INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 4, Pages 587–608

doi: 10.4208/ijnam2024-1023

THE WEAK GALERKIN FINITE ELEMENT METHOD FOR THE DUAL-POROSITY-STOKES MODEL

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Abstract. In this paper, we introduce a weak Galerkin finite element method for the dual-porosity-Stokes model. The dual-porosity-Stokes model couples the dual-porosity equations with the Stokes equations through four interface conditions. In this method, we define several weak Galerkin finite element spaces and weak differential operators. We provide the weak Galerkin scheme for the model, and establish the well-posedness of the numerical scheme. The optimal convergence orders of errors in the energy norm are derived. Finally, we verify the effectiveness of the numerical method with different weak Galerkin elements on different meshes.

Key words. Dual-porosity-Stokes model, weak Galerkin finite element method, discrete weak gradient, discrete weak divergence.

1. Introduction

In practical production and daily life, the coupling models of porous media flow and free flow are widely used, such as groundwater system [16], industrial filtration [13], oil exploitation [8], and biochemical transportation [11], etc. The classic Stokes-Darcy model is used frequently in the coupling flow problems [3, 5, 6, 12, 20]. However, Darcy's law is only available for single-pore model and cannot accurately describe complex porous medium model with multiple porosities, which arises, such as the hydrology and geothermal systems. Therefore, Hou et al. proposed a dual-porosity-Stokes model in [17], the authors used the matrix pressure equation to characterize the flow in the matrix medium and the microfracture pressure equation in the microfractures medium, respectively. The free flow in conduits and microfractures are governed by Stokes equations. For appropriate coupling, four physical conditions are imposed on the interface: the no-exchange condition, mass conservation condition, force balance condition, and the Beavers-Joseph-Saffman (BJS) condition.

Some efforts have been made to numerically solve the dual-porosity-Stokes model. In [1], Al Mahbub et al. proposed and analyzed two stabilized mixed finite element methods for the nonstationary dual-porosity-Stokes model: the coupled method in traditional formulation and the decoupled method based on the partition time stepping method. Then, in [2], they developed a stabilized mixed finite element method for the stationary dual-porosity-Stokes model. This method only needs to add a mesh-dependent stabilization term to ensure the numerical stability of the algorithm and does not introduce any Lagrange multiplier. Combining the IPDG method and mixed finite element method, Wen et al. [37] designed a monolithic scheme with strong mass conservation for the stationary coupled model. Gao et al. considered the Navier-Stokes equation in the free flow region to couple the microfracture-matrix system in [14]. Yang et al gave a prior estimate of the discrete solutions to the stationary dual-porosity Navier-Stokes model by constructing an

Received by the editors on January 16, 2024 and, accepted on May 14, 2024.
2000 *Mathematics Subject Classification.* 65M60, 65M12, 65M15, 35M10, 35Q35, 76D07, 76S05.
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auxiliary problem and proved the existence and uniqueness of the discrete solutions in [38].

In this paper, we introduce the weak Galerkin (WG) finite element method for dual-porosity-Stokes model. The WG method was proposed by Wang and Ye in [31] for solving the second-order elliptic problem. The key idea of this method is that the solutions are approximated by discontinuous weak functions and the classical derivative operators in variational formulation are replaced by weakly defined derivative operators. At present it has many applications including parabolic equation [21, 40], Darcy equation [24, 25], Stokes equations [28, 33], Brinkman equations [26, 39], linear elasticity equations [35, 36], and so on.

For the coupled problem, as far as we know, there is some work for Stokes-Darcy model. In [9], WG finite element discretization was constructed for the Stokes equations with symmetric stress tensor and the Darcy equation in the mixed formulation. In [22, 23], the Stokes equations coupled with the Darcy equation in the primal formulation were investigated. The model is discretized by piecewise constants in [23] and high order polynomials in [22], yielding stable numerical schemes with optimal error estimates. Some methods combining the WG elements with other finite elements are discussed in [15, 29, 30].

The dual-porosity-Stokes model consists of two second-order elliptic equations in the dual-porosity domain and Stokes equation in the free flow region. The existing work has verified the efficiency of the individual Stokes equations and elliptic equations. Therefore, in this paper, we develop the WG method for the coupled model. We establish the stability of the WG scheme and prove the existence and uniqueness of the numerical solutions. Furthermore, the optimal convergence orders for the errors are obtained. The results of numerical experiments are consistent with the theoretical analysis.

The rest of the paper is organized as follows. In Section 2, we introduce the dual-porosity-Stokes model and present its variational form. In Section 3, some definitions of the weak Galerkin finite element spaces are given and then the WG numerical scheme for the coupled model is established. In Section 4, we prove the existence and uniqueness of the WG numerical solutions. In Section 5, the error equations and the corresponding optimal order error estimates are obtained. Finally, in Section 6, we present some numerical examples to verify the effectiveness of the WG method.

2. Preliminaries

In this section, we introduce the dual-porosity-Stokes model and present the corresponding variational formulation.

2.1. Dual-porosity-Stokes Model.

Let Ω be a bounded domain in $\mathbb{R}^N$, $(N = 2, 3)$, which is divided into two subdomains, the dual-porosity domain Ω_d and the conduit domain Ω_c (see Figure 1). Let $\Gamma = \partial\Omega_c \cap \partial\Omega_d$ be the interface between two subdomains. Denote the boundaries of Ω_d and Ω_c by $\Gamma_d = \partial\Omega_d \backslash \Gamma$ and $\Gamma_c = \partial\Omega_c \backslash \Gamma$, respectively. In addition, $\mathbf{n}_{cd}$ is the unit normal vector on Γ which points from Ω_c into Ω_d and $\boldsymbol{\tau}_j,\ j = 1, 2, ..., N-1$ are unit tangent vectors on Γ.

The flow in the dual-porosity domain Ω_d is governed by the traditional dual-porosity model [18], which consists of matrix equation and microfracture equation.

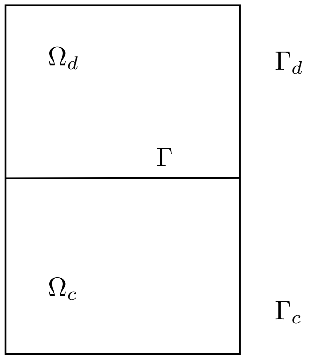

FIGURE 1. A sketch of the dual-porosity-Stokes model.

$$-\nabla\cdot\left(\frac{k_m}{\mu}\nabla p_m\right)=-Q, \tag{1}$$

$$-\nabla\cdot\left(\frac{k_f}{\mu}\nabla p_f\right)=Q+q_p, \tag{2}$$

where $Q=\frac{\sigma k_m}{\mu}(p_m-p_f)$ is mass exchange term between the matrix and microfracture, σ is a shape factor, p_m and p_f are the pressure functions in matrix and microfracture, respectively. k_m and k_f are the intrinsic permeability in the matrix and microfracture, respectively. μ is the dynamic viscosity and q_p is the source term.

In the conduit domain Ω_c, the flow is described by the Stokes equations,

$$-\nabla\cdot\mathbb{T}(\mathbf{u},p)=\mathbf{f}, \tag{3}$$

$$\nabla\cdot\mathbf{u}=0, \tag{4}$$

where $\mathbf{u}$ is the flow velocity function and p is the pressure function. $\mathbb{T}(\mathbf{u},p):=2\nu\mathbb{D}(\mathbf{u})-\frac{1}{\rho}p\mathbb{I}$ is the stress tensor, where $\mathbb{D}(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+\nabla^{\mathrm{T}}\mathbf{u})$ is the deformation tensor. $\mathbb{I}$ is the identity matrix, ρ is the fluid density and ν is the viscosity coefficient of the fluid. $\mathbf{f}$ is the given external force.

On the interface Γ, we consider the following four interface conditions,

$$-\frac{k_m}{\mu}\nabla p_m\cdot(-\mathbf{n}_{cd})=0, \tag{5}$$

$$-\frac{k_f}{\mu}\nabla p_f\cdot\mathbf{n}_{cd}=\mathbf{u}\cdot\mathbf{n}_{cd}, \tag{6}$$

$$-\mathbf{n}_{cd}^{\mathrm{T}}\left(\mathbb{T}(\mathbf{u},p)\mathbf{n}_{cd}\right)=\frac{p_f}{\rho}, \tag{7}$$

$$-P_\tau\left(\mathbb{T}(\mathbf{u},p)\mathbf{n}_{cd}\right)=\frac{\alpha\nu\sqrt{N}}{\sqrt{trace(\Pi)}}P_\tau(\mathbf{u}), \tag{8}$$

where $P_\tau(\mathbf{u})=\sum\limits_{j=1}^{N-1}(\mathbf{u}\cdot\boldsymbol{\tau}_j)\boldsymbol{\tau}_j$ represents the projection on the local tangential plane of Γ and α is a constant parameter. $\Pi=k_f\mathbb{I}$ is the intrinsic permeability of the microfracture. The interface condition (5) represents that there is no exchange between matrix and microfracture. Eq.(6) describes the mass conservation between microfracture and conduits. Eq.(7) represents the balance of two driving forces [7, 10], and Eq.(8) is the Beavers-Joseph-Saffman interface condition [19].

For simplicity, the Dirichlet boundary conditions are applied to the boundary of the domain Ω.

$$p_m = p_m^{dir},\ p_f = p_f^{dir} \quad \text{on } \Gamma_d, \tag{9}$$

$$\mathbf{u} = \mathbf{u}^{dir}, \qquad \text{on } \Gamma_c. \tag{10}$$

2.2. Variational Formulation. First, we define the following Sobolev spaces.

$$H^1_{0,\Gamma_d}(\Omega_d) = \{\psi \in H^1(\Omega_d) : \psi = 0 \text{ on } \Gamma_d\},$$

and

$$[H^1_{0,\Gamma_c}(\Omega_c)]^N = \{\mathbf{v} \in [H^1(\Omega_c)]^N : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_c\}.$$

Then, we give the variational formulation of the dual-porosity-Stokes model: find $p_m \in H^1(\Omega_d)$, $p_f \in H^1(\Omega_d)$, $\mathbf{u} \in [H^1(\Omega_c)]^N$ and $p \in L^2(\Omega_c)$ to satisfy $p_m = p_m^{dir}$, $p_f = p_f^{dir}$ on Γ_d, $\mathbf{u} = \mathbf{u}^{dir}$ on Γ_c and the equations

$$\begin{aligned}
&\left(\frac{k_m}{\mu}\nabla p_m, \nabla\psi_m\right)_{\Omega_d} + \left(\frac{\sigma k_m}{\mu}(p_m - p_f), \psi_m\right)_{\Omega_d} + \left(\frac{k_f}{\mu}\nabla p_f, \nabla\psi_f\right)_{\Omega_d} \\
&+\left(\frac{\sigma k_m}{\mu}(p_f - p_m), \psi_f\right)_{\Omega_d} + \left\langle \frac{p_f}{\rho}, \mathbf{v}\cdot\mathbf{n}_{cd}\right\rangle_\Gamma - \left\langle \mathbf{u}\cdot\mathbf{n}_{cd}, \psi_f\right\rangle_\Gamma \\
&+(2\nu\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v}))_{\Omega_c} + \left\langle \frac{\alpha\nu\sqrt{N}}{\sqrt{trace(\Pi)}} P_\tau(\mathbf{u}), P_\tau(\mathbf{v})\right\rangle_\Gamma - \frac{1}{\rho}(\nabla\cdot\mathbf{v}, p)_{\Omega_c} \\
&=(q_p, \psi_f)_{\Omega_d} + (\mathbf{f}, \mathbf{v})_{\Omega_c}, \qquad \forall\, \psi_m, \psi_f \in H^1_{0,\Gamma_d}(\Omega_d),\ \mathbf{v} \in [H^1_{0,\Gamma_c}(\Omega_c)]^N,
\end{aligned} \tag{11}$$

$$-(\nabla\cdot\mathbf{u}, q)_{\Omega_c} = 0, \qquad \forall\, q \in L^2(\Omega_c). \tag{12}$$

3. The Weak Galerkin Finite Element Method

In this section, the WG method is applied to the dual-porosity-Stokes model. To this end, we first give definitions of WG spaces and weak differential operators. Then the WG scheme for this model is proposed.

