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A HYBRID STRESS FINITE ELEMENT METHOD FOR INTEGRO-DIFFERENTIAL EQUATIONS MODELLING DYNAMIC FRACTIONAL ORDER VISCOELASTICITY

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Abstract. We consider a semi-discrete finite element method for a dynamic model for linear viscoelastic materials based on the constitutive law of fractional order. The corresponding integro-differential equation is of a Mittag-Leffler type convolution kernel. A 4-node hybrid stress quadrilateral finite element is used for the spatial discretization. We show the existence and uniqueness of the semi-discrete solution, then derive some error estimates. Finally, we provide several numerical examples to verify the theoretical results.

Key words. Integro-differential equation, fractional order viscoelasticity, hybrid stress finite element, error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with boundary Γ, and let T be a positive constant. We consider a hyperbolic type integro-differential system arising in the theory of linear and fractional-order viscoelasticity:

$$
\begin{cases}
\rho \mathbf{u}_{tt} - \mathrm{div}\boldsymbol{\sigma} = \mathbf{f}(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times (0,T], \\
\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 - \int_0^t K(t-s)\boldsymbol{\sigma}_0(\cdot,s)\mathrm{d}s, & (\mathbf{x},t) \in \Omega \times (0,T], \\
\mathbf{u} = 0, & (\mathbf{x},t) \in \Gamma \times (0,T], \\
\mathbf{u}(\mathbf{x},0) = \varphi_0(\mathbf{x}), \mathbf{u}_t(\mathbf{x},0) = \varphi_1(\mathbf{x}), & \mathbf{x} \in \Omega.
\end{cases} \tag{1}
$$

Here $\rho > 0$ is the (constant) mass density, $\mathbf{u}(\mathbf{x},t) = (u_1,u_2)^T$ the displacement field, $\boldsymbol{\sigma}(\mathbf{x},t) = (\sigma_{ij})_{2\times 2}$ the symmetric stress tensor, $\mathbf{f}(\mathbf{x},t)$ the body force, and $\varphi_0(\mathbf{x}), \varphi_1(\mathbf{x})$ the initial data. $\boldsymbol{\sigma}_0(\mathbf{x},t)$ denotes the elastic stress tensor,

$$\boldsymbol{\sigma}_0 := 2\mu\epsilon(\mathbf{u}) + \lambda \mathrm{tr}(\epsilon(\mathbf{u}))\mathbf{I},$$

with $\lambda, \mu > 0$ being the Lamé constants, $\epsilon(\mathbf{u}) = \dfrac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ the strain tensor, $\mathrm{tr} \doteq \mathrm{tr}(\cdot)$ the trace of a matrix, and $\mathbf{I}$ the 2×2 identity. For $0 < \nu < 1, 0 < \alpha < 1$, the convolution kernel

$$K(t) := -\nu \frac{d}{dt} E_{\alpha,1}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right) = \nu \frac{t^{\alpha-1}}{\tau^{\alpha}} E_{\alpha,\alpha}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right), \tag{2}$$

where $\tau > 0$ is the relaxation time, and

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

denotes the two-parameter Mittag-Leffler function.

Fractional order viscoelastic models are capable of accurately describing memory and non-locality properties of viscoelastic materials [3, 4, 5, 6, 10, 12, 13, 15, 16, 21, 34, 39]. In fact, the second equation of (1), which involves a convolution integral

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and is an explicit expression for the stress tensor in terms of the strain tensor, originates from the fractional-order viscoelastic constitutive law

$$\sigma + \tau^{\alpha} D_t^{\alpha}\sigma = (1-\nu)\boldsymbol{\sigma}_0 + \tau^{\alpha} D_t^{\alpha}\boldsymbol{\sigma}_0, \tag{3}$$

where

$$D_t^{\alpha} f(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha} f(s)\mathrm{d}s.$$

denotes the left Riemann-Liouville operator of fractional differentiation of order α. The explicit expression is obtained by using Laplace transform techniques on (3), and the use of the convolution integral formulation avoids the difficulties concerning the physical interpretation, justification and verification of fractional order initial conditions; see [1, 11, 13, 14].

There are many works on the numerical analysis of related displacement models of (1) where the stress tensor $\boldsymbol{\sigma}$ does not appear as an independent variable; see, e.g. [2, 22, 23, 35, 37, 38]. Adolfsson et al. [2] and Saedpanah [35, 38] studied spatial semi-discrete continuous Galerkin finite element methods and gave optimal a priori error estimates. Larsson et al. [22] analyzed a temporal semi-discrete discontinuous Galerkin method based on piecewise constant polynomials. In [23, 37] Larsson and Saedpanah used the continuous space-time linear finite element method to formulate the full discretizations, and derived optimal error estimates. We also refer to [18, 19, 32, 33] for some literature on numerical treatment of linear viscoelasticity problems with exponential kernels in the constitute equation.

In the numerical analysis of elasticity, the hybrid stress finite element method, pioneered by Pian [28], is known to be an efficient approach to improve the performance of the standard 4-node compatible displacement quadrilateral (bilinear) element (cf. [28, 29, 30, 41, 42, 43, 45, 49, 50]). This method is based on the domain-decomposed Hellinger-Reissner variational principle, which includes the displacement and stress variables. Since the stress parameters can be eliminated at the element level, only the unknowns of the displacements will remain in the resulting final discrete system. In [30] Pian and Sumihara proposed a robust 4-node hybrid stress quadrilateral element by using a rational choice of the 5-parameter stress mode. In [43, 42, 47, 49] optimal stress modes were studied for two- and three-dimensional hybrid stress elements. We refer to [24, 45, 50] for the stability and convergence analysis of 4-node hybrid stress quadrilateral elements. In [44, 46] and [40] semi-discrete and fully discrete hybrid stress methods were proposed and analyzed for linear elastodynamic problems and Maxwell viscoelastic problems, respectively.

In this paper, we apply the hybrid stress finite element method to discretize the viscoelastic model (1) to obtain a spatial semi-discrete scheme. The standard isoparametric bilinear interpolation is used for the displacement approximation, and the Pian-Sumihara's 5-parameters stress mode is used for the stress approximation. We prove the existence and uniqueness of the semi-discrete solution, and derive optimal error estimates.

The rest of the paper is organized as follows. Section 2 introduces notations and weak formulations. Section 3 gives the semi-discrete hybrid stress scheme and carries out the error analysis. Finally, Section 4 provides some numerical results.

2. Notations and weak formulations

Throughout this paper,we use $H^r(\Omega)$ to denote the standard Sobolev spaces with norm $\|\cdot\|_r$ and semi-norm $|\cdot|_r$. And $H^0(\Omega) = L^2(\Omega)$ is the space of square integral functions. We use $(\cdot,\cdot)$ to denote the inner product of $L^2(\Omega)$. For a time dependent function $v(\mathbf{x},t)$, sometimes we simply write it as v or $v(t)$.

For any Hilbert space X defined on Ω, with norm $\|\cdot\|_X$ and semi-norm $|\cdot|_X$, let $C^k([0,T];X)$ be the space of k times continuously differentiable maps of $[0,T]$ into X. For $1\le p<+\infty$, we set

$$L^p([0,T];X) := \left\{ v:[0,T]\to X; \|v\|_{L^p(X)}<\infty \right\},$$

where

$$\|v\|_{L^p(X)} = \begin{cases} \left(\int_0^T \|v(\cdot,t)\|_X^p \,\mathrm{d}t\right)^{1/p} & \text{if } 1\le p<+\infty,\\ \text{ess sup}_{t\in[0,T]} \|v(\cdot,t)\|_X & \text{if } p=+\infty. \end{cases}$$

For convenience, we use the notation $a\lesssim b$ to represent $a\le Cb$, where C is a generic positive constant independent of the spatial mesh size h.

For the Mittag-Leffler function $E_{\alpha,\beta}(z)$, we have the following lemma (cf. [20, 31]).

Lemma 2.1. (1) *For* $0<\alpha<2, \beta\in\mathbb{R}$ *and* $\frac{\alpha}{2}\pi<\varphi<min(\pi,\alpha\pi)$ *, there holds*

$$|E_{\alpha,\beta}(z)|\lesssim (1+|z|)^{-1}, \quad \varphi\le|arg(z)|\le\pi.$$

(2) *For* $\lambda>0,\alpha>0$ *and positive integer* $m\in\mathbb{N}$*, there holds*

$$\frac{d^m}{dt^m}E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m}E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t>0.$$

(3) *For* $0<\alpha<1$*, there holds* $E_{\alpha,\alpha}(-t)>0$ *on* $\mathbb{R}_+$.

The following lemma gives a convolution inequality (cf. [7]):

Lemma 2.2. *If* $g\in L^2(0,T)$ *and* $\phi\in L^1(0,T)$ *, then*

$$\|\phi * g\|_{L^2(0,T)} \le \|\phi\|_{L^1(0,T)} \|g\|_{L^2(0,T)},$$

where

$$\phi * g(t) := \int_0^t \phi(t-r)g(r)\mathrm{d}r.$$

We also need the following two different versions of Gronwall's inequality:

Lemma 2.3 ([17])**.** *Let* $\phi\in L^\infty[0,T]$ *be nonnegative almost everywhere satisfying*

$$\phi(t)\le c_0(t)+\int_0^t c_1(s)\phi(s)ds, \ \ for\ a.e.\ t\in[0,T],$$

where $c_1\in L^1[0,T]$ *and* $c_0\in L^\infty[0,T]$ *are nonnegative almost everywhere. Then*

$$\phi(t)\le \|c_0\|_{L^\infty[0,t]}\exp\left(\int_0^t c_1(s)ds\right), \ for\ a.e.\ t\in[0,T].$$

Furthermore, if c_0 *is non-decreasing, then*

$$\phi(t)\le c_0(t)\exp\left(\int_0^t c_1(s)ds\right), \ for\ a.e.\ t\in[0,T].$$

We introduce two spaces as follows:

$$V := (H_0^1(\Omega))^2 = \left\{\mathbf{u}\in(H^1(\Omega))^2 : \mathbf{u}|_\Gamma=0\right\},$$

$$\Sigma := \left\{\boldsymbol{\tau}\in L^2(\Omega;R^{2\times2}_{sym}); \int_\Omega tr\boldsymbol{\tau}\mathrm{d}\mathbf{x}=0\right\},$$

where $L^2(\Omega; R^{2\times 2}_{sym})$ denotes the space of square-integrable symmetric tensors with the norm $\|\cdot\|_0$ defined by $\|\boldsymbol{\tau}\|_0^2 := (\boldsymbol{\tau}, \boldsymbol{\tau})$, and $tr\boldsymbol{\tau} := \tau_{11} + \tau_{22}$ represents the trace of tensor $\boldsymbol{\tau}$.

