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CONTINUOUS/DISCONTINUOUS FINITE ELEMENT APPROXIMATION OF A 2D NAVIER-STOKES PROBLEM ARISING IN FLUID CONFINEMENT

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Abstract. In this work, a stationary 2d Navier-Stokes problem with nonlinear feedback forces field is considered in the stream-function formulation. We use the Continuous/Discontinuous Finite Element Method (CD-FEM), with interior penalty terms, to numerically solve the associated boundary-value problem. For the associated continuous and discrete problems, we prove the existence of weak solutions and establish the conditions for their uniqueness. Consistency, stability and convergence of the method are also shown analytically. To validate the numerical model regarding its applicability and robustness, several test cases are carried out.

Key words. 2d Navier-Stokes, feedback forces, CD-FEM, existence and uniqueness, consistency and stability, error analysis.

1. Introduction

Given a bounded domain $\Omega := (0,K)\times(0,L)$ of $\mathbb{R}^2$, where L and K are positive constants, let us consider the following problem for the Navier-Stokes equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}(\mathbf{x},\mathbf{u}) - \nabla p \qquad \text{in} \quad \Omega, \tag{1}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in} \quad \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{u}_* \qquad \text{on} \quad x = 0, \tag{3}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on} \quad y = 0,\ L \quad \text{and on} \quad x = K. \tag{4}$$

Here, $\mathbf{u} = (u,v)$ denotes the velocity field, p accounts for the pressure, ν is the kinematics viscosity, $\mathbf{f} = (f_1, f_2)$ is a feedback forces field (divided by the constant density ρ that is supposed to be $\rho = 1$), $\mathbf{u}_* := (u_*, v_*)$ is the prescribed velocity at the strip entrance $x = 0$, and by $\mathbf{x} = (x,y)$ we denote a generic element of $\mathbb{R}^2$.

Problem (1)-(4) can be used to describe a planar steady flow of a viscous fluid that is controlled by a feedback forces field. This type of forces field plays a central role, for instance, in the confinement of magnetic fluids in Tokamaks and Stellarators (ball- or torus-shaped devices used to produce controlled thermonuclear fusion energy). In these devices, a powerful magnetic field is used to confine very hot plasmas far enough away from the boundaries to prevent damage. The governing equations of this real-world problem consist of the Navier-Stokes equations coupled to Maxwell's equations via the Lorentz force – the feedback forces field [16]. For the sake of mathematical simplification, we only consider the Navier-Stokes problem (1)-(4) and proceed to characterize the type of forces field that can confine the fluid. The fluid confinement property we are interested in for the problem (1)-(4) can be read mathematically as follows:

$$\exists\ a \in (0,K)\ :\ \mathbf{u} = \mathbf{0} \quad \text{a.e. in } \Omega_a := (a,K)\times(0,L). \tag{5}$$

In the works [3, 4, 5, 6] we undertook a project to characterize the nature of the forces field that can confine a fluid governed by a system of equations of the type

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(1)-(4). There we look at forces fields, the notation of which in its simplest form is:

$$\mathbf{f}(\mathbf{x},\mathbf{u}) := -\delta\left(|u|^{\sigma-2}u, 0\right), \tag{6}$$

where $\delta > 0$ is a constant that accounts for the intensity of the forces field, and $\sigma > 1$ is another constant that characterizes the flow. The presence of feedback forces field of the type (6) can also be justified in other applications, such as in porous media flows and in continuous electromagnetic media. In porous media it is known as the Forchheimer term and it is important to characterize the flow resistance created by the skeleton of the porous medium, specially when the pore Reynolds number exceeds 10 [29]. This term is also considered for some quasi-stationary processes, in crystalline semiconductors, to model the density of sources or sinks of free electrons in the semiconductor lattice [2].

For the different flow conditions considered in [3, 4, 5, 6], we have shown that a feedback forces field of the type (6) can confine the fluid flow as long as the exponent of nonlinearity σ satisfies

$$1 < \sigma < 2. \tag{7}$$

The fluid confinement property was proved analytically in [3, 4, 5, 6] by considering the problem (1)-(4) in the stream-function formulation. This formulation is obtained by taking the curl of the momentum equation (1), where the forces field is given by (6), and by introducing the stream function

$$\psi \ : \ u = \psi_y \quad \text{and} \quad v = -\psi_x \quad \text{in } \Omega, \tag{8}$$

which exists in view of (2) [17, Theorem I.3.1]. By this procedure, we obtain the following fourth-order nonlinear boundary-value problem

$$\nu\Delta^2\psi + J(\psi,\Delta\psi) = \delta\left(|\psi_y|^{\sigma-2}\psi_y\right)_y \qquad \text{in} \quad \Omega, \tag{9}$$

$$\psi = f_* \quad \text{and} \quad \frac{\partial\psi}{\partial\mathbf{n}} = g_* \quad \text{on} \ \overline{\Gamma}^*, \tag{10}$$

$$\psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad \text{on} \ \overline{\Gamma}^0, \tag{11}$$

where $\Delta^2\psi$ accounts for the bi-Laplacian of ψ, $J(\psi,\Delta\psi)$ denotes the Jacobian

$$J(\psi,\Delta\psi) := \det\begin{bmatrix} \psi_x & \psi_y \\ \Delta\psi_x & \Delta\psi_y \end{bmatrix} = \psi_x\Delta\psi_y - \psi_y\Delta\psi_x,$$

$\frac{\partial\psi}{\partial\mathbf{n}}$ is the normal derivative of ψ, with $\mathbf{n} = (n_1, n_2)$ standing for the outward unit normal to the boundary $\partial\Omega$ of the domain Ω, which in turn is decomposed into the following two disjoint subsets

$$\overline{\Gamma}^* := \{(x,y) \in [0,K]\times[0,L] : x = 0\}, \tag{12}$$

$$\overline{\Gamma}^0 := \{(x,y) \in (0,K]\times[0,L] : y = 0 \vee y = L \vee x = K\}. \tag{13}$$

Data f_* and g_*, given in (10), can be written in terms of the prescribed velocity (u_*, v_*) at the strip entrance $x = 0$ as follows

$$f_*(y) = \int_0^y u_*(s)\,\mathrm{d}s, \qquad g_*(y) = v_*(y), \qquad y \in (0,L), \tag{14}$$

and are assumed to satisfy the following compatibility conditions

$$f_*(0) = f_*(L) = 0, \qquad g_*(0) = g_*(L) = 0. \tag{15}$$

Moreover, the feedback forces field (6), written in terms of the stream function, reads as

$$\mathbf{f}(\mathbf{x},\psi_y,-\psi_x) = -\delta\left(|\psi_y|^{\sigma-2}\psi_y,0\right). \tag{16}$$

The aim of the present work is to study the numerical solution of the boundary-value problem (9)-(11) by using the the Continuous/Discontinuous Finite Element Method. As mentioned above, our motivation to perform the numerical analysis of the problem (9)-(11) is mainly concerned with the confinement property (5), which does only hold if (7) is verified [3, 4]. This work is part of a project we developed to study, both from an analytical and numerical point of view, the 2d Navier-Stokes equations with feedback forces fields that are able to confine the fluid. With this regard, it should be mentioned that the transient version of the Navier-Stokes problem (1)-(4) is still open. But we think the numerical method studied here is very promising. In the first output [23] of this project, we have studied the Stokes version of the problem (9)-(11) with respect to the existence and uniqueness of weak solutions for both continuous and discrete problems, and we have shown analytically the consistency, stability and convergence of the method. To show the applicability and robustness of the numerical model, several test cases have been also performed in [23]. In the present work, we proceed with the same numerical study for the stream-function formulation (9)-(11) of the Navier-Stokes problem (1)-(4). Contrary to the Stokes version of the problem (9)-(11), whose main difficulty lies in the presence of the nonlinear feedback forces field, in the Navier-Stokes problem (9)-(11) studied here we also have to take into account the nonlinearity due to the convective term, represented here by the Jacobian $J(\psi,\Delta\psi)$. Our next goal will be to prove the numerical confinement property. For this purpose we have to prove first discrete versions of some Sobolev and Hardy type inequalities with weights.

With respect to the Continuous/Discontinuous Finite Element Method (CD-FEM), introduced in [14] and used in our work for the numerical study, it has been widely used over the last 20 years, especially to numerically study higher order PDEs. It is worth mentioning that the main feature of this method is the weak enforcement of continuity of first and higher-order derivatives through stabilization terms on interior boundaries. Further developments of this method, were subsequently carried out by many authors, among whom [7, 8, 13, 24, 28]. We emphasize that, unlike other variants of the Finite Element Method (FEM), such as the Discontinuous Finite Element Method [15], the Weak Galerkin FEM [27] (D-FEM), or the Mixed Finite Element Method [10] (M-FEM), the CD-FEM uses the same number of degrees of freedom as the standard continuous FEM. Furthermore, the CD-FEM scheme is formulated on the primary variable ψ, and therefore no auxiliary variables or systems are required as in the mixed formulations of the FEM.

To find works that are somewhat related to ours, from either the perspective of the numerical scheme or the analytical model, we can go to the original paper [14] on the CD-FEM, where the authors studied fourth-order linear elliptic problems that arise in Structural and Continuum Mechanics. A little bit later, the authors of [25, 26] have considered the D-FEM to approximate the 2d Navier-Stokes equations written in the stream-function formulation. More recently, in [21, 30], the Stokes and Navier-Stokes equations with damping have been considered, using the D-FEM and M-FEM, respectively. In the context of the continuous mathematical model, the main difference between our work and [21, 30] is that, in our case, the damping term is unidirectional and appears as a feedback forces field that acts on the system and, more importantly, is capable of confining the fluid flow. On the other hand, our approach is completely different, because when we introduce the stream function (8), we get rid of the pressure of the system (1)-(4), which makes it easier to prove the confinement property (5). Most importantly, and contrary to [21, 30], in our work we also consider the case $1 < \sigma < 2$, which, has we have mentioned above, is the most important for our project. Furthermore, our strategy for addressing the fourth-order nonlinear boundary-value problem (9)-(11) involves employing CD-FEM alongside a Picard-type iterative procedure to handle the nonlinear nature of the model. To the best of our knowledge, this numerical approach is not commonly used in the literature for solving such fluid-flow problems. Additionally, we also consider non-homogeneous boundary conditions that are generally avoided in this type of numerical studies.

2. Weak formulation of the continuous problem

To define the notion of solutions to the problem (9)-(11) we are interested in, let us introduce the following notation

$$D^2\psi : D^2\phi := \nabla\psi_x \cdot \nabla\phi_x + \nabla\psi_y \cdot \nabla\phi_y.$$

For the definition of the function spaces considered throughout this work to handle the continuous problem, we address the reader to the monographs [9, 13, 15]. We just introduce the following function space

$$V := \text{ closure of } C_0^\infty(\Omega) \text{ in the norm of } H^2(\Omega) \cap W^{1,\sigma}(\Omega).$$

Note that both V and $H^2(\Omega) \cap W^{1,\sigma}(\Omega)$ are equipped with the norm

$$\sum_{|\alpha|\leq 2} \|D^\alpha \phi\|_2 + \sum_{|\alpha|\leq 1} \|D^\alpha \phi\|_\sigma, \quad D^\alpha \phi := \frac{\partial^{|\alpha|}\phi}{\partial^{\alpha_1}x\partial^{\alpha_2}y}, \quad \alpha = (\alpha_1, \alpha_2),$$

where by $\|\cdot\|_q$ we denote the L^q-norm of the Lebesgue space $L^q(\Omega)$.

Definition 2.1. Let $\sigma > 1$. A function ψ is called a weak solution to the problem (9)-(11), if (10) and (11) are satisfied in the trace sense and

$$(17) \qquad \begin{cases} \psi \in \mathrm{H}^2(\Omega) \cap W^{1,\sigma}(\Omega), \\ \nu \displaystyle\int_\Omega D^2\psi : D^2\phi \, \mathrm{d}\mathbf{x} + \int_\Omega \Delta\psi \left(\psi_y\phi_x - \psi_x\phi_y\right) \, \mathrm{d}\mathbf{x} \\ \qquad\qquad + \delta \displaystyle\int_\Omega |\psi_y|^{\sigma-2}\psi_y \, \phi_y \, \mathrm{d}\mathbf{x} = 0, \quad \forall \, \phi \in H_0^2(\Omega). \end{cases}$$

It is worth mentioning that, once the space dimension we are considering is $d = 2$, the continuous imbedding $H^2(\Omega) \hookrightarrow W^{1,\sigma}(\Omega)$ holds for any $\sigma \geq 1$. This justifies why that in (17) we are considering test functions that are just in $H_0^2(\Omega)$.

Let us define the following trilinear form, which corresponds to the convective term in the Navier-Stokes equations,

$$(18) \quad \mathcal{J}(\psi, \omega, \phi) := \int_\Omega \Delta\omega \left(\psi_y\phi_x - \psi_x\phi_y\right) \mathrm{d}\mathbf{x}, \quad \omega \in H^2(\Omega), \quad \psi, \ \phi \in W^{1,4}(\Omega).$$

In some parts of this work, it will be more convenient to write this term as follows,

$$(19) \qquad \mathcal{J}(\psi, \omega, \phi) = \int_\Omega \Delta\omega \nabla\psi \cdot \nabla^\perp\phi \, \mathrm{d}\mathbf{x},$$

where $(a, b)^\perp$ denotes the perpendicular vector to (a, b), i.e. $(-b, a)$, and hence $\nabla^\perp\phi = (-\phi_y, \phi_x)$.

From (18), it is clear that

$$\mathcal{J}(\psi,\omega,\psi)=0 \qquad \forall\, \omega\in H^2(\Omega),\quad \forall\, \psi\in W^{1,4}(\Omega), \tag{20}$$

$$\mathcal{J}(\psi,\omega,\phi)=-\mathcal{J}(\phi,\omega,\psi) \qquad \forall\, \omega\in H^2(\Omega),\quad \forall\, \psi,\ \phi\in W^{1,4}(\Omega). \tag{21}$$

We recall that the continuous imbedding $H^1(\Omega)\hookrightarrow L^q(\Omega)$ holds for any $q\geq 1$ and $d=2$. Therefore there exists a positive constant $C=C(q,\Omega)$ such that

$$\|\phi_y\|_q\leq\|\nabla\phi\|_q\leq C\|\phi\|_{H^2(\Omega)} \qquad \forall\, \phi\in H^2(\Omega). \tag{22}$$

As $\|D^2\phi\|_2$ is an equivalent norm to $\|\phi\|_{H^2(\Omega)}$ in $H_0^2(\Omega)$, we also have

$$\|\phi_y\|_q\leq\|\nabla\phi\|_q\leq C\|D^2\phi\|_2 \qquad \forall\, \phi\in H_0^2(\Omega). \tag{23}$$

Moreover, we can easily see that

$$\|\Delta\psi\|_2\leq\sqrt{2}\|D^2\psi\|_2 \qquad \forall\, \phi\in H^2(\Omega). \tag{24}$$

Hence, by combining the Hölder inequality with (22) and (23), in the case of $q=4$, and with (24), we can show that

$$\big|\mathcal{J}(\psi,\omega,\phi)\big|\leq C\|\psi\|_{H^2(\Omega)}\|D^2\omega\|_2\|\phi\|_{H^2(\Omega)} \qquad \forall\, \psi,\ \omega,\ \phi\in H^2(\Omega),$$

$$\big|\mathcal{J}(\psi,\omega,\phi)\big|\leq C\|D^2\psi\|_2\|D^2\omega\|_2\|\phi\|_{H^2(\Omega)} \qquad \forall\, \omega,\ \phi\in H^2(\Omega),\quad \forall\, \psi\in H_0^2(\Omega).$$

We also define the following nonlinear form for any $\sigma\geq 2$,

$$\mathcal{I}(\psi,\omega,\phi):=\int_\Omega|\omega_y|^{\sigma-2}\psi_y\phi_y\ \mathrm{d}\mathbf{x}, \qquad \psi\in W^{1,r}(\Omega),\quad \omega\in W^{1,s}(\Omega),\quad \phi\in W^{1,t}(\Omega), \tag{25}$$

where $r,\ s,\ t>1$ and $r^{-1}+(\sigma-2)s^{-1}+t^{-1}=1$. In this case, one easily realizes that $\mathcal{I}(\psi,\omega,\phi)=\mathcal{I}(\phi,\omega,\psi)$. Combining the Hölder inequality with (22) and (23), now in the case of $q=\sigma$, we can show that for all $\sigma\geq 2$ one has

$$\big|\mathcal{I}(\psi,\omega,\phi)\big|\leq C\|\psi\|_{H^2(\Omega)}\|\omega\|^{\sigma-2}_{H^2(\Omega)}\|\phi\|_{H^2(\Omega)} \qquad \forall\, \psi,\ \omega,\ \phi\in H^2(\Omega),$$

$$\big|\mathcal{I}(\psi,\omega,\phi)\big|\leq C\|\psi\|_{H^2(\Omega)}\|D^2\omega\|_2^{\sigma-2}\|\phi\|_{H^2(\Omega)} \qquad \forall\, \omega\in H_0^2(\Omega),\quad \forall\, \psi,\ \phi\in H^2(\Omega).$$

Note that $\mathcal{I}(\psi,\psi,\phi)$ is meaningful for any $\sigma>1$,

$$\mathcal{I}(\psi,\psi,\phi):=\int_\Omega|\psi_y|^{\sigma-2}\psi_y\phi_y\ \mathrm{d}\mathbf{x}, \qquad \psi\in W^{1,\sigma}(\Omega),\quad \phi\in W^{1,\sigma'}(\Omega).$$

In this case, we can also show that

$$\big|\mathcal{I}(\psi,\psi,\phi)\big|\leq C\|\psi\|^{\sigma-1}_{H^2(\Omega)}\|\phi\|_{H^2(\Omega)} \qquad \forall\, \psi,\ \phi\in H^2(\Omega),$$

$$\big|\mathcal{I}(\psi,\psi,\phi)\big|\leq C\|D^2\psi\|_2^{\sigma-1}\|\phi\|_{H^2(\Omega)} \qquad \forall\, \psi\in H_0^2(\Omega),\quad \forall\, \phi\in H^2(\Omega).$$

Later on, we will prove an existence result to the problem (9)-(11) by assuming the compatibility conditions (15) hold and that $u_*,\ v_*\in H^{\frac{1}{2}}(0,L)$, which means that, in view of (14),

$$f_*\in H^{\frac{3}{2}}(0,L), \qquad g_*\in H^{\frac{1}{2}}(0,L). \tag{26}$$

Using assumptions (15) and (26), we can extend by zero the functions f_* and g_* to all $\partial\Omega$,

$$F_*=\begin{cases} f_* & \text{in } \{0\}\times(0,L) \\ 0 & \text{in } \partial\Omega\setminus\{0\}\times(0,L) \end{cases}, \qquad G_*=\begin{cases} g_* & \text{in } \{0\}\times(0,L) \\ 0 & \text{in } \partial\Omega\setminus\{0\}\times(0,L) \end{cases},$$

so that $F_*\in H^{\frac{3}{2}}(\partial\Omega)$ and $G_*\in H^{\frac{1}{2}}(\partial\Omega)$. Hence, by the Trace theorem, there exists a function $\vartheta\in H^2(\Omega)$ such that

$$\vartheta=F_* \quad\text{and}\quad \frac{\partial\vartheta}{\partial\mathbf{n}}=G_* \quad \text{on } \partial\Omega \tag{27}$$

in the traces sense, and
(28)
$$\|\vartheta\|_{H^2(\Omega)} \leq C\left(\|F_*\|_{H^{\frac{3}{2}}(\partial\Omega)} + \|G_*\|_{H^{\frac{1}{2}}(\partial\Omega)}\right) \leq C\left(\|f_*\|_{H^{\frac{3}{2}}(0,L)} + \|g_*\|_{H^{\frac{1}{2}}(0,L)}\right).$$

For such ϑ, we shall look for solutions $\psi = \omega + \vartheta$ to the problem (17), which in turn is equivalent to look for solutions to the following problem,

$$\begin{cases} \omega \in H^2_0(\Omega) \cap W^{1,\sigma}(\Omega), \\ \nu \displaystyle\int_\Omega D^2(\omega+\vartheta) : D^2\phi \,\mathrm{d}\mathbf{x} + \mathcal{J}(\omega+\vartheta, \omega+\vartheta, \phi) \\ \qquad\qquad + \delta\mathcal{I}(\omega+\vartheta, \omega+\vartheta, \phi) = 0 \qquad \forall\ \phi \in H^2_0(\Omega), \end{cases} \tag{29}$$

for the forms $\mathcal{J}$ and $\mathcal{I}$ defined above at (18) and (25).

