Q_h u-U\|$	rate	CPU time	$\|u-U\|$	rate	$\|Q_h u-U\|$	rate	CPU time
4×4	1.7641E-02	-	1.4017E-01	-	11.44 s	1.5191E-02	-	1.3023E-01	-	10.29 s
8×8	4.6419E-03	1.93	6.9838E-02	1.01	44.58 s	4.1371E-03	1.88	6.6864E-02	0.96	44.31 s
16×16	1.1346E-03	2.03	3.4373E-02	1.02	203.79 s	1.0869E-03	1.93	3.4484E-02	0.96	254.24 s
32×32	2.7147E-04	2.06	1.6915E-02	1.02	864.21 s	2.7602E-04	1.98	1.7481E-02	0.98	2748.94 s
64×64	6.9140E-05	1.97	8.5624E-03	0.98	7037.13 s	6.9253E-05	1.99	8.7832E-03	0.99	95894.49 s
4×4	1.6074E-03	-	3.2798E-02	-	23.20 s	1.6081E-03	-	3.2789E-02	-	25.91 s
8×8	1.7086E-04	3.23	8.4032E-03	1.96	95.37 s	1.7092E-04	3.23	8.4039E-03	1.96	133.94 s
16×16	1.8797E-05	3.18	2.0867E-03	2.01	442.34 s	1.8798E-05	3.18	2.0867E-03	2.01	1339.29 s
32×32	2.1724E-06	3.11	5.1834E-04	2.01	2570.49 s	2.1724E-06	3.11	5.1834E-04	2.01	28113.18 s

TABLE 6. Error rate and CPU time of two methods for Example 4.3 on Nonconv mesh. Top half: $r = 1$, Bottom half: $r = 2$.

	ROWG method					WG method				
$\mathcal{T}_h$	$\|u-U\|$	rate	$\|Q_h u-U\|$	rate	CPU time	$\|u-U\|$	rate	$\|Q_h u-U\|$	rate	CPU time
4×4	6.6245E-02	-	5.7702E-01	-	12.73 s	6.1225E-02	-	5.1646E-01	-	11.83 s
8×8	1.5900E-02	2.06	2.5459E-01	1.18	45.17 s	1.4771E-02	2.05	2.4381E-01	1.08	47.59 s
16×16	4.0388E-03	1.98	1.2514E-01	1.02	196.14 s	3.7181E-03	1.99	1.2173E-01	1.00	287.43 s
32×32	1.0278E-03	1.97	6.2078E-02	1.01	870.26 s	9.3822E-04	1.99	6.1056E-02	1.00	2742.40 s
64×64	2.8588E-04	1.85	3.1254E-02	0.99	8116.43 s	2.4122E-04	1.96	3.0590E-02	1.00	95012.61 s
4×4	8.2697E-03	-	1.6620E-01	-	23.30 s	8.2293E-03	-	1.6634E-01	-	27.04 s
8×8	1.1066E-03	2.90	3.9073E-02	2.09	98.00 s	9.1325E-04	3.17	4.0298E-02	2.05	154.01 s
16×16	1.0890E-04	3.35	9.6076E-03	2.02	461.01 s	1.0406E-04	3.13	9.7244E-03	2.05	1370.65 s
32×32	1.4810E-05	2.88	2.3616E-03	2.02	2642.94 s	1.3257E-05	2.97	2.3868E-03	2.03	28487.63 s

Example 4.1. *In equation* (1)*, let the nonlinear capacity term* $g(u) = \exp(0.1u)$*, and* $D = \begin{pmatrix} x^2+1 & 0 \\ 0 & y^2+1 \end{pmatrix}$*. The exact solution is* $u(x,t) = \exp(-t)\sin(\pi x)\sin(\pi y)$ *, and the right-hand side term can be calculated accordingly.*

Example 4.2. *In equation* (1)*, let the nonlinear capacity term* $g(u) = u^2 + 1$*, and* $D = \begin{pmatrix} x+1 & 0 \\ 0 & y+1 \end{pmatrix}$*. The exact solution is* $u(x,t) = tx(x-1)y(y-1)$*, and the right-hand term can be calculated accordingly.*

Example 4.3. *In equation* (1)*, let the nonlinear capacity term* $g(u,x,y) = \cos(\pi^2 t) \cdot \cos(\pi x)\cos(\pi y)u + 1$*, and* $D = \begin{pmatrix} \cos(\pi^2 t)+2 & \cos(\pi^2 t)+1 \\ \cos(\pi^2 t)+1 & \cos(\pi^2 t)+2 \end{pmatrix}$*. The exact solution is* $u(x,t) = \exp(-t)\sin(\pi x)\sin(\pi y)$*, and the right-hand term can be calculated accordingly.*

In order to illustrate the robustness of numerical schemes, we perform numerical simulations on two types of meshes–Voronoi and nonconv meshes (see Fig. 1). Note that there are hanging points in the latter one. Both the WG and ROWG schemes perform well on both meshes, reaching desired error rates under L^2 norm and the discrete H^1 norm. A comparison of the numerical solutions of WG, ROWG methods and the exact solution for Example 4.2 on Voronoi mesh are shown in Fig. 2.