The weak problem of the model (1) reads: For $\mathbf{f} \in C([0,T]; (H^{-1}(\Omega))^2)$, $\varphi_0(\mathbf{x}), \varphi_1(\mathbf{x}) \in (H_0^1(\Omega))^2$, find $(\boldsymbol{\sigma}, \mathbf{u}) \in C([0,T]; \Sigma) \times C^2([0,T]; V)$ such that for any $t \in (0,T]$,

$$\begin{cases} a(\boldsymbol{\sigma}(t), \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u}(t)) - \int_0^t K(t-s) b(\boldsymbol{\tau}, \mathbf{u}(s)) \mathrm{d}s = 0, & \forall \boldsymbol{\tau} \in \Sigma, \\ \rho(\mathbf{u}_{tt}(t), \mathbf{v}) - b(\boldsymbol{\sigma}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v}), & \forall \mathbf{v} \in V, \\ \mathbf{u}(0) = \varphi_0, \ \mathbf{u}_t(0) = \varphi_1, \end{cases} \tag{4}$$

where

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \frac{1}{2\mu}(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \frac{\lambda}{4\mu(\mu+\lambda)}(tr\boldsymbol{\sigma}, tr\boldsymbol{\tau}),$$
$$b(\boldsymbol{\tau}, \mathbf{v}) := -(\boldsymbol{\tau}, \epsilon(\mathbf{v})).$$

Introduce an energy norm $\|\cdot\|_a$ on Σ, with $\|\cdot\|_a^2 := a(\cdot,\cdot)$. Then it holds for any $\boldsymbol{\tau} \in \Sigma$,

$$\frac{1}{\sqrt{2(\lambda+\mu)}} \|\boldsymbol{\tau}\|_0 \le \|\boldsymbol{\tau}\|_a \le \frac{1}{\sqrt{2\mu}} \|\boldsymbol{\tau}\|_0 , \tag{5}$$

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_a^2 \ge \frac{1}{2(\lambda+\mu)} \|\boldsymbol{\tau}\|_0^2 . \tag{6}$$

It is easy to see that the following continuity conditions hold:

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})| \lesssim \|\boldsymbol{\sigma}\|_0 \|\boldsymbol{\tau}\|_0 , \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \Sigma, \tag{7}$$

$$|b(\boldsymbol{\tau}, \mathbf{v})| \lesssim \|\boldsymbol{\tau}\|_0 |\mathbf{v}|_1 , \quad \forall \boldsymbol{\tau} \in \Sigma, \ \forall \mathbf{v} \in V. \tag{8}$$

3. Semi-discrete hybrid stress finite element method

3.1. Semi-discrete scheme. Let $\mathscr{T}_h$ be a conventional quadrilateral mesh of the polygonal domain Ω. Let $K \in \mathscr{T}_h$ be arbitrary quadrilateral with diameter h_K. Denote $h := \max_{K\in\mathscr{T}_h}\{h_K\}$. Let $Z_i(x_i, y_i), 1 \le i \le 4$ be four vertices of K (Figure 1) and T_i denotes the sub-triangle of K with vertices Z_{i-1}, Z_i and Z_{i+1} (the index on Z_i is module 4). Define

$$\varrho_K = \min_{1\le i\le 4} \text{diameter of circle inscribed in } T_i.$$

Throughout the paper, we assume that the partition $\mathscr{T}_h$ satisfies the following "shape-regularity" hypothesis [48]: there exists a constant $\theta > 2$, independent of h, such that

$$h_K \le \theta \varrho_K, \quad \forall K \in \mathscr{T}_h.$$

Let $\Sigma_h \subset \Sigma$ and $V_h \subset V$ be two finite dimensional spaces for stress and displacement approximations, respectively. Let $\varphi_{0,h}, \varphi_{1,h} \in V_h$ be the approximations of initial data φ_0 and φ_1, respectively. Then the corresponding semi-discrete scheme for the problem (4) reads as follows: Find $(\boldsymbol{\sigma}_h(t), \mathbf{u}_h(t)) \in C([0,T]; \Sigma_h) \times C^2([0,T]; V_h)$ such that

$$\begin{cases} a(\boldsymbol{\sigma}_h(t), \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{u}_h(t)) - \int_0^t K(t-s) b(\boldsymbol{\tau}_h, \mathbf{u}_h(s)) \mathrm{d}s = 0, & \forall \boldsymbol{\tau}_h \in \Sigma_h, \\ \rho(\mathbf{u}_{h,tt}(t), \mathbf{v}_h) - b(\boldsymbol{\sigma}_h(t), \mathbf{v}_h) = (\mathbf{f}(t), \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h, \\ \mathbf{u}_h(0) = \varphi_{0,h}, \ \mathbf{u}_{h,t}(0) = \varphi_{1,h}. \end{cases} \tag{9}$$

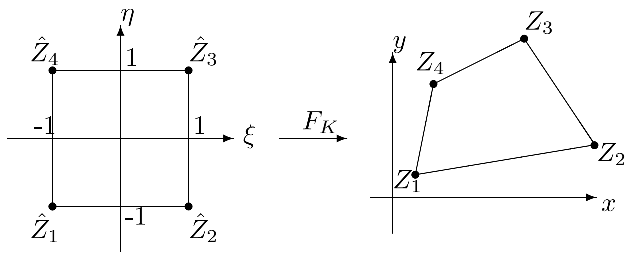


FIGURE 1. The mapping F_K.

Let $F_K : \hat{K} = [-1,1]^2 \longrightarrow K$ be the isoparametric bilinear mapping (cf. Figure 1) given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = F_K(\zeta,\eta) = \begin{pmatrix} \sum_{i=1}^4 x_i N_i(\zeta,\eta) \\ \sum_{i=1}^4 y_i N_i(\zeta,\eta) \end{pmatrix}, \tag{10}$$

where $-1 \le \zeta, \eta \le 1$ are the local isoparametric coordinates, and

$$N_1 = \frac{1}{4}(1-\zeta)(1-\eta), \quad N_2 = \frac{1}{4}(1+\zeta)(1-\eta),$$
$$N_3 = \frac{1}{4}(1+\zeta)(1+\eta), \quad N_4 = \frac{1}{4}(1-\zeta)(1+\eta).$$

For the Pian-Sumihara (PS) element [30], the isoparametric bilinear interpolation is used for the displacement approximation, i.e. the displacement space V_h is chosen as

$$V_h := \left\{ \mathbf{v} \in V : \hat{\mathbf{v}} = \mathbf{v}|_K \circ F_K \in span\{1,\zeta,\eta,\zeta\eta\}^2 \text{ for all } K \in \mathscr{T}_h \right\}.$$

The 5-parameters stress mode on K for the PS element takes the form

$$\boldsymbol{\tau} = P\gamma \text{ for } \gamma := (\gamma_1, \ldots, \gamma_5)^T \in \mathbb{R}^5, \tag{11}$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & \eta & \frac{a_2^2}{b_2^2}\zeta \\ 0 & 1 & 0 & \frac{b_1^2}{a_1^2}\eta & \zeta \\ 0 & 0 & 1 & \frac{b_1}{a_1}\eta & \frac{a_2}{b_2}\zeta \end{pmatrix},$$

and for simplicity the symmetric stress tensor $\boldsymbol{\tau} = (\tau_{ij})_{2\times 2}$ is abbreviated as $\boldsymbol{\tau} = (\tau_{11}, \tau_{22}, \tau_{12})^T$. Then the corresponding stress space for the PS element is

$$\Sigma_h := \{ \boldsymbol{\tau} \in \Sigma : \hat{\boldsymbol{\tau}} = \boldsymbol{\tau}|_K \circ F_K \text{ is of form (11) for all } K \in \mathscr{T}_h \}.$$

From [45] there holds the following inf-sup inequality for the stress and displacement approximation spaces Σ_h and V_h.

Lemma 3.1. *For any $\mathbf{v}_h \in V_h$, there holds*

$$|\mathbf{v}_h|_1 \lesssim \sup_{0 \ne \tau_h \in \Sigma_h} \frac{(\tau_h, \epsilon(\mathbf{v}_h))}{\|\tau_h\|_0}.$$

Theorem 3.1. *The semi-discrete scheme (9) admits a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$.*

Proof. For any $K \in \mathscr{T}_h$, let

$$\mathbf{u}_h(t)|_K = N\theta_K(t), \quad \theta_K(t) := (\theta_1^K(t), \cdots, \theta_8^K(t))^T,$$
$$\boldsymbol{\sigma}_h(t)|_K = P\gamma_K(t), \quad \gamma_K(t) := (\gamma_1^K(t), \cdots, \gamma_5^K(t))^T,$$

and

$$N := \begin{pmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{pmatrix},$$

where P is given in (11) and $N_i (i = 1, \cdots, 4)$ are the same as in (10).