3. Description of the numerical method

For each $h > 0$, let

$$\mathcal{T}^h(\Omega) = \{\Omega_j\}_{j=1}^{N_e} \tag{30}$$

be a partition of the computational domain Ω, into N_e finite triangular elements Ω_j with diameters bounded by h,

$$h_j := \mathrm{diam}(\Omega_j) \leq h \qquad \forall\ j \in \{1, \dots, N_e\}.$$

For each triangular element Ω_j, we denote by Γ_j^l, with $l \in \{1, 2, 3\}$, any of its three boundaries (mesh edges), which are supposed to be open. We assume the partition $\mathcal{T}^h(\Omega)$ is shape-and uniformly regular, as $h \to 0$, in the usual sense [17, Definition II-2.3]. In particular, we assume the existence of positive constants α and β, which are supposed to be independent of h_j, such that

$$\alpha h \leq \rho_j \leq h_j \qquad \forall\ \Omega_j \in \mathcal{T}^h(\Omega), \tag{31}$$

$$\beta h \leq \partial_j \leq h_j \qquad \forall\ \Omega_j \in \mathcal{T}^h(\Omega), \tag{32}$$

where $\rho_j := \sup\{\mathrm{diam}(B) : B \subset \Omega_j \text{ is a ball}\}$, $\partial_j := \sup\{\mathrm{diam}(\Gamma_j^l) : l \in \{1, 2, 3\}\}$. Observe that the total number, say N_t, of mesh edges Γ_j^l is larger than

$$N_e := \text{ total number of triangular elements } \Omega_j.$$

Mesh edges Γ_j^l are rearranged into a family $\{\Gamma_k\}_{k=1}^{N_t}$ so that, for each $k \in \{1, \dots, N_t\}$, Γ_k denotes a unique mesh edge in the partition $\mathcal{T}^h(\Omega)$. $\{\Gamma_k\}_{k=1}^{N_t}$ is decomposed into the family of all mesh edges that lie in the interior of Ω: $\widetilde{\Gamma}_k \in \{\Gamma_k\}_{k=1}^{N_t}$, with $\widetilde{\Gamma}_k \cap \mathrm{int}(\Omega) \neq \emptyset$; and into the family of all mesh edges that lie over the boundary $\partial\Omega$: $\overline{\Gamma}_k \in \{\Gamma_k\}_{k=1}^{N_t}$, with $\overline{\Gamma}_k \cap \mathrm{int}(\Omega) = \emptyset$; so that $\{\Gamma_k\}_{k=1}^{N_t} = \{\widetilde{\Gamma}_k\}_{k=1}^{N_i} \cup \{\overline{\Gamma}_k\}_{k=1}^{N_b}$ and $\{\widetilde{\Gamma}_k\}_{k=1}^{N_i} \cap \{\overline{\Gamma}_k\}_{k=1}^{N_b} = \emptyset$, where

$$N_i := \text{ total number of mesh edges } \widetilde{\Gamma}_k \text{ in the interior of } \Omega.$$
$$N_b := \text{ total number of mesh edges } \overline{\Gamma}_k \text{ over the boundary } \partial\Omega.$$

We define

$$\widetilde{\Omega} := \bigcup_{j=1}^{N_e} \Omega_j, \quad \Gamma := \bigcup_{k=1}^{N_t} \Gamma_k, \quad \widetilde{\Gamma} := \bigcup_{k=1}^{N_i} \widetilde{\Gamma}_k, \quad \overline{\Gamma} := \bigcup_{k=1}^{N_b} \overline{\Gamma}_k. \tag{33}$$

Moreover, $\{\overline{\Gamma}_k\}_{k=1}^{N_b}$ is decomposed into the family of mesh edges over the strip entrance $\overline{\Gamma}^*$: $\overline{\Gamma}_k^* \in \{\overline{\Gamma}_k\}_{k=1}^{N_b}$, with $\overline{\Gamma}_k^* \cap \overline{\Gamma}^* \neq \emptyset$; and into the family of all mesh edges

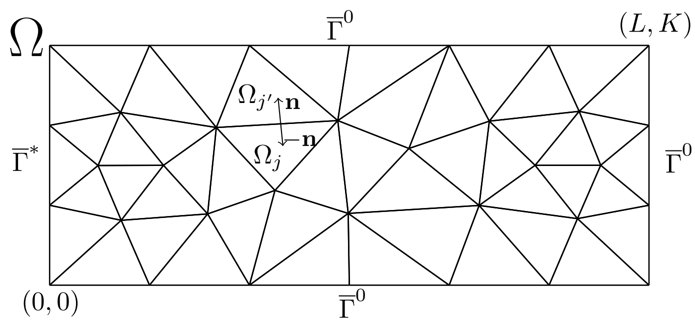


FIGURE 1. Computational domain Ω and its partition into triangular elements.

that lie in the rest of the strip boundary $\overline{\Gamma}^0$: $\overline{\Gamma}^0_k \in \{\overline{\Gamma}_k\}_{k=1}^{N_b}$, with $\overline{\Gamma}^0_k \cap \overline{\Gamma}^0 \neq \emptyset$; so that $\{\overline{\Gamma}_k\}_{k=1}^{N_t} = \{\overline{\Gamma}^*_k\}_{k=1}^{N_b^*} \cup \{\overline{\Gamma}^0_k\}_{k=1}^{N_b^0}$ and $\{\overline{\Gamma}^*_k\}_{k=1}^{N_b^*} \cap \{\overline{\Gamma}^0_k\}_{k=1}^{N_b^0} = \emptyset$. Here,

$$N_b^* := \text{ total number of mesh edges over the strip entrance } \overline{\Gamma}^*.$$

$$N_b^0 := \text{ total number of mesh edges over the rest of the boundary } \overline{\Gamma}^0.$$

As a consequence, we have $\mathrm{cl}(\overline{\Gamma}^0) = \cup_{k=1}^{N_b^0} \mathrm{cl}(\overline{\Gamma}^0_k)$ and $\mathrm{cl}(\overline{\Gamma}^*) = \cup_{k=1}^{N_b^*} \mathrm{cl}(\overline{\Gamma}^*_k)$. Moreover, $\Gamma = \widetilde{\Gamma} \cup \overline{\Gamma}$ and $\overline{\Gamma} = \overline{\Gamma}^* \cup \overline{\Gamma}^0 = \partial\Omega$. A schematic representation of the computational domain Ω and its partition into triangular elements, with the interior triangular elements Ω_j and $\Omega_{j'}$, interior boundaries $\widetilde{\Gamma}_k \subset \widetilde{\Gamma}$ and the exterior boundaries $\overline{\Gamma}_k \subset \overline{\Gamma}$, is drawn in Fig. 1.

Let $k \geq 0$ be an integer. The finite-dimensional space of all polynomials, of total degree at most k, defined in Ω_j, is denoted by $\mathbb{P}_k(\Omega_j)$. By

$$\mathcal{P}_k^h := \{\phi^h \in H^1(\Omega) \cap L^\sigma(\Omega) : \phi^h|_{\Omega_j} \in \mathbb{P}_k(\Omega_j) \quad \forall\ \Omega_j \in \mathcal{T}^h(\Omega)\}, \tag{34}$$

we denote the broken polynomial space associated with the partition $\mathcal{T}^h(\Omega)$. For $m \geq 2$, we introduce the broken Sobolev space associated with the partition $\mathcal{T}^h(\Omega)$,

$$H^m(\mathcal{T}^h(\Omega)) := \{\phi^h \in L^2(\Omega) : \phi^h|_{\Omega_j} \in H^m(\Omega_j) \cap W^{1,\sigma}(\Omega_j) \quad \forall\ \Omega_j \in \mathcal{T}^h(\Omega)\}.$$

According to [8, § 3], the inclusion $\mathcal{P}_k^h \subset C^0(\overline{\Omega}) \cap H^2(\mathcal{T}^h(\Omega))$ holds true. Therefore the approximation functions chosen in $\mathcal{P}_k^h$ are continuous in Ω, and up to the boundary $\partial\Omega$, and are in $H^2(\Omega_j) \cap W^{1,\sigma}(\Omega_j)$ for every element $\Omega_j \in \mathcal{T}^h(\Omega)$. However, the approximations chosen in $\mathcal{P}_k^h$ might be discontinuous in the first and higher-order derivatives across the boundaries of the elements Ω_j in the interior of Ω. In view of this, on each interior mesh edge $\widetilde{\Gamma}_k$ separating two triangular elements of $\mathcal{T}^h(\Omega)$, say Ω_j and $\Omega_{j'}$, we define the jump of the normal and of the tangential derivatives of $\phi^h \in \mathcal{P}_k^h$ across $\widetilde{\Gamma}_k$ by

$$\left[\!\left[\frac{\partial \phi^h}{\partial \mathbf{n}}\right]\!\right] := \frac{\partial \phi^h|_{\Omega_j}}{\partial \mathbf{n}} - \frac{\partial \phi^h|_{\Omega_{j'}}}{\partial \mathbf{n}} \qquad \text{and} \qquad \left[\!\left[\frac{\partial \phi^h}{\partial \boldsymbol{\tau}}\right]\!\right] := \frac{\partial \phi^h|_{\Omega_j}}{\partial \boldsymbol{\tau}} - \frac{\partial \phi^h|_{\Omega_{j'}}}{\partial \boldsymbol{\tau}}, \tag{35}$$

where $\mathbf{n}$ is assumed here to be the unit normal to $\widetilde{\Gamma}_j$ pointing from Ω_j to $\Omega_{j'}$ (see Fig. 1), whereas $\boldsymbol{\tau}$ is the unit tangent to $\widetilde{\Gamma}_j$ with respect to Ω_j. These notations are extended to mesh edges $\overline{\Gamma}_k$ over $\partial\Omega$ with the following meaning,

$$\left[\!\left[\frac{\partial \phi^h}{\partial \mathbf{n}}\right]\!\right] := \frac{\partial \phi^h|_{\Omega_j}}{\partial \mathbf{n}} \qquad \text{and} \qquad \left[\!\left[\frac{\partial \phi^h}{\partial \boldsymbol{\tau}}\right]\!\right] := \frac{\partial \phi^h|_{\Omega_j}}{\partial \boldsymbol{\tau}}. \tag{36}$$

Here, $\mathbf{n}$ and $\boldsymbol{\tau}$ denote the outward unit normal and the unit tangent to $\overline{\Gamma}_k$, respectively. Similarly, we define the average of the (second) normal derivative of $\phi^h \in \mathcal{P}_k^h$ across an interior mesh edge $\widetilde{\Gamma}_j$ by

$$\left\langle \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \right\rangle := \frac{1}{2}\left(\frac{\partial^2 \phi^h|_{\Omega_j}}{\partial \mathbf{n}^2} + \frac{\partial^2 \phi^h|_{\Omega_{j'}}}{\partial \mathbf{n}^2} \right), \tag{37}$$

and on mesh edges $\overline{\Gamma}_k$ over $\partial\Omega$ by

$$\left\langle \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \right\rangle := \frac{\partial^2 \phi^h|_{\Omega_j}}{\partial \mathbf{n}^2}. \tag{38}$$

Observe that the first equations in both (35) and (36) along with (37) and (38) are independent of the choice of the normal $\mathbf{n}$.

In order to approximate the solutions of our problem, we redefine the finite-dimensional space of our discontinuous Galerkin approximations associated with $\mathcal{T}^h(\Omega)$ as

$$\mathcal{V}^h := \mathcal{P}_k^h \cap H_0^1(\Omega),$$

for the space $\mathcal{P}_k^h$ defined in (34). We introduce the space of trial solutions,

$$\begin{aligned}\mathcal{S}^h := \{\psi^h \in H^1(\Omega) \cap L^\sigma(\Omega) : & \psi^h\big|_{\Omega_j} \in \mathbb{P}_k(\Omega_j) \quad \forall\, \Omega_j \in \mathcal{T}^h(\Omega), \quad \psi^h = 0 \text{ on } \overline{\Gamma}^0, \\ & \psi^h = f_* \text{ on } \overline{\Gamma}^*\},\end{aligned} \tag{39}$$

for $k \geq 2$. For each h, we consider the projection operator from the broken Sobolev space $H^m(\mathcal{T}^h(\Omega))$ onto the finite element space $\mathcal{S}^h$,

$$\Pi^h : H^m(\mathcal{T}^h(\Omega)) \longrightarrow \mathcal{S}^h, \tag{40}$$

associated with the scalar product

$$(\psi,\phi)_h = \sum_{j=1}^{N_e} (\psi,\phi)_{\Omega_j}, \qquad (\psi,\phi)_{\Omega_j} := \sum_{|\alpha|\leq m} \int_{\Omega_j} D^\alpha\psi \cdot D^\alpha\phi \,\mathrm{d}\mathbf{x}, \tag{41}$$

and such that, for each $\psi \in H^m(\mathcal{T}^h(\Omega))$,

$$\begin{cases} \Pi^h(\psi) \in \mathcal{S}^h, \\ \left(\psi - \Pi^h(\psi), \phi\right)_{\Omega_j} = 0 \quad \forall\, \Omega_j \in \mathcal{T}^h(\Omega), \quad \forall\, \phi \in \mathcal{S}^h. \end{cases} \tag{42}$$

We proceed with the derivation of the weak formulation of the problem (9)-(11) in the partition $\mathcal{T}^h(\Omega)$. Let ψ be an exact solution to the problem (9)-(11) and assume that ψ has all the required regularity. For the purposes of this section, it is enough to consider $\psi \in H^4(\mathcal{T}^h(\Omega))$ [8, § 4, p. 99]. For such ψ, we multiply (9) by $\phi^h \in \mathcal{V}^h$ and integrate by parts, the resulting equation, over an arbitrary element $\Omega_j \in \mathcal{T}^h(\Omega)$. We thus obtain

$$\begin{aligned} &\nu \int_{\Omega_j} D^2\psi : D^2\phi^h \,\mathrm{d}\mathbf{x} - \nu \int_{\partial\Omega_j} \left(\nabla\psi_x \cdot \mathbf{n}\,\phi^h_x + \nabla\psi_y \cdot \mathbf{n}\,\phi^h_y\right) \mathrm{d}\mathbf{x} \\ &+ \nu \int_{\partial\Omega_j} \nabla(\Delta\psi)\cdot\mathbf{n}\,\phi^h \,\mathrm{d}s - \int_{\Omega_j} \Delta\psi \left(\psi_x\phi^h_y - \psi_y\phi^h_x\right) \mathrm{d}\mathbf{x} \\ &+ \int_{\partial\Omega_j} \Delta\psi\,\nabla^\perp\psi\cdot\mathbf{n}\,\phi^h \,\mathrm{d}s = \int_{\Omega_j} \mathbf{f}^\perp\cdot\nabla\phi^h \,\mathrm{d}\mathbf{x} - \int_{\partial\Omega_j} \mathbf{f}^\perp\cdot\mathbf{n}\,\phi^h \,\mathrm{d}s, \end{aligned} \tag{43}$$

where $\mathbf{f}$ is the feedback field defined in (16). Throughout the text, it is also implied that the elements $\phi^h \in \mathcal{V}^h$, or $\phi^h \in \mathcal{S}^h$, should be evaluated by using their restrictions to the considered mesh element Ω_j. Summing up (43) from $j = 1$ till $j = N_e$, we get

$$\begin{aligned}
&\nu \sum_{j=1}^{N_e} \int_{\Omega_j} D^2\psi : D^2\phi^h \,\mathrm{d}\mathbf{x} - \nu \sum_{j=1}^{N_e} \int_{\partial\Omega_j} \left(\nabla\psi_x \cdot \mathbf{n}\,\phi_x^h + \nabla\psi_y \cdot \mathbf{n}\,\phi_y^h\right) \mathrm{d}\mathbf{x} \\
&+ \nu \sum_{j=1}^{N_e} \int_{\partial\Omega_j} \nabla(\Delta\psi) \cdot \mathbf{n}\,\phi^h \;\mathrm{d}s + \sum_{j=1}^{N_e} \int_{\Omega_j} \int_{\Omega_j} \Delta\psi \left(\psi_y\phi_x^h - \psi_x\phi_y^h\right) \mathrm{d}\mathbf{x} \\
&+ \sum_{j=1}^{N_e} \int_{\partial\Omega_j} \Delta\psi\, \nabla^{\perp}\psi \cdot \mathbf{n}\,\phi^h \;\mathrm{d}s = \sum_{j=1}^{N_e} \int_{\Omega_j} \mathbf{f}^{\perp} \cdot \nabla\phi^h \,\mathrm{d}\mathbf{x} - \sum_{j=1}^{N_e} \int_{\partial\Omega_j} \mathbf{f}^{\perp} \cdot \mathbf{n}\,\phi^h \;\mathrm{d}s.
\end{aligned}$$

Proceeding as we did in [23, Lemmas 3.1-3.2], and using [23, Lemma 2.1], we obtain

$$\begin{aligned}
&\nu \sum_{j=1}^{N_e} \int_{\Omega_j} D^2\psi : D^2\phi^h \,\mathrm{d}\mathbf{x} + \sum_{j=1}^{N_e} \int_{\Omega_j} \Delta\psi \left(\psi_y\phi_x^h - \psi_x\phi_y^h\right) \phi \,\mathrm{d}\mathbf{x} \\
&- \nu \sum_{j=1}^{N_i} \int_{\widetilde{\Gamma}_j} \left(\left\langle \frac{\partial^2\psi}{\partial\mathbf{n}^2} \right\rangle \left[\!\!\left[\frac{\partial\phi^h}{\partial\mathbf{n}} \right]\!\!\right] + \left[\!\!\left[\frac{\partial\psi}{\partial\mathbf{n}} \right]\!\!\right] \left\langle \frac{\partial^2\phi^h}{\partial\mathbf{n}^2} \right\rangle \right) \mathrm{d}s \\
&- \nu \sum_{j=1}^{N_b} \int_{\overline{\Gamma}_j} \left(\frac{\partial^2\psi}{\partial\mathbf{n}^2} \frac{\partial\phi}{\partial\mathbf{n}} + \frac{\partial\psi}{\partial\mathbf{n}} \frac{\partial^2\phi}{\partial\mathbf{n}^2} \right) \mathrm{d}s \\
&+ \delta \sum_{j=1}^{N_e} \int_{\Omega_j} |\psi_y|^{\sigma-2}\psi_y\phi_y^h \,\mathrm{d}\mathbf{x} = -\nu \sum_{j=1}^{N_b^*} \int_{\Gamma_j^*} g_* \frac{\partial^2\phi^h}{\partial\mathbf{n}^2} \,\mathrm{d}s.
\end{aligned} \tag{44}$$