From Table 1–Table 6, one can observe that for all three examples, the errors of the ROWG algorithm are roughly the same order of magnitude as that of WG scheme, while the ROWG scheme greatly reduces the CPU time without affecting the error accuracy. Moreover, the effectiveness of ROWG algorithm becomes more obvious as the size of algebraic equation gets larger. The convergence orders are consistent with the theoretical results given in previous sections. More precisely, when $r = 1$, the convergence orders under the L^2 norm and the discrete H^1 norm are 2 and 1, respectively; when $r = 2$, the convergence orders under the L^2 norm and the discrete H^1 norm are 3 and 2, respectively.

Acknowledgments

The authors' research is supported in part by National Natural Science Foundation of China grant no. 11871038, 12131014 and 12271214, and Guangdong Basic and Applied Basic Research Foundation, grant no. 2023A1515012209.

References

[1] An, J., Luo, Z.D., A reduced finite difference scheme based on pod bases and posteriori error estimation for the three dimensional parabolic equation. Acta Mathematica Scientia 31, (2011) 769–775.

[2] Chen, L., Wu, F., Zeng, G., Two-grid weak Galerkin method for semilinear elliptic differential equations. Mathematical Methods in the Applied Sciences 46, (2023) 423–437.

[3] Chen, W., Wang, F., Wang, Y., Weak Galerkin method for the coupled darcy–stokes flow. IMA Journal of Numerical Analysis 36, (2016) 897–921.

[4] Chen, Y., Chen, L., Zhang, X., Two-grid method for nonlinear parabolic equations by expanded mixed finite element methods. Numerical Methods for Partial Differential Equations 29, (2013) 1238–1256.

[5] Di, Z., Luo, Z., Xie, Z., Wang, A., A reduced finite element scheme based on pod two-dimensional unsaturated soil water flow equation. Beijing Jiaotong Daxue Xuebao(Journal of Beijing Jiaotong University) 35, (2011) 142–149.

[6] Hong, L., Zhendong, L., Junqiang, G., A new reduced-order fve algorithm based on pod method for viscoelastic equations. Acta Mathematica Scientia 33, (2013) 1076–1098.

[7] Kunisch, K., Volkwein, S., Galerkin proper orthogonal decomposition methods for parabolic problems. Numerische Mathematik 90, (2001) 117–148.

[8] Lin, G., Liu, J., Mu, L., Ye, X., Weak Galerkin finite element methods for darcy flow: Anisotropy and heterogeneity. Journal of Computational Physics 276, (2014) 422–437.

[9] Luo, Z., A reduced-order extrapolation algorithm based on sfve method and pod technique for non-stationary stokes equations. Applied Mathematics and Computation 247, (2014) 976–995.

[10] Luo, Z., Chen, G., Proper orthogonal decomposition methods for partial differential equations. Academic Press, 2018.

[11] Mu, L., Wang, J., Ye, X., Weak Galerkin finite element methods on polytopal meshes. (2012) arXiv preprint arXiv:1204.3655 .

[12] Mu, L., Wang, J., Ye, X., A weak Galerkin finite element method with polynomial reduction. Journal of Computational and Applied Mathematics 285, (2015) 45–58.

[13] Song, J., Rui, H., A reduced-order finite element method based on pod for the incompressible miscible displacement problem. Computers & Mathematics with Applications 98, (2021) 99–117.

[14] Wang, J., Ye, X., A weak Galerkin finite element method for second-order elliptic problems. Journal of Computational and Applied Mathematics 241, (2013) 103–115.

[15] Wang, J., Ye, X., A weak Galerkin mixed finite element method for second order elliptic problems. Mathematics of Computation 83, (2014) 2101–2126.

[16] Zhang, H., Yang, M., Wei, J., Nie, Y., Stability and error analysis of reduced-order methods based on pod with finite element solutions for nonlocal diffusion problems. Digital Engineering and Digital Twin 2, (2024) 49–3277.

[17] Zhao, J., Rui, H., Song, J., A reduced-order weak Galerkin finite element algorithm based on pod technique for parabolic problem on polytopal mesh. Applied Mathematics Letters 127, (2022) 107842.

[18] Zhu, P., Xie, S., A weak Galerkin method and its two-grid algorithm for the quasi-linear elliptic problems of non-monotone type. Numerical Methods for Partial Differential Equations 39, (2023) 1042–1066.

School of Mathematics, Shandong University, Jinan, Shandong 250100, China

Department of Mathematics, Jinan University, Guangzhou 510632, China
E-mail: cuijintao@jnu.edu.cn