Taking $\boldsymbol{\tau}_h|_K = P\tilde{\gamma}_K$ with an arbitrary $\tilde{\gamma}_K \in \mathbb{R}^5$ in the first equation of (9), we obtain

$$\begin{aligned} \int_K P^T \mathbb{C}^{-1} P \mathrm{d}\mathbf{x} \gamma_K(t) - \int_K P^T \epsilon(N) \mathrm{d}\mathbf{x} \theta_K(t) \\ + \int_K P^T \epsilon(N) \mathrm{d}\mathbf{x} \int_0^t K(t-s)\theta_K(s)\mathrm{d}s = 0, \end{aligned}$$

where $\epsilon(N)$ is the strain on N. Denote

$$B_K := -\int_K P^T \epsilon(N)\mathrm{d}\mathbf{x}, \qquad A_K := \int_K P^T \mathbb{C}^{-1} P \mathrm{d}\mathbf{x},$$

then the above relation yields

$$\gamma_K(t) = A_K^{-1} B_K \theta_K(t) - A_K^{-1} B_K \int_0^t K(t-s)\theta_K(s)\mathrm{d}s. \tag{12}$$

Let $\{v_i\}_{i=1}^m$ and $\{w_i\}_{i=1}^r$ be the basis functions of the finite element spaces V_h and Σ_h, respectively. We write

$$\mathbf{u}_h(t) = (v_1, \cdots, v_m)\theta(t), \qquad \boldsymbol{\sigma}_h(t) = (w_1, \cdots, w_r)\gamma(t),$$

where $\theta(t) := (\theta_1(t), \cdots, \theta_m(t))^T, \gamma(t) := (\gamma_1(t), \cdots, \gamma_r(t))^T$. Then the system (9) can be written as the following matrix forms:

$$\mathbb{A}\gamma(t) + \mathbb{B}\theta(t) - \int_0^t K(t-s)\mathbb{B}\theta(s)\mathrm{d}s = 0, \tag{13}$$

$$\mathbb{D}\theta_{tt}(t) - \mathbb{B}^T\gamma(t) = \mathbb{F}(t), \tag{14}$$

where

$\mathbb{A} := [a(w_i, w_j)]_{r\times r}$, $\mathbb{B} := [b(w_j, v_i)]_{r\times m}$, $\mathbb{D} := [\rho(v_i, v_j)]_{m\times m}$, $\mathbb{F}(t) := [(\mathbf{f}(t), v_j)]_{m\times 1}$.

Since A is symmetric positive definite, we can eliminate $\gamma(t)$ from (13)- (14) to get

$$\mathbb{D}\theta_{tt}(t) + \mathbb{B}^T\mathbb{A}^{-1}\mathbb{B}\theta(t) - \int_0^t K(t-s)\mathbb{B}^T\mathbb{A}^{-1}\mathbb{B}\theta(s)\mathrm{d}s = \mathbb{F}(t),$$

which can be rewritten as a linear system of first order:

$$\tilde{\theta}_t(t) + \tilde{\mathbb{D}}^{-1}\mathbb{G}\tilde{\theta}(t) - \tilde{\mathbb{D}}^{-1}\tilde{\mathbb{G}}\int_0^t K(t-s)\tilde{\theta}(s)\mathrm{d}s = \tilde{\mathbb{D}}^{-1}\tilde{\mathbb{F}}(t), \tag{15}$$

where

$$\tilde{\theta}(t) := \begin{bmatrix} \theta(t) \\ \theta_t(t) \end{bmatrix}, \qquad \tilde{\mathbb{F}}(t) := \begin{bmatrix} 0 \\ \mathbb{F}(t) \end{bmatrix}, \qquad \tilde{\mathbb{D}} := \begin{bmatrix} \mathbb{D} & 0 \\ 0 & \mathbb{D} \end{bmatrix},$$

$$\mathbb{G} := \begin{bmatrix} 0 & -\mathbb{D} \\ \mathbb{B}^T\mathbb{A}^{-1}\mathbb{B} & 0 \end{bmatrix}, \qquad \tilde{\mathbb{G}} := \begin{bmatrix} 0 & 0 \\ \mathbb{B}^T\mathbb{A}^{-1}\mathbb{B} & 0 \end{bmatrix}.$$

Integrate (15) with respect to t and interchange the order of integrals for the convolution term, then we have

$$\tilde{\theta}(t) = \int_0^t \tilde{\mathbb{D}}^{-1}\left(-\mathbb{G} + \int_s^t K(r-s)\mathrm{d}r\tilde{\mathbb{G}}\right)\tilde{\theta}(s)\mathrm{d}s + \int_0^t \tilde{\mathbb{D}}^{-1}\tilde{\mathbb{F}}(s)\mathrm{d}s + \tilde{\theta}(0).$$

Denote

$$\mathcal{K}(t,s) := \tilde{\mathbb{D}}^{-1}\left(-\mathbb{G} + \int_s^t K(r-s)\mathrm{d}r\tilde{\mathbb{G}}\right), \quad \mathcal{F}(t) := \int_0^t \tilde{\mathbb{D}}^{-1}\tilde{\mathbb{F}}(s)\mathrm{d}s + \tilde{\theta}(0),$$

then we get the following linear Volterra equation of second kind:

$$\tilde{\theta}(t) = \int_0^t \mathcal{K}(t,s)\tilde{\theta}(s)\mathrm{d}s + \mathcal{F}(t), \quad t \in [0.T]. \tag{16}$$

Since $\mathcal{K}(t,s)$ is continuous, by using the Picards method [8, Theorem 2.1.1.], we know that the system (16) has a unique solution $\tilde{\theta}(t)$. Hence, we obtain the existence and uniqueness of $\theta(t)$. And the existence and uniqueness of $\gamma(t)$ follow from (13). This completes the proof. ∎

3.2. Error estimation. To derive error estimates for the semi-discrete problem (9), we first define the "elliptic projection" of $\boldsymbol{\sigma}$ and $\mathbf{u}$ as follows: For $\mathbf{u}(t) \in V$ and $\boldsymbol{\sigma}(t) \in \Sigma$, $t \in [0,T]$, find $(\widehat{\boldsymbol{\sigma}_h}(t), \widehat{\mathbf{u}_h}(t)) \in \Sigma_h \times V_h$ such that

$$\begin{cases} a(\widehat{\boldsymbol{\sigma}_h}(t), \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \widehat{\mathbf{u}_h}(t)) = a(\boldsymbol{\sigma}(t), \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{u}(t)), & \forall \boldsymbol{\tau}_h \in \Sigma_h, \\ b(\widehat{\boldsymbol{\sigma}_h}(t), \mathbf{v}_h) = b(\boldsymbol{\sigma}(t), \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h. \end{cases} \tag{17}$$

From [46], we have the following error estimates.

Lemma 3.2. *Let $(\boldsymbol{\sigma}(t), \mathbf{u}(t))$ and $(\widehat{\boldsymbol{\sigma}_h}(t), \widehat{\mathbf{u}_h}(t))$ be the solutions to the weak problem (4) and the discrete "elliptic projection" problem (17), respectively. If $\mathbf{u}(t) \in (H^2(\Omega))^2 \cap (H_0^1(\Omega))^2, \boldsymbol{\sigma}(t) \in (H^1(\Omega))^{2\times 2}$ for $t \in [0,T]$, then there hold*

$$\|\boldsymbol{\sigma}(t) - \widehat{\boldsymbol{\sigma}_h}(t)\|_0 + \|\mathbf{u}(t) - \widehat{\mathbf{u}_h}(t)\|_1 \lesssim h(\|\boldsymbol{\sigma}(t)\|_1 + \|\mathbf{u}(t)\|_2), \tag{18}$$

$$\|\mathbf{u}(t) - \widehat{\mathbf{u}_h}(t)\|_0 \lesssim h^2(\|\boldsymbol{\sigma}(t)\|_1 + \|\mathbf{u}(t)\|_2). \tag{19}$$

Additionally, if $\mathbf{u}_t(t) \in (H^2(\Omega))^2, \boldsymbol{\sigma}_t(t) \in (H^1(\Omega))^{2\times 2}$ for $t \in [0,T]$, then

$$\left\|\boldsymbol{\sigma}_t(t) - \widehat{\boldsymbol{\sigma}}_{h,t}(t)\right\|_0 + \left\|\mathbf{u}_t(t) - \widehat{\mathbf{u}}_{h,t}(t)\right\|_1 \lesssim h(\|\boldsymbol{\sigma}_t(t)\|_1 + \|\mathbf{u}_t(t)\|_2), \tag{20}$$

$$\left\|\mathbf{u}_t(t) - \widehat{\mathbf{u}}_{h,t}(t)\right\|_0 \lesssim h^2(\|\boldsymbol{\sigma}_t(t)\|_1 + \|\mathbf{u}_t(t)\|_2). \tag{21}$$

and if $\mathbf{u}_{tt}(t) \in (H^2(\Omega))^2, \boldsymbol{\sigma}_{tt}(t) \in (H^1(\Omega))^{2\times 2}$ for $t \in [0,T]$, then

$$\left\|\boldsymbol{\sigma}_{tt}(t) - \widehat{\boldsymbol{\sigma}}_{h,tt}(t)\right\|_0 + \left\|\mathbf{u}_{tt}(t) - \widehat{\mathbf{u}}_{h,tt}(t)\right\|_1 \lesssim h(\|\boldsymbol{\sigma}_{tt}(t)\|_1 + \|\mathbf{u}_{tt}(t)\|_2), \tag{22}$$

$$\left\|\mathbf{u}_{tt}(t) - \widehat{\mathbf{u}}_{h,tt}(t)\right\|_0 \lesssim h^2(\|\boldsymbol{\sigma}_{tt}(t)\|_1 + \|\mathbf{u}_{tt}(t)\|_2). \tag{23}$$

By [36] the following lemma holds.

Lemma 3.3. *Let $K(t)$ be the convolution kernel given in (2), then the function $\kappa(t) := \nu - \int_0^t K(s)\mathrm{d}s$ satisfies (1) and (2):*

(1) $\frac{d}{dt}\kappa(t) = -K(t) < 0, \quad \kappa(0) = \nu, \quad 0 < \kappa(t) \le \nu;$
(2) *κ is a positive type kernel, that is, for any $T \ge 0$ and $\phi \in C([0,T])$,*

$$\int_0^T \int_0^t \kappa(t-s)\phi(t)\phi(s)\mathrm{d}s \ \mathrm{d}t \ge 0. \tag{24}$$

Let $Q_h : V \to V_h$ be the standard L^2 projection operator. We are now at a position to state one main result of error estimation.