Following the approach performed in [23], jump terms in (44), across interior mesh edges, are penalized to approximately enforce continuity of the normal derivatives across element interfaces [23, Lemma 2.1-(2)]. This method also takes into account the penalization of normal fluxes on the exterior mesh edges. Therefore (44) is rewritten as follows,

$$\begin{aligned}
&\nu \sum_{j=1}^{N_e} \int_{\Omega_j} D^2\psi : D^2\phi^h \,\mathrm{d}\mathbf{x} + \sum_{j=1}^{N_e} \int_{\Omega_j} \Delta\psi \left(\psi_y\phi_x^h - \psi_x\phi_y^h\right) \phi \,\mathrm{d}\mathbf{x} \\
&- \nu \sum_{j=1}^{N_i} \int_{\widetilde{\Gamma}_j} \left(\left\langle \frac{\partial^2\psi}{\partial\mathbf{n}^2} \right\rangle \left[\!\!\left[\frac{\partial\phi^h}{\partial\mathbf{n}} \right]\!\!\right] + \left[\!\!\left[\frac{\partial\psi}{\partial\mathbf{n}} \right]\!\!\right] \left\langle \frac{\partial^2\phi^h}{\partial\mathbf{n}^2} \right\rangle \right) \mathrm{d}s \\
&- \nu \sum_{j=1}^{N_b} \int_{\overline{\Gamma}_j} \left(\frac{\partial^2\psi}{\partial\mathbf{n}^2} \frac{\partial\phi}{\partial\mathbf{n}} + \frac{\partial\psi}{\partial\mathbf{n}} \frac{\partial^2\phi}{\partial\mathbf{n}^2} \right) \mathrm{d}s \\
&+ \sum_{j=1}^{N_i} \frac{\tau^i}{\partial_j^i} \int_{\widetilde{\Gamma}_j} \left[\!\!\left[\frac{\partial\psi}{\partial\mathbf{n}} \right]\!\!\right] \left[\!\!\left[\frac{\partial\phi^h}{\partial\mathbf{n}} \right]\!\!\right] \mathrm{d}s + \sum_{j=1}^{N_b} \frac{\tau^b}{\partial_j^b} \int_{\overline{\Gamma}_j} \frac{\partial\psi}{\partial\mathbf{n}} \frac{\partial\phi^h}{\partial\mathbf{n}} \,\mathrm{d}s \\
&+ \delta \sum_{j=1}^{N_e} \int_{\Omega_j} |\psi_y|^{\sigma-2}\psi_y\phi_y^h \,\mathrm{d}\mathbf{x} = -\nu \sum_{j=1}^{N_b^*} \int_{\Gamma_j^*} g_* \frac{\partial^2\phi^h}{\partial\mathbf{n}^2} \,\mathrm{d}s + \sum_{j=1}^{N_b^*} \frac{\tau^b}{\partial_j^{b^*}} \int_{\Gamma_j^*} g_* \frac{\partial\phi^h}{\partial\mathbf{n}} \,\mathrm{d}s,
\end{aligned} \tag{45}$$

where, similarly to (32), $\partial_j^i := \text{diam}(\widetilde{\Gamma}_j)$, $\partial_j^b := \text{diam}(\overline{\Gamma}_j)$ and $\partial_j^{b^*} := \text{diam}(\overline{\Gamma}_j^*)$. The notations τ^i and τ^b stand for stabilization parameters in the interior mesh edges $\widetilde{\Gamma}$ and on the exterior mesh edges $\overline{\Gamma}$, respectively.

Let us assume that stabilization parameters

$$\tau^i > 0 \quad \text{and} \quad \tau^b > 0 \tag{46}$$

are given. The method proposed here for approximating the problem (9)-(11) requires the computation of $\psi^h \in \mathcal{S}^h$ so that

$$B_d(\psi^h, \phi^h) + B_a(\psi^h, \psi^h, \phi^h) + \delta B_f(\psi^h, \psi^h, \phi^h) = F(\phi^h) \qquad \forall\ \phi^h \in \mathcal{V}^h, \tag{47}$$

where $B_d(\cdot,\cdot)$ is the bilinear form defined by

$$\begin{aligned} B_d(\psi^h, \phi^h) := {}& \nu \sum_{j=1}^{N_e} \int_{\Omega_j} D^2\psi^h : D^2\phi^h \,\mathrm{d}\mathbf{x} \\ & - \nu \sum_{j=1}^{N_i} \int_{\widetilde{\Gamma}_j} \left(\left\langle \frac{\partial^2 \psi^h}{\partial \mathbf{n}^2} \right\rangle \left[\!\left[\frac{\partial \phi^h}{\partial \mathbf{n}} \right]\!\right] + \left[\!\left[\frac{\partial \psi^h}{\partial \mathbf{n}} \right]\!\right] \left\langle \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \right\rangle \right) \mathrm{d}s \\ & - \nu \sum_{j=1}^{N_b} \int_{\overline{\Gamma}_j} \left(\frac{\partial^2 \psi^h}{\partial \mathbf{n}^2} \frac{\partial \phi^h}{\partial \mathbf{n}} + \frac{\partial \psi^h}{\partial \mathbf{n}} \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \right) \mathrm{d}s \\ & + \sum_{j=1}^{N_i} \frac{\tau^i}{\partial_j^i} \int_{\widetilde{\Gamma}_j} \left[\!\left[\frac{\partial \psi^h}{\partial \mathbf{n}} \right]\!\right] \left[\!\left[\frac{\partial \phi^h}{\partial \mathbf{n}} \right]\!\right] \mathrm{d}s + \sum_{j=1}^{N_b} \frac{\tau^b}{\partial_j^b} \int_{\overline{\Gamma}_j} \frac{\partial \psi^h}{\partial \mathbf{n}} \frac{\partial \phi^h}{\partial \mathbf{n}} \,\mathrm{d}s, \end{aligned} \tag{48}$$

$B_a(\cdot,\cdot,\cdot)$ is the convective form defined by

$$B_a(\psi^h, \omega^h, \phi^h) := \sum_{j=1}^{N_e} \mathcal{J}_j(\psi^h, \omega^h, \phi^h), \tag{49}$$

$B_f(\cdot,\cdot,\cdot)$ is the nonlinear form defined by

$$B_f(\psi^h, \omega^h, \phi^h) := \sum_{j=1}^{N_e} \mathcal{I}_j(\psi^h, \omega^h, \phi^h), \tag{50}$$

and $F(\cdot)$ is the linear form defined by

$$F(\phi^h) := -\nu \sum_{j=1}^{N_b^*} \int_{\overline{\Gamma}_j^*} g_* \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \,\mathrm{d}s + \sum_{j=1}^{N_b^*} \frac{\tau^b}{\partial_j^{b^*}} \int_{\overline{\Gamma}_j^*} g_* \frac{\partial \phi^h}{\partial \mathbf{n}} \,\mathrm{d}s, \tag{51}$$

where, adapting the notations (19) and (25) to the present situation,

$$\begin{aligned} \mathcal{J}_j(\psi^h, \omega^h, \phi^h) &:= \int_{\Omega_j} \Delta\omega^h \nabla\psi^h \cdot \nabla^\perp \phi^h \,\mathrm{d}\mathbf{x}, \\ \mathcal{I}_j(\psi^h, \omega^h, \phi^h) &:= \int_{\Omega_j} |\omega_y^h|^{\sigma-2} \psi_y^h \phi_y^h \,\mathrm{d}\mathbf{x}. \end{aligned} \tag{52}$$

Similarly to the continuous problem, let also $\vartheta \in H^2(\Omega)$ be the extension function considered in (27)-(28). For such ϑ, we shall look for solutions $\psi^h \in \mathcal{S}^h$ to the problem (47) in the form

$$\psi^h = \omega^h + \Pi^h(\vartheta), \tag{53}$$

where $\Pi^h(\vartheta)$ is the projection of the function ϑ into $\mathcal{S}^h$ (see (40)). In our case, we just need Π^h to be the projection operator, associated to the scalar product (41), from the Sobolev space $H^2(\Omega)$ onto the finite element space $\mathcal{S}^h$. From (47), we can see that our discrete problem is thus equivalent to the following one,

$$(54)\qquad \begin{cases} \omega^h \in \mathcal{V}^h, \\ B_d(\omega^h + \Pi^h(\vartheta), \phi^h) + B_a(\omega^h + \Pi^h(\vartheta), \omega^h + \Pi^h(\vartheta), \phi^h) \\ \qquad\qquad + \delta B_f(\omega^h + \Pi^h(\vartheta), \omega^h + \Pi^h(\vartheta), \phi^h) = F(\phi^h) \qquad \forall\, \phi^h \in \mathcal{V}^h. \end{cases}$$

4. Auxiliary results

In this section, we recall several auxiliary results that will be used below. We start by recalling the following elementary algebraic inequality,

$$(55)\qquad (a+b)^p \le 2^{p-1}(a^p + b^p) \qquad \forall\, p \ge 1, \quad \forall\, a,\ b \ge 0,$$

and the Hölder inequality in the form

$$(56)\qquad \sum_{j=1}^{N} |a_j b_j| \le \left(\sum_{j=1}^{N} |a_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{N} |b_j|^q\right)^{\frac{1}{q}}, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

for some $N \in \mathbb{N}$, and where $a_j,\ b_j \in \mathbb{R}$ for all $j \in \{1, \ldots, N\}$. As usual, the case $p = q = 2$ shall be denoted in the sequel as the Cauchy-Schwarz inequality.

We now recall some useful inequalities of Sobolev type for finite elements Ω_j of the partition $\mathcal{T}^h(\Omega)$. The following results are stated for any space dimension $d \ge 2$, although the scope of our work is $d = 2$.

To show the discrete problem (47) is consistent with the stream-function problem (9)-(11), we shall need the following inverse Sobolev inequality.

Lemma 4.1. *Let $\Omega_j \in \mathcal{T}^h(\Omega)$ with diameter h_j satisfying (31). Assume that $\mathcal{X}$ is a finite-dimensional subspace of $W^{m,p}(\Omega_j) \cap W^{s,q}(\Omega_j)$, with $1 \le p,\ q \le \infty$ and $0 \le s \le m$. Then, there exists a positive constant C depending only on k, p and q such that*

$$(57)\qquad \|D^m u\|_{L^p(\Omega_j)} \le C h_j^{s-m+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|D^s u\|_{L^q(\Omega_j)} \qquad \forall\, u \in \mathcal{X}.$$

Proof. This result is a particular case of a more general one established in [9, Lemma 4.5.3]. □

The next auxiliary result shows that the trace is well defined under suitable conditions.

Lemma 4.2. *Let $\Omega_j \in \mathcal{T}^h(\Omega)$ with diameter h_j. If $u \in H^s(\Omega_j)$, $s \in \mathbb{N}$, then the trace of $D^\alpha u$ on $\partial\Omega_j$ is a well defined $L^2(\partial\Omega_j)$–function for any $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha| < s - \frac{1}{2}$. Moreover, there exists a positive constant C, independent of h_j, such that*

$$\|u\|^2_{L^2(\partial\Omega_j)} \le C \sum_{l=0}^{r} h_j^{2l-1} \|D^l u\|^2_{L^2(\Omega_j)} \qquad \text{for} \quad \frac{1}{2} < r \le s.$$

Proof. See [1, Theorem 3.10]. □

The proof of the existence of a weak solution to the discrete problem (47), shall be carried out by the application of the following result.

Lemma 4.3. *Let V be a finite-dimensional Hilbert space whose scalar product is denoted by $(\cdot,\cdot)$ and the correspondingly norm by $\|\cdot\|$. Let P be a continuous mapping from V into itself and such that*

$$\exists\ R > 0\ :\ (P(\xi), \xi) > 0 \quad \forall\ \xi \in V \quad \text{with} \quad \|\xi\| = R.$$

Then, there exists $\psi \in V$, with $\|\psi\| \leq R$, such that $P(\psi) = 0$.

Proof. The proof of Lemma 4.3 proceeds by contradiction and uses Brouwer's fixed point theorem [17, Corollary IV.1.1]. □

To establish the uniqueness of weak solutions to the discrete problem (47), it is of the utmost importance the following auxiliary result to handle the nonlinear term (50).

Lemma 4.4. *For all $\xi,\ \eta \in \mathbb{R}$, the following assertions hold true:*

$$2 \leq \sigma < \infty \quad \Rightarrow \quad C_1|\xi-\eta|^\sigma \leq \left(|\xi|^{\sigma-2}\xi - |\eta|^{\sigma-2}\eta\right)\cdot(\xi-\eta); \tag{58}$$

$$1 < \sigma < 2 \quad \Rightarrow \quad C_2|\xi-\eta|^2 \leq \left(|\xi|^{\sigma-2}\xi - |\eta|^{\sigma-2}\eta\right)\cdot(\xi-\eta)\left(|\xi|^\sigma + |\eta|^\sigma\right)^{\frac{2-\sigma}{\sigma}}; \tag{59}$$

for some positive constants C_1 and C_2 depending only on σ.

Proof. For the proof of Lemma 4.4, we address the reader to [19]. □

Also to handle the nonlinear term (50), but now in the stability analysis, we shall perform, it is very important the result of the following lemma.

Lemma 4.5. *For all $\xi,\ \eta \in \mathbb{R}$, the following assertions hold true:*

$$2 \leq \sigma < \infty \quad \Rightarrow \quad \left||\xi|^{\sigma-2}\xi - |\eta|^{\sigma-2}\eta\right| \leq C_1\left(|\xi|+|\eta|\right)^{\sigma-2}|\xi-\eta|;$$
$$1 < \sigma < 2 \quad \Rightarrow \quad \left||\xi|^{\sigma-2}\xi - |\eta|^{\sigma-2}\eta\right| \leq C_2|\xi-\eta|^{\sigma-1};$$

for some positive constants C_1 and C_2 depending only on σ.

Proof. The proof of Lemma 4.5 is also addressed for [19]. □

For the study of the stability analysis, the next result takes a major role.

Lemma 4.6. *Assume that the partition $\mathcal{T}^h(\Omega)$ is regular in the sense of (31)-(32). Let Π^h be the projection operator defined in (40)-(42) and assume that $1 \leq p,\ q \leq \infty$. Then, for all $s \in \{0,1,\ldots,k+1\}$ and all $\psi \in W^{s,q}(\Omega_j)$, there holds*

$$\|D^m\left(\psi - \Pi^h(\psi)\right)\|_{L^p(\Omega_j)} \leq C h_j^{s-m+d\left(\frac{1}{p}-\frac{1}{q}\right)}\|D^s\psi\|_{L^q(\Omega_j)} \quad \forall\ m \in \{0,1,\ldots,s\}, \tag{60}$$

for some positive constant C that does not depend on h_j.

Proof. See [11, Theorem 15.3] and [13, Lemma 1.58]. □

An immediate consequence of (60) is that

$$\|\Pi^h(\psi)\|_{L^p(\Omega_j)} \leq C\|\psi\|_{L^p(\Omega_j)}. \tag{61}$$

5. Consistency, stability and continuity

In this section, we start by proving that the discrete problem (47) is (strong) consistent with the problem (9)-(11), *i.e.*, that a sufficiently smooth solution to the problem (9)-(11) solves the discrete problem (47).

Theorem 5.1. *The discrete weak formulation (47) of the problem (9)-(11) is consistent in the space $H^4(\mathcal{T}^h(\Omega))$,* i.e. *any solution $\psi \in H^4(\mathcal{T}^h(\Omega))$ to the problem (9)-(11) satisfies*

$$B_d(\psi,\phi^h) + B_a(\psi,\psi,\phi^h) + \delta B_f(\psi,\psi,\phi^h) = F(\phi^h) \qquad \forall\ \phi^h \in \mathcal{V}^h. \tag{62}$$

Reciprocally, any solution $\psi^h \in \mathcal{S}^h \cap H^4(\mathcal{T}^h(\Omega))$ to the discrete problem (47) satisfies

$$\begin{aligned}
&\sum_{j=1}^{N_e} \int_{\Omega_j} \left(\nu\, \Delta^2 \psi^h + \psi^h_x \Delta \psi^h_y - \psi^h_y \Delta \psi^h_x - \delta \left(|\psi^h_y|^{\sigma-2} \psi^h_y\right)_y\right) \phi^h \, \mathrm{d}\mathbf{x} + \\
&\tau^b \sum_{j=1}^{N_b^*} \int_{\overline{\Gamma}_j^*} \left(\frac{\partial \psi^h}{\partial \mathbf{n}} - g_*\right) \frac{\partial \phi^h}{\partial \mathbf{n}} \, \mathrm{d}s + \tau^b \sum_{j=1}^{N_b^0} \int_{\overline{\Gamma}_j^o} \frac{\partial \psi^h}{\partial \mathbf{n}} \frac{\partial \phi^h}{\partial \mathbf{n}} \, \mathrm{d}s \\
&= \nu \sum_{j=1}^{N_b^*} \int_{\overline{\Gamma}_j^*} \left(\frac{\partial \psi^h}{\partial \mathbf{n}} - g_*\right) \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \, \mathrm{d}s + \nu \sum_{j=1}^{N_b^0} \int_{\overline{\Gamma}_j^o} \frac{\partial \psi^h}{\partial \mathbf{n}} \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \, \mathrm{d}s,
\end{aligned} \tag{63}$$

for all $\phi^h \in \mathcal{V}^h$.

Proof. Let ψ be a strong solution to the problem (9)-(11). If we assume that $\psi \in H^4(\mathcal{T}^h(\Omega))$, then we can use (44) together with (47)-(50) so that for every $\phi^h \in \mathcal{V}^h$ one has

(64)
$$\begin{aligned}
&B_d(\psi, \phi^h) + B_a(\psi, \psi, \phi^h) + \delta B_f(\psi, \psi, \phi^h) \\
&= \sum_{j=1}^{N_i} \frac{\tau^i}{\partial^i_j} \int_{\widetilde{\Gamma}_j} \left[\!\!\left[\frac{\partial \psi}{\partial \mathbf{n}}\right]\!\!\right] \left[\!\!\left[\frac{\partial \phi^h}{\partial \mathbf{n}}\right]\!\!\right] \mathrm{d}s + \sum_{j=1}^{N_b} \frac{\tau^b}{\partial^b_j} \int_{\overline{\Gamma}_j} \frac{\partial \psi}{\partial \mathbf{n}} \frac{\partial \phi^h}{\partial \mathbf{n}} \, \mathrm{d}s - \nu \sum_{j=1}^{N_b^*} \int_{\overline{\Gamma}_j^*} g_* \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \, \mathrm{d}s.
\end{aligned}$$

Since $\psi \in H^4(\mathcal{T}^h(\Omega))$, $\left[\!\!\left[\frac{\partial \psi}{\partial \mathbf{n}}\right]\!\!\right] = 0$ on each interior mesh edge $\widetilde{\Gamma}_j$. On the other hand, by (10)-(11), $\frac{\partial \psi}{\partial \mathbf{n}} = 0$ on each mesh edge $\overline{\Gamma}_j^o$ and $\frac{\partial \psi}{\partial \mathbf{n}} = g_*$ on each mesh edge $\overline{\Gamma}_j^*$. Therefore, the right-hand side of (64) reduces to (51), which proves (62).