Let $\mathcal{T}_{h,d}$ and $\mathcal{T}_{h,c}$ be the shape regular partitions [33] of the domain Ω_d and Ω_c, respectively. The sets of all edges or flat faces in $\mathcal{T}_{h,d}$ and $\mathcal{T}_{h,c}$ are denoted by $\mathcal{E}_{h,d}$ and $\mathcal{E}_{h,c}$, respectively. The set of all edges or flat faces on the interface Γ is denoted by $\mathcal{E}_{h,I}$. For $T \in \mathcal{T}_{h,d} \cup \mathcal{T}_{h,c}$, define the diameter of T as h_T. $h_d = \max_{T\in\mathcal{T}_{h,d}} h_T$ and $h_c = \max_{T\in\mathcal{T}_{h,c}} h_T$ are the mesh sizes in the dual-porosity domain Ω_d and the conduit domain Ω_c, respectively. $h = \max_{T\in\mathcal{T}_{h,d}\cup\mathcal{T}_{h,c}} h_T$ is the mesh size. For simplicity, we use the following notations to represent the inner products:

$$(v, w)_{\mathcal{T}_{h,i}} = \sum_{T_i\in\mathcal{T}_{h,i}} (v, w)_{T_i},\ \langle v, w\rangle_{\partial\mathcal{T}_{h,i}} = \sum_{T_i\in\mathcal{T}_{h,i}} \langle v, w\rangle_{\partial T_i},\ \langle v, w\rangle_{\mathcal{E}_{h,I}} = \sum_{e\in\mathcal{E}_{h,I}} \langle v, w\rangle_e.$$

Next, we define the following WG spaces and weak differential operators.

$$\begin{aligned}
V_{h,d} &= \{p = \{p_0, p_b\} : p_0|_T \in P_k(T), T \in \mathcal{T}_{h,d};\ p_b|_e \in P_{k-1}(e), e \in \mathcal{E}_{h,d}\}, \\
V^0_{h,d} &= \{p = \{p_0, p_b\} \in V_{h,d} : p_b|_e = 0, e \in \mathcal{E}_{h,d} \cap \Gamma_d\}, \\
V_{h,c} &= \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0|_T \in [P_k(T)]^N, T \in \mathcal{T}_{h,c};\ \mathbf{v}_b|_e \in [P_k(e)]^N,\ e \in \mathcal{E}_{h,c}\}, \\
V^0_{h,c} &= \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_{h,c} : \mathbf{v}_b|_e = \mathbf{0}, e \in \mathcal{E}_{h,c} \cap \Gamma_c\}, \\
W_{h,c} &= \{q : q \in L^2(\Omega_c),\ q|_T \in P_{k-1}(T), T \in \mathcal{T}_{h,c}\},
\end{aligned}$$

where $P_k(T)$ denotes the space of polynomials on $T \in \mathcal{T}_{h,d} \cup \mathcal{T}_{h,c}$ with degree no more than k, and $P_{k-1}(e)$ represents the space of polynomials on e with degree no more than $k-1$.

Definition 3.1. [33, 34] *For any scalar-valued function $p \in V_{h,d}$ and $T \in \mathcal{T}_{h,d}$, the discrete weak gradient $\nabla_w p \in [P_{k-1}(T)]^N$ satisfies*

$$(\nabla_w p, \mathbf{q})_T = -(p_0, \nabla \cdot \mathbf{q})_T + \langle p_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{q} \in [P_{k-1}(T)]^N. \tag{13}$$

Similarly, the discrete weak gradient and the discrete weak divergence of a vector-valued function are defined as follows.

Definition 3.2. [33, 34] *For any vector-valued function $\mathbf{v} \in V_{h,c}$ and $T \in \mathcal{T}_{h,c}$, the discrete weak gradient $\nabla_w \mathbf{v} \in [P_{k-1}(T)]^{N\times N}$ satisfies*

$$(\nabla_w \mathbf{v}, \psi)_T = -(\mathbf{v}_0, \nabla \cdot \psi)_T + \langle \mathbf{v}_b, \psi \mathbf{n} \rangle_{\partial T}, \quad \forall\, \psi \in [P_{k-1}(T)]^{N\times N}. \tag{14}$$

According to the above definition, we give the definition of the discrete weak deformation tensor:

$$\mathbb{D}_w(\mathbf{v}) = \frac{1}{2}\left(\nabla_w \mathbf{v} + (\nabla_w \mathbf{v})^{\mathrm{T}}\right), \forall\, \mathbf{v} \in V_{h,c}.$$

Definition 3.3. [33] *For any vector-valued function $\mathbf{v} \in V_{h,c}$ and $T \in \mathcal{T}_{h,c}$, the discrete weak divergence $\nabla_w \cdot \mathbf{v} \in P_{k-1}(T)$ satisfies*

$$(\nabla_w \cdot \mathbf{v}, \varphi)_T = -(\mathbf{v}_0, \nabla \varphi)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, \varphi \rangle_{\partial T}, \quad \forall\, \varphi \in P_{k-1}(T). \tag{15}$$

To obtain the WG scheme, we define some projection operators. For $T_d \in \mathcal{T}_{d,h}$ and each edge $e_d \in \mathcal{E}_{d,h}$, define

$$Q_{0,d} : L^2(T_d) \to P_k(T_d), \quad Q_{b,d} : L^2(e_d) \to P_{k-1}(e_d).$$

For $T_c \in \mathcal{T}_{c,h}$ and $e_c \in \mathcal{E}_{c,h}$, define

$$Q_{0,c} : [L^2(T_c)]^N \to [P_k(T_c)]^N, \quad Q_{b,c} : [L^2(e_c)]^N \to [P_k(e_c)]^N.$$

Set $Q_{h,d} = \{Q_{0,d}, Q_{b,d}\}$ and $Q_{h,c} = \{Q_{0,c}, Q_{b,c}\}$.

We also need to define some bilinear forms for any p_m^h, p_f^h, ψ_m^h, $\psi_f^h \in V_{h,d}$, $\mathbf{u}_h$, $\mathbf{v}_h \in V_{h,c}$ and $q_h \in W_{h,c}$,

$$\begin{aligned}
a_{s,m}(p_m^h, \psi_m^h) &= \left(\frac{k_m}{\mu} \nabla_w p_m^h, \nabla_w \psi_m^h\right)_{\mathcal{T}_{h,d}} + s(p_m^h, \psi_m^h),\\
a_{s,f}(p_f^h, \psi_f^h) &= \left(\frac{k_f}{\mu} \nabla_w p_f^h, \nabla_w \psi_f^h\right)_{\mathcal{T}_{h,d}} + s(p_f^h, \psi_f^h),\\
a_{s,c}(\mathbf{u}_h, \mathbf{v}_h) &= (2\nu \mathbb{D}_w(\mathbf{u}_h), \mathbb{D}_w(\mathbf{v}_h))_{\mathcal{T}_{h,c}} + s_c(\mathbf{u}_h, \mathbf{v}_h)\\
&\quad + \left\langle \frac{\alpha\nu\sqrt{N}}{\sqrt{trace(\Pi)}} P_\tau(\mathbf{u}_b), P_\tau(\mathbf{v}_b) \right\rangle_{\mathcal{E}_{h,I}},\\
a_d(p_m^h, p_f^h, \psi_m^h, \psi_f^h) &= \left(\frac{\sigma k_m}{\mu}(p_m^0 - p_f^0), \psi_m^0\right)_{\mathcal{T}_{h,d}} + \left(\frac{\sigma k_m}{\mu}(p_f^0 - p_m^0), \psi_f^0\right)_{\mathcal{T}_{h,d}},\\
a_\Gamma\langle \mathbf{u}_h, p_f^h, \mathbf{v}_h, \psi_f^h \rangle &= \left\langle \frac{1}{\rho} p_f^b, \mathbf{v}_b \cdot \mathbf{n}_{cd} \right\rangle_{\mathcal{E}_{h,I}} - \left\langle \psi_f^b, \mathbf{u}_b \cdot \mathbf{n}_{cd} \right\rangle_{\mathcal{E}_{h,I}},\\
b_c(\mathbf{v}_h, q_h) &= \frac{1}{\rho}(\nabla_w \cdot \mathbf{v}_h, q_h)_{\mathcal{T}_{h,c}},
\end{aligned}$$

where

$$s(p_i^h,\psi_i^h) = \sum_{T_d\in\mathcal{T}_{h,d}} h_{T_d}^{-1}\langle Q_{b,d}p_i^0 - p_i^b, Q_{b,d}\psi_i^0 - \psi_i^b\rangle_{\partial T_d},\ i = m, f,$$

$$s_c(\mathbf{u}_h,\mathbf{v}_h) = \sum_{T_c\in\mathcal{T}_{h,c}} h_{T_c}^{-1}\langle \mathbf{u}_0 - \mathbf{u}_b, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T_c}.$$

According to the above definitions and the variational formulation (11)-(12), we give the WG scheme of the dual-porosity-Stokes coupling problems (1)-(10).

Algorithm 1 Weak Galerkin Scheme

Find $p_m^h = \{p_m^0, p_m^b\}$, $p_f^h = \{p_f^0, p_f^b\} \in V_{h,d}$, $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} \in V_{h,c}$ and $p_h \in W_{h,c}$ such that $p_m^b = Q_{b,d}p_m^{dir}$, $p_f^b = Q_{b,d}p_f^{dir}$ on Γ_d, $\mathbf{u}_b = Q_{b,c}\mathbf{u}^{dir}$ on Γ_c, and

$$\begin{aligned}(16)\quad & a_{s,m}(p_m^h,\psi_m^h) + a_{s,f}(p_f^h,\psi_f^h) + a_{s,c}(\mathbf{u}_h,\mathbf{v}_h) + a_d(p_m^h,p_f^h,\psi_m^h,\psi_f^h) \\ & + a_\Gamma\langle \mathbf{u}_h, p_f^h, \mathbf{v}_h, \psi_f^h\rangle - b_c(\mathbf{v}_h,p_h) = (q_p,\psi_f^0)_{\mathcal{T}_{h,d}} + (\mathbf{f},\mathbf{v}_0)_{\mathcal{T}_{h,c}},\end{aligned}$$

$$(17)\quad b_c(\mathbf{u}_h,q_h) = 0,$$

for any ψ_m^h, $\psi_f^h \in V_{h,d}^0$, $\mathbf{v}_h \in V_{h,c}^0$ and $q_h \in W_{h,c}$.

4. Existence and Uniqueness

In this section, we discuss the well-posedness of the WG scheme (16)-(17). Define the following semi-norms in $V_{h,d}$ and $V_{h,c}$

Definition 4.1. *For any $\psi_h \in V_{h,d}$, we define the semi-norms,*

$$|||\psi_h|||_i^2 = a_{s,i}(\psi_h,\psi_h), \qquad i = m,\ f,$$

where

$$a_{s,i}(\psi_h,\psi_h) = \left\|(\frac{k_i}{\mu})^{\frac12}\nabla_w\psi_h\right\|_{\mathcal{T}_{h,d}}^2 + \sum_{T_d\in\mathcal{T}_{h,d}} h_{T_d}^{-1}\left\|Q_{b,d}\psi_0 - \psi_b\right\|_{\partial T_d}^2.$$

Definition 4.2. *For any $\mathbf{v}_h \in V_{h,c}$, we define the semi-norm,*

$$\begin{aligned}|||\mathbf{v}_h|||^2 =& a_{s,c}(\mathbf{v}_h,\mathbf{v}_h)\\ =& \|(2\nu)^{\frac12}\mathbb{D}_w(\mathbf{v}_h)\|_{\mathcal{T}_{h,c}}^2 + \sum_{j=1}^{N-1}\left\|\Big(\frac{\alpha\nu\sqrt{N}}{\sqrt{trace(\Pi)}}\Big)^{\frac12}\mathbf{v}_b\cdot\boldsymbol{\tau}_j\right\|_{\mathcal{E}_{h,I}}^2\\ & + \sum_{T_c\in\mathcal{T}_{h,c}} h_{T_c}^{-1}\|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T_c}^2.\end{aligned}$$

Lemma 4.1. [27] $|||\cdot|||_i (i = m, f)$ *are the norms in the* $V_{h,d}^0$.