Theorem 3.2. *Let $(\boldsymbol{\sigma}_h(t), \mathbf{u}_h(t))$ be the solution to the semi-discrete scheme (9) with the initial condition*

$$\varphi_{0,h} = \widehat{\mathbf{u}_h}(0), \quad \varphi_{1,h} = Q_h \varphi_1, \tag{25}$$

and $(\boldsymbol{\sigma}(t), \mathbf{u}(t))$ be the solution to the weak problem (4). Suppose that $\boldsymbol{\sigma} \in L^\infty([0,T];(H^1(\Omega))^{2\times 2}), \boldsymbol{\sigma}_t, \boldsymbol{\sigma}_{tt} \in L^2([0,T];(H^1(\Omega))^{2\times 2})$ and $\mathbf{u} \in L^\infty([0,T];(H^2(\Omega))^2 \cap (H_0^1(\Omega))^2), \mathbf{u}_t, \mathbf{u}_{tt} \in L^2([0,T];(H^2(\Omega))^2)$, then there holds the following error estimate:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^\infty((L^2(\Omega))^{2\times 2})} + \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty((H^1(\Omega))^2)} \lesssim C_1(\boldsymbol{\sigma}, \mathbf{u})h,$$

where

$$\begin{aligned} C_1(\boldsymbol{\sigma}, \mathbf{u}) := &\|\boldsymbol{\sigma}(0)\|_1 + \|\varphi_0\|_2 + \|\boldsymbol{\sigma}_t(0)\|_1 + \|\varphi_1\|_2 + \|\boldsymbol{\sigma}\|_{L^\infty((H^1(\Omega))^{2\times 2})} \\ &+ \|\mathbf{u}\|_{L^\infty((H^2(\Omega))^2)} + \|\boldsymbol{\sigma}_t\|_{L^2((H^1(\Omega))^{2\times 2})} + \|\mathbf{u}_t\|_{L^2((H^2(\Omega))^2)} \\ &+ \|\boldsymbol{\sigma}_{tt}\|_{L^2((H^1(\Omega))^{2\times 2})} + \|\mathbf{u}_{tt}\|_{L^2((H^2(\Omega))^2)} . \end{aligned} \tag{26}$$

Proof. Using the elliptic projection defined in (17), we write

$$\begin{aligned} \mathbf{e}_{\mathbf{u}} :=& \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \widehat{\mathbf{u}_h}) - (\mathbf{u}_h - \widehat{\mathbf{u}_h}) =: \boldsymbol{\gamma}_{\mathbf{u}} - \boldsymbol{\xi}_{\mathbf{u}}, \\ \mathbf{e}_{\boldsymbol{\sigma}} :=& \boldsymbol{\sigma} - \boldsymbol{\sigma}_h = (\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}_h}) - (\boldsymbol{\sigma}_h - \widehat{\boldsymbol{\sigma}_h}) =: \boldsymbol{\gamma}_{\boldsymbol{\sigma}} - \boldsymbol{\xi}_{\boldsymbol{\sigma}}. \end{aligned}$$

Since the estimates of $\boldsymbol{\gamma}_{\mathbf{u}}$ and $\boldsymbol{\gamma}_{\boldsymbol{\sigma}}$ are known from Lemma 3.2, it is sufficient to estimate $\boldsymbol{\xi}_{\mathbf{u}}$ and $\boldsymbol{\xi}_{\boldsymbol{\sigma}}$.

Note that $(\mathbf{e}_{\mathbf{u}}, \mathbf{e}_{\boldsymbol{\sigma}})$ satisfies that

$$\begin{aligned} a(\mathbf{e}_{\boldsymbol{\sigma}}(t), \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{e}_{\mathbf{u}}(t)) - \int_0^t K(t-s) b(\boldsymbol{\tau}_h, \mathbf{e}_{\mathbf{u}}(s)) \mathrm{d}s = 0, \quad \forall \boldsymbol{\tau}_h \in \Sigma_h, \\ b(\mathbf{e}_{\boldsymbol{\sigma}}(t), \mathbf{v}_h) = \rho(\mathbf{e}_{\mathbf{u},tt}(t), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h, \end{aligned}$$

which, together with (17), imply that

$$\begin{aligned} a(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t), \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{\xi}_{\mathbf{u}}(t)) - \int_0^t K(s) b(\boldsymbol{\tau}_h, \boldsymbol{\xi}_{\mathbf{u}}(t-s)) \mathrm{d}s \\ = - \int_0^t K(s) b(\boldsymbol{\tau}_h, \boldsymbol{\gamma}_{\mathbf{u}}(t-s)) \mathrm{d}s, \quad \forall \boldsymbol{\tau}_h \in \Sigma_h, \end{aligned} \tag{27a}$$

$$b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t), \mathbf{v}_h) = \rho(\boldsymbol{\xi}_{\mathbf{u},tt}(t) - \boldsymbol{\gamma}_{\mathbf{u},tt}(t), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h. \tag{27b}$$

Differentiating (27a) with respect to t and taking $\boldsymbol{\tau}_h = \boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)$, we obtain

$$\begin{aligned} &a(\boldsymbol{\xi}_{\boldsymbol{\sigma},t}(t), \boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)) + b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t), \boldsymbol{\xi}_{\mathbf{u},t}(t)) - \int_0^t K(t-s) b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t), \boldsymbol{\xi}_{\mathbf{u},t}(s)) \mathrm{d}s \\ =& K(t) b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t), \boldsymbol{\xi}_{\mathbf{u}}(0)) - K(t) b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t), \boldsymbol{\gamma}_{\mathbf{u}}(0)) - \int_0^t K(t-s) b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t), \boldsymbol{\gamma}_{\mathbf{u},t}(s)) \mathrm{d}s. \end{aligned} \tag{28}$$

Taking $\mathbf{v}_h = \boldsymbol{\xi}_{\mathbf{u},t}(t)$ in (27b), we have

$$b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t), \boldsymbol{\xi}_{\mathbf{u},t}(t)) = \rho(\boldsymbol{\xi}_{\mathbf{u},tt}(t), \boldsymbol{\xi}_{\mathbf{u},t}(t)) - \rho(\boldsymbol{\gamma}_{\mathbf{u},tt}(t), \boldsymbol{\xi}_{\mathbf{u},t}(t)). \tag{29}$$

Combining (28) with (29), using the Cauchy-Schwarz inequality and (8) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)\|_a^2+\rho\left\|\boldsymbol{\xi}_{\mathbf{u},t}(t)\right\|_0^2)-\rho\int_0^t K(t-s)(\boldsymbol{\xi}_{\mathbf{u},tt}(t),\boldsymbol{\xi}_{\mathbf{u},t}(s))\mathrm{d}s\\
=&\rho(\boldsymbol{\gamma}_{\mathbf{u},tt}(t),\boldsymbol{\xi}_{\mathbf{u},t}(t))-\rho\int_0^t K(t-s)(\boldsymbol{\gamma}_{\mathbf{u},tt}(t),\boldsymbol{\xi}_{\mathbf{u},t}(s))\mathrm{d}s\\
&-K(t)b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t),\boldsymbol{e}_{\mathbf{u}}(0))-\int_0^t K(t-s)b(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t),\boldsymbol{\gamma}_{\mathbf{u},t}(s))\mathrm{d}s\\
\le&\rho\left\|\boldsymbol{\gamma}_{\mathbf{u},tt}\right\|_0\left\|\boldsymbol{\xi}_{\mathbf{u},t}\right\|_0+\rho\left\|\boldsymbol{\gamma}_{\mathbf{u},tt}\right\|_0\int_0^t K(t-s)\left\|\boldsymbol{\xi}_{\mathbf{u},t}(s)\right\|_0\mathrm{d}s\\
&+CK(t)\left\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\right\|_0|\boldsymbol{e}_{\mathbf{u}}(0)|_1+C\left\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\right\|_0\int_0^t K(t-s)\left|\boldsymbol{\gamma}_{\mathbf{u},t}(s)\right|_1\mathrm{d}s,
\end{aligned}$$

where C is a positive constant independent of h. Integrate the above inequality from 0 to t, and we obtain

$$\begin{aligned}
&\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)\|_a^2+\rho\left\|\boldsymbol{\xi}_{\mathbf{u},t}(t)\right\|_0^2-2\rho\int_0^t\int_0^r K(r-s)(\boldsymbol{\xi}_{\mathbf{u},tt}(r),\boldsymbol{\xi}_{\mathbf{u},t}(s))\mathrm{d}s\mathrm{d}r\\
\le&\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(0)\|_a^2+\rho\left\|\boldsymbol{\xi}_{\mathbf{u},t}(0)\right\|_0^2+2\rho\int_0^t\left\|\boldsymbol{\gamma}_{\mathbf{u},tt}\right\|_0^2\mathrm{d}s+\rho\int_0^t\left\|\boldsymbol{\xi}_{\mathbf{u},t}\right\|_0^2\mathrm{d}s\\
&+C\int_0^t K(s)\left\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\right\|_0^2\mathrm{d}s+C\int_0^t K(s)\left|\boldsymbol{e}_{\mathbf{u}}(0)\right|_1^2\mathrm{d}s+C\int_0^t\left\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\right\|_0^2\mathrm{d}s\\
&+C\int_0^t\left(\int_0^r K(r-s)\left|\boldsymbol{\gamma}_{\mathbf{u},t}(s)\right|_1\mathrm{d}s\right)^2\mathrm{d}r\\
&+\rho\int_0^t\left(\int_0^r K(r-s)\left\|\boldsymbol{\xi}_{\mathbf{u},t}(s)\right\|_0\mathrm{d}s\right)^2\mathrm{d}r.
\end{aligned}\tag{30}$$

For the last term on the right hand side of (30), using Lemma 2.2 we have

$$\int_0^t\left(\int_0^r K(r-s)\left\|\boldsymbol{\xi}_{\mathbf{u},s}(s)\right\|_0\mathrm{d}s\right)^2\mathrm{d}r\le\|K(s)\|_{L^1(0,t)}^2\int_0^t\left\|\boldsymbol{\xi}_{\mathbf{u},t}(s)\right\|_0^2\mathrm{d}s.\tag{31}$$