We now assume that $\psi^h \in \mathcal{S}^h$ is a solution to the discrete problem (47). In view of (45) and (48)-(51), the discrete weak formulation (47) can be written as follows,

(65)
$$\begin{aligned}
0 &= B_d(\psi^h, \phi^h) + B_a(\psi^h, \psi^h, \phi^h) + \delta B_f(\psi^h, \psi^h, \phi^h) - F(\phi^h) \\
&= \nu \sum_{j=1}^{N_e} \int_{\Omega_j} D^2 \psi^h : D^2 \phi^h \, \mathrm{d}\mathbf{x} + \sum_{j=1}^{N_e} \int_{\Omega_j} \Delta \psi^h \left(\psi^h_y \phi^h_x - \psi^h_x \phi^h_y\right) \mathrm{d}\mathbf{x} \\
&\quad - \nu \sum_{j=1}^{N_i} \int_{\widetilde{\Gamma}_j} \left(\left\langle \frac{\partial^2 \psi^h}{\partial \mathbf{n}^2} \right\rangle \left[\!\!\left[\frac{\partial \phi^h}{\partial \mathbf{n}}\right]\!\!\right] + \left[\!\!\left[\frac{\partial \psi^h}{\partial \mathbf{n}}\right]\!\!\right] \left\langle \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \right\rangle\right) \mathrm{d}s \\
&\quad - \nu \sum_{j=1}^{N_b} \int_{\overline{\Gamma}_j} \left(\frac{\partial^2 \psi^h}{\partial \mathbf{n}^2} \frac{\partial \phi^h}{\partial \mathbf{n}} + \frac{\partial \psi^h}{\partial \mathbf{n}} \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2}\right) \mathrm{d}s \\
&\quad + \sum_{j=1}^{N_i} \frac{\tau^i}{\partial^i_j} \int_{\widetilde{\Gamma}_j} \left[\!\!\left[\frac{\partial \psi^h}{\partial \mathbf{n}}\right]\!\!\right] \left[\!\!\left[\frac{\partial \phi^h}{\partial \mathbf{n}}\right]\!\!\right] \mathrm{d}s + \sum_{j=1}^{N_b} \frac{\tau^b}{\partial^b_j} \int_{\overline{\Gamma}_j} \frac{\partial \psi^h}{\partial \mathbf{n}} \frac{\partial \phi^h}{\partial \mathbf{n}} \, \mathrm{d}s \\
&\quad + \delta \sum_{j=1}^{N_e} \int_{\Omega_j} |\psi^h_y|^{\sigma-2} \psi^h_y \phi^h_y \, \mathrm{d}\mathbf{x} + \nu \sum_{j=1}^{N_b^*} \int_{\overline{\Gamma}_j^*} g_* \frac{\partial^2 \phi^h}{\partial \mathbf{n}^2} \, \mathrm{d}s - \sum_{j=1}^{N_b^*} \frac{\tau^b}{\partial^{b*}_j} \int_{\overline{\Gamma}_j^*} g_* \frac{\partial \phi^h}{\partial \mathbf{n}} \, \mathrm{d}s.
\end{aligned}$$

Since $\psi^h \in \mathcal{S}^h \cap H^4(\mathcal{T}^h(\Omega))$, we can integrate by parts the first, second and seventh terms of the right-hand side of (65). After some simplifications, we get

$$\begin{aligned}
&\sum_{j=1}^{N_e} \int_{\Omega_j} D^2\psi^h : D^2\phi^h \,\mathrm{d}\mathbf{x} = \sum_{j=1}^{N_e} \int_{\Omega_j} \Delta^2\psi^h \phi^h \,\mathrm{d}\mathbf{x} + \\
&\sum_{j=1}^{N_i} \int_{\widetilde{\Gamma}_j} \left(\left\langle \frac{\partial^2\psi^h}{\partial \mathbf{n}^2} \right\rangle \left[\!\left[\frac{\partial \phi^h}{\partial \mathbf{n}} \right]\!\right] + \left[\!\left[\frac{\partial \psi^h}{\partial \mathbf{n}} \right]\!\right] \left\langle \frac{\partial^2\phi^h}{\partial \mathbf{n}^2} \right\rangle \right) \mathrm{d}s \\
&+ \sum_{j=1}^{N_b} \int_{\overline{\Gamma}_j} \left(\frac{\partial^2\psi^h}{\partial \mathbf{n}^2} \frac{\partial \phi^h}{\partial \mathbf{n}} + \frac{\partial \psi^h}{\partial \mathbf{n}} \frac{\partial^2\phi^h}{\partial \mathbf{n}^2} \right) \mathrm{d}s \\
&- \sum_{j=1}^{N_b^*} \int_{\overline{\Gamma}_j^*} \frac{\partial \psi^h}{\partial \mathbf{n}} \frac{\partial^2\phi^h}{\partial \mathbf{n}^2} \,\mathrm{d}s - \sum_{j=1}^{N_b^0} \int_{\overline{\Gamma}_j^o} \frac{\partial \psi^h}{\partial \mathbf{n}} \frac{\partial^2\phi^h}{\partial \mathbf{n}^2} \,\mathrm{d}s,
\end{aligned} \tag{66}$$

$$\sum_{j=1}^{N_e} \int_{\Omega_j} \Delta\psi^h \left(\psi_y^h \phi_x^h - \psi_x^h \phi_y^h \right) \mathrm{d}\mathbf{x} = \sum_{j=1}^{N_e} \int_{\Omega_j} \left(\psi_x^h \Delta\psi_y^h - \psi_y^h \Delta\psi_x^h \right) \phi^h \,\mathrm{d}\mathbf{x}, \tag{67}$$

$$\sum_{j=1}^{N_e} \int_{\Omega_j} |\psi_y^h|^{\sigma-2} \psi_y^h \phi_y^h \,\mathrm{d}\mathbf{x} = -\sum_{j=1}^{N_e} \int_{\Omega_j} \left(|\psi_y^h|^{\sigma-2} \psi_y^h \right)_y \phi^h \,\mathrm{d}\mathbf{x}. \tag{68}$$

Then, plugging (66)-(68) into (65), and observing that $\psi^h \in H^4(\mathcal{T}^h(\Omega))$ implies $\left[\!\left[\frac{\partial \psi^h}{\partial \mathbf{n}} \right]\!\right] = 0$, we get (63). □

Recalling the definitions of $\overline{\Gamma}^*$, $\overline{\Gamma}^0$ and $\widetilde{\Omega}$ (see (12)-(13) and (33)), we can see that a sufficient condition for the validity of (63) is that

$$\nu\, \Delta^2\psi^h + \psi_x^h \Delta\psi_y^h - \psi_y^h \Delta\psi_x^h = \delta \left(|\psi_y^h|^{\sigma-2} \psi_y^h \right)_y \quad \text{in } \widetilde{\Omega}, \tag{69}$$

$$\frac{\partial \psi^h}{\partial \mathbf{n}} = g_* \quad \text{on } \overline{\Gamma}^*, \tag{70}$$

$$\frac{\partial \psi^h}{\partial \mathbf{n}} = 0 \quad \text{on } \overline{\Gamma}^0. \tag{71}$$

On the other hand, since the Galerkin approximations ψ^h are sought in the function space $\mathcal{S}^h$ (see (39)), we also should have

$$\psi^h = 0 \quad \text{on } \overline{\Gamma}^0, \tag{72}$$

$$\psi^h = f_* \quad \text{on } \overline{\Gamma}^*. \tag{73}$$

Observe that (69) denotes the enforcement of the governing partial differential equation (9) on each element Ω_j of the partition $\mathcal{T}^h(\Omega)$, whereas (70)-(71) and (72)-(73) account for the enforcement of the boundary conditions (10)-(11). Moreover, the underlying assumption

$$\left[\!\left[\frac{\partial \psi^h}{\partial \mathbf{n}} \right]\!\right] = 0 \quad \text{on } \widetilde{\Gamma},$$

ensures the continuity of $\frac{\partial \psi^h}{\partial \mathbf{n}}$ across interior mesh edges $\widetilde{\Gamma}_j$.

We now define the norm associated with the discrete problem (47), by the following identity

(74)

$$|||\phi^h|||^2 := \nu \sum_{j=1}^{N_e} \|D^2\phi^h\|^2_{L^2(\Omega_j)} + \sum_{j=1}^{N_i} \frac{\tau^i}{\partial^i_j} \left\| \left[\!\left[\frac{\partial \phi^h}{\partial \mathbf{n}} \right]\!\right] \right\|^2_{L^2(\widetilde{\Gamma}_j)} + \sum_{j=1}^{N_b} \frac{\tau^b}{\partial^b_j} \left\| \frac{\partial \phi^h}{\partial \mathbf{n}} \right\|^2_{L^2(\overline{\Gamma}_j)},$$

for $\phi^h \in \mathcal{V}^h$, where $\|\cdot\|_{L^2(\cdot)}$ denotes the Lebesgue L^2-norm. In the next lemma, we set the conditions under which $|||\cdot|||$ is a norm in $\mathcal{V}^h$.

Lemma 5.1. *If $\tau^i > 0$ and $\tau^b > 0$, then $|||\cdot|||$ is a norm on $\mathcal{V}^h$ and on $\mathcal{S}^h$ i.*

Proof. We address the proof that $|||\cdot|||$ is a norm on $\mathcal{V}^h$ for [23, Lemma 5.1]. In view of that, one can easily see that $|||\cdot|||$ is also a norm on $\mathcal{S}^h$, in the same conditions. □

The following result states that the bilinear form $B_d(\cdot,\cdot)$, defined in (48), is coercive with respect to the norm (74).

Theorem 5.2. *Let $|||\cdot|||$ be the norm defined in (74). Assume that (31)-(32) hold and that there exists positive constants C_1 and C_2, which are independent of h, such that*

$$\tau^i \geq C_1\nu \qquad \text{and} \qquad \tau^b \geq C_2\nu. \tag{75}$$

Then there exists a positive constant m such that

$$B_d(\phi^h,\phi^h) \geq m|||\phi^h|||^2 \qquad \forall\ \phi^h \in \mathcal{P}^h_k. \tag{76}$$

Proof. The proof is performed in [23, Theorem 5.2], using Lemmas 4.5-4.6. □

Since the coercivity property (76) implies the discrete inf-sup condition, Theorem 5.2 asserts that the bilinear form $B_d(\cdot,\cdot)$ enjoys the discrete stability property on $\mathcal{P}^h_k$. Therefore, the continuous/discontinuous method, introduced in (47), is stable in the norm (74). If we define the global energy associated to the discrete problem (47) by

$$\mathcal{E}(\phi^h) := |||\phi^h|||^2 + \delta \sum_{j=1}^{N_e} \|\phi^h_y\|^{\sigma}_{L^\sigma(\Omega_j)}, \tag{77}$$

where $|||\cdot|||$ denotes the norm defined in (74), the aforementioned result also asserts that the discrete problem (47) is stable in the energy (77).

We now are going to state a discrete version of the continuous Sobolev inequality that holds in the broken polynomial space $\mathcal{P}^h_k$. Firstly, let us introduce the norm

$$|||\phi^h|||^p_{dG,p} := \sum_{j=1}^{N_e} \|\nabla\phi^h\|^p_{L^p(\Omega_j)} + \sum_{j=1}^{N_t} \frac{1}{\partial_j^{p-1}} \left\| [\![\phi^h]\!] \right\|^p_{L^p(\Gamma_j)},$$

for $1 \leq p < \infty$.

Lemma 5.2. *Assume $d = 2$ and let p be an arbitrary real number such that $1 \leq p < \infty$. Then there holds*

$$\|\phi^h\|_{L^p(\Omega)} \leq C|||\phi^h|||_{dG,2} \qquad \forall\ \phi^h \in \mathcal{P}^h_k,$$

for some positive constant C that depends on k, Ω and p.

Proof. We address the proof to [23, Lemma 6.1] (see also [12, 13]). □

Corollary 5.1. *Let $1 \leq p < \infty$ and $k \in \mathbb{N}_0$, and assume the partition $\mathcal{T}^h(\Omega)$, defined at (30), satisfies (31)-(32). If (75) holds, then there exists a positive constant C, depending only on p and Ω, such that*

$$\|\nabla\phi^h\|_{L^p(\Omega)} \leq \frac{C}{\sqrt{\nu}}|||\phi^h||| \qquad \forall\ \phi^h \in \mathcal{P}_k^h. \tag{78}$$

Proof. The proof is an immediate consequence of Lemma 5.2 (see also [23, Lemma 6.2]). □

The next result is devoted to prove that the forms $B_d(\cdot,\cdot)$, $B_a(\cdot,\cdot,\cdot)$, $B_f(\cdot,\cdot,\cdot)$ and $F(\cdot)$, defined at (48)-(51), are continuous, with respect to the discrete norm (74).

Theorem 5.3. *Assume that (31)-(32) and (75) hold. Then there exist positive constants C_1, C_2, C_3 and C_4, independent of h, such that*

$$B_d(\psi^h,\phi^h) \leq C_1|||\psi^h|||\,|||\phi^h|||, \tag{79}$$

$$B_a(\psi^h,\omega^h,\phi^h) \leq C_2|||\psi^h|||\,|||\omega^h|||\,|||\phi^h|||, \tag{80}$$

$$B_f(\psi^h,\psi^h,\phi^h) \leq C_3|||\psi^h|||^{\sigma-1}|||\phi^h||| \qquad \forall\sigma > 1, \tag{81}$$

$$B_f(\psi^h,\omega^h,\phi^h) \leq C_4|||\psi^h|||^{\sigma-2}|||\omega^h|||\,|||\phi^h||| \qquad \forall\sigma \geq 2, \tag{82}$$

for all $\psi^h,\ \omega^h,\ \phi^h \in \mathcal{P}_k^h$.

If, in addition, (26) holds, then there also exists a positive constant C_5 such that

$$F(\phi^h) \leq C_5|||\phi^h|||, \tag{83}$$

for all $\psi^h \in \mathcal{P}_k^h$.

Proof. The proof of (79), (81)-(82) and (83) are addressed to [23, Theorem 6.1]. To prove (80), we use (49), (52) and (74), together with the Hölder inequality (56) and with the discrete Sobolev inequality (78), so that

$$\begin{aligned} &B_a(\psi^h,\omega^h,\phi^h) \\ \leq &\left(\sum_{j=1}^{N_e}\|\Delta\omega^h\|^2_{L^2(\Omega_j)}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{N_e}\|\nabla\psi^h\|^4_{L^4(\Omega_j)}\right)^{\frac{1}{4}}\left(\sum_{j=1}^{N_e}\|\nabla^\perp\phi^h\|^4_{L^4(\Omega_j)}\right)^{\frac{1}{4}} \\ \leq &\sqrt{\frac{2N_e}{\nu}}\left(\nu\sum_{j=1}^{N_e}\|D^2\omega^h\|^2_{L^2(\Omega_j)}\right)^{\frac{1}{2}}\|\nabla\psi^h\|_{L^4(\Omega)}\|\nabla\phi^h\|_{L^4(\Omega)} \\ \leq &C_2|||\omega^h|||\,|||\psi^h|||\,|||\phi^h|||. \end{aligned} \tag{84}$$

□

6. Existence and uniqueness for the continuous problem

In this section, we prove the existence and uniqueness of solutions for the continuous problem. As we have mentioned in the final part of Section 2, searching for a solution ψ to the problem (17) is equivalent to look for a solution ω to the problem (29). We start by considering the following auxiliary problem

$$
\begin{cases}
\omega \in H_0^2(\Omega), \\
\nu \int_\Omega D^2\omega : D^2\phi \,\mathrm{d}\mathbf{x} + \mathcal{J}(\xi+\vartheta, \omega+\vartheta, \phi) = \delta\mathcal{I}(\omega+\vartheta, \omega+\vartheta, \phi) \\
-\nu \int_\Omega D^2\vartheta : D^2\phi \,\mathrm{d}\mathbf{x} \qquad \forall\, \phi \in H_0^2(\Omega),
\end{cases} \tag{85}
$$

where ξ is a given function, ϑ is the extension function satisfying (27)-(28), and ω is the solution to the problem (29). Recall that $\mathcal{J}$ and $\mathcal{I}$ are the forms defined in (18) and (25).

Lemma 6.1. *Let be given $\vartheta \in H^2(\Omega)$ and assume that $\xi \in H^2(\Omega)$ is fixed but arbitrarily given. Then for any $\sigma > 1$ exists a unique solution to the problem (85).*

Proof. (Lemma 6.1) Observe that for a given $\vartheta \in H^2(\Omega)$ and for a fixed $\xi \in H^2(\Omega)$, $\mathcal{J}(\xi, \omega+\vartheta, \phi)$ defines a linear functional for $\phi \in H_0^2(\Omega)$. Therefore the proof can be carried out by using a similar reasoning to the proof of [23, Theorem 7.1]. □

Lemma 6.1 allows us now to prove the main result of this section.

Theorem 6.1. *Let $\sigma > 1$ and assume that (15) and (26) hold. Then there exists, at least, a solution to the problem (17). Moreover, if ψ_1 and ψ_2 are two solutions to the problem (17) and if there exists a positive constant $C = C(\Omega)$ such that*

$$\nu > C\|\psi_2\|_{H^2(\Omega)}, \tag{86}$$

then $\psi_1 = \psi_2$.

Proof. As we have mentioned above, searching for a solution ψ to the problem (17) is equivalent to look for a solution ω to the problem (29). To prove the existence of a solution to the problem (29), we shall apply the Schauder fixed point theorem. At the moment, we know from Lemma 6.1 that for each $\xi \in H^2(\Omega)$ there exists a unique $\omega = \omega_\xi$ solution to the problem (85). In particular,

$$\|D^2\omega\|_2 \le C\left(\|\xi\|_{H^2(\Omega)} + \|f_*\|_{H^{\frac{3}{2}}(0,L)} + \|g_*\|_{H^{\frac{1}{2}}(0,L)}\right) := C_0, \tag{87}$$

for some positive constant $C = C(\nu, \sigma, \delta)$. Hence, by using the Sobolev inequality, we obtain

$$\|\omega\|_{H^2(\Omega)} \le C\left(\|\xi\|_{H^2(\Omega)} + \|f_*\|_{H^{\frac{3}{2}}(0,L)} + \|g_*\|_{H^{\frac{1}{2}}(0,L)}\right) := C_{00},$$

for another positive constant $C = C(\nu, \sigma, \delta, \Omega)$.

Let us now consider the mapping

$$B \ni \xi \longmapsto \omega_\xi \in B,$$

where $B := \{\phi \in H^2(\Omega) : \|\phi\|_{H^2(\Omega)} \le C_{00}\}$. From the Schauder fixed point theorem, it is clear that this mapping will have a fixed point provided it is continuous. To prove this, let us assume that ξ_n is a sequence in $H^2(\Omega)$ such that

$$\xi_n \xrightarrow[n\to\infty]{} \xi \quad \text{in } H^2(\Omega). \tag{88}$$

For any $n \in \mathbb{N}$, let ω_n be the solution to the problem (85) associated to ξ_n. Due to (87), one also has

$$\|D^2\omega_n\|_{L^2(\Omega)} \le C_0.$$

In view of the Banach-Alaoglu theorem and the Sobolev compact imbedding theorem, we have for some subsequence, still labeled by n, and some ω

$$\omega_n \xrightarrow[n\to\infty]{} \omega \quad \text{in } H_0^2(\Omega), \tag{89}$$

$$\omega_n \xrightarrow[n\to\infty]{} \omega \quad \text{in } L^2(\Omega). \tag{90}$$

Writing (85) with ω_n and ξ_n in the places of ω and ξ, there holds

$$\begin{aligned} &\nu \int_\Omega D^2\omega_n : D^2\phi \,\mathrm{d}\mathbf{x} + \mathcal{J}(\xi_n+\vartheta, \omega_n+\vartheta, \phi) \\ =&\delta\mathcal{I}(\omega_n+\vartheta, \omega_n+\vartheta, \phi) - \nu \int_\Omega D^2\vartheta : D^2\phi \,\mathrm{d}\mathbf{x} \qquad \forall\ \phi \in H_0^2(\Omega). \end{aligned} \tag{91}$$

Using the convergence results (88) and (89)-(90), we can pass (91) to the limit to get (85) for the function ω found in (89)-(90). This proves that $\omega = \omega_\xi$. Since the limit is uniquely determined, we can see that, in view of (90), one has

$$\omega_{\xi_n} \xrightarrow[n\to\infty]{} \omega_\xi \quad \text{in } L^2(\Omega),$$

which proves the continuity of the mapping. Hence problem (29) has, at least, a solution. Since $\omega = \psi - \vartheta$, this also shows that problem (17) has a solution as well.