Lemma 4.2. $|||\cdot|||$ *is a norm in* $V_{h,c}^0$.

Proof. It's obvious to get $|||\mathbf{v}_h||| = 0$ with $\mathbf{v}_h = \mathbf{0}$. Let $|||\mathbf{v}_h||| = 0$ for some $\mathbf{v}_h \in V_{h,c}$. Using the definition, we have

$$\|(2\nu)^{\frac12}\mathbb{D}_w(\mathbf{v}_h)\|_{\mathcal{T}_{h,c}}^2 + \sum_{T_c\in\mathcal{T}_{h,c}} h_{T_c}^{-1}\|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T_c}^2 + \sum_{j=1}^{N-1}\left\|\Big(\frac{\alpha\nu\sqrt{N}}{\sqrt{trace(\Pi)}}\Big)^{\frac12}\mathbf{v}_b\cdot\boldsymbol{\tau}_j\right\|_{\mathcal{E}_{h,I}}^2 = 0.$$

Hence we have $\mathbb{D}_w(\mathbf{v}_h) = 0$ in $T_c \in \mathcal{T}_{h,c}$, $\mathbf{v}_0 = \mathbf{v}_b$ on $e \in \mathcal{E}_{h,c}\cup\mathcal{E}_{h,I}$, and $\mathbf{v}_b\cdot\boldsymbol{\tau}_j = 0$ on $e \in \mathcal{E}_{h,I}$. From [29, Lemma 4.1], we know that $\nabla\mathbf{v}_0 = 0$ in $T_c \in \mathcal{T}_{h,c}$. Combining with $\mathbf{v}_0 = \mathbf{v}_b$ on $e \in \mathcal{E}_{h,c} \cup \mathcal{E}_{h,I}$ and $\mathbf{v}_b = \mathbf{0}$ on Γ_c, we get $\mathbf{v}_h = \mathbf{0}$. Hence, $|||\cdot|||$ is a norm in $V^0_{h,c}$. □

Besides the projection operators defined earlier, we define some other projection operators. Let $\mathbf{Q}_{h,d}$ be the projection operator from $[L^2(T_d)]^N$ onto $[P_{k-1}(T_d)]^N$, $T_d \in \mathcal{T}_{h,d}$, $\mathbf{Q}_{h,c}$ be the projection operator from $[L^2(T_c)]^{N\times N}$ onto $[P_{k-1}(T_c)]^{N\times N}$ and $\mathbb{Q}_{h,c}$ be the projection operator from $L^2(T_c)$ onto $P_{k-1}(T_c)$, $T_c \in \mathcal{T}_{h,c}$.

Lemma 4.3. *The projection operators $Q_{h,d}, \mathbf{Q}_{h,d}, Q_{h,c}, \mathbf{Q}_{h,c}$ and $\mathbb{Q}_{h,c}$ satisfy the following properties.*

$$\nabla_w(Q_{h,d}\phi) = \mathbf{Q}_{h,d}(\nabla\phi), \quad \forall\phi \in H^1(\Omega_d), \tag{18}$$

$$\nabla_w(Q_{h,c}\boldsymbol{\kappa}) = \mathbf{Q}_{h,c}(\nabla\boldsymbol{\kappa}), \quad \forall\boldsymbol{\kappa} \in [H^1(\Omega_c)]^N, \tag{19}$$

$$\nabla_w \cdot (Q_{h,c}\boldsymbol{\varphi}) = \mathbb{Q}_{h,c}(\nabla\cdot\boldsymbol{\varphi}), \ \forall\boldsymbol{\varphi} \in [H(div,\Omega_c)]^N. \tag{20}$$

Proof. For any $T_d \in \mathcal{T}_{h,d}$ and $\boldsymbol{q} \in [P_{k-1}(T_d)]^N$, according to the definition of discrete weak gradient operator, the property of the L^2 projection operator and integration by parts, we have

$$\begin{aligned}(\nabla_w(Q_{h,d}\phi), \boldsymbol{q})_{T_d} &= -(Q_{0,d}\phi, \nabla\cdot\boldsymbol{q})_{T_d} + \langle Q_{b,d}\phi, \boldsymbol{q}\cdot\mathbf{n}\rangle_{\partial T_d} \\ &= -(\phi, \nabla\cdot\boldsymbol{q})_{T_d} + \langle \phi, \boldsymbol{q}\cdot\mathbf{n}\rangle_{\partial T_d} \\ &= (\nabla\phi, \boldsymbol{q})_{T_d} \\ &= (\mathbf{Q}_{h,d}(\nabla\phi), \boldsymbol{q})_{T_d}.\end{aligned}$$

By taking $\boldsymbol{q} = \nabla_w(Q_{h,d}\phi) - \mathbf{Q}_{h,d}(\nabla\phi)$ in the above equation, Eq.(18) holds true. The proof of Eqs.(19)-(20) is similar. □

With these preparations, we prove the well-posedness of the WG scheme.

Lemma 4.4. *(Inf-Sup Condition) There is a positive constant β independent of h_c such that*

$$\sup_{\mathbf{v}_h \in V^0_{h,c}} \frac{b_c(\mathbf{v}_h, \tilde{\rho}_h)}{|||\mathbf{v}_h|||} \geq \beta\|\tilde{\rho}_h\|_{\mathcal{T}_{h,c}}, \tag{21}$$

for any $\tilde{\rho}_h \in W_{h,c} \cap L^2_0(\Omega_c)$.

Proof. From [4], we know that for any $\tilde{\rho}_h \in W_{h,c} \cap L^2_0(\Omega_c)$, there exists a $\mathbf{w} \in [H^1_0(\Omega_c)]^N$ such that $\nabla\cdot\mathbf{w} = \tilde{\rho}_h$ and the following inequality holds true

$$\|\mathbf{w}\|_{1,\Omega_c} \leq C\|\tilde{\rho}_h\|_{\Omega_c}.$$

Taking $\mathbf{v}_h = Q_{h,c}\mathbf{w} \in V^0_{h,c}$ and using Eq.(20), we have

$$\begin{aligned}&(\nabla_w\cdot\mathbf{v}_h, \tilde{\rho}_h)_{\mathcal{T}_{h,c}} = (\nabla_w\cdot Q_{h,c}\mathbf{w}, \tilde{\rho}_h)_{\mathcal{T}_{h,c}} \\ =&(\mathbb{Q}_{h,c}(\nabla\cdot\mathbf{w}), \tilde{\rho}_h)_{\mathcal{T}_{h,c}} = (\nabla\cdot\mathbf{w}, \tilde{\rho}_h)_{\mathcal{T}_{h,c}} = \|\tilde{\rho}_h\|^2_{\mathcal{T}_{h,c}}.\end{aligned}$$

Next, we shall estimate the three terms of $|||\cdot|||$. For the first term, by Eq.(19), we have

$$\|\mathbb{D}_w(\mathbf{v}_h)\|_{\mathcal{T}_{h,c}} \leq \|\nabla_w\mathbf{v}_h\|_{\mathcal{T}_{h,c}} = \|\nabla_w(Q_{h,c}\mathbf{w})\|_{\mathcal{T}_{h,c}} = \|\mathbf{Q}_{h,c}(\nabla\mathbf{w})\|_{\mathcal{T}_{h,c}} \leq \|\nabla\mathbf{w}\|_{\Omega_c}.$$

For the second term, by the trace inequality and Poincaré inequality, we get

$$\sum_{j=1}^{N-1}\left\|\left(\frac{\alpha\nu\sqrt{N}}{\sqrt{trace(\Pi)}}\right)^{\frac{1}{2}}\mathbf{v}_b\cdot\boldsymbol{\tau}_j\right\|^2_{\mathcal{E}_{h,I}} \leq C\|Q_{b,c}\mathbf{w}\|^2_{\mathcal{E}_{h,I}} \leq C\|\mathbf{w}\|^2_{\Gamma} \leq C\|\nabla\mathbf{w}\|^2_{\Omega_c}.$$

Finally, it follows from the triangle inequality, trace inequality and projection inequality (A.3) that

$$\begin{aligned}\sum_{T_c\in\mathcal{T}_{h,c}} h_{T_c}^{-\frac{1}{2}}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T_c} &= \sum_{T_c\in\mathcal{T}_{h,c}} h_{T_c}^{-\frac{1}{2}}\|Q_{0,c}\mathbf{w}-Q_{b,c}\mathbf{w}\|_{\partial T_c}\\ &\le \sum_{T_c\in\mathcal{T}_{h,c}} h_{T_c}^{-\frac{1}{2}}(\|Q_{0,c}\mathbf{w}-\mathbf{w}\|_{\partial T_c}+\|\mathbf{w}-Q_{b,c}\mathbf{w}\|_{\partial T_c})\\ &\le C\sum_{T_c\in\mathcal{T}_{h,c}} h_{T_c}^{-\frac{1}{2}}\|Q_{0,c}\mathbf{w}-\mathbf{w}\|_{\partial T_c}\\ &\le C\|\nabla\mathbf{w}\|_{\Omega_c}.\end{aligned}$$

Combining all these inequalities yields $|||\mathbf{v}_h||| \le C\|\nabla\mathbf{w}\|_{\Omega_c}$. Therefore, we get

$$\sup_{\mathbf{v}_h\in V_{h,c}^0}\frac{b_c(\mathbf{v}_h,\tilde{\rho}_h)}{|||\mathbf{v}_h|||}\ge\frac{(\nabla_w\cdot\mathbf{v}_h,\tilde{\rho}_h)_{\mathcal{T}_{h,c}}}{|||\mathbf{v}_h|||}\ge C\frac{\|\tilde{\rho}_h\|^2_{\mathcal{T}_{h,c}}}{\|\nabla\mathbf{w}\|_{\Omega_c}}\ge\beta\|\tilde{\rho}_h\|_{\mathcal{T}_{h,c}}.$$

The proof of the lemma is complete. □

Lemma 4.5. *The WG scheme (16)-(17) has a unique solution.*

Proof. Since (16)-(17) are finite-dimensional square linear equations, the existence is equivalent to the uniqueness. Consider the homogeneous case, i.e. $q_p = 0$ and $\mathbf{f} = \mathbf{0}$.

Taking $\psi_m^h = \frac{1}{\rho}p_m^h$, $\psi_f^h = \frac{1}{\rho}p_f^h$, $\mathbf{v}_h = \mathbf{u}_h$ and $q_h = p_h$ in Eqs.(16)-(17). And summing them together yields

$$\begin{aligned}0 =&\frac{1}{\rho}a_{s,m}(p_m^h,p_m^h)+a_{s,f}(p_f^h,p_f^h)+\frac{1}{\rho}a_{s,c}(\mathbf{u}_h,\mathbf{u}_h)+\frac{1}{\rho}a_d(p_m^h,p_f^h,p_m^h,p_f^h)\\ \ge&\frac{1}{\rho}|||p_m^h|||_m^2+\frac{1}{\rho}|||p_f^h|||_f^2+|||\mathbf{u}_h|||^2.\end{aligned}$$

Hence, we have $\mathbf{u}_h = \mathbf{0}$, $p_m^h = 0$ and $p_f^h = 0$. Moreover, we get $b_c(\mathbf{v}_h, p_h) = 0$ for any $\mathbf{v}_h \in V_{h,c}^0$.