On the other hand, recalling that $K(r-s)=\frac{d}{ds}\kappa(r-s),\kappa(0)=\nu$ from Lemma 3.3, we apply integration by parts to the third term on the left hand side of (30) to find that

$$\begin{aligned}
&-2\int_0^t\int_0^r K(r-s)(\boldsymbol{\xi}_{\mathbf{u},tt}(r),\boldsymbol{\xi}_{\mathbf{u},t}(s))\mathrm{d}s\mathrm{d}r\\
=&-2\int_0^t\int_0^r\frac{d}{ds}\kappa(r-s)(\boldsymbol{\xi}_{\mathbf{u},tt}(r),\boldsymbol{\xi}_{\mathbf{u},t}(s))\mathrm{d}s\mathrm{d}r\\
=&-2\int_0^t\kappa(0)(\boldsymbol{\xi}_{\mathbf{u},tt}(r),\boldsymbol{\xi}_{\mathbf{u},t}(r))\mathrm{d}r+2\int_0^t\kappa(r)(\boldsymbol{\xi}_{\mathbf{u},tt}(r),\boldsymbol{\xi}_{\mathbf{u},t}(0))\mathrm{d}r\\
&+2\int_0^t\int_0^r\kappa(r-s)(\boldsymbol{\xi}_{\mathbf{u},tt}(r),\boldsymbol{\xi}_{\mathbf{u},tt}(s))\mathrm{d}s\mathrm{d}r\\
=&-\nu\left\|\boldsymbol{\xi}_{\mathbf{u},t}(t)\right\|_0^2+2\kappa(t)(\boldsymbol{\xi}_{\mathbf{u},t}(t),\boldsymbol{\xi}_{\mathbf{u},t}(0))-2\kappa(0)\|\boldsymbol{\xi}_{\mathbf{u},t}(0)\|_0^2\\
&+2\int_0^t K(r)(\boldsymbol{\xi}_{\mathbf{u},t}(r),\boldsymbol{\xi}_{\mathbf{u},t}(0))\mathrm{d}r+2\int_0^t\int_0^r\kappa(r-s)(\boldsymbol{\xi}_{\mathbf{u},tt}(r),\boldsymbol{\xi}_{\mathbf{u},tt}(s))\mathrm{d}s\mathrm{d}r.
\end{aligned}$$

This equation, together with (31), (30), (24) and the Young inequality, gives

$$\begin{aligned}
&\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)\|_a^2+\rho(1-\nu)\left\|\boldsymbol{\xi}_{\mathbf{u},t}(t)\right\|_0^2\\
\le&\frac{\rho\nu}{q}\left\|\boldsymbol{\xi}_{\mathbf{u},t}(t)\right\|_0^2+\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(0)\|_a^2+\rho(1+3\nu+\nu q)\left\|\boldsymbol{\xi}_{\mathbf{u},t}(0)\right\|_0^2\\
&+C\nu\left|\boldsymbol{e}_{\mathbf{u}}(0)\right|_1^2+2\rho\int_0^t\left\|\boldsymbol{\gamma}_{\mathbf{u},tt}\right\|_0^2\mathrm{d}s+C\nu^2\int_0^t\left|\boldsymbol{\gamma}_{\mathbf{u},t}\right|_1^2\mathrm{d}s\\
&+\int_0^t C(K(s)+1)(2\lambda+\mu)\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_a^2\mathrm{d}s+\rho\int_0^t(1+\nu^2+K(s))\left\|\boldsymbol{\xi}_{\mathbf{u},t}\right\|_0^2\mathrm{d}s.
\end{aligned}$$

Here the constant $q>\frac{\nu}{1-\nu}>0$. Thus, we get

$$\begin{aligned}
&\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)\|_a^2+\rho(1-\nu-\frac{\nu}{q})\left\|\boldsymbol{\xi}_{\mathbf{u},t}(t)\right\|_0^2\\
\le&\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(0)\|_a^2+\rho(1+3\nu+\nu q)\left\|\boldsymbol{\xi}_{\mathbf{u},t}(0)\right\|_0^2\\
&+C\nu\left|\boldsymbol{e}_{\mathbf{u}}(0)\right|_1^2+2\rho\int_0^t\left\|\boldsymbol{\gamma}_{\mathbf{u},tt}\right\|_0^2\mathrm{d}s+C\nu^2\int_0^t\left|\boldsymbol{\gamma}_{\mathbf{u},t}\right|_1^2\mathrm{d}s\\
&+\int_0^t C(K(s)+1)(2\lambda+\mu)\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_a^2\mathrm{d}s+\rho\int_0^t(1+\nu^2+K(s))\left\|\boldsymbol{\xi}_{\mathbf{u},t}\right\|_0^2\mathrm{d}s.
\end{aligned}$$

By (25) it holds $\boldsymbol{\xi}_{\mathbf{u}}(0)=0$, then by (27a) we have $\boldsymbol{\xi}_{\boldsymbol{\sigma}}(0)=0$. Moreover, from Lemma 3.2 it follows that

$$\begin{aligned}
|\boldsymbol{e}_{\mathbf{u}}(0)|_1&=|\mathbf{u}(0)-\widehat{\mathbf{u}_h}(0)|_1\lesssim h(\|\boldsymbol{\sigma}(0)\|_1+\|\varphi_0\|_2),\\
\|\boldsymbol{\xi}_{\mathbf{u},t}(0)\|_0&=\|\mathbf{u}_{h,t}(0)-\widehat{\mathbf{u}_{h}}_{,t}(0)\|_0\\
&\le\|\mathbf{u}_{h,t}(0)-\mathbf{u}_t(0)\|_0+\|\mathbf{u}_t(0)-\widehat{\mathbf{u}_{h}}_{,t}(0)\|_0\\
&\lesssim h^2(\|\boldsymbol{\sigma}_t(0)\|_1+\|\varphi_1\|_2).
\end{aligned}$$

Then, using Lemmas 2.3 and 3.2 and the estimates (20) and (23), we have

$$\begin{aligned}
&\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)\|_a^2+\rho(1-\nu-\frac{\nu}{q})\left\|\boldsymbol{\xi}_{\mathbf{u},t}(t)\right\|_0^2\\
\le&C_{(\alpha,\nu,\rho,T,\lambda,\mu)}(|\boldsymbol{e}_{\mathbf{u}}(0)|_1^2+\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(0)\|_a^2+\left\|\boldsymbol{\xi}_{\mathbf{u},t}(0)\right\|_0^2+\int_0^t\left\|\boldsymbol{\gamma}_{\mathbf{u},tt}\right\|_0^2\mathrm{d}s+\int_0^t\left|\boldsymbol{\gamma}_{\mathbf{u},t}\right|_1^2\mathrm{d}s)\\
\lesssim&h^2\Big(\|\boldsymbol{\sigma}(0)\|_1^2+\|\varphi_0\|_2^2+\|\boldsymbol{\sigma}_t(0)\|_1^2+\|\varphi_1\|_2^2+\|\boldsymbol{\sigma}_t\|^2_{L^2((H^1(\Omega))^{2\times2})}+\|\mathbf{u}_t\|^2_{L^2((H^2(\Omega))^2)}\\
&+\|\boldsymbol{\sigma}_{tt}\|^2_{L^2((H^1(\Omega))^{2\times2})}+\|\mathbf{u}_{tt}\|^2_{L^2((H^2(\Omega))^2)}\Big),
\end{aligned}$$

which plus (5) yields

$$\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)\|_0+\left\|\boldsymbol{\xi}_{\mathbf{u},t}(t)\right\|_0\lesssim C_1(\boldsymbol{\sigma},\mathbf{u})h \tag{32}$$

with $C_1(\boldsymbol{\sigma},\mathbf{u})$ being given in (26).

We now turn to estimate $\|\boldsymbol{\xi}_{\mathbf{u}}(t)\|_1$. From Lemma 3.1, (27a), and (5)- (8), it follows that

$$\begin{aligned}
&\|\boldsymbol{\xi}_{\mathbf{u}}(t)\|_1\lesssim|\boldsymbol{\xi}_{\mathbf{u}}(t)|_1\lesssim\sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h}\frac{-b(\boldsymbol{\tau}_h,\boldsymbol{\xi}_{\mathbf{u}})}{\|\boldsymbol{\tau}_h\|_0}\\
=&\sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h}\frac{a(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\tau}_h)-\int_0^t K(t-s)b(\boldsymbol{\tau}_h,\boldsymbol{\xi}_{\mathbf{u}}(s))\mathrm{d}s+\int_0^t K(t-s)b(\boldsymbol{\tau}_h,\boldsymbol{\gamma}_{\mathbf{u}}(s))\mathrm{d}s}{\|\boldsymbol{\tau}_h\|_0}\\
\lesssim&\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_0+\int_0^t(t-s)^{\alpha-1}(|\boldsymbol{\gamma}_{\mathbf{u}}(s)|_1+|\boldsymbol{\xi}_{\mathbf{u}}(s)|_1)\mathrm{d}s,
\end{aligned}$$

which, together with Lemma 2.3, (18) and (32), indicates

$$\|\boldsymbol{\xi}_{\mathbf{u}}(t)\|_1 \lesssim C_1(\boldsymbol{\sigma}, \mathbf{u})h. \tag{33}$$

Finally, in view of Lemma 3.2, (32)- (33), and the triangle inequality, we obtain the desired estimates. ∎

Remark 3.1. *From the estimate (32) and Lemma 3.2, we can easily derive*

$$\|\mathbf{u}_t(t) - \mathbf{u}_{h,t}(t)\|_{L^\infty((L^2(\Omega))^2)} \lesssim C_2(\boldsymbol{\sigma}, \mathbf{u})h,$$

where

$$\begin{aligned} C_2(\boldsymbol{\sigma}, \mathbf{u}) := {}& \|\boldsymbol{\sigma}(0)\|_1 + \|\varphi_0\|_2 + \|\boldsymbol{\sigma}_t(0)\|_1 + \|\varphi_1\|_2 + \|\boldsymbol{\sigma}_t\|_{L^\infty((H^1(\Omega))^{2\times 2})} \\ & + \|\mathbf{u}_t\|_{L^\infty((H^2(\Omega))^2)} + \|\boldsymbol{\sigma}_{tt}\|_{L^\infty((H^1(\Omega))^{2\times 2})} + \|\mathbf{u}_{tt}\|_{L^\infty((H^2(\Omega))^2)}, \end{aligned}$$

provided that $\mathbf{u}_t \in L^\infty([0,T];(H^2(\Omega))^2)$ *and* $\boldsymbol{\sigma}_t \in L^\infty([0,T];(H^1(\Omega))^{2\times 2})$.