To prove the uniqueness, let ψ_1 and ψ_2 be two solutions, corresponding to the same data, of the problem (17). Subtracting the equation (17) for ψ_2 to the one for ψ_1, and taking for test function $\phi = \psi_1 - \psi_2$, observing that (10)-(11) being satisfied in the trace sense imply $\psi_1 - \psi_2 \in H_0^2(\Omega)$, we obtain

$$\begin{aligned} &\nu\|D^2\psi\|^2_{L^2(\Omega)} + \mathcal{J}(\psi_1,\psi_1,\psi) - \mathcal{J}(\psi_2,\psi_2,\psi) \\ &+ \delta \int_\Omega \left(|\psi_{1,y}|^{\sigma-2}\psi_{1,y} - |\psi_{2,y}|^{\sigma-2}\psi_{2,y}\right)(\psi_{1,y}-\psi_{2,y}) \,\mathrm{d}\mathbf{x} = 0, \end{aligned} \tag{92}$$

where we have set $\psi := \psi_1 - \psi_2$, and $\psi_{1,y}$, $\psi_{2,y}$ denote the partial derivatives of ψ_1 and ψ_2 with respect to y.

By Lemma 4.4, there exist positive constants C_1 and C_2, depending only on σ, such that

$$\left(|\psi_{1,y}|^{\sigma-2}\psi_{1,y} - |\psi_{2,y}|^{\sigma-2}\psi_{2,y}\right)(\psi_{1,y}-\psi_{2,y}) \geq C_1 \left|\psi_{1,y}-\psi_{2,y}\right|^\sigma, \tag{93}$$

if $\sigma \geq 2$, and

$$\begin{aligned} &\left(|\psi_{1,y}|^{\sigma-2}\psi_{1,y} - |\psi_{2,y}|^{\sigma-2}\psi_{2,y}\right)(\psi_{1,y}-\psi_{2,y})\left(|\psi_{1,y}|^\sigma + |\psi_{2,y}|^\sigma\right)^{\frac{2-\sigma}{\sigma}} \geq \\ &C_2 \left|\psi_{1,y}-\psi_{2,y}\right|^2, \end{aligned}$$

if $1 < \sigma < 2$. From here, we can easily derive

$$\left(|\psi_{1,y}|^{\sigma-2}\psi_{1,y} - |\psi_{2,y}|^{\sigma-2}\psi_{2,y}\right)(\psi_{1,y}-\psi_{2,y}) \geq C_2 \frac{\left|\psi_{1,y}-\psi_{2,y}\right|^2}{1+\left(|\psi_{1,y}|^\sigma + |\psi_{2,y}|^\sigma\right)^{\frac{2-\sigma}{\sigma}}}. \tag{94}$$

On the other hand, using (19) and (20), we can show that

$$\mathcal{J}(\psi_1,\psi_1,\psi) - \mathcal{J}(\psi_2,\psi_2,\psi) = \mathcal{J}(\psi,\psi_1,\psi) - \mathcal{J}(\psi_2,\psi,\psi) = -\mathcal{J}(\psi_2,\psi,\psi). \tag{95}$$

Then, combining (93) and (94) with (95), we obtain from (92)

$$\nu\|D^2\psi\|^2_{L^2(\Omega)} \leq \mathcal{J}(\psi_2,\psi,\psi).$$

By the Hölder inequality and (22)-(24), one has

$$\mathcal{J}(\psi_2,\psi,\psi) \leq C\|\psi_2\|_{H^2(\Omega)}\|D^2\psi\|^2_{L^2(\Omega)}$$

for some positive constant $C = C(\Omega)$. As a consequence,

$$\left(\nu - C\|\psi_2\|_{H^2(\Omega)}\right)\|D^2\psi\|^2_{L^2(\Omega)} \leq 0.$$

In view of the assumption (86), $\|D^2\psi\|_{L^2(\Omega)} \leq 0$, and thus, by the continuous Sobolev inequality, $\|\psi\|_{L^2(\Omega)} \leq 0$. Consequently $\psi_1 = \psi_2$ a.e. in Ω. □

Observe that, by taking $\phi = \omega$ in (29), one can show that

$$\|D^2\omega\|_{L^2(\Omega)} \leq C\left(\|f_*\|_{H^{\frac{3}{2}}(0,L)}, \|g_*\|_{H^{\frac{1}{2}}(0,L)}, \nu, \sigma, \delta\right) := C_{00},$$

and consequently

$$\|\psi\|_{H^2(\Omega)} \leq C\left(\|f_*\|_{H^{\frac{3}{2}}(0,L)}, \|g_*\|_{H^{\frac{1}{2}}(0,L)}, \nu, \sigma, \delta\right) := C_0,$$

for distinct positive constants C_0 and C_{00}.

Therefore assumption (86) can be replaced by

$$\nu > C_{00},$$

which means the uniqueness is attained only for large viscosity or small data.

7. Existence and uniqueness for the discrete problem

To prove the existence of solutions to the discrete problem (47), we shall use the Brouwer fixed point theorem. In view of (53), proving the existence of a weak solution $\psi^h \in \mathcal{S}^h$ to the discrete problem (47) is equivalent to prove the existence of a solution $\omega^h \in \mathcal{V}^h$ to the problem (54). For this purpose, let us consider for the extension function

$$\vartheta_\varepsilon = \eta_\varepsilon \vartheta, \tag{96}$$

where ϑ is the function given by (27)-(28) and η_ε, with $\varepsilon > 0$, is a cut-off function in the spirit of [18, Lemmas III.6.1-2]. In particular, $\eta_\varepsilon \in C^\infty(\overline{\Omega})$ and

$$|\eta_\varepsilon(x)| \leq 1 \qquad \forall\, x \in \Omega, \tag{97}$$

$$|\eta_\varepsilon(x)| \leq 1 \qquad \forall\, x \in \Omega, \tag{98}$$

$$\eta_\varepsilon(x) = 0 \quad \text{if} \quad \operatorname{dist}(x, \partial\Omega) \geq 2e^{-\frac{1}{\varepsilon}}, \tag{99}$$

$$|D^\alpha \eta_\varepsilon(x)| \leq \frac{\varepsilon}{\operatorname{dist}^{|\alpha|}(x, \partial\Omega)} \qquad \forall\, x \in \Omega, \quad |\alpha| \geq 1. \tag{100}$$

In the auxiliary result below we estimate $\Pi^h(\vartheta)$ in the discrete norm (74).

Lemma 7.1. *Let $\vartheta \in H^2(\Omega)$ be the function considered in (27)-(28), ϑ_ε the function defined in* (96)*, and Π^h the projection operator, associated to the scalar product (41), from the Sobolev space $H^2(\Omega)$ onto the finite element space $\mathcal{S}^h$. If the assumptions (15), (26) and (31)-(32) hold, then there exists an independent of h_j positive constant C such that*

$$|||\Pi^h(\vartheta_\varepsilon)||| \leq C\left(\|f_*\|_{H^{\frac{3}{2}}(0,L)} + \|g_*\|_{H^{\frac{1}{2}}(0,L)}\right) := C_0. \tag{101}$$

Proof. Observing the properties (97)-(100) of the cut-off function considered above, we can use reasoning very similar to the proof of [23, Lemma 7.1] to prove Lemma 7.1. □

We are now in conditions to establish the main result of this section.

Theorem 7.1. *Let $N_e \in \mathbb{N}$ be given and assume the hypotheses (15), (26), (31)-(32) and (75) hold. Then problem (47) has, at least, a solution $\psi^h \in \mathcal{S}^h$.*

Proof. For each $h > 0$ let us consider the mapping $P_h : \mathcal{V}^h \longrightarrow \mathcal{V}^h$ defined by

$$\begin{aligned}(P_h(\omega^h), \phi^h) &:= B_d(\omega^h + \Pi^h(\vartheta_\varepsilon), \phi^h) + B_a(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \phi^h)\\ &+ \delta B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \phi^h) - F(\phi^h) \qquad \forall\ \phi^h \in \mathcal{V}^h,\end{aligned}$$

for the forms B_d, B_a, B_f and F defined in (48)-(51). It is clear that this mapping is continuous. In particular, we have

$$\begin{aligned}(P_h(\omega^h), \omega^h) &= B_d(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h) + B_a(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h)\\ &+ \delta B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h) - F(\omega^h).\end{aligned}$$

Observe that, due to (20), (49) and (52),

$$B_a(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h) = B_a(\Pi^h(\vartheta_\varepsilon), \omega^h, \omega^h) + B_a(\Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon), \omega^h).$$

Hence the previous identity reads

$$\begin{aligned}(P_h(\omega^h), \omega^h) =& B_d(\omega^h, \omega^h) + B_d(\Pi^h(\vartheta_\varepsilon), \omega^h)\\ &+ B_a(\Pi^h(\vartheta_\varepsilon), \omega^h, \omega^h) + B_a(\Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon), \omega^h)\\ &+ \delta B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h) - F(\omega^h).\end{aligned}$$

This identity, in turn, can be written as follows,

$$(P_h(\omega^h), \omega^h) = B_d(\omega^h, \omega^h) + B_a(\Pi^h(\vartheta_\varepsilon), \omega^h, \omega^h) - H(\omega^h), \tag{102}$$

where

$$\begin{aligned}H(\omega^h) := F(\omega^h) &- B_d(\Pi^h(\vartheta_\varepsilon), \omega^h) - B_a(\Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon), \omega^h)\\ &- \delta B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h).\end{aligned} \tag{103}$$

Due to assumptions (31) and (75), we can use Theorem 5.2 to assure the existence of a positive constant $m \in (0, 1)$ such that

$$B_d(\omega^h, \omega^h) \geq m|||\omega^h|||^2. \tag{104}$$

Using Hölder's inequality, Sobolev's inequality (23), (24), the inverse Sobolev inequality (57), and Lemma 4.5, we can show that

$$\begin{aligned}\left|B_a(\Pi^h(\vartheta_\varepsilon), \omega^h, \omega^h)\right| \leq& \sum_{j=1}^{N_e} \int_{\Omega_j} \|\Delta\omega^h\|_{L^2(\Omega_j)} \|\nabla\omega^h\|_{L^4(\Omega_j)} \|\nabla(\Pi^h(\vartheta_\varepsilon))\|_{L^4(\Omega_j)}\\ \leq& C_1 \sum_{j=1}^{N_e} \|D^2\omega^h\|^2_{L^2(\Omega_j)} \|\nabla(\vartheta_\varepsilon)\|_{L^4(\Omega_j)}\\ \leq& C_2|||\omega^h|||^2 \sum_{j=1}^{N_e} \|\nabla(\vartheta_\varepsilon)\|_{L^4(\Omega)}.\end{aligned}$$

In view of the properties (97)-(100) of the cut-off function η_ε, there holds

$$\|\nabla(\vartheta_\varepsilon)\|_{L^4(\Omega)} \xrightarrow[\varepsilon \to 0^+]{} 0.$$

Hence,

$$-B_a(\Pi^h(\vartheta_\varepsilon), \omega^h, \omega^h) \leq \rho|||\omega^h|||^2 \tag{105}$$

for any $\rho > 0$. Plugging (104) and (105) into (106), and then choosing $\rho = \frac{m}{2}$, we obtain

$$(P_h(\omega^h), \omega^h) \geq \frac{m}{2}|||\omega^h|||^2 - H(\omega^h).$$

Due to the properties of the absolute value,

$$(P_h(\omega^h), \omega^h) \geq \frac{m}{2}|||\omega^h|||^2 - \left|H(\omega^h)\right|, \tag{106}$$

and

$$\begin{aligned} \left|H(\omega^h)\right| \leq |F(\omega^h)| + |B_d(\Pi^h(\vartheta_\varepsilon), \omega^h)| + \left|B_a(\Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon), \omega^h)\right| \\ + \delta|B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h)|. \end{aligned} \tag{107}$$

From the application of (79)-(81) and (83) there exist independent of h positive constants C_1, C_2 and C_3 such that

$$|B_d(\Pi^h(\vartheta_\varepsilon), \omega^h)| \leq C_1|||\Pi^h(\vartheta_\varepsilon)|||\,|||\omega^h|||, \tag{108}$$

$$\left|B_a(\Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon), \omega^h)\right| \leq C_2|||\Pi^h(\vartheta_\varepsilon)|||^2\,|||\omega^h|||, \tag{109}$$

$$\begin{aligned} |B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h)| &\leq C_3'|||\omega^h + \Pi^h(\vartheta_\varepsilon)|||^{\sigma-1}|||\omega^h||| \\ &\leq C_3\left(|||\omega^h|||^{\sigma-1} + |||\Pi^h(\vartheta_\varepsilon)|||^{\sigma-1}\right)|||\omega^h|||, \end{aligned} \tag{110}$$

$$|F(\omega^h)| \leq C_4|||\omega^h|||. \tag{111}$$

Note that, in (110), we also have used Minkowski's inequality and the algebraic inequality (55), and thus the constant C_3 depends also on σ. Then, combining (107) with (108)-(111), and still using (101), there holds

$$|H(\omega^h)| \leq \left[C_4 + \delta C_3\left(|||\omega^h|||^{\sigma-1} + C_0^{\sigma-1}\right) + C_2C_0^2 + C_1C_0\right]|||\omega^h|||, \tag{112}$$

where C_0 is the constant defined in (101). Now, plugging (112) into (106), we have

$$(P_h(\omega^h), \omega^h) \geq \left(\frac{m}{2}|||\omega^h||| - K_1|||\omega^h|||^{\sigma-1} - K_2\right)|||\omega^h|||,$$

where $K_1 := \delta C_3$ and $K_2 := C_4 + \delta C_3 C_0^{\sigma-1} + C_2C_0^2 + C_1C_0$.

If $1 < \sigma < 2$, we can use Young's inequality so that

$$(P_h(\omega^h), \omega^h) \geq \left(\frac{m}{4}|||\omega^h||| - K\right)|||\omega^h||| \tag{113}$$

for another positive constant $K = C(K_1, K_2, \sigma, m)$ that does not depend on $|||\omega^h|||$. Defining the following sphere $\mathcal{V}_R^h$ in $\mathcal{V}^h$,

$$\mathcal{V}_R^h := \{\omega^h \in \mathcal{V}^h : |||\omega||| = R\} \tag{114}$$

with $R = \frac{4K}{m}$, we can see, from (113), that

$$(P_h(\omega^h), \omega^h) \geq 0 \qquad \forall\ \omega^h \in \mathcal{V}_R^h. \tag{115}$$

If $\sigma \geq 2$, we go back to (102)-(103) and rewrite these expressions as follows

$$\begin{aligned} (P_h(\omega^h), \omega^h) = B_d(\omega^h, \omega^h) + B_a(\Pi^h(\vartheta_\varepsilon), \omega^h, \omega^h) \\ + \delta B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon)) - H(\omega^h), \end{aligned} \tag{116}$$

where

$$\begin{aligned} H(\omega^h) := F(\omega^h) + \delta B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon)) \\ - B_d(\Pi^h(\vartheta_\varepsilon), \omega^h) - B_a(\Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon), \omega^h). \end{aligned}$$

In this case, (106) still holds, but (107) has to be replaced by

$$\begin{aligned} \left|H(\omega^h)\right| \leq |F(\omega^h)| + \delta|B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon))| \\ + |B_d(\Pi^h(\vartheta_\varepsilon), \omega^h)| + |B_a(\Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon), \omega^h)|. \end{aligned} \tag{117}$$

The only change here lies in the estimate of the term $B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon))$. Proceeding as we did for (110), we obtain

$$\begin{aligned}
&|B_f(\omega^h + \Pi^h(\vartheta_\varepsilon), \omega^h + \Pi^h(\vartheta_\varepsilon), \Pi^h(\vartheta_\varepsilon))| \\
\leq& C_2' |||\omega^h + \Pi^h(\vartheta_\varepsilon)|||^{\sigma-1} |||\Pi^h(\vartheta_\varepsilon)||| \\
\leq& C_2 \left(|||\omega^h|||^{\sigma-1} + |||\Pi^h(\vartheta_\varepsilon)|||^{\sigma-1} \right) |||\Pi^h(\vartheta_\varepsilon)|||.
\end{aligned} \tag{118}$$

Using now (118) instead of (110), and observing that $B_f(\omega^h+\Pi^h(\vartheta_\varepsilon), \omega^h+\Pi^h(\vartheta_\varepsilon), \omega^h+\Pi^h(\vartheta_\varepsilon)) \geq 0$, we get from (116)-(117) and (101) the counterpart of (112),

$$|H(\omega^h)| \leq \left(C_4 + \delta C_3 C_0 |||\omega^h|||^{\sigma-2} + C_2 C_0^2 + C_1 C_0 \right) |||\omega^h||| + \delta C_3 C_0^\sigma. \tag{119}$$

Plugging (119) into (106), we get

$$(P_h(\omega^h), \omega^h) \geq \left(\frac{m}{2} |||\omega^h||| - K_1 |||\omega^h|||^{\sigma-2} - K_2 \right) |||\omega^h||| - K_3,$$

where now $K_1 := \delta C_3 C_0$, $K_2 := C_4 + C_2 C_0^2 + C_1 C_0$ and $K_3 := \delta C_2 C_0^\sigma$. In this case, we also can use Young's inequality so that

$$(P_h(\omega^h), \omega^h) \geq \frac{m}{4} |||\omega^h|||^2 - K_4 |||\omega^h||| - K_3, \tag{120}$$

for another positive constant $K_4 = C(K_1, K_2, \sigma, m)$ that also does not depend on $|||\omega^h|||$. For sure that the right-hand side polynomial has a positive root, namely

$$a := \frac{2}{m} \left(K_4 + \sqrt{K_4^2 + m K_3} \right).$$

Thus, considering the sphere $\mathcal{V}_R^h$ with, for instance, $R = a + 1$ in (114), we can see, from (120), that (115) holds in this case as well.

For either $1 < \sigma < 2$ or $\sigma \geq 2$, with the corresponding choices of R, Lemma 4.3 assures the existence of a function $\omega^h \in \mathcal{V}^h$ such that $P_h(\omega^h) = 0$ for $|||\omega^h||| \leq R$. In particular,

$$\left(P_h(\omega^h), \phi^h \right) = 0 \qquad \forall \phi^h \in \mathcal{V}^h,$$

which proves the existence for the problem (54). The existence of a solution to the problem (47) is now a consequence of this and of (53). □

In the following result, we establish the uniqueness of the solution to the discrete problem (47), which proof is be based on Lemma 4.4.

Theorem 7.2. *Let ψ_1^h and ψ_2^h be two solutions to the discrete problem (47) and assume conditions (31) and (75) hold. If there exists a positive constant $C = C(\widetilde{\Omega}, h)$ such that*

$$\nu > C \|D^2 \psi_2^h\|_{L^2(\widetilde{\Omega})}, \tag{121}$$

then $\psi_1^h = \psi_2^h$.