Let $p_h = \overline{p}_h + \widetilde{p}_h$ with $\overline{p}_h = \frac{\int_{\Omega_c} p_h dx}{|\Omega_c|}$ and $\widetilde{p}_h = W_{h,c} \cap L_0^2(\Omega_c)$. There is a $\widetilde{\mathbf{v}} \in [H_0^1(\Omega_c)]^N$ such that $\nabla\cdot\widetilde{\mathbf{v}} = \widetilde{p}_h$. By Eq.(20) and taking $\mathbf{v}_h = Q_{h,c}\widetilde{\mathbf{v}}$, we have

$$\begin{aligned}0 = b_c(\mathbf{v}_h,p_h) &= \frac{1}{\rho}(\nabla_w\cdot Q_{h,c}\widetilde{\mathbf{v}},\overline{p}_h)_{\mathcal{T}_{h,c}}+\frac{1}{\rho}(\nabla_w\cdot Q_{h,c}\widetilde{\mathbf{v}},\widetilde{p}_h)_{\mathcal{T}_{h,c}}\\ &= \frac{1}{\rho}(\mathbb{Q}_{h,c}(\nabla\cdot\widetilde{\mathbf{v}}),\overline{p}_h)_{\mathcal{T}_{h,c}}+\frac{1}{\rho}(\mathbb{Q}_{h,c}(\nabla\cdot\widetilde{\mathbf{v}}),\widetilde{p}_h)_{\mathcal{T}_{h,c}}\\ &= \frac{1}{\rho}\|\widetilde{p}_h\|^2_{\mathcal{T}_{h,c}},\end{aligned}$$

which implies that $\widetilde{p}_h = 0$. Thus, $b_c(\mathbf{v}_h, p_h) = b_c(\mathbf{v}_h, \overline{p}_h + \widetilde{p}_h) = b_c(\mathbf{v}_h, \overline{p}_h) = 0$. Taking any $\mathbf{v}_h$ to make $\int_{\Omega_c}\nabla_w\cdot\mathbf{v}_h dx \ne 0$, we obtain $\overline{p}_h = 0$. To sum up, it yields $p_h = 0$. The proof of the lemma is complete. □

5. Error Analysis

In this section, we are going to derive the error equations and give the corresponding error estimates.

5.1. Error Equations. Assume that $p_m \in H^1(\Omega_d)$, $p_f \in H^1(\Omega_d)$, $\mathbf{u} \in [H^1(\Omega_c)]^N$ and $p \in L^2(\Omega_c)$ are the solutions of the model (1)-(10), $p_m^h \in V_{h,d}$, $p_f^h \in V_{h,d}$, $\mathbf{u}_h \in V_{h,c}$ and $p_h \in W_{h,c}$ are the numerical solutions of the WG scheme (16)-(17). We define the following errors.

$$\mathrm{e}_{h,m} = Q_{h,d}p_m - p_m^h, \qquad \mathrm{e}_{h,f} = Q_{h,d}p_f - p_f^h,$$
$$\mathbf{e}_{h,c} = Q_{h,c}\mathbf{u} - \mathbf{u}_h, \qquad \varepsilon_{h,c} = \mathbb{Q}_{h,c}p - p_h.$$

Then the following lemma holds true.

Lemma 5.1. *For any $\phi \in H^1(\Omega_d)$, $\psi_h \in V_{h,d}$, $\mathbf{w} \in [H^1(\Omega_c)]^N$, $\rho \in H^1(\Omega_c)$ and $\mathbf{v}_h \in V_{h,c}$, we have*

$$(\nabla_w(Q_{h,d}\phi), \nabla_w\psi_h)_{\mathcal{T}_{h,d}} = (\nabla\phi, \nabla\psi_0)_{\mathcal{T}_{h,d}} - \langle \psi_0 - \psi_b, (\mathbf{Q}_{h,d}(\nabla\phi)) \cdot \mathbf{n}\rangle_{\partial\mathcal{T}_{h,d}}, \tag{22}$$
$$(\nabla_w(Q_{h,c}\mathbf{w}), \nabla_w\mathbf{v}_h)_{\mathcal{T}_{h,c}} = (\nabla\mathbf{w}, \nabla\mathbf{v}_0)_{\mathcal{T}_{h,c}} - \langle \mathbf{v}_0 - \mathbf{v}_b, (\mathbf{Q}_{h,c}(\nabla\mathbf{w}))\mathbf{n}\rangle_{\partial\mathcal{T}_{h,c}}, \tag{23}$$
$$(\nabla_w \cdot \mathbf{v}_h, \mathbb{Q}_{h,c}\rho)_{\mathcal{T}_{h,c}} = (\nabla \cdot \mathbf{v}_0, \rho)_{\mathcal{T}_{h,c}} - \langle \mathbf{v}_0 - \mathbf{v}_b, (\mathbb{Q}_{h,c}\rho)\mathbf{n}\rangle_{\partial\mathcal{T}_{h,c}}. \tag{24}$$

The above properties can be derived from the definition of the weak differential operators, integration by parts and Lemma 4.3. Based on the above lemma, we establish the following error equations.

Lemma 5.2. *(Error Equations) Let p_m, $p_f \in H^1(\Omega_d)$, $\mathbf{u} \in [H^1(\Omega_c)]^2$, and $p \in L^2(\Omega_c)$ be sufficiently smooth, for any $\psi_m^h \in V_{h,d}^0$, $\psi_f^h \in V_{h,d}^0$, $\mathbf{v}_h \in V_{h,c}^0$, $q_h \in W_{h,c}$, we have*

$$\begin{aligned} &a_{s,m}(\mathrm{e}_{h,m}, \psi_m^h) + a_d(\mathrm{e}_{h,m}, \mathrm{e}_{h,f}, \psi_m^h, \psi_f^h) + a_\Gamma\langle \mathbf{e}_{h,c}, \mathrm{e}_{h,f}, \mathbf{v}_h, \psi_f^h\rangle \\ &+a_{s,f}(\mathrm{e}_{h,f}, \psi_f^h) + a_{s,c}(\mathbf{e}_{h,c}, \mathbf{v}_h) - b_c(\mathbf{v}_h, \varepsilon_{h,c}) = \varphi_{p_m,p_f,\mathbf{u},p}(\psi_m^h, \psi_f^h, \mathbf{v}_h), \end{aligned} \tag{25}$$

$$b_c(\mathbf{e}_{h,c}, q_h) = 0, \tag{26}$$

where

$$\begin{aligned} \varphi_{p_m,p_f,\mathbf{u},p}(\psi_m^h, \psi_f^h, \mathbf{v}_h) =&\ell_1(p_m, \psi_m^h) + \ell_2(p_f, \psi_f^h) + \ell_3(\mathbf{u}, \mathbf{v}) + \ell_4(p, \mathbf{v}) \\ &+ s(Q_{h,d}p_m, \psi_m^h) + s(Q_{h,d}p_f, \psi_f^h) + s_c(Q_{h,c}\mathbf{u}, \mathbf{v}_h), \end{aligned}$$

with

$$\begin{aligned} \ell_1(p_m, \psi_m^h) &= \frac{k_m}{\mu}\langle(\nabla p_m - \mathbf{Q}_{h,d}(\nabla p_m)) \cdot \mathbf{n}, \psi_m^0 - \psi_m^b\rangle_{\partial\mathcal{T}_{h,d}}, \\ \ell_2(p_f, \psi_f^h) &= \frac{k_f}{\mu}\langle(\nabla p_f - \mathbf{Q}_{h,d}(\nabla p_f)) \cdot \mathbf{n}, \psi_f^0 - \psi_f^b\rangle_{\partial\mathcal{T}_{h,d}}, \\ \ell_3(\mathbf{u}, \mathbf{v}) &= 2\nu\langle(\mathbb{D}(\mathbf{u}) - \mathbf{Q}_{h,c}\mathbb{D}(\mathbf{u}))\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}, \\ \ell_4(p, \mathbf{v}) &= \frac{1}{\rho}\langle(p - \mathbb{Q}_{h,c}p)\mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}. \end{aligned}$$

Proof. First, we consider the dual-porosity equations. According to integration by parts and the definition of Q, we have

$$\begin{aligned} (q_p, \psi_f^0)_{\mathcal{T}_{h,d}} =&(\frac{k_m}{\mu}\nabla p_m, \nabla\psi_m^0)_{\mathcal{T}_{h,d}} - \langle\frac{k_m}{\mu}\nabla p_m \cdot \mathbf{n}, \psi_m^0\rangle_{\partial\mathcal{T}_{h,d}} + (\frac{k_f}{\mu}\nabla p_f, \nabla\psi_f^0)_{\mathcal{T}_{h,d}} \\ &- \langle\frac{k_f}{\mu}\nabla p_f \cdot \mathbf{n}, \psi_f^0\rangle_{\partial\mathcal{T}_{h,d}} + (\frac{\sigma k_m}{\mu}(p_m - p_f), \psi_m^0 - \psi_f^0)_{\mathcal{T}_{h,d}} \end{aligned}$$

$$
\begin{aligned}
&=\frac{k_m}{\mu}(\nabla p_m, \nabla \psi_m^0)_{\mathcal{T}_{h,d}} - \frac{k_m}{\mu}\langle \nabla p_m \cdot \mathbf{n}, \psi_m^0 - \psi_m^b\rangle_{\partial\mathcal{T}_{h,d}} \\
&\quad - \frac{k_m}{\mu}\langle \nabla p_m \cdot \mathbf{n}, \psi_m^b\rangle_{\partial\mathcal{T}_{h,d}} + \frac{k_f}{\mu}(\nabla p_f, \nabla \psi_f^0)_{\mathcal{T}_{h,d}} \\
&\quad - \frac{k_f}{\mu}\langle \nabla p_f \cdot \mathbf{n}, \psi_f^0 - \psi_f^b\rangle_{\partial\mathcal{T}_{h,d}} - \frac{k_f}{\mu}\langle \nabla p_f \cdot \mathbf{n}, \psi_f^b\rangle_{\partial\mathcal{T}_{h,d}} \\
&\quad + \frac{\sigma k_m}{\mu}(p_m - p_f, \psi_m^0 - \psi_f^0)_{\mathcal{T}_{h,d}}.
\end{aligned} \tag{27}
$$

By Eq.(22), we obtain

$$
\begin{aligned}
&\frac{k_m}{\mu}(\nabla p_m, \nabla \psi_m^0)_{\mathcal{T}_{h,d}} - \frac{k_m}{\mu}\langle \nabla p_m \cdot \mathbf{n}, \psi_m^0 - \psi_m^b\rangle_{\partial\mathcal{T}_{h,d}} \\
=&\frac{k_m}{\mu}(\nabla_w(Q_{h,d}p_m), \nabla_w \psi_m^h)_{\mathcal{T}_{h,d}} \\
&- \frac{k_m}{\mu}\ \langle (\nabla p_m - \mathbf{Q}_{h,d}(\nabla p_m)) \cdot \mathbf{n}, \psi_m^0 - \psi_m^b\rangle_{\partial\mathcal{T}_{h,d}}.
\end{aligned} \tag{28}
$$

Similarly,

$$
\begin{aligned}
&- \frac{k_f}{\mu}\langle \nabla p_f \cdot \mathbf{n}, \psi_f^0 - \psi_f^b\rangle_{\partial\mathcal{T}_{h,d}} + \frac{k_f}{\mu}(\nabla p_f, \nabla \psi_f^0)_{\mathcal{T}_{h,d}} \\
=&\frac{k_f}{\mu}(\nabla_w(Q_{h,d}p_f), \nabla_w \psi_f^h)_{\mathcal{T}_{h,d}} \\
&- \frac{k_f}{\mu}\langle (\nabla p_f - \mathbf{Q}_{h,d}(\nabla p_f)) \cdot \mathbf{n}, \psi_f^0 - \psi_f^b\rangle_{\partial\mathcal{T}_{h,d}}.
\end{aligned} \tag{29}
$$

Using the interface conditions (5)-(6), we get

$$
\begin{aligned}
&- \frac{k_m}{\mu}\langle \nabla p_m \cdot \mathbf{n}, \psi_m^b\rangle_{\partial\mathcal{T}_{h,d}} - \frac{k_f}{\mu}\langle \nabla p_f \cdot \mathbf{n}, \psi_f^b\rangle_{\partial\mathcal{T}_{h,d}} \\
&= - \frac{k_m}{\mu}\langle \nabla p_m \cdot (-\mathbf{n}_{cd}), \psi_m^b\rangle_{\mathcal{E}_{h,I}} - \frac{k_f}{\mu}\langle \nabla p_f \cdot (-\mathbf{n}_{cd}), \psi_f^b\rangle_{\mathcal{E}_{h,I}} \\
&= - \langle Q_{b,c}\mathbf{u} \cdot \mathbf{n}_{cd}, \psi_f^b\rangle_{\mathcal{E}_{h,I}}.
\end{aligned} \tag{30}
$$