3.3. $L^\infty(L^2)$ error estimate of the displacement approximation. In Theorem 3.2 we have obtained the estimate of $\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty((H^1(\Omega))^2)}$. We devote this section to the estimation of the error $\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty((L^2(\Omega))^2)}$, and we have the following theorem:

Theorem 3.3. *Let $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ be the solution to the semi-discrete scheme (9) with the initial condition (25) and $(\boldsymbol{\sigma}, \mathbf{u})$ be the solution to the weak problem (4), and suppose that $\boldsymbol{\sigma}, \boldsymbol{\sigma}_t \in L^\infty([0,T];(H^1(\Omega))^{2\times 2})$ and $\mathbf{u} \in L^\infty([0,T];(H^2(\Omega))^2 \cap (H_0^1(\Omega))^2), \mathbf{u}_t \in L^\infty([0,T];(H^2(\Omega))^2)$. Then there holds the following error estimate:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty((L^2(\Omega))^2)} \lesssim C_3(\boldsymbol{\sigma}, \mathbf{u})h^2,$$

where

$$\begin{aligned} C_3(\boldsymbol{\sigma}, \mathbf{u}) := {}& \|\boldsymbol{\sigma}(0)\|_1 + \|\varphi_0\|_2 + \|\boldsymbol{\sigma}\|_{L^\infty((H^1(\Omega))^{2\times 2})} + \|\mathbf{u}\|_{L^\infty((H^2(\Omega))^2)} \\ & + \|\boldsymbol{\sigma}_t\|_{L^\infty((H^1(\Omega))^{2\times 2})} + \|\mathbf{u}_t\|_{L^\infty((H^2(\Omega))^2)}. \end{aligned}$$

To prove this theorem, we follow [25] to introduce the extended mixed Ritz-Volterra projection, $(\tilde{\boldsymbol{\sigma}}_h(t), \tilde{\mathbf{u}}_h(t))$, of the continuous solution $(\boldsymbol{\sigma}(t), \mathbf{u}(t)) \in \Sigma \times V$: find $(\tilde{\boldsymbol{\sigma}}_h(t), \tilde{\mathbf{u}}_h(t)) \in \Sigma_h \times V_h$ such that

$$\begin{cases} a(\tilde{\boldsymbol{\sigma}}_h(t), \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \tilde{\mathbf{u}}_h(t)) - \int_0^t K(t-s)b(\boldsymbol{\tau}_h, \tilde{\mathbf{u}}_h(s))\mathrm{d}s = 0, & \forall \boldsymbol{\tau}_h \in \Sigma_h, \\ b(\tilde{\boldsymbol{\sigma}}_h(t), \mathbf{v}_h) = b(\boldsymbol{\sigma}(t), \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h. \end{cases} \tag{34}$$

Lemma 3.4. *The system (34) admits a unique solution $(\tilde{\boldsymbol{\sigma}}_h(t), \tilde{\mathbf{u}}_h(t))$. Moreover, if $\mathbf{u}(t) \in C([0,T];(H^2(\Omega))^2 \cap (H_0^1(\Omega))^2), \boldsymbol{\sigma}(t) \in C([0,T];(H^1(\Omega))^{2\times 2})$, then there hold that*

$$\|\boldsymbol{\sigma}(t) - \tilde{\boldsymbol{\sigma}}_h(t)\|_0 + \|\mathbf{u}(t) - \tilde{\mathbf{u}}_h(t)\|_1 \lesssim h \sup_{s\le t}(\|\boldsymbol{\sigma}(s)\|_1 + \|\mathbf{u}(s)\|_2), \tag{35}$$

$$\|\mathbf{u}(t) - \tilde{\mathbf{u}}_h(t)\|_0 \lesssim h^2 \sup_{s\le t}(\|\boldsymbol{\sigma}(s)\|_1 + \|\mathbf{u}(s)\|_2). \tag{36}$$

Proof. We first follow the same procedure as in the proof of Theorem 3.1 to show the existence and uniqueness of the mixed Ritz-Volterra projection.

Let

$$\tilde{\boldsymbol{u}}_h(t) = (v_1, \cdots, v_m)\tilde{\theta}(t), \qquad \tilde{\boldsymbol{\sigma}}_h(t) = (w_1, \cdots, w_r)\tilde{\gamma}(t).$$

Then we rewrite (34) as a Volterra system:

$$\mathbb{A}\tilde{\gamma}(t) + \mathbb{B}\tilde{\theta}(t) - \mathbb{B}\int_0^t K(t-s)\tilde{\theta}(s)\mathrm{d}s = 0, \tag{37}$$

$$-\mathbb{B}^T\tilde{\gamma}(t) = F(t), \tag{38}$$

where the matrices $\mathbb{A}$ and $\mathbb{B}$ are as same as in the proof of Theorem 3.1 matrices, and F is a vector associated with $(\boldsymbol{\sigma}, \mathbf{u})$. We eliminate $\tilde{\gamma}(t)$ from (37)- (38) to get

$$\tilde{\theta}(t) - \int_0^t K(t-s)\tilde{\theta}(s)\mathrm{d}s = G^{-1}F(t),$$

with $G = \mathbb{B}^T\mathbb{A}^{-1}\mathbb{B}$. Notice that

$$K(t) = \nu\frac{t^{\alpha-1}}{\tau^\alpha}E_{\alpha,\alpha}\left(-\left(\frac{t}{\tau}\right)^\alpha\right) =: \frac{k(t)}{t^{1-\alpha}}.$$

Since $k(t)$ is continuous, from [9, Theorem 1, p.46] it follows that $\tilde{\theta}(t)$ exists uniquely. And the existence and uniqueness of $\tilde{\gamma}(t)$ follow from (37).

Next let us prove the estimate (35). Denote $\boldsymbol{\eta}_{\mathbf{u}} := \mathbf{u}(t) - \tilde{\boldsymbol{u}}_h(t)$ and $\boldsymbol{\eta}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma}(t) - \tilde{\boldsymbol{\sigma}}_h(t)$, then the following error equations hold:

$$a(\boldsymbol{\eta}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{\eta}_{\mathbf{u}}) - \int_0^t K(t-s)b(\boldsymbol{\tau}_h, \boldsymbol{\eta}_{\mathbf{u}}(s))\mathrm{d}s = 0, \quad \forall \boldsymbol{\tau}_h \in \Sigma_h, \tag{39a}$$

$$b(\boldsymbol{\eta}_{\boldsymbol{\sigma}}, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in V_h. \tag{39b}$$

Let $I_h : V \to V_h$ be the standard isoparametric bilinear interpolation operator. We rewrite $\boldsymbol{\eta}_{\mathbf{u}}$ and $\boldsymbol{\eta}_{\boldsymbol{\sigma}}$ as

$$\begin{aligned}\boldsymbol{\eta}_{\mathbf{u}} &= (\mathbf{u} - I_h\mathbf{u}) - (\tilde{\boldsymbol{u}}_h - I_h\mathbf{u}) =: \boldsymbol{\theta}_{\mathbf{u}} - \boldsymbol{\rho}_{\mathbf{u}},\\ \boldsymbol{\eta}_{\boldsymbol{\sigma}} &= (\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}_h}) - (\tilde{\boldsymbol{\sigma}}_h - \widehat{\boldsymbol{\sigma}_h}) =: \boldsymbol{\theta}_{\boldsymbol{\sigma}} - \boldsymbol{\rho}_{\boldsymbol{\sigma}}.\end{aligned}$$

Because the estimates of $\boldsymbol{\theta}_{\mathbf{u}}$ and $\boldsymbol{\theta}_{\boldsymbol{\sigma}}$ are known, it suffices to estimate $\boldsymbol{\rho}_{\mathbf{u}}$ and $\boldsymbol{\rho}_{\boldsymbol{\sigma}}$. Now rewrite (39a) as

$$\begin{aligned}&a(\boldsymbol{\rho}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{\rho}_{\mathbf{u}}) - \int_0^t K(t-s)b(\boldsymbol{\tau}_h, \boldsymbol{\rho}_{\mathbf{u}}(s))\mathrm{d}s\\ =&a(\boldsymbol{\theta}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{\theta}_{\mathbf{u}}) - \int_0^t K(t-s)b(\boldsymbol{\tau}_h, \boldsymbol{\theta}_{\mathbf{u}}(s))\mathrm{d}s, \quad \forall \boldsymbol{\tau}_h \in \Sigma_h,\end{aligned} \tag{40}$$

and take $\boldsymbol{\tau}_h = \boldsymbol{\rho}_{\boldsymbol{\sigma}}$ to obtain

$$\begin{aligned}a(\boldsymbol{\rho}_{\boldsymbol{\sigma}}, \boldsymbol{\rho}_{\boldsymbol{\sigma}}) = -b(\boldsymbol{\rho}_{\boldsymbol{\sigma}}, \boldsymbol{\rho}_{\mathbf{u}}) &+ \int_0^t K(t-s)b(\boldsymbol{\rho}_{\boldsymbol{\sigma}}(t), \boldsymbol{\rho}_{\mathbf{u}}(s))\mathrm{d}s\\ &+ a(\boldsymbol{\theta}_{\boldsymbol{\sigma}}, \boldsymbol{\rho}_{\boldsymbol{\sigma}}) + b(\boldsymbol{\rho}_{\boldsymbol{\sigma}}, \boldsymbol{\theta}_{\mathbf{u}}) - \int_0^t K(t-s)b(\boldsymbol{\rho}_{\boldsymbol{\sigma}}(t), \boldsymbol{\theta}_{\mathbf{u}}(s))\mathrm{d}s.\end{aligned}$$

Notice that by (39b) and (17) we have

$$b(\boldsymbol{\rho}_{\boldsymbol{\sigma}}, \boldsymbol{\rho}_{\mathbf{u}}) = b(\boldsymbol{\theta}_{\boldsymbol{\sigma}}, \boldsymbol{\rho}_{\mathbf{u}}) = 0.$$

Thus,

$$a(\boldsymbol{\rho}_{\boldsymbol{\sigma}}, \boldsymbol{\rho}_{\boldsymbol{\sigma}}) = a(\boldsymbol{\theta}_{\boldsymbol{\sigma}}, \boldsymbol{\rho}_{\boldsymbol{\sigma}}) + b(\boldsymbol{\rho}_{\boldsymbol{\sigma}}, \boldsymbol{\theta}_{\mathbf{u}}) - \int_0^t K(t-s)b(\boldsymbol{\rho}_{\boldsymbol{\sigma}}(t), \boldsymbol{\theta}_{\mathbf{u}}(s))\mathrm{d}s,$$

which, together with (5)- (8), yields

$$\begin{aligned}\|\boldsymbol{\rho}_{\boldsymbol{\sigma}}\|_0^2 &\lesssim \|\boldsymbol{\rho}_{\boldsymbol{\sigma}}\|_a^2 = a(\boldsymbol{\rho}_{\boldsymbol{\sigma}}, \boldsymbol{\rho}_{\boldsymbol{\sigma}}) \\ &\lesssim \|\boldsymbol{\rho}_{\boldsymbol{\sigma}}\|_0 \|\boldsymbol{\theta}_{\boldsymbol{\sigma}}\|_0 + \|\boldsymbol{\rho}_{\boldsymbol{\sigma}}\|_0 |\boldsymbol{\theta}_{\mathbf{u}}|_1 + \|\boldsymbol{\rho}_{\boldsymbol{\sigma}}\|_0 \int_0^t K(t-s) |\boldsymbol{\theta}_{\mathbf{u}}(s)|_1 \,\mathrm{d}s.\end{aligned}$$