Proof. Let ψ_1^h and ψ_2^h be two solutions to the problem (47) corresponding to the same data. Subtracting (47) with $\psi^h = \psi_1^h$ to (47) with $\psi^h = \psi_2^h$, and then taking $\phi^h = \psi_2^h - \psi_1^h := \psi^h$ in the resulting equation, we get, after some algebraic manipulations and using (20),

$$
\begin{aligned}
&\nu\sum_{j=1}^{N_e}\int_{\Omega_j}|D^2\psi^h|^2\,\mathrm{d}\mathbf{x}+\sum_{j=1}^{N_i}\frac{\tau^i}{\partial_j^i}\int_{\widetilde{\Gamma}_j}\left[\!\left[\frac{\partial\psi^h}{\partial\mathbf{n}}\right]\!\right]^2\,\mathrm{d}s+\sum_{j=1}^{N_b}\frac{\tau^b}{\partial_j^b}\int_{\overline{\Gamma}_j}\left(\frac{\partial\psi^h}{\partial\mathbf{n}}\right)^2\,\mathrm{d}s\\
&+B_a(\psi_2^h,\psi^h,\psi^h)+\delta\sum_{j=1}^{N_e}\int_{\Omega_j}\left(|\psi_{2,y}^h|^{\sigma-2}\psi_{2,y}^h-|\psi_{1,y}^h|^{\sigma-2}\psi_{1,y}^h\right)\psi_y^h\,\mathrm{d}\mathbf{x}\\
=&2\nu\sum_{j=1}^{N_i}\int_{\widetilde{\Gamma}_j}\left\langle\frac{\partial^2\psi^h}{\partial\mathbf{n}^2}\right\rangle\left[\!\left[\frac{\partial\psi^h}{\partial\mathbf{n}}\right]\!\right]\,\mathrm{d}s+2\nu\sum_{j=1}^{N_b}\int_{\overline{\Gamma}_j}\frac{\partial^2\psi^h}{\partial\mathbf{n}^2}\frac{\partial\psi^h}{\partial\mathbf{n}}\,\mathrm{d}s.
\end{aligned}\tag{122}
$$

For $\sigma\geq 2$, we can use (58) of Lemma 4.4 so that

$$
C_1\int_{\Omega_j}\left|\psi_{2,y}^h-\psi_{1,y}^h\right|^\sigma\,\mathrm{d}\mathbf{x}\leq\int_{\Omega_j}\left(|\psi_{2,y}^h|^{\sigma-2}\psi_{2,y}^h-|\psi_{1,y}^h|^{\sigma-2}\psi_{1,y}^h\right)\psi_y^h\,\mathrm{d}\mathbf{x}.\tag{123}
$$

If $1<\sigma<2$, we use (59) of Lemma 4.4 to get

(124)

$$
C_2\int_{\Omega_j}\frac{\left|\psi_{2,y}^h-\psi_{1,y}^h\right|^2}{1+\left(|\psi_{2,y}^h|^\sigma+|\psi_{1,y}^h|^\sigma\right)^{\frac{2-\sigma}{\sigma}}}\,\mathrm{d}\mathbf{x}\leq\int_{\Omega_j}\left(|\psi_{2,y}^h|^{\sigma-2}\psi_{2,y}^h-|\psi_{1,y}^h|^{\sigma-2}\psi_{1,y}^h\right)\psi_y^h\,\mathrm{d}\mathbf{x}.
$$

In any case, for either $\sigma\geq 2$ or $1<\sigma<2$, (123)-(124) imply

$$
\int_{\Omega_j}\left(|\psi_{2,y}^h|^{\sigma-2}\psi_{2,y}^h-|\psi_{1,y}^h|^{\sigma-2}\psi_{1,y}^h\right)\psi_y^h\,\mathrm{d}\mathbf{x}\geq 0\qquad\forall\, j\in\{1,\dots,N_e\}.\tag{125}
$$

Using (125), we immediately get from (122)

(126)

$$
\begin{aligned}
&\nu\sum_{j=1}^{N_e}\int_{\Omega_j}|D^2\psi^h|^2\,\mathrm{d}\mathbf{x}+\sum_{j=1}^{N_i}\frac{\tau^i}{\partial_j^i}\int_{\widetilde{\Gamma}_j}\left[\!\left[\frac{\partial\psi^h}{\partial\mathbf{n}}\right]\!\right]^2\,\mathrm{d}s+\sum_{j=1}^{N_b}\frac{\tau^b}{\partial_j^b}\int_{\overline{\Gamma}_j}\left(\frac{\partial\psi^h}{\partial\mathbf{n}}\right)^2\,\mathrm{d}s\\
&\leq 2\nu\sum_{j=1}^{N_i}\int_{\widetilde{\Gamma}_j}\left\langle\frac{\partial^2\psi^h}{\partial\mathbf{n}^2}\right\rangle\left[\!\left[\frac{\partial\psi^h}{\partial\mathbf{n}}\right]\!\right]\,\mathrm{d}s+2\nu\sum_{j=1}^{N_b}\int_{\overline{\Gamma}_j}\frac{\partial^2\psi^h}{\partial\mathbf{n}^2}\frac{\partial\psi^h}{\partial\mathbf{n}}\,\mathrm{d}s-B_a(\psi_2^h,\psi^h,\psi^h).
\end{aligned}
$$

Proceeding for the terms of the last row as we did in the proof of [23, Theorem 5.2], we obtain from (126)

$$
\begin{aligned}
&\nu\sum_{j=1}^{N_e}\left(1-2\varepsilon_j\frac{C_j}{h_j}\right)\left\|D^2\psi^h\right\|_{L^2(\Omega_j)}^2+\sum_{j=1}^{N_i}\frac{\tau^i}{\partial_j^i}\left(1-\frac{\nu\,\partial_j^i}{2\varepsilon_j\tau^i}\right)\left\|\left[\!\left[\frac{\partial\psi^h}{\partial\mathbf{n}}\right]\!\right]\right\|_{L^2(\widetilde{\Gamma}_j)}^2\\
&+\sum_{j=1}^{N_b}\frac{\tau^b}{\partial_j^b}\left(1-\frac{\nu\,\partial_j^b}{2\varepsilon_j\tau^b}\right)\left\|\frac{\partial\psi^h}{\partial\mathbf{n}}\right\|_{L^2(\overline{\Gamma}_j)}^2\leq -B_a(\psi_2^h,\psi^h,\psi^h),
\end{aligned}\tag{127}
$$

where C_j are positive constants that result by the application of Lemmas 4.1 and 4.6, and therefore do not depend on h_j. Here ε_j are positive constant to be chosen later on.

On the other hand, combining Hölder's inequality (56) with Sobolev's inequality (23), (24), and Corollary 5.1, we can show that

$$
\left|B_a(\psi_2^h,\psi^h,\psi^h)\right|\leq\frac{C}{\nu}\sum_{j=1}^{N_e}\|\psi_2^h\|_{H^2(\Omega_j)}\|D^2\psi^h\|_{L^2(\Omega_j)}^2\tag{128}
$$

for some independent of h_j positive constants K_j.

Since $-B_a(\psi_2^h,\psi^h,\psi^h)\leq \left|B_a(\psi_2^h,\psi^h,\psi^h)\right|$, we can plug (128) into (127) so that

(129)
$$\sum_{j=1}^{N_e}\left[\nu\left(1-2\varepsilon_j\frac{C_j}{h_j}\right)-\sqrt{2}K_jh_j\|D^2\psi_2^h\|_{L^2(\Omega_j)}\right]\left\|D^2\psi^h\right\|^2_{L^2(\Omega_j)}$$
$$+\sum_{j=1}^{N_i}\frac{\tau^i}{\partial_j^i}\left(1-\frac{\nu\,\partial_j^i}{2\varepsilon_j\tau^i}\right)\left\|\left[\!\left[\frac{\partial\psi^h}{\partial\mathbf{n}}\right]\!\right]\right\|^2_{L^2(\widetilde{\Gamma}_j)}+\sum_{j=1}^{N_b}\frac{\tau^b}{\partial_j^b}\left(1-\frac{\nu\,\partial_j^b}{2\varepsilon_j\tau^b}\right)\left\|\frac{\partial\psi^h}{\partial\mathbf{n}}\right\|^2_{L^2(\overline{\Gamma}_j)}\leq 0.$$

From (129), we can get $|||\psi^h|||=0$ as long as

$$1-2\varepsilon_j\frac{C_j}{h_j}>0,$$
$$\nu\left(1-2\varepsilon\frac{C_j}{h_j}\right)-\sqrt{2}K_jh_j\|D^2\psi_2^h\|_{L^2(\Omega_j)}\Leftrightarrow\nu>\frac{\sqrt{2}K_jh_j}{1-2\varepsilon\frac{C_j}{h_j}}\|D^2\psi_2^h\|_{L^2(\Omega_j)},$$
$$1-\frac{\nu\,\partial_j^i}{2\varepsilon\,\tau^i}\geq 0\Leftrightarrow\varepsilon\geq\frac{\nu\,\partial_j^i}{2\tau^i},\qquad 1-\frac{\nu\,\partial_j^b}{2\varepsilon\,\tau^b}\geq 0\Leftrightarrow\varepsilon\geq\frac{\nu\,\partial_j^b}{2\tau^b},$$

for all $j\in\{1,\dots,N_e\}$ and for some positive constants C_j not depending on h_j. The range of the positive ε is nonempty if

$$\tau^i\geq\nu C_j\frac{\partial_j^i}{h_j}\qquad\text{and}\qquad\tau^b\geq\nu C_j\frac{\partial_j^b}{h_j}\qquad\forall\, j\in\{1,\dots,N_e\},$$

and these relation are true provided hypothesis (32) is valid and $C_1=C_2=\max_{j\in\{1,\dots,N_e\}}C_j$ in assumption (75). Now, using assumption (121) with $C=\max_{j\in\{1,\dots,N_e\}}\frac{\sqrt{2}K_jh_j}{1-2\varepsilon\frac{C_j}{h_j}}$ and ε chosen as mentioned above, we can see that it must be $\psi_2^h=\psi_1^h$ a.e. in $\widetilde{\Omega}$. □

8. Error analysis and convergence of the method

Let us now analyze the error of the finite element solution

$$e:=\psi-\psi^h,\tag{130}$$

where ψ is a solution to the continuous problem (17) and ψ^h is a solution to the discrete problem (47). Due to the presence of the nonlinear terms (49) and (50) in the discrete problem (47), we cannot expect that the usual Galerkin error orthogonality property for the finite element space is valid here.

To perform the error analysis, we decompose the global error (130) into a sufficiently smooth part, say η, and another one that is in the finite element space $\mathcal{P}_k^h$, say e^h,

$$e:=\eta+e^h,\qquad\eta:=\psi-\Pi^h(\psi),\quad e^h:=\Pi^h(\psi)-\psi^h,\tag{131}$$

where Π^h is the projection operator from the broken Sobolev space $H^4(\mathcal{T}^h(\Omega))$ onto the finite element space $\mathcal{S}^h$.

The next result is about the convergence, in the discrete norm (74), of the method introduced in (46)-(51).

Theorem 8.1. *Let ψ be a solution to the continuous problem (17) so that $\psi\in H^4(\mathcal{T}^h(\Omega))$ and (121) is satisfied. Let also $\psi^h\in\mathcal{S}^h$ be a solution to the discrete* *problem (47). If (31)-(32) and (75) hold, and $\psi \in H^{k+1}(\mathcal{T}^h(\Omega))$, then there exist positive constants C_1 and C_2, that do not depend on h_j, such that*

(132)
$$|||e||| \le C_1 \sum_{j=1}^{N_e} h_j^{k-1}\Big(1 + \|\psi\|^{\sigma-2}_{H^2(\Omega_j)} + h_j^{(k-1)(\sigma-2)}\|D^{k+1}\psi\|^{\sigma-2}_{L^2(\Omega_j)} + h_j^{k-1}\|D^{k+1}\psi\|_{L^2(\Omega_j)} + \|\psi\|_{H^2(\Omega_j)}\Big)\|D^{k+1}\psi\|_{L^2(\Omega_j)}, \qquad \text{when} \quad \sigma \ge 2,$$

(133)
$$|||e||| \le C_2 \sum_{j=1}^{N_e} h_j^{(k-1)(\sigma-1)}\Big[\left(1 + \|\psi\|_{H^2(\Omega_j)}\right) h_j^{(k-1)(2-\sigma)}\|D^{k+1}\psi\|^{2-\sigma}_{L^2(\Omega_j)} + h_j^{(k-1)(3-\sigma)}\|D^{k+1}\psi\|^{3-\sigma}_{L^2(\Omega_j)}\Big]\|D^{k+1}\psi\|^{\sigma-1}_{L^2(\Omega_j)}, \qquad \text{when} \quad 1 < \sigma < 2,$$

for all $k \ge 2$.

Proof. From Theorem 5.1, any solution ψ of the continuous problem (17), with $\psi \in H^4(\mathcal{T}^h(\Omega))$, satisfies (62). For ϕ^h arbitrarily chosen in $\mathcal{V}^h$, we subtract (47) to (62) so that, after using the linear properties of $B_d(\cdot,\cdot)$ and $B_a(\cdot,\cdot,\cdot)$ (see (48)-(49)), and (20), we have

$$B_d(\psi - \psi^h, \phi^h) + B_a(\psi - \psi^h, \psi, \phi^h) + B_a(\psi^h, \psi - \psi^h, \phi^h) + \delta\left[B_f(\psi,\psi,\phi^h) - B_f(\psi^h,\psi^h,\phi^h)\right] = 0.$$

Using the notation (130), we can see this identity immediately implies

(134)
$$B_d(e, \phi^h) + B_a(e, \psi, \phi^h) + B_a(\psi^h, e, \phi^h) + \delta\left[B_f(\psi,\psi,\phi^h) - B_f(\psi^h,\psi^h,\phi^h)\right] = 0$$

for all $\phi^h \in \mathcal{V}^h$. Since $e^h \in \mathcal{V}^h$, we can take $\phi^h = e^h$ in (134) so that

(135)
$$B_d(e, e^h) + B_a(e, \psi, e^h) + B_a(\psi^h, e, e^h) + \delta\left[B_f(\psi,\psi,e^h) - B_f(\psi^h,\psi^h,e^h)\right] = 0.$$

We use (20)-(21) and (131) to write the diffusion and convective terms as follows,

$$B_d(e, e^h) = B_d(e^h, e^h) + B_d(\eta, e^h),$$

$$B_a(e, \psi, e^h) + B_a(\psi^h, e, e^h) = B_a(\Pi^h(\psi), e^h, e^h) + B_a(\eta, \psi, e^h) + B_a(\Pi^h(\psi), \eta, e^h).$$

Combining this with (135), we can write

$$B_d(e^h, e^h) + B_a(\Pi^h(\psi), e^h, e^h) = -B_d(\eta, e^h) - B_a(\eta, \psi, e^h) - B_a(\eta, \psi, e^h) - B_a(\Pi^h(\psi), \eta, e^h) - \delta\left[B_f(\psi,\psi,e^h) - B_f(\psi^h,\psi^h,e^h)\right]. \tag{136}$$

Theorem 5.2 assures the existence of $m \in (0,1)$ such that

$$B_d(e^h, e^h) \ge m|||e^h|||^2. \tag{137}$$

On the other hand, using Hölder's inequality (56), (24), Lemmas 4.6 and 5.2, and (22), one can prove that

$$\left|B_a(\Pi^h(\psi), e^h, e^h)\right| \le C\left(\sum_{j=1}^{N_e}\|D^2 e^h\|^2_{L^2(\Omega_j)}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{N_e}\|\nabla\psi\|^4_{L^4(\Omega_j)}\right)^{\frac{1}{4}}\left(\sum_{j=1}^{N_e}\|\nabla e^h\psi\|^4_{L^4(\Omega_j)}\right)^{\frac{1}{4}} \le \frac{C_0}{\nu}|||e^h|||^2\|\psi\|_{H^2(\Omega)}, \tag{138}$$

for some positive constants C and C_0 that do not depend on h.

Similarly to (125), Lemma 4.4 implies

$$B_f(\Pi^h(\psi), \Pi^h(\psi), e^h) - B_f(\psi^h, \psi^h, e^h) \geq 0,$$

and, as a consequence,

$$B_f(\psi, \psi, e^h) - B_f(\psi^h, \psi^h, e^h) \geq B_f(\psi, \psi, e^h) - B_f(\Pi^h(\psi), \Pi^h(\psi), e^h). \tag{139}$$

Using the information of (137), (138) and (139) in (136), we get

$$\begin{aligned}
&\left(m - \frac{C_0}{\nu}\|\psi\|_{H^2(\Omega)}\right)|||e^h|||^2 \leq -B_d(\eta, e^h) - B_a(\eta, \psi, e^h) \\
&- B_a(\Pi^h(\psi), \eta, e^h) - \delta\left[B_f(\psi, \psi, e^h) - B_f(\Pi^h(\psi), \Pi^h(\psi), e^h)\right] \leq \\
&\left|B_d(\eta, e^h)\right| + \left|B_a(\eta, \psi, e^h)\right| + \left|B_a(\Pi^h(\psi), \eta, e^h)\right| \\
&+ \delta\left|B_f(\psi, \psi, e^h) - B_f(\Pi^h(\psi), \Pi^h(\psi), e^h)\right|.
\end{aligned} \tag{140}$$

From (79) of Theorem 5.3, there exists an independent of h positive constant C such that

$$|B_d(\eta, e^h)| \leq C|||\eta|||\,|||e^h|||. \tag{141}$$

Proceeding as we did for (84), we can show that

$$\begin{aligned}
&\left|B_a(\eta, \psi, e^h)\right| \leq C\left(\sum_{j=1}^{N_e}\|D^2\psi\|^2_{L^2(\Omega_j)}\right)^{\frac{1}{2}}|||\eta|||\,|||e^h||| \leq \\
&C\sum_{j=1}^{N_e}\|D^2\psi\|_{L^2(\Omega_j)}|||\eta|||\,|||e^h|||,
\end{aligned} \tag{142}$$

for some positive constant C not depending on h_j. Similarly, using, in addition, (22), Corollary 5.1 and (131), we can also show that

$$\begin{aligned}
&\left|B_a(\Pi^h(\psi), \eta, e^h)\right| \leq C_1|||\eta|||\left(\sum_{j=1}^{N_e}\|\nabla\left(\psi - \Pi^h(\psi)\right)\|^4_{L^4(\Omega_j)}\right)^{\frac{1}{4}}|||e^h||| \\
&+ C_2|||\eta|||\left(\sum_{j=1}^{N_e}\|\nabla\psi\|^4_{L^4(\Omega_j)}\right)^{\frac{1}{4}}|||e^h||| \leq \\
&C_1|||\eta|||^2|||e^h||| + C_2|||\eta|||\left(\sum_{j=1}^{N_e}\|\psi\|_{H^2(\Omega_j)}\right)|||e^h|||,
\end{aligned} \tag{143}$$

for distinct positive constants C_1 and C_2 that also do not depend on h_j.

On the one hand, by using Lemmas 4.5 and 5.1, we can proceed as in the proof of [23, Theorem 8.1] to show that

$$\begin{aligned}
&\left|B_f(\psi, \psi, e^h) - B_f(\Pi^h(\psi), \Pi^h(\psi), e^h)\right| \leq \\
&C_1\sum_{j=1}^{N_e}\left(\|\psi_y\|^{\sigma-2}_{L^\sigma(\Omega_j)} + |||\eta|||^{\sigma-2}\right)|||\eta|||\,|||e^h|||, \quad \sigma \geq 2,
\end{aligned} \tag{144}$$

$$\left|B_f(\psi, \psi, e^h) - B_f(\Pi^h(\psi), \Pi^h(\psi), e^h)\right| \leq C_2|||\eta|||^{\sigma-1}|||e^h|||, \quad 1 < \sigma < 2, \tag{145}$$

for some positive constants C_1 and C_2 that do not depend on h_j.