By the property of the L^2 projection operator, we have

$$
\begin{aligned}
&\frac{\sigma k_m}{\mu}(p_m - p_f, \psi_m^0 - \psi_f^0)_{\mathcal{T}_{h,d}} \\
=&\frac{\sigma k_m}{\mu}(Q_{h,0}p_m - Q_{h,0}p_f, \psi_m^0)_{\mathcal{T}_{h,d}} + \frac{\sigma k_m}{\mu}(Q_{h,0}p_f - Q_{h,0}p_m, \psi_f^0)_{\mathcal{T}_{h,d}}.
\end{aligned} \tag{31}
$$

Substituting (28)-(31) into (27) leads to

$$
\begin{aligned}
&(q_p, \psi_f^0)_{\mathcal{T}_{h,d}} \\
=&\frac{k_m}{\mu}(\nabla_w(Q_{h,d}p_m), \nabla_w \psi_m^h)_{\mathcal{T}_{h,d}} + \frac{k_f}{\mu}(\nabla_w(Q_{h,d}p_f), \nabla_w \psi_f^h)_{\mathcal{T}_{h,d}} \\
&- \frac{k_m}{\mu}\langle (\nabla p_m - \mathbf{Q}_{h,d}(\nabla p_m)) \cdot \mathbf{n}, \psi_m^0 - \psi_m^b\rangle_{\partial\mathcal{T}_{h,d}} \\
&- \frac{k_f}{\mu}\langle (\nabla p_f - \mathbf{Q}_{h,d}(\nabla p_f)) \cdot \mathbf{n}, \psi_f^0 - \psi_f^b\rangle_{\partial\mathcal{T}_{h,d}} - \langle Q_{b,c}\mathbf{u} \cdot \mathbf{n}_{cd}, \psi_f^b\rangle_{\mathcal{E}_{h,I}} \\
&+ \frac{\sigma k_m}{\mu}(Q_{h,0}p_m - Q_{h,0}p_f, \psi_m^0)_{\mathcal{T}_{h,d}} + \frac{\sigma k_m}{\mu}(Q_{h,0}p_f - Q_{h,0}p_m, \psi_f^0)_{\mathcal{T}_{h,d}}.
\end{aligned} \tag{32}
$$

Now, we consider the Stokes equations. According to integration by parts and the definition of $\mathbb{T}(\mathbf{u},p)$, we obtain

$$
\begin{aligned}
(\mathbf{f},\mathbf{v}_0)_{\mathcal{T}_{h,c}} =&(2\nu\mathbb{D}(\mathbf{u})-\frac{1}{\rho}p\mathbb{I},\nabla\mathbf{v}_0)_{\mathcal{T}_{h,c}}-\langle(2\nu\mathbb{D}(\mathbf{u})-\frac{1}{\rho}pI)\mathbf{n},\mathbf{v}_0\rangle_{\partial\mathcal{T}_{h,c}}\\
=&2\nu(\mathbb{D}(\mathbf{u}),\nabla\mathbf{v}_0)_{\mathcal{T}_{h,c}}-\frac{1}{\rho}(\nabla\cdot\mathbf{v}_0,p)_{\mathcal{T}_{h,c}}\\
&-2\nu\sum_{T_c\in\mathcal{T}_{h,c}}\langle\mathbb{D}(\mathbf{u})\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}-2\nu\langle\mathbb{D}(\mathbf{u})\mathbf{n},\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}\\
&+\frac{1}{\rho}\langle p\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}+\frac{1}{\rho}\langle p\mathbf{n},\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}.
\end{aligned}
\tag{33}
$$

Using Eq.(23) yields

$$
\begin{aligned}
&-2\nu\langle\mathbb{D}(\mathbf{u})\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}+2\nu(\mathbb{D}(\mathbf{u}),\nabla\mathbf{v}_0)_{\mathcal{T}_{h,c}}\\
=&-2\nu\langle(\mathbf{Q}_{h,c}\mathbb{D}(\mathbf{u}))\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}+2\nu(\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v}_0))_{\mathcal{T}_{h,c}}\\
&-2\nu\langle(\mathbb{D}(\mathbf{u})-\mathbf{Q}_{h,c}\mathbb{D}(\mathbf{u}))\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}\\
=&-2\nu\langle(\mathbb{D}(\mathbf{u})-\mathbf{Q}_{h,c}\mathbb{D}(\mathbf{u}))\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}\\
&+2\nu(\mathbb{D}_w(Q_{h,c}\mathbf{u}),\mathbb{D}_w(\mathbf{v}_h))_{\mathcal{T}_{h,c}}.
\end{aligned}
\tag{34}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{\rho}\langle p\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}-\frac{1}{\rho}(\nabla\cdot\mathbf{v}_0,p)_{T_c}\\
=&-\frac{1}{\rho}(\nabla_w\cdot\mathbf{v}_h,\mathbb{Q}_{h,c}p)_{\mathcal{T}_{h,c}}+\frac{1}{\rho}\langle(p-\mathbb{Q}_{h,c}p)\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}.
\end{aligned}
\tag{35}
$$

By the interface conditions (7)-(8) and the property of the L^2 projection operator, we get

$$
\begin{aligned}
&-2\nu\langle\mathbb{D}(\mathbf{u})\mathbf{n},\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}+\frac{1}{\rho}\langle p\mathbf{n},\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}\\
=&-2\nu\langle\mathbb{D}(\mathbf{u})\mathbf{n}_{cd},\mathbf{v}_b\rangle_{\mathcal{E}_{h,I}}+\frac{1}{\rho}\langle p\mathbf{n}_{cd},\mathbf{v}_b\rangle_{\mathcal{E}_{h,I}}\\
=&-\langle\mathbb{T}(\mathbf{u},p)\mathbf{n}_{cd}\cdot\mathbf{n}_{cd},\mathbf{v}_b\cdot\mathbf{n}_{cd}\rangle_{\mathcal{E}_{h,I}}-\langle\mathbb{T}(\mathbf{u},p)\mathbf{n}_{cd}\cdot\boldsymbol{\tau},\mathbf{v}_b\cdot\boldsymbol{\tau}\rangle_{\mathcal{E}_{h,I}}\\
=&\frac{1}{\rho}\langle Q_{b,d}p_f,\mathbf{v}_b\cdot\mathbf{n}_{cd}\rangle_{\mathcal{E}_{h,I}}+\frac{\alpha\nu\sqrt{N}}{\sqrt{trace(\Pi)}}\langle\mathbf{u}\cdot\boldsymbol{\tau},\mathbf{v}_b\cdot\boldsymbol{\tau}\rangle_{\mathcal{E}_{h,I}}.
\end{aligned}
\tag{36}
$$

Substituting (34)-(36) into (33), we obtain

$$
\begin{aligned}
(\mathbf{f},\mathbf{v}_0)_{\mathcal{T}_{h,c}} =&2\nu(\mathbb{D}_w(Q_{h,c}\mathbf{u}),\mathbb{D}_w(\mathbf{v}_h))_{\mathcal{T}_{h,c}}\\
&-2\nu\langle(\mathbb{D}(\mathbf{u})-\mathbf{Q}_{h,c}\mathbb{D}(\mathbf{u}))\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}\\
&-(\nabla_w\cdot\mathbf{v}_h,\mathbb{Q}_{h,c}p)_{\mathcal{T}_{h,c}}+\frac{1}{\rho}\langle Q_{b,d}p_f,\mathbf{v}_b\cdot\mathbf{n}_{cd}\rangle_{\mathcal{E}_{h,I}}\\
&+\frac{1}{\rho}\langle(p-\mathbb{Q}_{h,c}p)\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial\mathcal{T}_{h,c}}+\frac{\alpha\nu\sqrt{N}}{\sqrt{trace(\Pi)}}\langle\mathbf{u}\cdot\boldsymbol{\tau},\mathbf{v}_b\cdot\boldsymbol{\tau}\rangle_{\mathcal{E}_{h,I}}.
\end{aligned}
\tag{37}
$$

Adding (37) to (32) leads to

$$\begin{aligned}(38)\qquad &(q_p,\psi_f^0)_{\mathcal{T}_{h,d}}+(\mathbf{f},\mathbf{v}_0)_{\mathcal{T}_{h,c}}\\ =&a_{s,m}(Q_{h,d}p_m,\psi_m^h)+a_{s,f}(Q_{h,d}p_f,\psi_f^h)+a_{s,c}(Q_{h,c}\mathbf{u},\mathbf{v}_h)\\ &+a_d(Q_{h,d}p_m,Q_{h,d}p_f,\psi_m^h,\psi_f^h)+a_\Gamma\langle Q_{h,c}\mathbf{u},Q_{h,d}p_f,\mathbf{v}_h,\psi_f^h\rangle\\ &-b_c(\mathbf{v}_h,\mathbb{Q}_{h,c}p)-\varphi_{p_m,p_f,\mathbf{u},p}(\psi_m^h,\psi_f^h,\mathbf{v}_h).\end{aligned}$$

Subtracting (38) from the WG scheme (16), the proof of Eq.(25) is complete.

For any $q_h\in W_{h,c}$, using Eq.(4), we have

$$(39)\qquad \frac{1}{\rho}(\nabla_w\cdot Q_{h,c}\mathbf{u},q_h)_{\mathcal{T}_{h,c}}=\frac{1}{\rho}(\mathbb{Q}_{h,c}(\nabla\cdot\mathbf{u}),q_h)_{\mathcal{T}_{h,c}}=\frac{1}{\rho}(\nabla\cdot\mathbf{u},q_h)_{\mathcal{T}_{h,c}}=0.$$

Subtracting (39) from (17), Eq.(26) holds true. □

5.2. Error Estimates. In this subsection, we give the error estimates in the energy norm.

Theorem 5.1. *Let $p_m\in H^{k+1}(\Omega_d)$, $p_f\in H^{k+1}(\Omega_d)$, $\mathbf{u}\in[H^{k+1}(\Omega_c)]^N$ and $p\in H^k(\Omega_c)$ be the exact solutions of the model (1)-(10). Let $p_m^h\in V_{h,d}$, $p_f^h\in V_{h,d}$, $\mathbf{u}_h\in V_{h,c}$ and $p_h\in W_{h,c}$ be the numerical solutions of the WG scheme (16)-(17), then we have*

$$\begin{aligned}(40)\qquad &|\!|\!|\mathrm{e}_{h,m}|\!|\!|_m+|\!|\!|\mathrm{e}_{h,f}|\!|\!|_f+|\!|\!|\mathbf{e}_{h,c}|\!|\!|+\|\varepsilon_{h,c}\|_{\mathcal{T}_{h,c}}\\ \le&Ch^k\Big(\|p_m\|_{k+1,\Omega_d}+\|p_f\|_{k+1,\Omega_d}+\|\mathbf{u}\|_{k+1,\Omega_c}+\|p\|_{k,\Omega_c}\Big),\end{aligned}$$

where C is independent of h.

Proof. For simplicity, we use δ_m, δ_f and δ_c to represent $h^k\|p_m\|_{k+1,\Omega_d}$, $h^k\|p_f\|_{k+1,\Omega_d}$ and $h^k(\|\mathbf{u}\|_{k+1,\Omega_c}+\|p\|_{k,\Omega_c})$, respectively.