By Lemma 2.1, we have

$$\int_0^t K(t-s)\mathrm{d}s = -\nu \int_0^{\frac{t}{\tau}} \frac{d}{ds} E_{\alpha,1}\left(-s^\alpha\right)\mathrm{d}s = \nu(1 - E_{\alpha,1}\left(-\left(t/\tau\right)^\alpha\right)) < \nu,$$

and hence,

$$\begin{aligned}\|\boldsymbol{\eta}_{\boldsymbol{\sigma}}\|_0 &\lesssim \|\boldsymbol{\theta}_{\boldsymbol{\sigma}}\|_0 + |\boldsymbol{\theta}_{\mathbf{u}}|_1 + \sup_{s\le t} |\boldsymbol{\theta}_{\mathbf{u}}(s)|_1 \int_0^t K(t-s)\mathrm{d}s \\ &\lesssim h(\|\boldsymbol{\sigma}(t)\|_1 + \sup_{s\le t} \|\mathbf{u}(s)\|_2).\end{aligned} \tag{41}$$

In order to estimate $\|\boldsymbol{\rho}_{\mathbf{u}}(t)\|_1$, take $\mathbf{v}_h = \boldsymbol{\rho}_{\mathbf{u}}(t)$ in Lemma 3.1 and apply (40), (7)- (8), Lemma 2.1, the approximation property of projection I_h and the estimate of $\boldsymbol{\eta}_{\boldsymbol{\sigma}}$ to arrive at

$$\begin{aligned}\|\boldsymbol{\rho}_{\mathbf{u}}(t)\|_1 \lesssim |\boldsymbol{\rho}_{\mathbf{u}}(t)|_1 &\lesssim \sup_{0\ne\boldsymbol{\tau}_h\in\Sigma_h} \frac{-b(\boldsymbol{\tau}_h, \boldsymbol{\rho}_{\mathbf{u}})}{\|\boldsymbol{\tau}_h\|_0} \\ &\lesssim \int_0^t K(t-s) |\boldsymbol{\theta}_{\mathbf{u}}(s)|_1 \,\mathrm{d}s + \int_0^t K(t-s) |\boldsymbol{\rho}_{\mathbf{u}}(s)|_1 \,\mathrm{d}s + \|\boldsymbol{\eta}_{\boldsymbol{\sigma}}(t)\|_0 + |\boldsymbol{\theta}_{\mathbf{u}}(t)|_1 \\ &\lesssim h(\|\boldsymbol{\sigma}(t)\|_1 + \sup_{s\le t} \|\mathbf{u}(s)\|_2) + \int_0^t (t-s)^{\alpha-1} |\boldsymbol{\rho}_{\mathbf{u}}(s)|_1 \,\mathrm{d}s,\end{aligned}$$

which plus Lemma 2.3 gives

$$|\boldsymbol{\rho}_{\mathbf{u}}(t)|_1 \lesssim h \sup_{s\le t} (\|\boldsymbol{\sigma}(s)\|_1 + \|\mathbf{u}(s)\|_2). \tag{42}$$

Using (41)- (42) and the triangle inequality, we obtain the desired estimate (35).

We next use a duality argument to estimate $\boldsymbol{\eta}_{\mathbf{u}}(t) = \|\mathbf{u}(t) - \tilde{\mathbf{u}}_h(t)\|_0$. Consider the auxiliary elliptic problem:

$$\begin{cases} -\mathrm{div}(\mathbb{C}\epsilon(\boldsymbol{\phi})) = \boldsymbol{\eta}_{\mathbf{u}} & \text{in } \Omega, \\ \boldsymbol{\phi} = 0 & \text{on } \Gamma. \end{cases}$$

Since Ω is a convex polygon, we have the following elliptic regularity result:

$$\|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\psi}\|_1 \lesssim \|\boldsymbol{\eta}_{\mathbf{u}}\|_0, \tag{43}$$

where $\boldsymbol{\psi} := \mathbb{C}\epsilon(\boldsymbol{\phi})$. Clearly, there hold

$$a(\boldsymbol{\psi}(t), \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{\phi}(t)) = 0, \quad \forall \boldsymbol{\tau} \in \Sigma, \tag{44}$$

$$b(\boldsymbol{\psi}(t), \mathbf{v}) + (\boldsymbol{\eta}_{\mathbf{u}}(t), \mathbf{v}) = 0, \quad \forall \mathbf{v} \in V. \tag{45}$$

Let $P_h : \Sigma \to \Sigma_h$ be the L^2 projection operator. Take $\boldsymbol{\tau} = \boldsymbol{\eta}_{\boldsymbol{\sigma}}(t)$ and $\mathbf{v} = \boldsymbol{\eta}_{\mathbf{u}}(t)$ in (44)- (45), respectively, and add the resulting equations to get

$$\begin{aligned}\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|_0^2 &= -b(\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\mathbf{u}}(t)) - a(\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\boldsymbol{\sigma}}(t)) - b(\boldsymbol{\eta}_{\boldsymbol{\sigma}}(t), \boldsymbol{\phi}(t)) \\ &= -b(\boldsymbol{\psi} - P_h\boldsymbol{\psi}, \boldsymbol{\eta}_{\mathbf{u}}) - a(\boldsymbol{\psi} - P_h\boldsymbol{\psi}, \boldsymbol{\eta}_{\boldsymbol{\sigma}}) - b(\boldsymbol{\eta}_{\boldsymbol{\sigma}}, \boldsymbol{\phi} - I_h\boldsymbol{\phi}) \\ &\quad - b(P_h\boldsymbol{\psi}, \boldsymbol{\eta}_{\mathbf{u}}) - a(P_h\boldsymbol{\psi}, \boldsymbol{\eta}_{\boldsymbol{\sigma}}) - b(\boldsymbol{\eta}_{\boldsymbol{\sigma}}, I_h\boldsymbol{\phi}).\end{aligned} \tag{46}$$

Take $\boldsymbol{\tau}_h = P_h\boldsymbol{\psi}$ and $\mathbf{v}_h = I_h\boldsymbol{\phi}$ in (39a)- (39b), substitute them into (46), and use (7)- (8), (43) and (35) to obtain

$$\begin{aligned}\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|_0^2 &= -b(\boldsymbol{\psi} - P_h\boldsymbol{\psi}, \boldsymbol{\eta}_{\mathbf{u}}) - a(\boldsymbol{\psi} - P_h\boldsymbol{\psi}, \boldsymbol{\eta}_{\boldsymbol{\sigma}}) - b(\boldsymbol{\eta}_{\boldsymbol{\sigma}}, \boldsymbol{\phi} - I_h\boldsymbol{\phi}) \\ &\quad - \int_0^t K(t-s)b(P_h\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\mathbf{u}}(s))\mathrm{d}s \\ &\lesssim h^2(\sup_{s\le t}\|\boldsymbol{\sigma}(s)\|_1 + \sup_{s\le t}\|\mathbf{u}(s)\|_2)\,\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|_0 \\ &\quad + \int_0^t K(t-s)|b(P_h\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\mathbf{u}}(s))|\mathrm{d}s. \end{aligned} \tag{47}$$

Note that for fixed t, we have

$$\begin{aligned} &\int_0^t K(t-s)|b(P_h\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\mathbf{u}}(s))|\mathrm{d}s \\ \le &\int_0^t K(t-s)|b(\boldsymbol{\psi}(t) - P_h\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\mathbf{u}}(s))|\mathrm{d}s + \int_0^t K(t-s)|b(\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\mathbf{u}}(s))|\mathrm{d}s \\ \lesssim &\|\boldsymbol{\psi} - P_h\boldsymbol{\psi}\|_0 \int_0^t K(t-s)\,|\boldsymbol{\eta}_{\mathbf{u}}(s)|_1\,\mathrm{d}s + \|\mathrm{div}\boldsymbol{\psi}\|_0 \int_0^t K(t-s)\,\|\boldsymbol{\eta}_{\mathbf{u}}(s)\|_0\,\mathrm{d}s \\ \lesssim &\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|_0 \left(h^2(\sup_{s\le t}\|\boldsymbol{\sigma}(s)\|_1 + \sup_{s\le t}\|\mathbf{u}(s)\|_2) + \int_0^t (t-s)^{\alpha-1}\,\|\boldsymbol{\eta}_{\mathbf{u}}(s)\|_0\,\mathrm{d}s.\right) \end{aligned} \tag{48}$$

Combining (47) and (48), we get

$$\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|_0 \lesssim h^2(\sup_{s\le t}\|\boldsymbol{\sigma}(s)\|_1 + \sup_{s\le t}\|\mathbf{u}(s)\|_2) + \int_0^t (t-s)^{\alpha-1}\,\|\boldsymbol{\eta}_{\mathbf{u}}(s)\|_0\,\mathrm{d}s.$$

Finally, apply Lemma 2.3 to obtain

$$\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|_0 \lesssim h^2(\sup_{s\le t}\|\boldsymbol{\sigma}(s)\|_1 + \sup_{s\le t}\|\mathbf{u}(s)\|_2),$$

and this finishes the proof of (36).

■

The following lemma gives estimates for time derivatives of the Ritz-Volterra projection errors.