So, combining (140) with (141), (142)-(143) and (144), or (145), and still using assumption (121) and Sobolev's inequality (22), we obtain

(146)
$$|||e^h||| \leq C_1 \sum_{j=1}^{N_e} \left(\|\psi\|^{\sigma-2}_{H^2(\Omega_j)} + |||\eta|||^{\sigma-2} + 1 + |||\eta||| + \|\psi\|_{H^2(\Omega_j)} \right) |||\eta|||, \ \sigma \geq 2,$$

(147)
$$|||e^h||| \leq C_2 \left[\left(1 + \sum_{j=1}^{N_e} \|\psi\|_{H^2(\Omega_j)} \right) |||\eta|||^{2-\sigma} + 1 + |||\eta|||^{3-\sigma} \right] |||\eta|||^{\sigma-1}, \ 1 < \sigma < 2.$$

On the other hand, we can take $p = q = d = 2$ and $s = k+1$ in Lemma 4.6 so that

$$\|D^2\eta\|^2_{L^2(\Omega_j)} \leq C h_j^{2(k-1)} \|D^{k+1}\psi\|^2_{L^2(\Omega_j)}. \tag{148}$$

Next, using Lemma 4.2, with $r = 1$, and Lemma 4.6, with $p = q = d = 2$ and $s = k+1$, first with $m = 1$ and then with $m = 2$, we show that

(149)
$$\left\| \frac{\partial \eta}{\partial \mathbf{n}} \right\|^2_{L^2(\partial\Omega_j)} \leq C \left(h_j^{-1} \|\nabla\eta\|^2_{L^2(\Omega_j)} + h_j \|D^2\eta\|^2_{L^2(\Omega_j)} \right) \leq C h_j^{2k-1} \|D^{k+1}\psi\|^2_{L^2(\Omega_j)}.$$

Inserting (148)-(149) into the discrete norm (74) of η, we can prove that

$$\begin{aligned} |||\eta|||^2 &\leq \nu \sum_{j=1}^{N_e} \|D^2\eta\|^2_{L^2(\Omega_j)} + \frac{\tau}{\beta} \sum_{j=1}^{N_e} \frac{1}{h_j} \left\| \frac{\partial \eta}{\partial \mathbf{n}} \right\|^2_{L^2(\partial\Omega_j)} \leq \\ & C \sum_{j=1}^{N_e} h_j^{2(k-1)} \|D^{k+1}\psi\|^2_{L^2(\Omega_j)} \end{aligned} \tag{150}$$

for some positive constant C_1 and C_2 that do not depend on h_j, and where $\tau := \max\{2\tau^i, \tau^b\}$ and β is the constant that results by the application of (32).

We introduce the information of (150) in (146) and (147) so that

(151)
$$\begin{aligned} |||e^h||| \leq & C_1 \sum_{j=1}^{N_e} h_j^{k-1} \Big(1 + \|\psi\|^{\sigma-2}_{H^2(\Omega_j)} + h_j^{(k-1)(\sigma-2)} \|D^{k+1}\psi\|^{\sigma-2}_{L^2(\Omega_j)} \\ & + h_j^{k-1} \|D^{k+1}\psi\|_{L^2(\Omega_j)} + \|\psi\|_{H^2(\Omega_j)} \Big) \|D^{k+1}\psi\|_{L^2(\Omega_j)}, \qquad \text{when} \quad \sigma \geq 2, \end{aligned}$$

(152)
$$\begin{aligned} |||e^h||| \leq & C_2 \sum_{j=1}^{N_e} h_j^{(k-1)(\sigma-1)} \Big[\left(1 + \|\psi\|_{H^2(\Omega_j)}\right) h_j^{(k-1)(2-\sigma)} \|D^{k+1}\psi\|^{2-\sigma}_{L^2(\Omega_j)} \\ & + h_j^{(k-1)(3-\sigma)} \|D^{k+1}\psi\|^{3-\sigma}_{L^2(\Omega_j)} \Big] \|D^{k+1}\psi\|^{\sigma-1}_{L^2(\Omega_j)} \qquad \text{when} \quad 1 < \sigma < 2, \end{aligned}$$

and for distinct positive constants C_1 and C_2 that do not depend on h_j.

Finally, using (151)-(152) for $|||e^h|||$ and (150) for $|||\eta|||$, together with the triangular inequality and (131), we achieve to (132)-(133). □

Our next aim is to prove the convergence of the method (47) in the L^2-norm, in the case of zero Dirichlet boundary conditions, that is when $f_* = g_* = 0$ in (10).

For this, we consider the following linear dual problem

$$\nu\Delta^2\psi_d = f \qquad \text{in} \quad \Omega, \tag{153}$$

$$\psi_d = \frac{\partial \psi_d}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega. \tag{154}$$

For any $f \in H^{-2}(\Omega)$, it is well-known that the problem

$$\nu \left\langle D^2\phi_d, D^2\phi \right\rangle = \langle f, \phi\rangle \qquad \forall \phi \in H^{-2}(\Omega) \tag{155}$$

has a unique solution $\psi_d \in H_0^2(\Omega)$. Since $d = 2$, $H_0^2(\Omega) \hookrightarrow W^{1,\sigma}(\Omega)$ for any $\sigma \geq 1$, and so we also have $\psi_d \in W^{1,\sigma}(\Omega)$. Just like in (17), we can also write (155) in the equivalent form

$$\nu \int_\Omega D^2\psi_d : D^2\phi \; \mathrm{d}\mathbf{x} = \int_\Omega f\phi \; \mathrm{d}\mathbf{x} \qquad \forall\, \phi \in H_0^2(\Omega). \tag{156}$$

Beyond that, one can prove the existence of a positive constant C, depending only on $|\Omega|$, such that

$$\|D^2\psi_d\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{157}$$

Theorem 8.2. *Let ψ be a solution to the continuous problem (17) and ψ^h be a solution to the discrete problem (47) in the conditions of Theorem 8.1. Let also ψ_d be a solution to the dual problem (156). Assume that (31)-(32) and (75) are verified, $\psi \in H^{k+1}(\Omega)$, and suppose that*

$$\|D^4\psi_d\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \tag{158}$$

for some positive constant C. Then there exists positive constants C_1 and C_2, not depending on h, such that

(159)
$$\|e\|_{L^2(\Omega)} \leq C_1 \left(1 + \|\psi\|_{H^2(\Omega)} + \|\psi\|^{\sigma-2}_{H^2(\Omega)} + A_k\right)(h^2+1)A_k \qquad \text{if } \sigma \geq 2,$$

(160)
$$\|e\|_{L^2(\Omega)} \leq C_2 \left[\left(1 + \|\psi\|_{H^2(\Omega)} + B_k\right) B_k^{2-\sigma}\right]\left(h^2+1\right) B_k^{\sigma-1} \qquad \text{if } 1 < \sigma < 2,$$

for all $k \geq 3$, and where

$$\begin{aligned} A_k &:= C_1 h^{k-1}\Big(1 + \|\psi\|^{\sigma-2}_{H^2(\Omega)} + h^{(k-1)(\sigma-2)}\|D^{k+1}\psi\|^{\sigma-2}_{L^2(\Omega)} \\ &+ h^{k-1}\|D^{k+1}\psi\|_{L^2(\Omega)} + \|\psi\|_{H^2(\Omega)}\Big)\|D^{k+1}\psi\|_{L^2(\Omega)}, \\ B_k &:= C_2 h^{(k-1)(\sigma-1)}\Big[\left(1 + \|\psi\|_{H^2(\Omega)}\right) h^{(k-1)(2-\sigma)}\|D^{k+1}\psi\|^{2-\sigma}_{L^2(\Omega)} \\ &+ h^{(k-1)(3-\sigma)}\|D^{k+1}\psi\|^{3-\sigma}_{L^2(\Omega)}\Big]\|D^{k+1}\psi\|^{\sigma-1}_{L^2(\Omega)}. \end{aligned}$$

Proof. Using the zero Dirichlet boundary conditions (154), we can proceed as we did for (62) to show that any solution $\psi_d \in H^4(\Omega)$ to the linear dual problem (153)-(154) satisfies

$$B_d(\psi_d, \phi^h) = \sum_{j=1}^{N_e} \int_{\Omega_j} f\, \phi^h \; \mathrm{d}\mathbf{x} \qquad \forall\, \phi^h \in \mathcal{V}^h. \tag{161}$$

By (130), $e \in \mathcal{V}^h$ and so we can take $\phi^h = e$ in (161) such that

$$\int_\Omega f\, e \; \mathrm{d}\mathbf{x} = B_d(\psi_d, e). \tag{162}$$

In view of (40)-(42), $\Pi^h(\psi_d) \in \mathcal{V}^h$ and therefore we can take $\phi^h = \Pi^h(\psi_d)$ in (134) so that

$$\begin{aligned}0 = &B_d(e, \Pi^h(\psi_d)) + B_a(e, \psi, \Pi^h(\psi_d)) + B_a(\psi^h, e, \Pi^h(\psi_d)) \\ &+ \delta\left[B_f(\psi, \psi, \Pi^h(\psi_d)) - B_f(\psi^h, \psi^h, \Pi^h(\psi_d))\right].\end{aligned}$$

Subtracting this equation from (162), using the linearity and symmetry of $B_d(\cdot,\cdot)$, as well as the linearity of $B_a(\cdot,\cdot,\cdot)$, and observing (131), we obtain

$$\begin{aligned}&\int_\Omega f\, e \;\mathrm{d}\mathbf{x} = B_d(\eta_d, e) - B_a(e, \psi, \Pi^h(\psi_d)) - B_a(\psi^h, e, \Pi^h(\psi_d)) \\ &- \delta\left[B_f(\psi, \psi, \Pi^h(\psi_d)) - B_f(\psi^h, \psi^h, \Pi^h(\psi_d))\right] \le \\ &|B_d(\eta_d, e)| + |B_a(e, \psi, \Pi^h(\psi_d))| + |B_a(\psi^h, e, \Pi^h(\psi_d))| \\ &+ \delta\left|B_f(\psi, \psi, \Pi^h(\psi_d)) - B_f(\psi^h, \psi^h, \Pi^h(\psi_d))\right|.\end{aligned} \tag{163}$$

From Theorem 5.3, one has

$$|B_d(\eta_d, e)| \le C|||\eta_d|||\,|||e|||, \tag{164}$$

for a positive constant C that does not depend on h.

Arguing as we did for (142)-(143), we can show that

$$\begin{aligned}&\left|B_a(e, \psi, \Pi^h(\psi_d))\right| \le C_1\|D^2\psi\|_{L^2(\Omega)}\|D^2\psi_d\|_{L^2(\Omega)}|||e||| \le \\ &C_1\|\psi\|_{H^2(\Omega)}\|D^2\psi_d\|_{L^2(\Omega)}|||e|||\end{aligned} \tag{165}$$

$$\begin{aligned}&\left|B_a(\psi^h, e, \Pi^h(\psi_d))\right| \le \left|B_a(\psi, e, \Pi^h(\psi_d))\right| + \left|B_a(e, e, \Pi^h(\psi_d))\right| \le \\ &C_1\|\psi\|_{H^2(\Omega)}\|D^2\psi_d\|_{L^2(\Omega)}|||e||| + C_2\|D^2\psi_d\|_{L^2(\Omega)}|||e|||^2,\end{aligned} \tag{166}$$

for distinct, and independent of h, positive constants C_1 and C_2.

The last term of (163) is estimated by proceeding as we did for (144) and (145) (see the proof of [23, Theorem 8.2]),

$$\begin{aligned}&\left|B_f(\psi, \psi, \Pi^h(\psi_d)_y) - B_f(\psi^h, \psi^h, \Pi^h(\psi_d)_y)\right| \le \\ &C_1\left(\|\psi_y\|^{\sigma-2}_{L^\sigma(\Omega)} + |||e|||^{\sigma-2}\right)|||e|||\left(|||\eta_d||| + \|\psi_{d,y}\|_{L^\sigma(\Omega)}\right), \qquad \sigma \ge 2,\end{aligned} \tag{167}$$

$$\begin{aligned}&\left|B_f(\psi, \psi, \Pi^h(\psi_d)_y) - B_f(\psi^h, \psi^h, \Pi^h(\psi_d)_y)\right| \le \\ &C_2|||e|||^{\sigma-1}\left(|||\eta_d||| + \|\psi_{d,y}\|_{L^\sigma(\Omega)}\right), \qquad 1 < \sigma < 2,\end{aligned} \tag{168}$$

for some positive constants C_1 and C_2 that do not depend on h. The notation $\psi_{d,y}$ and $\Pi^h(\psi_d)_y$ is used here for the partial derivatives of ψ_d and $\Pi^h(\psi_d)$, respectively, with respect to y. Moreover, for the term with $\Pi^h(\psi_d)_y$, we also have used $(131)_2$.

Regrouping the information from (163), (164), (165)-(166), and (167), or (168), we obtain

$$\begin{aligned}\int_\Omega f\, e \;\mathrm{d}\mathbf{x} \le &C_1\Big[|||\eta_d||| + \|\psi\|_{H^2(\Omega)}\|D^2\psi_d\|_{L^2(\Omega)} + \|D^2\psi_d\|_{L^2(\Omega)}|||e||| \\ &+ \left(\|\psi_y\|^{\sigma-2}_{L^\sigma(\Omega)} + |||e|||^{\sigma-2}\right)\left(|||\eta_d||| + \|\psi_{d,y}\|_{L^\sigma(\Omega)}\right)\Big]|||e|||\end{aligned} \tag{169}$$

for $\sigma \ge 2$, and

$$\begin{aligned}\int_\Omega f\, e \;\mathrm{d}\mathbf{x} \le &C_2\Big[\left(|||\eta_d||| + \|\psi\|_{H^2(\Omega)}\|D^2\psi_d\|_{L^2(\Omega)} + \|D^2\psi_d\|_{L^2(\Omega)}\,|||e|||\right)|||e||| \\ &+ \left(|||\eta_d||| + \|\psi_{d,y}\|_{L^\sigma(\Omega)}\right)|||e|||^{\sigma-1}\Big]\end{aligned} \tag{170}$$

for $1<\sigma<2$, where C_1, C_2 and C_3 are positive constants that do not depend on h.

It was already established (see the proof of [23, Theorem 8.2]) that

(171)
$$\|D^2\eta_d\|^2_{L^2(\Omega_j)} \leq Ch_j^4\|D^4\psi_d\|^2_{L^2(\Omega_j)},$$

(172)
$$\left\|\frac{\partial\eta_d}{\partial\mathbf{n}}\right\|^2_{L^2(\partial\Omega_j)} \leq C\left(h_j^{-1}\|\nabla\eta_d\|^2_{L^2(\Omega_j)} + h_j\|D^2\eta_d\|^2_{L^2(\Omega_j)}\right) \leq Ch_j^5\|D^4\psi_d\|^2_{L^2(\Omega_j)}.$$

Using (171)-(172) in the discrete norm (74) of η_d, we have

$$|||\eta_d||| \leq Ch^2\|D^4\psi_d\|_{L^2(\Omega)} \leq Ch^2\|f\|_{L^2(\Omega)}, \tag{173}$$

where in the last part we still have used assumption (158). Furthermore, combining Sobolev's continuous inequality with (157), one has

$$\|\psi_{d,y}\|_{L^\sigma(\Omega)} \leq C_1\|D^2\psi_d\|_{L^2(\Omega)} \leq C_2\|f\|_{L^2(\Omega)}. \tag{174}$$

Here, it should be stressed that none of the positive constants of (173) and (174) depend on h.

Combining (173) and (174) with (169), and still using Sobolev's inequality (22), and next taking $f=e$ in the resulting inequality, we obtain

(175)
$$\begin{aligned}\|e\|_{L^2(\Omega)} \leq& C_1\left(h^2+\|\psi\|_{H^2(\Omega)}+|||e|||+\left(\|\psi_y\|^{\sigma-2}_{L^\sigma(\Omega)}+|||e|||^{\sigma-2}\right)(h^2+1)\right]|||e|||\\ \leq& C_2\left(1+\|\psi\|_{H^2(\Omega)}+\|\psi\|^{\sigma-2}_{H^2(\Omega)}+|||e|||+|||e|||^{\sigma-2}\right)(h^2+1)|||e|||,\end{aligned}$$

for $\sigma\geq 2$. In the case of $1<\sigma<2$, we combine (173) and (174) with (170). We then use (22) and take $f=e$ in the resulting inequality. By this procedure, we have

$$\begin{aligned}\|e\|_{L^2(\Omega)} \leq& C_1\left[\left(h^2+\|\psi\|_{H^2(\Omega)}+|||e|||\right)|||e|||+\left(h^2+1\right)|||e|||^{\sigma-1}\right]\\ \leq& C_2\left[\left(1+\|\psi\|_{H^2(\Omega)}+|||e|||\right)|||e|||^{2-\sigma}\right]\left(h^2+1\right)|||e|||^{\sigma-1}.\end{aligned} \tag{176}$$

From Theorem 8.1, one has

(177)
$$\begin{aligned}|||e||| \leq A_k :=& C_1h^{k-1}\Big(1+\|\psi\|^{\sigma-2}_{H^2(\Omega)}+h^{(k-1)(\sigma-2)}\|D^{k+1}\psi\|^{\sigma-2}_{L^2(\Omega)}\\ &+h^{k-1}\|D^{k+1}\psi\|_{L^2(\Omega)}+\|\psi\|_{H^2(\Omega)}\Big)\|D^{k+1}\psi\|_{L^2(\Omega)}, \qquad \sigma\geq 2,\end{aligned}$$

(178)
$$\begin{aligned}|||e||| \leq B_k :=& C_2h^{(k-1)(\sigma-1)}\Big[\left(1+\|\psi\|_{H^2(\Omega)}\right)h^{(k-1)(2-\sigma)}\|D^{k+1}\psi\|^{2-\sigma}_{L^2(\Omega)}\\ &+h^{(k-1)(3-\sigma)}\|D^{k+1}\psi\|^{3-\sigma}_{L^2(\Omega)}\Big]\|D^{k+1}\psi\|^{\sigma-1}_{L^2(\Omega)}, \qquad 1<\sigma<2,\end{aligned}$$

for all $k\geq 2$. Using the information of (177) and (178) into (175) and (176), we obtain

$$\|e\|_{L^2(\Omega)} \leq C_2\left(1+\|\psi\|_{H^2(\Omega)}+\|\psi\|^{\sigma-2}_{H^2(\Omega)}+A_k+A_k^{\sigma-2}\right)(h^2+1)A_k,$$

for $\sigma\geq 2$, and

$$\|e\|_{L^2(\Omega)} \leq C_2\left[\left(1+\|\psi\|_{H^2(\Omega)}+B_k\right)B_k^{2-\sigma}\right]\left(h^2+1\right)B_k^{\sigma-1},$$

for $1 < \sigma < 2$. Finally, observing that $A_k^{\sigma-2} \le A_k$, we arrive at (159)-(160). □

9. Numerical validation

To validate the numerical model, two distinct test cases are employed. The initial test case involves a problem for which an exact solution is established. This test offers numerical evidence showcasing the convergence of the proposed numerical method. Additionally, it is also shown that the convergence rate depends on the value of ν, i.e., decreasing the kinematics viscosity decreases the convergence rates.

In the subsequent test, the effects related to the advection and feedback forces field are shown (see [5])

The simulations in this section were made using the FEniCS libraries (see *e.g.* the FEniCS monograph [22]), on a computer with and AMD Ryzen 9 3950X processor running Linux Ubuntu. Moreover, in order to approximate the exact solution ψ^h of the nonlinear problem stated in (47)–(51), we have used a Picard-type iterative method (see *e.g.* the monograph by Langtangen [20, § 4]). This iterative procedure is stopped when the relative error (calculated in the $L^2(\Omega)$, $H^1(\Omega)$ and $l^\infty(\Omega)$ norms) between two consecutive approximations reaches a certain threshold value $\varepsilon > 0$.