Taking $\psi_m^h=\mathrm{e}_{h,m}$, $\psi_f^h=\mathrm{e}_{h,f}$, and $\mathbf{v}_h=\mathbf{e}_{h,c}$ in Eq.(25) and $q_h=\varepsilon_{h,c}$ in Eq.(26), we have

$$\begin{aligned}(41)\qquad &|\!|\!|\mathrm{e}_{h,m}|\!|\!|_m^2+|\!|\!|\mathrm{e}_{h,f}|\!|\!|_f^2+|\!|\!|\mathbf{e}_{h,c}|\!|\!|^2\\ =&\varphi_{p_m,p_f,\mathbf{u},p}(\mathrm{e}_{h,m},\mathrm{e}_{h,f},\mathbf{e}_{h,c})-a_d(\mathrm{e}_{h,m},\mathrm{e}_{h,f},\mathrm{e}_{h,m},\mathrm{e}_{h,f})\\ &-a_\Gamma\langle\mathbf{e}_{h,c},\mathrm{e}_{h,f},\mathbf{e}_{h,c},\mathrm{e}_{h,f}\rangle.\end{aligned}$$

By the definition of the $a_d(\mathrm{e}_{h,m},\mathrm{e}_{h,f},\mathrm{e}_{h,m},\mathrm{e}_{h,f})$, we get

$$|\!|\!|\mathrm{e}_{h,m}|\!|\!|_m^2+|\!|\!|\mathrm{e}_{h,f}|\!|\!|_f^2+|\!|\!|\mathbf{e}_{h,c}|\!|\!|^2+\frac{\sigma k_m}{\mu}\|\mathrm{e}_{h,m}-\mathrm{e}_{h,f}\|_{\mathcal{T}_{h,d}}^2=\varphi_{p_m,p_f,\mathbf{u},p}(\mathrm{e}_{h,m},\mathrm{e}_{h,f},\mathbf{e}_{h,c}).$$

Thus,

$$(42)\qquad |\!|\!|\mathrm{e}_{h,m}|\!|\!|_m^2+|\!|\!|\mathrm{e}_{h,f}|\!|\!|_f^2+|\!|\!|\mathbf{e}_{h,c}|\!|\!|^2\le\varphi_{p_m,p_f,\mathbf{u},p}(\mathrm{e}_{h,m},\mathrm{e}_{h,f},\mathbf{e}_{h,c}).$$

According to Lemma A.5 , we obtain

$$(43)\qquad \varphi_{p_m,p_f,\mathbf{u},p}(\mathrm{e}_{h,m},\mathrm{e}_{h,f},\mathbf{e}_{h,c})\le C(\delta_m|\!|\!|\mathrm{e}_{h,m}|\!|\!|_m+\delta_f|\!|\!|\mathrm{e}_{h,f}|\!|\!|_f+\delta_c|\!|\!|\mathbf{e}_{h,c}|\!|\!|).$$

Substituting (43) into (42) and using the Young inequality, we have

$$(44)\qquad |\!|\!|\mathrm{e}_{h,m}|\!|\!|_m+|\!|\!|\mathrm{e}_{h,f}|\!|\!|_f+|\!|\!|\mathbf{e}_{h,c}|\!|\!|\le C(\delta_m+\delta_f+\delta_c).$$

Next let $\varepsilon_{h,c}=\overline{\varepsilon}_{h,c}+\widetilde{\varepsilon}_{h,c}$, where $\overline{\varepsilon}_{h,c}=\frac{\int_{\Omega_c}\varepsilon_{h,c}dx}{|\Omega_c|}$ and $\widetilde{\varepsilon}_{h,c}\in L_0^2(\Omega_c)$.

First, we estimate $\widetilde{\varepsilon}_{h,c}$. From error equation (25), we get

$$\begin{aligned} b_c(\mathbf{v}_h, \varepsilon_{h,c}) =& a_{s,m}(\mathrm{e}_{h,m}, \psi_m^h) + a_{s,f}(\mathrm{e}_{h,f}, \psi_f^h) + a_{s,c}(\mathbf{e}_{h,c}, \mathbf{v}_h) \\ &+ a_d(\mathrm{e}_{h,m}, \mathrm{e}_{h,f}, \psi_m^h, \psi_f^h) + a_\Gamma\langle \mathbf{e}_{h,c}, \mathrm{e}_{h,f}, \mathbf{v}_h, \psi_f^h\rangle \\ &- \varphi_{p_m,p_f,\mathbf{u},p}(\psi_m^h, \psi_f^h, \mathbf{v}_h). \end{aligned}$$

From [4], we know that for $\widetilde{\varepsilon}_{h,c} \in W_{h,c} \cap L_0^2(\Omega_c)$, there exists a $\widetilde{\mathbf{v}} \in [H_0^1(\Omega_c)]^N$ such that $\nabla \cdot \widetilde{\mathbf{v}} = \widetilde{\varepsilon}_{h,c}$. Choosing $\mathbf{v}_h = Q_{h,c}\widetilde{\mathbf{v}}$ and $\psi_m^h = \psi_f^h = 0$ in the error equation (25), it follows from the inequality (42), Eq.(20) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} |b_c(\mathbf{v}_h, \widetilde{\varepsilon}_{h,c})| =& \left|\frac{1}{\rho}(\mathbf{Q}_{h,c}(\nabla \cdot \widetilde{\mathbf{v}}), \widetilde{\varepsilon}_{h,c})_{\mathcal{T}_{h,c}}\right| \\ =& \left|\frac{1}{\rho}(\nabla \cdot \widetilde{\mathbf{v}}, \varepsilon_{h,c})_{\mathcal{T}_{h,c}}\right| \\ =& |b_c(\mathbf{v}_h, \varepsilon_{h,c})| \\ =& a_{s,c}(\mathbf{e}_{h,c}, \mathbf{v}_h) - \varphi_{p_m,p_f,\mathbf{u},p}(0,0,\mathbf{v}_h) \\ \le& |\!|\!|\mathbf{e}_{h,c}|\!|\!| \; |\!|\!|\mathbf{v}_h|\!|\!| + |\varphi_{p_m,p_f,\mathbf{u},p}(0,0,\mathbf{v}_h)| \\ \le& C(\delta_m + \delta_f + \delta_c)|\!|\!|\mathbf{v}_h|\!|\!|. \end{aligned} \tag{45}$$

By the inf-sup condition (21), we have

$$\beta\|\varepsilon_{h,c}\| \le \sup_{\mathbf{v}_h \in V_h^0} \frac{b_c(\mathbf{v}_h, \varepsilon_{h,c})}{|\!|\!|\mathbf{v}_h|\!|\!|}. \tag{46}$$

Combining Eq.(45) with Eq.(46), we have

$$\|\tilde{\varepsilon}_{h,c}\|_{\mathcal{T}_{h,c}} \le C(\delta_m + \delta_f + \delta_c).$$

Next we consider the estimate of $\overline{\varepsilon}_{h,c}$. Taking the smooth function $\boldsymbol{\zeta} \in [C_0^2(\Omega_c)]^N$ to satisfy the following equation,

$$\int_{\Omega_c} \nabla \cdot \boldsymbol{\zeta} dx = 1.$$

Choosing $\gamma = \|\boldsymbol{\zeta}\|_{1,\Omega_c}$ and $\overline{\mathbf{v}}_h = Q_{h,c}\boldsymbol{\zeta}$, and using Lemma 4.3, we have

$$\int_{\Omega_c} \nabla_w \cdot \overline{\mathbf{v}}_h dx = \int_{\Omega_c} \mathbb{Q}_{h,c}(\nabla \cdot \boldsymbol{\zeta}) dx = \int_{\Omega_c} \nabla \cdot \boldsymbol{\zeta} dx = 1,$$

and

$$\begin{aligned} |\!|\!|\overline{\mathbf{v}}_h|\!|\!| + \|\nabla_w \cdot \overline{\mathbf{v}}_h\|_{\mathcal{T}_{h,c}} &= |\!|\!|Q_{h,c}\boldsymbol{\zeta}|\!|\!| + \|\nabla_w \cdot \overline{\mathbf{v}}_h\|_{\mathcal{T}_{h,c}} \\ &\le C_0\|\boldsymbol{\zeta}\|_{1,\Omega_c} + \|\mathbb{Q}_{h,c}\nabla \cdot \boldsymbol{\zeta}\|_{\mathcal{T}_{h,c}} \le (C_0 + 1)\gamma. \end{aligned}$$

By the error equation (25), we get

$$\begin{aligned} |\overline{\varepsilon}_{h,c}| =& \frac{b_c(\overline{\mathbf{v}}_h, \overline{\varepsilon}_{h,c})}{\int_{\Omega_c} \nabla_w \cdot \overline{\mathbf{v}}_h dx} \\ =& a_c(\mathbf{e}_{h,c}, \overline{\mathbf{v}}_h) - \varphi_{p_m,p_f,\mathbf{u},p}(0,0,\overline{\mathbf{v}}_h) - b_c(\overline{\mathbf{v}}_h, \widetilde{\varepsilon}_{h,c}) \\ \le& |\!|\!|\mathbf{e}_{h,c}|\!|\!| \; |\!|\!|\overline{\mathbf{v}}_h|\!|\!| + |\varphi_{p_m,p_f,\mathbf{u},p}(0,0,\overline{\mathbf{v}}_h)| + \|\widetilde{\varepsilon}_{h,c}\|_{\mathcal{T}_{h,c}}\|\nabla_w \cdot \overline{\mathbf{v}}_h\|_{\mathcal{T}_{h,c}} \\ \le& C_1\delta_m + C_2\delta_f + C_3\delta_c. \end{aligned}$$

Thus, we obtain

$$\|\varepsilon_{h,c}\|_{\Omega_c} = \|\overline{\varepsilon}_{h,c} + \widetilde{\varepsilon}_{h,c}\|_{\Omega_c} \le \|\widetilde{\varepsilon}_{h,c}\|_{\Omega_c} + C|\overline{\varepsilon}_{h,c}|_{\Omega_c} \le C_1\delta_m + C_2\delta_f + C_3\delta_c. \tag{47}$$

From (44) and (47), the proof of the theorem is complete. □

6. Numerical Examples

In this section, we present some numerical examples to verify the efficiency of the WG method for solving the dual-porosity-Stokes model. In the examples, we calculate the relative errors in the energy norm

$$\|\!|\bar{\mathrm{e}}_{h,m}|\!\|_m := \frac{\|\!|Q_{h,d}p_m - p_m^h|\!\|_m}{\|\!|Q_{h,d}p_m|\!\|_m}, \quad \|\!|\bar{\mathrm{e}}_{h,f}|\!\|_f := \frac{\|\!|Q_{h,d}p_f - p_f^h|\!\|_f}{\|\!|Q_{h,d}p_f|\!\|_f},$$
$$\|\!|\bar{\mathbf{e}}_{h,c}|\!\| := \frac{\|\!|Q_{h,c}\mathbf{u} - \mathbf{u}_h|\!\|}{\|\!|Q_{h,c}\mathbf{u}|\!\|},$$

and the relative errors in the L^2 norm

$$\|\bar{\mathrm{e}}_{0,m}\|_{\mathcal{T}_{h,d}} := \frac{\|Q_{0,d}p_m - p_m^0\|_{\mathcal{T}_{h,d}}}{\|Q_{0,d}p_m\|_{\mathcal{T}_{h,d}}}, \quad \|\bar{\mathrm{e}}_{0,f}\|_{\mathcal{T}_{h,d}} := \frac{\|Q_{0,d}p_f - p_f^0\|_{\mathcal{T}_{h,d}}}{\|Q_{0,d}p_f\|_{\mathcal{T}_{h,d}}},$$
$$\|\bar{\mathbf{e}}_{0,c}\|_{\mathcal{T}_{h,c}} := \frac{\|Q_{0,c}\mathbf{u} - \mathbf{u}_0\|_{\mathcal{T}_{h,c}}}{\|Q_{0,c}\mathbf{u}\|_{\mathcal{T}_{h,c}}}, \quad \|\bar{\varepsilon}_{h,c}\|_{\mathcal{T}_{h,c}} := \frac{\|\mathbb{Q}_{h,c}p - p_h\|_{\mathcal{T}_{h,c}}}{\|\mathbb{Q}_{h,c}p\|_{\mathcal{T}_{h,c}}}.$$

We implement these examples on three types of meshes: the triangular meshes $\mathcal{T}_h^1$, the rectangular meshes $\mathcal{T}_h^2$ and the polygon meshes $\mathcal{T}_h^3$ (see Figure 2).