For the first test, we have considered the problem (9)–(10) in the computational domain $\Omega = [0,1]^2$ with $\nu = \delta = 1$ and $\sigma = \frac{3}{2}$. Furthermore, to work with a problem with an exact solution, we add a function g to the right side of (9), making the function

$$\bar{\psi}(x,y) := \begin{cases} \sin(\pi x) e^{5+\left(\frac{-8}{1-(2(y-\frac{1}{2}))^4} - \frac{1}{1-x^2}\right)}, & y \in (0,1), \\ 0, & y \in \{0,1\} \vee x \in \{0,1\}, \end{cases}$$

an exact solution to the problem

$$\nu\Delta^2\psi + \psi_x\Delta\psi_y - \psi_y\Delta\psi_x = \delta\left(|\psi_y|^{\sigma-2}\psi_y\right)_y + g \quad \text{in } \Omega, \tag{179}$$

$$\psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad \text{on } y = 0,\ y = 1 \text{ and on } x = 1, \tag{180}$$

$$\psi = f_* \quad \text{and} \quad \frac{\partial\psi}{\partial\mathbf{n}} = g_* \quad \text{on } x = 0. \tag{181}$$

We stress that the function g is obtained by replacing ψ by $\bar{\psi}$ in (179). In fact, the external force can be writen by $g = g_1 + g_2 + g_3$ where g_i $(i = 1, 2, 3)$ are the components of the external force related with the diffusion, convection and feedback forces, respectively:

$$g_1 = \nu\Delta^2\bar{\psi}, \qquad g_2 = \bar{\psi}_x\Delta\bar{\psi}_y - \bar{\psi}_y\Delta\bar{\psi}_x \qquad g_3 = -\delta\left(|\bar{\psi}_y|^{\sigma-2}\bar{\psi}_y\right)_y. \tag{182}$$

The appropriated boundary conditions stated in problem (179)-(181) are the following,

$$f_*(y) := \psi(0,y) = 0 \quad \text{and} \tag{183}$$

$$g_*(y) := \frac{\partial\psi}{\partial\mathbf{n}}(0,y) = -\pi e^{4-\frac{8}{1-(2(y-\frac{1}{2}))^4}} \qquad \text{for } y \in (0,1). \tag{184}$$

In Figs. 2 and 3 one can see the graph of the exact solution $\bar{\psi}$ as well as the graph of the function g.

Moreover, in Fig. 4 one can see the graphs of the 3 components g_1, g_2 and g_3 of the external force g. From these graphs, one concludes that the primary driver of this test is the diffusion term g_1, with convection and feedback force terms playing

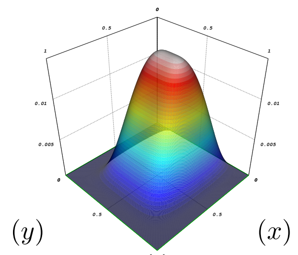


FIGURE 2. Exact solution $\bar{\psi}(\mathbf{x})$.

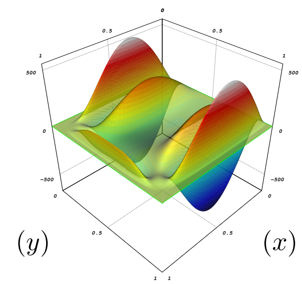
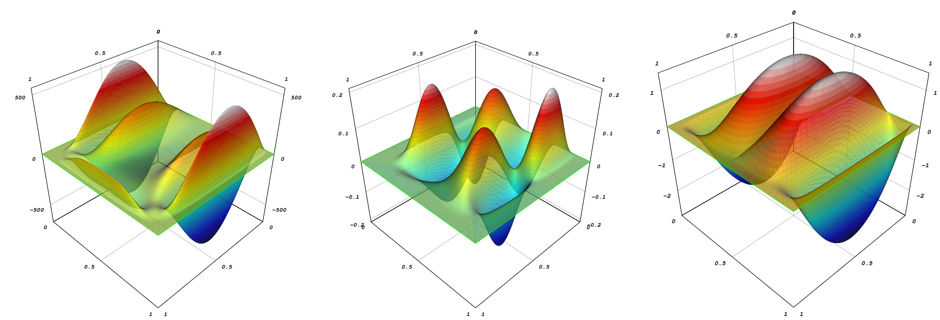


FIGURE 3. External force g.

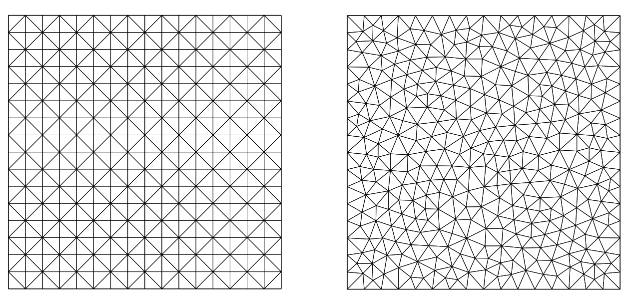


(A) Diffusion $g_1 : [0,1]^2 \to [-500, 500]$.

(B) Convection $g_2 : [0,1]^2 \to [-0.2, 0.2]$.

(C) Feedback force $g_3 : [0,1]^2 \to [0, 1]$.

FIGURE 4. Components of the external force g.



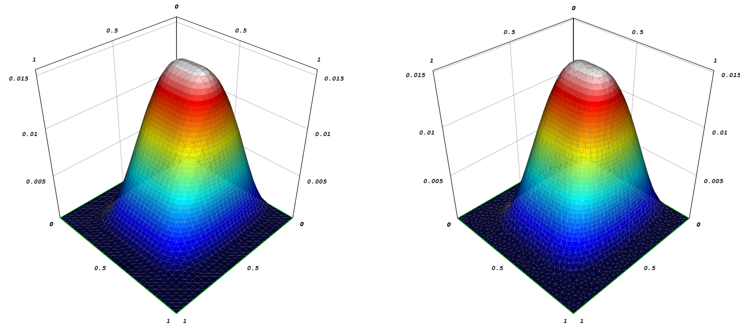
(A) Triangular symmetric mesh

(B) Triangular unstructured mesh

FIGURE 5. Meshes with equally spaced intervals along each of the boundary sides.

a comparatively minor role. However, it is important to note that the nonlinearity of the problem is a result of the convection effect along with the imposed feedback forces.

The computational domain Ω is discretized utilizing symmetric and unstructured triangular meshes (see Figure 5). More specifically, to illustrate the convergence of the method and to assess mesh independence, we consider ten distinct meshes. These consist of five symmetric uniform meshes and five unstructured meshes, each comprising an equal number of equally spaced intervals along all boundaries of the domain, with values of 16, 32, 64, 128, and 256 intervals.



(A) Symmetric mesh (B) Unstructured mesh

FIGURE 6. Graphs of the solutions ψ^h.

In Fig. 6 the numerical solutions obtained using the continuous/discontinuos finite element scheme are presented. In this case, the meshes (symmetric and unstructured), with 32 equally spaced intervals along the boundaries are assumed.

In Tables 1 and 2, we compare the absolute and relative errors between the numerical and exact solutions using the $L^2(\Omega)$, $H^1(\Omega)$, $l^\infty(\Omega)$ norms.

In the provided tables, we include estimations for the convergence rates (ECR) of errors, with the subscripts Δ and ε denoting absolute and relative errors, respectively. It is noteworthy that we evaluate the $l^\infty(\Omega)$ norm of the error across all mesh nodes corresponding to the degrees of freedom within the system. The column labeled *dofs* displays the number of degrees of freedom associated with the problem.

Furthermore, $\bar{\psi}^h$ represents the numerical solution, while $\bar{\psi}$ stands for the exact solution. To calculate the aforementioned errors, we approximate the exact solution using finer meshes consisting of 256 equally spaced intervals along the boundary sides. These finer solutions are then projected onto the higher-order polynomial space of degree 3 defined over the triangular elements present in both symmetric and unstructured meshes.

We denote the uniform and unstructured meshes as M_j, with j ranging from 1 to 5 based on the number of intervals utilized in their construction. Additionally, we provide information regarding the maximum ($r$max) and minimum ($r_{\min}$) radii of the inscribed circles for the triangular elements within each mesh.

In our numerical calculations, we assume $\tau^b = \tau^i = \frac{1}{\bar{h}_j}$, with $\bar{h}_j$ representing the average of the diameters of the two elements sharing an edge ($j \in 1, \ldots, N_e$).

The ECR values are calculated using the following expressions:

$$\mathrm{ECR}_j := \frac{\log \Delta_j^r}{\log r_{\max j}^r}, \quad (j = 2, \cdots, 5),$$

with $r_{\max j}^r = \frac{r_{\max j}}{r_{\max j-1}}$ standing for the ratio between the $r_{\max}$ values of two consecutive meshes. Similarly, $\Delta_r^j = \frac{\Delta_j}{\Delta_{j-1}}$, stands for the ratio between the absolute errors of two consecutive meshes. This final ratio is dependent on the selected norm, as specified in each respective ECR column. It is worth emphasizing that the construction of the meshes is designed to achieve an approximate value of $r_{\max j}^r \approx 0.5$.

Table 1. Errors for the symmetric meshes.

M_j	$r_{\max}$	$r_{\min}$	$l^\infty_\Delta(\Omega)$	$l^\infty_\varepsilon(\Omega)$	ECR l^∞	$L^2_\Delta(\Omega)$	$L^2_\varepsilon(\Omega)$	ECR $L^2(\Omega)$
16	1.83e-2	1.83e-2	4.67e-3	3.24e-1		1.82e-3	2.99e-1	
32	9.15e-3	9.15e-3	1.15e-3	7.97e-2	2.02	4.61e-4	7.57e-2	1.98
64	4.58e-3	4.58e-3	4.77e-4	3.31e-2	1.27	1.93e-4	3.17e-2	1.25
128	2.29e-3	2.29e-3	2.23e-4	1.55e-2	1.10	9.07e-5	1.49e-2	1.09
256	1.14e-3	1.14e-3	1.09e-4	7.60e-3	1.03	4.45e-5	7.31e-3	1.03

$H^1_\Delta(\Omega)$	$H^1_\varepsilon(\Omega)$	ECR $H^1(\Omega)$	dofs
9.99e-3	2.83e-1		1089
2.73e-3	7.73e-2	1.87	4225
1.18e-3	3.35e-2	1.21	16641
5.63e-4	1.60e-2	1.07	66049
2.78e-4	7.88e-3	1.02	263169

Table 2. Errors for the unstructured meshes.

M_j	$r_{\max}$	$r_{\min}$	$l^\infty_\Delta(\Omega)$	$l^\infty_\varepsilon(\Omega)$	ECR l^∞	$L^2_\Delta(\Omega)$	$L^2_\varepsilon(\Omega)$	ECR $L^2(\Omega)$
16	2.13e-2	1.29e-2	2.17e-3	1.51e-1		8.34e-4	1.37e-1	
32	1.16e-2	5.63e-3	7.50e-4	5.21e-2	1.74	2.92e-4	4.81e-2	1.72
64	5.64e-3	2.84e-3	3.26e-4	2.26e-2	1.15	1.29e-4	2.13e-2	1.13
128	3.00e-3	1.36e-3	1.55e-4	1.08e-2	1.18	6.17e-5	1.01e-2	1.17
256	1.55e-3	6.40e-4	7.73e-5	5.37e-3	1.05	3.11e-5	5.10e-3	1.04

$H^1_\Delta(\Omega)$	$H^1_\varepsilon(\Omega)$	ECR $H^1(\Omega)$	dofs
4.66e-3	1.32e-1		1417
1.66e-3	4.72e-2	1.69	5445
7.45e-4	2.11e-2	1.11	21785
3.63e-4	1.03e-2	1.14	86985
1.83e-4	5.19e-3	1.04	347417

From the numerical results presented in Tables 1 and 2, we can see that with an increasing number of mesh nodes, all the errors tend to diminish to zero. This trend serves as a strong indicator of the convergence behavior of the proposed method.

Thus far, we have followed the methodology outlined in our previous publication [23], and from a qualitative standpoint, the results obtained are similar.

However, it is important to note that in the current study, apart from the advection effects, we are also testing the method with nonhomogeneous Neumann conditions at the inflow boundary, while our prior work focused on homogeneous Dirichlet boundary conditions.

In what follows, we address the influence of the advection effect in the estimate for the convergence rate of the numerical model. This effect is shown by decreasing the value of the kinematics viscosity ν.

In Table 3, the expected convergence rates for the $L^2(\Omega)$ with $\nu \in \{0.5, 0.3, 0.1\}$ (with symmetric meshes) are shown. One can see that decreasing the value of ν leads to worst convergence rates and to instability.

TABLE 3. Estimates for the convergences rates for $\nu \in \{0.5, 0.3, 0.1\}$.

(A) $\nu = 0.5$

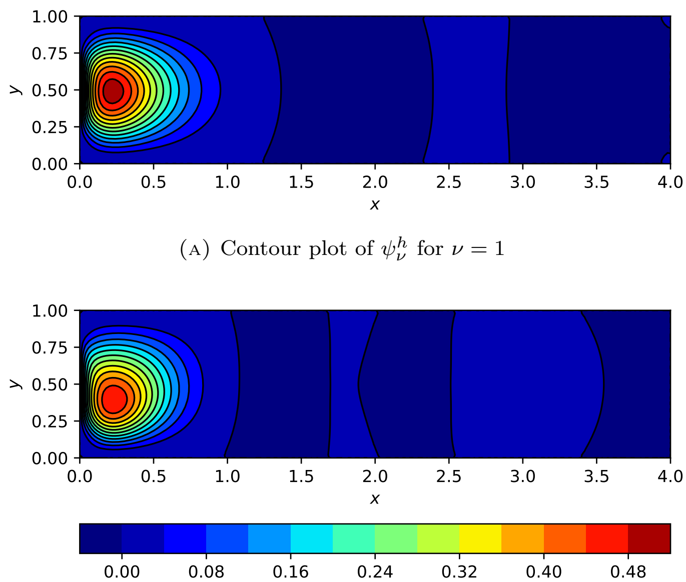
M_j	ECR $L^2(\Omega)$
32	1.45
64	1.02
128	9.87e-1
256	9.57e-1

(B) $\nu = 0.3$

M_j	ECR $L^2(\Omega)$
32	1.55
64	8.75e-1
128	9.10e-1
256	8.68e-1

(C) $\nu = 0.1$

M_j	ECR $L^2(\Omega)$
32	2.40
64	1.30
128	4.75e-1
256	1.24e-1



(A) Contour plot of ψ_ν^h for $\nu = 1$

(B) Contour plot of ψ_ν^h for $\nu = 15^{-1}$

FIGURE 7. Contour plots of ψ_ν^h with $\sigma = 1.5$ and $\delta = 1.0$ for $\nu^{-1} \in \{1, 15\}$.

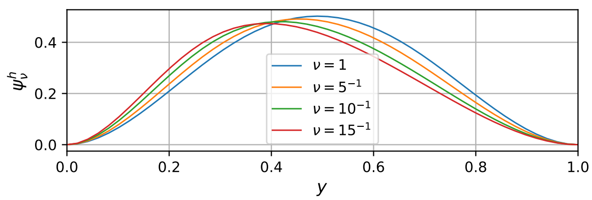
The first goal of the following test is to showcase how the solutions changes when varying the value of the viscosity in the model. By doing so, we can observe the direct influence of the convective terms within the model. The second goal is to show the damping of the solutions by decreasing the exponent σ of the feedback forces field. In this framework, let us consider the following problem:

$$\Delta^2\psi + \frac{1}{\nu}\left(\psi_x\Delta\psi_y - \psi_y\Delta\psi_x\right) - \frac{\delta}{\nu}\left(|\psi_y|^{\sigma-2}\psi_y\right)_y = 0 \text{ in } \Omega := (0,4)\times(0,1), \tag{185}$$

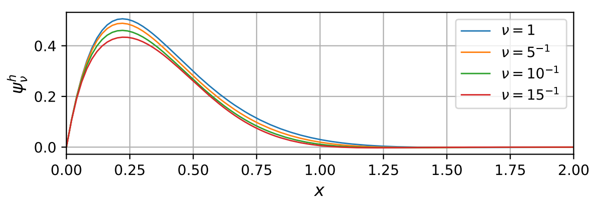
$$\psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\mathbf{n}} = 0 \text{ on } y = 0,\ y = 1 \text{ and on } x = 4, \tag{186}$$

$$\psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\mathbf{n}} = -100\pi e^{4-\frac{8}{1-(-2y+1)^4}} \text{ on } x = 0 \text{ for } y \in (0,1). \tag{187}$$

The computational domain is discretized using a symmetric triangular mesh with 200 and 50 equally spaced intervals along the x-axis and y-axis parallel boundaries, respectively.

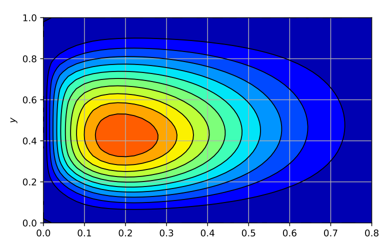


(A) Graphs of ψ_ν^h along $x = 0.25$ with $y \in (0,1)$

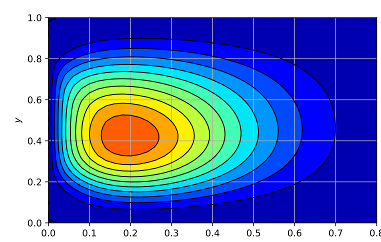


(B) Graphs of ψ_ν^h along $y = 0.5$ with $x \in (0,2)$

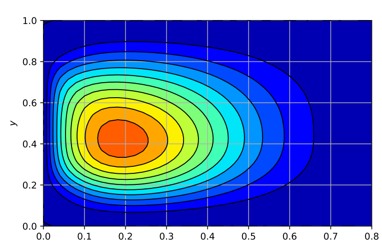
FIGURE 8. Graphs of ψ^h along the lines $x = 0.25$ and $y \in (0,1)$ and $y = 0.5$ with $x \in (0,2)$ for $\nu^{-1} \in \{1, 5, 10, 15\}$.



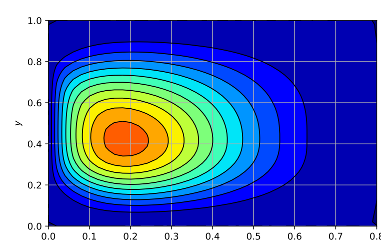
(A) Zoom of the contour plot of $\psi_{1.75}^h$



(B) Zoom of the contour plot of $\psi_{1.5}^h$



(C) Zoom of the contour plot of $\psi_{1.25}^h$



(D) Zoom of the contour plot of $\psi_{1.1}^h$

FIGURE 9. Contour plots of ψ_σ^h with $x \in (0, 0.8)$ for $\sigma \in \{1.1, 1.25, 1.5, 1.75\}$.

For this experiment, the exponent of nonlinearity σ and the magnitude δ of the feedback forces field are fixed at $\sigma = 1.5$ and $\delta = 1$, respectively. On the other hand, we vary the values for the kinematics viscosity ν. In Fig. 7 one can see the numerical solutions ψ_ν^h for the given test problem for $\nu^{-1} \in \{1, 15\}$. In Fig. 8 we show the variation of the solutions ψ_ν^h along the lines $x = 0.25$ for $y \in (0,1)$, and $y = 0.5$ for $x \in (0,2)$ for $\nu^{-1} \in \{1, 5, 10, 15\}$. As can be observed from these figures, decreasing the value of the kinematics viscosity produces two effects on the behavior of the solutions: it makes the solution asymmetric around the horizontal line with $y = 0.5$, pulling the maximum of the solution towards the x-axis (see Figs. 7 and 8(a)); it accentuates the decay of the solution along the x-axis (see Figs. 7 and 8(b)).

Finally, we show the damping of the solutions by decreasing the value of the exponent of the feedback forces field σ. We start by comparing the numerical solutions of (185)–(187) for different values of σ. More specifically, in what follows we consider $\sigma \in \{1.1, 1.25, 1.5, 1.75\}$. On the other hand, the kinematics viscosity and the intensity of the feedback forces field are assumed to be fixed: $\nu = 10^{-1}$ and $\delta = 4$. Here, we denote these numerical solutions by ψ_σ^h. In Fig. 9 we show of the contour lines for the solutions ψ_σ^h with values of σ decreasing from 1.75 to 1.1. We can observe the damping effect occurring as the parameter σ decreases. This effect is visually demonstrated through contour plots of the solutions, wherein decreasing values of σ consistently shift the isovalues of the solutions toward the y-axis.

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