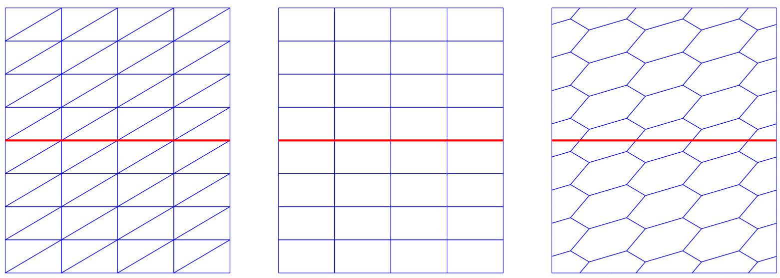

FIGURE 2. The first level grid of meshes, left: $\mathcal{T}_h^1$, middle: $\mathcal{T}_h^2$, right: $\mathcal{T}_h^3$.

Example 6.1. *Consider the dual-porosity-Stokes model in the rectangular domain* $\Omega = (0,\pi)\times(-1,1)$. *The dual-porosity domain, Stokes domain and the interface are* $\Omega_d = (0,\pi)\times(0,1)$, $\Omega_c = (0,\pi)\times(-1,0)$ *and* $\Gamma = (0,\pi)\times\{0\}$, *respectively. Choose* $k_m = 0.01$, $k_f = 1$, $\mu = 1$, $\nu = 1$, $\rho = 1$, $\sigma = 1$, *and* $\frac{\alpha\nu\sqrt{N}}{\sqrt{trace(\Pi)}} = 1$ *in the dual-porosity-Stokes model. The exact solutions are*

$$p_m = \sin(xy^2 - y^3), \qquad p_f = (\mathrm{e}^y - \mathrm{e}^{-y})\sin x,$$
$$\mathbf{u} = \begin{pmatrix} \frac{1}{\pi}\sin(2\pi y)\cos x \\ (-2 + \frac{1}{\pi^2}\sin^2(\pi y))\sin x \end{pmatrix}, \qquad p = 0.$$

In this example, we use the P_1 to P_2 WG elements to solve the dual-porosity-Stokes model on different meshes. The convergence results obtained from Example 6.1 are shown in Figures 3-8. As we can see, the errors of matrix pressure function p_m and microfracture pressure function p_f reach the optimal convergence orders in the energy norm and L^2 norm. In the conduit domain, the P_k WG elements show the convergence orders $O(h^k)$ and $O(h^{k+1})$ for the fluid velocity function in the energy norm and L^2 norm, respectively. For the fluid pressure function, the P_k WG elements achieve the convergence orders $O(h^k)$ in the L^2 norm. These numerical results are consistent with the theoretical analysis.

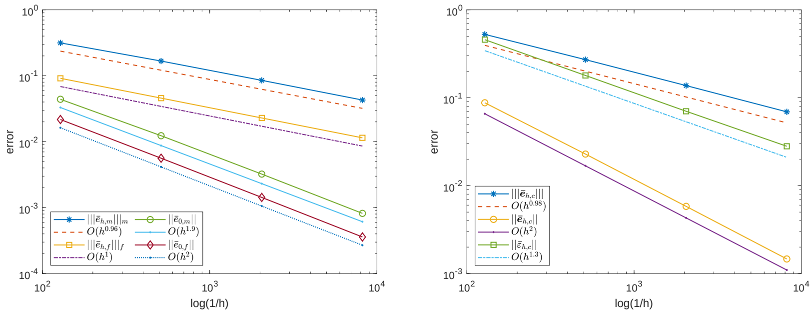

FIGURE 3. The convergence results for Example 6.1 on $\mathcal{T}_h^1$ with $k = 1$.

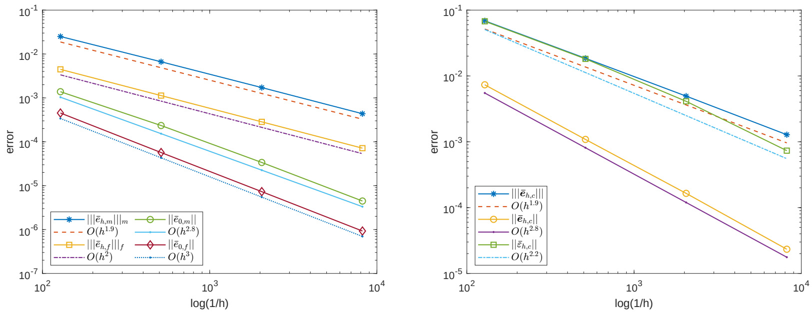

FIGURE 4. The convergence results for Example 6.1 on $\mathcal{T}_h^1$ with $k = 2$.

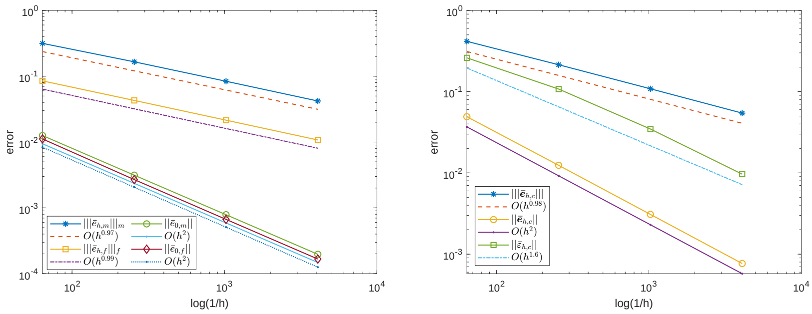

FIGURE 5. The convergence results for Example 6.1 on $\mathcal{T}_h^2$ with $k = 1$.

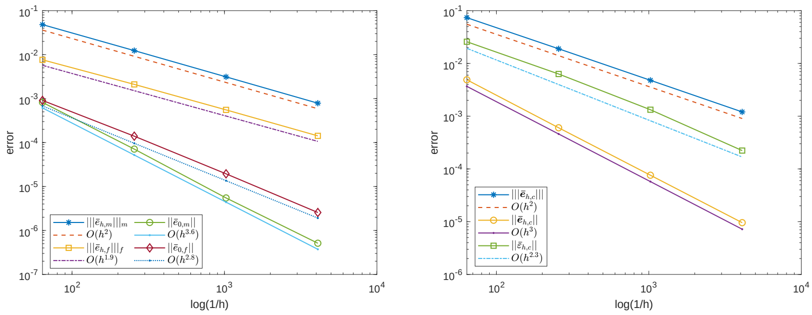

FIGURE 6. The convergence results for Example 6.1 on $\mathcal{T}_h^2$ with $k = 2$.

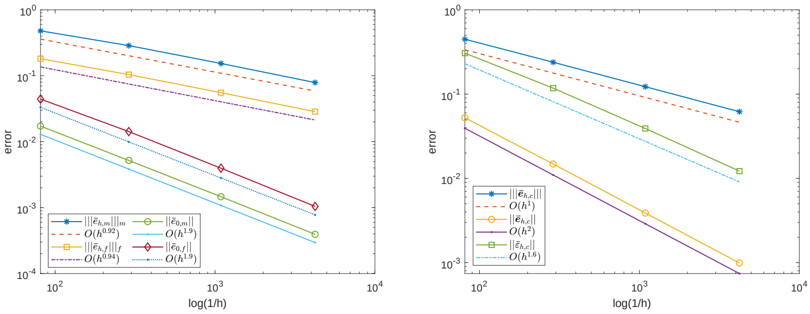

FIGURE 7. The convergence results for Example 6.1 on $\mathcal{T}_h^3$ with $k=1$.

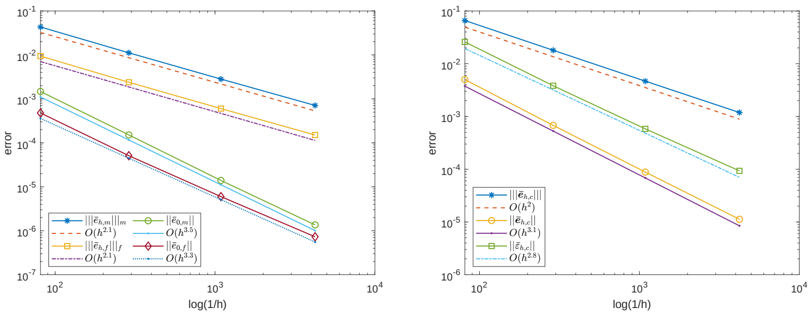

FIGURE 8. The convergence results for Example 6.1 on $\mathcal{T}_h^3$ with $k=2$.

Example 6.2. *The choice of the domain, the interface and model parameters are the same as in Example 6.1. The exact solution are as follows:*

$$p_m = \mathrm{e}^x + 2x^2, \qquad p_f = \Big(\frac{1}{2}y^2 + y\Big)\mathrm{e}^x + \frac{1}{3}x^3y + xy^2 + 2xy,$$

$$\mathbf{u} = \begin{pmatrix} \mathrm{e}^x + \mathrm{e}^y + x^2 + y^2 \\ (-y-1)\mathrm{e}^x - \frac{1}{3}x^3 - 2xy - 2x \end{pmatrix},$$

$$p = (\frac{1}{2}y^2 + y - 2)\mathrm{e}^x + \frac{1}{3}x^3y + xy^2 + 2xy - 4x.$$

In Figures 9-11, we present the convergence resluts of Example 6.2. From these figures, it becomes evident that the convergence orders of matrix pressure function p_m and microfracture pressure function p_f in the energy norm and the L^2 norm are $O(h^k)$ and $O(h^{k+1})$. Moreover, the convergence orders of fluid velocity function $\mathbf{u}$ in the energy norm and L^2 norm are$O(h^k)$ and $O(h^{k+1})$, and the convergence orders of fluid pressure function p in L^2 norm are $O(h^k)$. These results demonstrate that all numerical solutions converge at the optimal orders. Hence, the results of the above numerical examples show that it is effective to use the WG method to solve the dual-porosity-Stokes model.

7. Conclusion

In this paper, we use the weak Galerkin finite element method to solve the dual-porosity-Stokes model. We prove the stability of the numerical scheme and the existence and uniqueness of the numerical solutions. For the proposed WG scheme, the error equations are obtained. Based on the error equations, we give the optimal

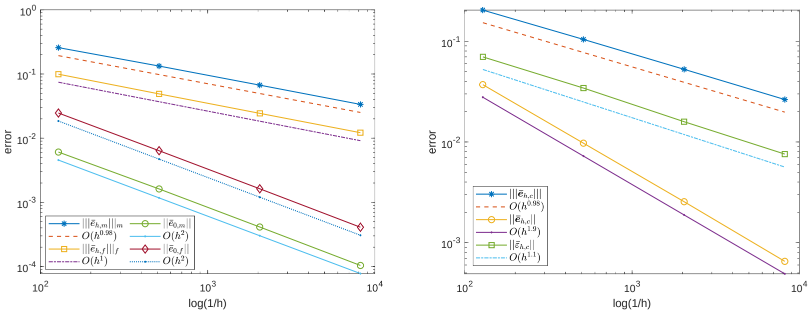

FIGURE 9. The convergence results for Example 6.2 on $\mathcal{T}_h^1$ with $k = 1$.

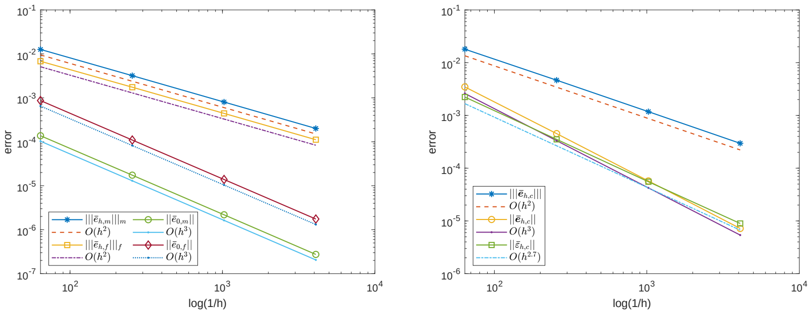

FIGURE 10. The convergence results for Example 6.2 on $\mathcal{T}_h^2$ with $k = 2$.

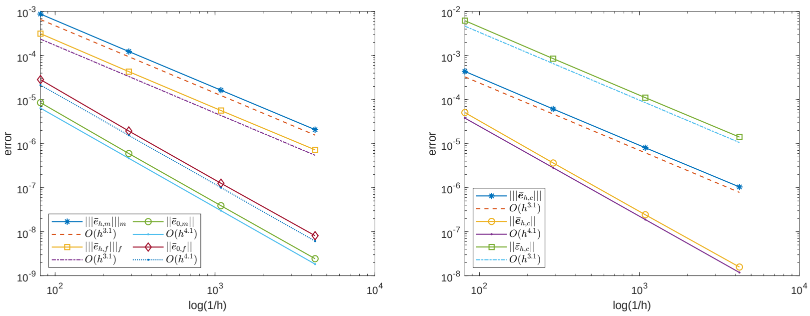

FIGURE 11. The convergence results for Example 6.2 on $\mathcal{T}_h^3$ with $k = 3$.

error estimates in the energy norm. Furthermore, numerical results demonstrate that the error convergence orders agree with theoretical analysis.

Appendix

Lemma A.1. [31] *(Trace inequality) For any $g \in H^1(T)$, we have*

$$\|g\|_e^2 \leq C(h_T^{-1}\|g\|_T^2 + h_T\|\nabla g\|_T^2). \tag{A.1}$$

Lemma A.2. [31] *(Inverse inequality) If g is a polynomial function on T, we have*

$$\|\nabla g\|_T^2 \leq Ch_T^{-2}\|g\|_T^2, \tag{A.2}$$

where C is a constant only related to the degree and dimension of the polynomial.

Lemma A.3. [32] *For any $\phi \in H^{r+1}(\Omega)$ with $1 \le r \le k$, we have*

$$\sum_{T\in\mathcal{T}_h} \|\phi - Q_0\phi\|_T^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|\nabla(\phi - Q_0\phi)\|_T^2 \le Ch^{2(r+1)}\|\phi\|_{r+1}^2, \tag{A.3}$$

$$\sum_{T\in\mathcal{T}_h} \|\nabla\phi - \mathbf{Q}_h(\nabla\phi)\|_T^2 \le Ch^{2r}\|\phi\|_{r+1}^2, \tag{A.4}$$

$$\sum_{T\in\mathcal{T}_h} \|\phi - Q_b\phi\|_{\partial T}^2 \le Ch^{2r}\|\nabla\phi\|_{\partial T,r}^2, \tag{A.5}$$

where C is a constant independent of the mesh size h and function ϕ.

Lemma A.4. *There exists a positive numbers C such that for any $\psi_h \in V_{h,d}$, we have*

$$\|\nabla\psi_0\| \le C|\!|\!|\psi_h|\!|\!|_i,\ i = m, f, \tag{A.6}$$

where C is independent of h.

Proof. For any $\psi_h \in V_{h,d}$ and $\mathbf{q} \in [P_{k-1}(T_d)]^N$ in $T_d \in \mathcal{T}_{h,d}$, according to the definition of weak gradient operator, integration by parts and the property of the L^2 projection operator, we have

$$\begin{aligned}(\nabla_w\psi_h, \mathbf{q})_{T_d} &= (\nabla\psi_0, \mathbf{q})_{T_d} + \langle\psi_b - \psi_0, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_d}\\ &= (\nabla\psi_0, \mathbf{q})_{T_d} + \langle\psi_b - Q_{b,d}\psi_0, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T_d}.\end{aligned} \tag{A.7}$$

Choosing $\mathbf{q} = \nabla\psi_0$ in Eq.(A.7) gives

$$(\nabla_w\psi_h, \nabla\psi_0)_{T_d} = (\nabla\psi_0, \nabla\psi_0)_{T_d} + \langle\psi_b - Q_{b,d}\psi_0, \nabla\psi_0\cdot\mathbf{n}\rangle_{\partial T_d}.$$

By the Cauchy-schwarz inequality, trace inequality, and inverse inequality, we obtain

$$\begin{aligned}&\|\nabla\psi_0\|_{T_d}^2\\ =&(\nabla\psi_0, \nabla\psi_0)_{T_d}\\ \le&\|\nabla_w\psi_h\|_{T_d}\|\nabla\psi_0\|_{T_d} + \|Q_{b,d}\psi_0 - \psi_b\|_{\partial T_d}\|\nabla\psi_0\|_{\partial T_d}\\ \le&\|\nabla_w\psi_h\|_{T_d}\|\nabla\psi_0\|_{T_d} + Ch_{T_d}^{-\frac{1}{2}}\|Q_{b,d}\psi_0 - \psi_b\|_{\partial T_d}\|\nabla\psi_0\|_{T_d}\\ \le&C\left(\|\nabla_w\psi_h\|_{T_d}^2 + h_{T_d}^{-\frac{1}{2}}\|Q_b\psi_0 - \psi_b\|_{\partial T_d}^2\right)^{\frac{1}{2}}\|\nabla\psi_0\|_{T_d},\end{aligned}$$

i.e.

$$\|\nabla\psi_0\|_{T_d} \le C|\!|\!|\psi_h|\!|\!|_i,\ i = m, f.$$

So the proof of Eq.(A.6) is complete. □

Lemma A.5. *Suppose p_m, $p_f \in H^{k+1}(\Omega_d)$, $\mathbf{u} \in [H^{k+1}(\Omega_c)]^N$ and $p \in H^k(\Omega_c)$, then we have following estimates*

$$|\ell_1(p_m, \psi_m^h)| \le Ch^k\|p_m\|_{k+1,\Omega_d}|\!|\!|\psi_m^h|\!|\!|_m, \tag{A.8}$$

$$|\ell_2((p_f, \psi_f^h)| \le Ch^k\|p_f\|_{k+1,\Omega_d}|\!|\!|\psi_f^h|\!|\!|_f, \tag{A.9}$$

$$|\ell_3(\mathbf{u}, \mathbf{v}_h)| \le Ch^k\|\mathbf{u}\|_{k+1,\Omega_c}|\!|\!|\mathbf{v}_h|\!|\!|, \tag{A.10}$$

$$|\ell_4(p, \mathbf{v}_h)| \le Ch^k\|p\|_{k,\Omega_c}|\!|\!|\mathbf{v}_h|\!|\!|, \tag{A.11}$$

$$|s(Q_{h,d}p_m, \psi_m^h)| \le Ch^k\|p_m\|_{k+1,\Omega_d}|\!|\!|\psi_m^h|\!|\!|_m, \tag{A.12}$$

$$|s(Q_{h,d}p_f, \psi_f^h)| \le Ch^k\|p_f\|_{k+1,\Omega_d}|\!|\!|\psi_f^h|\!|\!|_f, \tag{A.13}$$

$$|s_c(Q_{h,c}\mathbf{u}, \mathbf{v}_h)| \le Ch^k\|\mathbf{u}\|_{k+1}|\!|\!|\mathbf{v}_h|\!|\!|. \tag{A.14}$$

Proof. As to the estimate (A.8), according to the Cauchy-schwarz inequality, trace inequality, projection inequality (A.3) and Lemma A.4, we have

$$\begin{aligned}\left|\ell_1(p_m,\psi_m^h)\right| =& \left|\frac{k_m}{\mu}\sum_{T_d\in\mathcal{T}_{h,d}}\langle(\nabla p_m-\mathbf{Q}_{h,d}(\nabla p_m))\cdot\mathbf{n},\psi_m^0-\psi_m^b\rangle_{\partial T_d}\right|\\ \le& C\left(\sum_{T_d\in\mathcal{T}_{h,d}}\|\nabla p_m-\mathbf{Q}_{h,d}(\nabla p_m)\|_{\partial T_d}^2\right)^{\frac12}\left(\sum_{T_d\in\mathcal{T}_{h,d}}\|\psi_m^0-\psi_m^b\|_{\partial T_d}^2\right)^{\frac12}\\ \le& C\left(\sum_{T_d\in\mathcal{T}_{h,d}}\|\nabla p_m-\mathbf{Q}_{h,d}(\nabla p_m)\|_{\partial T_d}^2\right)^{\frac12}\\ &\left(\sum_{T_d\in\mathcal{T}_{h,d}}\|\psi_m^0-Q_{b,d}\psi_m^0\|_{\partial T_d}^2+\|Q_{b,d}\psi_m^0-\psi_m^b\|_{\partial T_d}^2\right)^{\frac12}\\ \le& Ch^k\|p_m\|_{k+1}|\!|\!|\psi_m^h|\!|\!|_m.\end{aligned}$$

The proof of the estimate (A.9) is similar to the estimate (A.8). For the estimate (A.10), it follows from the Cauchy-Schwarz inequality, trace inequality and projection inequality (A.3) that

$$\begin{aligned}&|\ell_3(\mathbf{u},\mathbf{v}_h)|\\ =&\left|2\mu\sum_{T_c\in\mathcal{T}_{h,c}}\langle(\mathbb{D}(\mathbf{u})-\mathbf{Q}_{h,c}\mathbb{D}(\mathbf{u}))\,\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T_c}\right|\\ \le& C\left(\sum_{T_c\in\mathcal{T}_{h,c}}h_T\|\mathbb{D}(\mathbf{u})-\mathbf{Q}_{h,c}\mathbb{D}(\mathbf{u})\|_{\partial T_c}^2\right)^{\frac12}\left(\sum_{T_c\in\mathcal{T}_{h,c}}h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T_c}^2\right)^{\frac12}\\ \le& Ch^k\|\mathbf{u}\|_{k+1}|\!|\!|\mathbf{v}_h|\!|\!|.\end{aligned}$$

For the estimate (A.11), by trace inequality, projection inequality (A.5), we get

$$\begin{aligned}&|\ell_4(p,\mathbf{v}_h)|\\ =&\left|\sum_{T_c\in\mathcal{T}_{h,c}}\langle(p-\mathbb{Q}_{h,c}p)\mathbf{n},\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T_c}\right|\\ \le&\left(\sum_{T_c\in\mathcal{T}_{h,c}}h_T\|p-\mathbb{Q}_{h,c}p\|_{\partial T_c}^2\right)^{\frac12}\left(\sum_{T_c\in\mathcal{T}_{h,c}}h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T_c}^2\right)^{\frac12}\\ \le& Ch^k\|p\|_k|\!|\!|\mathbf{v}_h|\!|\!|.\end{aligned}$$

We consider the estimate (A.12). Using the Cauchy-schwarz inequality, trace inequality and projection inequality (A.3), we obtain

$$
\begin{aligned}
&|s(Q_{h,d}p_m, \psi_m^h)| \\
=&\left|\sum_{T_d\in\mathcal{T}_{h,d}} h_{T_d}^{-1}\langle Q_{b,d}Q_{0,d}p_m - Q_{b,d}p_m, Q_{b,d}\psi_m^0 - \psi_m^b\rangle_{\partial T_d}\right| \\
\le& C\left(\sum_{T_d\in\mathcal{T}_{h,d}} h_{T_d}^{-1}\|Q_{b,d}Q_{0,d}p_m - Q_{b,d}p_m\|_{\partial T_d}^2\right)^{\frac{1}{2}} \\
&\left(\sum_{T_d\in\mathcal{T}_{h,d}} h_{T_d}^{-1}\|Q_{b,d}\psi_m^0 - \psi_m^b\|_{\partial T_d}^2\right)^{\frac{1}{2}} \\
\le& C\left(\sum_{T_d\in\mathcal{T}_{h,d}} h_{T_d}^{-1}\|Q_{0,d}p_m - p_m\|_{\partial T_d}^2\right)^{\frac{1}{2}} |||\psi_m^h|||_m \\
\le& Ch^k\|p_m\|_{k+1}|||\psi_m^h|||_m.
\end{aligned}
$$

The proof of the estimates (A.13)-(A.14) is similar to the estimate (A.12). □

Acknowledgments

This research was supported by the National Natural Science Foundation of China (grant No. 11971198, 11901015, 12271208, 12001232, 12301519), the National Key Research and Development Program of China (grant No. 2020YFA0713602), and the Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education of China housed at Jilin University.

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