Lemma 3.5. *If* $\mathbf{u}_t(t) \in L^1([0,T]; (H^2(\Omega))^2), \boldsymbol{\sigma}_t(t) \in L^1([0,T]; (H^1(\Omega))^{2\times 2})$, *then*

$$\begin{aligned} &\int_0^t \left(\|\boldsymbol{\sigma}_t(t) - \tilde{\boldsymbol{\sigma}}_{h,t}(t)\|_0 + \|\mathbf{u}_t(t) - \tilde{\mathbf{u}}_{h,t}(t)\|_1\right)\mathrm{d}s \\ \lesssim &h(\int_0^t (\|\boldsymbol{\sigma}_t\|_1 + \|\mathbf{u}_t\|_2)\,\mathrm{d}s + \|\boldsymbol{\sigma}(0)\|_1 + \|\varphi_0\|_2), \end{aligned}$$

$$\int_0^t \|\mathbf{u}_t(t) - \tilde{\mathbf{u}}_{h,t}(t)\|_0\,\mathrm{d}s \lesssim h^2(\int_0^t (\|\boldsymbol{\sigma}_t\|_1 + \|\mathbf{u}_t\|_2)\,\mathrm{d}s + \|\boldsymbol{\sigma}(0)\|_1 + \|\varphi_0\|_2).$$

Proof. First, differentiate (40) with respect to t to obtain

$$\begin{aligned} &a(\boldsymbol{\rho}_{\boldsymbol{\sigma},t}, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{\rho}_{\mathbf{u},t}) - K(t)b(\boldsymbol{\tau}_h, \boldsymbol{\rho}_{\mathbf{u}}(0)) - \int_0^t K(t-s)b(\boldsymbol{\tau}_h, \boldsymbol{\rho}_{\mathbf{u},s}(s))\mathrm{d}s \\ &= a(\boldsymbol{\theta}_{\boldsymbol{\sigma},t}, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{\theta}_{\mathbf{u},t}) - K(t)b(\boldsymbol{\tau}_h, \boldsymbol{\theta}_{\mathbf{u}}(0)) - \int_0^t K(t-s)b(\boldsymbol{\tau}_h, \boldsymbol{\theta}_{\mathbf{u},s}(s))\mathrm{d}s. \end{aligned} \tag{49}$$

Apply Lemma 2.1 and (5)- (8) to arrive at

$$\left\|\boldsymbol{\rho}_{\boldsymbol{\sigma},t}\right\|_0 \lesssim \|\boldsymbol{\theta}_{\boldsymbol{\sigma},t}\|_0 + |\boldsymbol{\theta}_{\mathbf{u},t}|_1 + t^{\alpha-1}\,|\boldsymbol{\eta}_{\mathbf{u}}(0)|_1 + \int_0^t K(t-s)\,|\boldsymbol{\theta}_{\mathbf{u},s}(s)|_1\,\mathrm{d}s.$$

Thus,

$$\begin{aligned}\left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 \lesssim & \|\boldsymbol{\theta}_{\boldsymbol{\sigma},t}\|_0 + |\boldsymbol{\theta}_{\mathbf{u},t}|_1 + t^{\alpha-1}\,|\boldsymbol{\eta}_{\mathbf{u}}(0)|_1 + \int_0^t K(t-s)\,|\boldsymbol{\theta}_{\mathbf{u},s}(s)|_1\,\mathrm{d}s \\ \lesssim & h(\|\boldsymbol{\sigma}_t(t)\|_1 + \|\mathbf{u}_t(t)\|_2 + t^{\alpha-1}\,\|\boldsymbol{\sigma}(0)\|_1 + t^{\alpha-1}\,\|\varphi_0\|_2) \\ & + h\int_0^t K(t-s)\,\|\mathbf{u}_s(s)\|_2\,\mathrm{d}s,\end{aligned}$$

which implies

$$\begin{aligned}\int_0^t \left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 \mathrm{d}s \lesssim & h(\int_0^t \|\boldsymbol{\sigma}_t\|_1\,\mathrm{d}s + \int_0^t \|\mathbf{u}_t\|_2\,\mathrm{d}s + \|\boldsymbol{\sigma}(0)\|_1 + \|\varphi_0\|_2) \\ & + h\int_0^t\int_s^t K(r-s)\,\|\mathbf{u}_s(s)\|_2\,\mathrm{d}r\mathrm{d}s \\ \lesssim & h(\int_0^t \|\boldsymbol{\sigma}_t\|_1\,\mathrm{d}s + \int_0^t \|\mathbf{u}_t\|_2\,\mathrm{d}s + \|\boldsymbol{\sigma}(0)\|_1 + \|\varphi_0\|_2).\end{aligned}$$

In order to estimate $\left\|\boldsymbol{\eta}_{\mathbf{u},t}(t)\right\|_1$, take $\mathbf{v}_h = \boldsymbol{\rho}_{\mathbf{u},t}(t)$ in Lemma 3.1, then apply (49), (7)- (8), Lemma 2.1, the approximation property of projection I_h and the estimate of $\boldsymbol{\eta}_{\boldsymbol{\sigma},t}$ to get

$$\left\|\boldsymbol{\rho}_{\mathbf{u},t}(t)\right\|_1 \lesssim \left|\boldsymbol{\rho}_{\mathbf{u},t}(t)\right|_1 \lesssim \sup_{0\neq\boldsymbol{\tau}_h\in\Sigma_h} \frac{-b(\boldsymbol{\tau}_h,\boldsymbol{\rho}_{\mathbf{u},t})}{\|\boldsymbol{\tau}_h\|_0},$$

where

$$\begin{aligned}-b(\boldsymbol{\tau}_h,\boldsymbol{\rho}_{\mathbf{u},t}) = & K(t)b(\boldsymbol{\tau}_h,\boldsymbol{\eta}_{\mathbf{u}}(0)) - a(\boldsymbol{\eta}_{\boldsymbol{\sigma},t},\boldsymbol{\tau}_h) - b(\boldsymbol{\tau}_h,\boldsymbol{\theta}_{\mathbf{u},t}) \\ & + \int_0^t K(t-s)b(\boldsymbol{\tau}_h,\boldsymbol{\theta}_{\mathbf{u},s}(s))\mathrm{d}s - \int_0^t K(t-s)b(\boldsymbol{\tau}_h,\boldsymbol{\rho}_{\mathbf{u},s}(s))\mathrm{d}s \\ \lesssim & \|\boldsymbol{\tau}_h\|_0\,(t^{\alpha-1}\,|\boldsymbol{\eta}_{\mathbf{u}}(0)|_1 + \left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 + |\boldsymbol{\theta}_{\mathbf{u},t}|_1) \\ & + \|\boldsymbol{\tau}_h\|_0\,(\int_0^t K(t-s)\,|\boldsymbol{\theta}_{\mathbf{u},s}(s)|_1\,\mathrm{d}s + \int_0^t K(t-s)\,\left|\boldsymbol{\rho}_{\mathbf{u},s}(s)\right|_1\mathrm{d}s),\end{aligned}$$

which indicates

$$\begin{aligned}\left\|\boldsymbol{\rho}_{\mathbf{u},t}(t)\right\|_1 \lesssim & t^{\alpha-1}\,|\boldsymbol{\eta}_{\mathbf{u}}(0)|_1 + \left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 + h\,\|\mathbf{u}_t\|_2 + h\int_0^t (t-s)^{\alpha-1}\,\|\mathbf{u}_s(s)\|_2\,\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1}\,\left|\boldsymbol{\rho}_{\mathbf{u},s}(s)\right|_1\mathrm{d}s.\end{aligned}$$

Then, using Lemma 2.3 and the triangle inequality, we have

$$\begin{aligned}\left\|\boldsymbol{\eta}_{\mathbf{u},t}(t)\right\|_1 \lesssim & t^{\alpha-1}\,|\boldsymbol{\eta}_{\mathbf{u}}(0)|_1 + \left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 + h\,\|\mathbf{u}_t(t)\|_2 + h\int_0^t (t-s)^{\alpha-1}\,\|\mathbf{u}_s(s)\|_2\,\mathrm{d}s \\ & + h\int_0^t (t-s)^{2\alpha-1}\,\|\mathbf{u}_s(s)\|_2\,\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}\,\left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}(s)\right\|_0\mathrm{d}s,\end{aligned}$$

and finally

$$\begin{aligned}
&\int_0^t \left\|\boldsymbol{\eta}_{\mathbf{u},t}\right\|_1 \mathrm{d}s \\
\lesssim &\left|\boldsymbol{\eta}_{\mathbf{u}}(0)\right|_1 + \int_0^t \left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 \mathrm{d}s + h\int_0^t \left\|\mathbf{u}_t\right\|_2 \mathrm{d}s + h\int_0^t \int_0^r (r-s)^{\alpha-1}\left\|\mathbf{u}_s(s)\right\|_2 \mathrm{d}s\mathrm{d}r \\
&+ h\int_0^t \int_0^r (r-s)^{2\alpha-1}\left\|\mathbf{u}_s(s)\right\|_2 \mathrm{d}s\mathrm{d}r + \int_0^t \int_0^r (r-s)^{\alpha-1}\left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}(s)\right\|_0 \mathrm{d}s\mathrm{d}r \\
\lesssim &h(\left\|\boldsymbol{\sigma}(0)\right\|_1 + \left\|\varphi_0\right\|_2 + \int_0^t \left\|\boldsymbol{\sigma}_t\right\|_1 \mathrm{d}s + \int_0^t \left\|\mathbf{u}_t\right\|_2 \mathrm{d}s).
\end{aligned}$$

In order to show the estimate $\left\|\boldsymbol{\eta}_{\mathbf{u},t}\right\|_0$. Following the previous steps for deriving the estimate of $\left\|\boldsymbol{\eta}_{\mathbf{u}}\right\|_0$, we write, with the notation of Lemma 3.4 and using (49),

$$\begin{aligned}
\left\|\boldsymbol{\eta}_{\mathbf{u},t}(t)\right\|_0^2 &= -b(\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\mathbf{u},t}(t)) - a(\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\boldsymbol{\sigma},t}(t)) - b(\boldsymbol{\eta}_{\boldsymbol{\sigma},t}(t), \boldsymbol{\phi}(t)) \\
&= -b(\boldsymbol{\psi} - P_h\boldsymbol{\psi}, \boldsymbol{\eta}_{\mathbf{u},t}) - a(\boldsymbol{\psi} - P_h\boldsymbol{\psi}, \boldsymbol{\eta}_{\boldsymbol{\sigma},t}) - b(\boldsymbol{\eta}_{\boldsymbol{\sigma},t}, \boldsymbol{\phi} - I_h\boldsymbol{\phi}) \\
&\quad - \int_0^t K(t-s) b(P_h\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\mathbf{u},t}(s))\mathrm{d}s - K(t)b(P_h\boldsymbol{\psi}(t), \boldsymbol{\eta}_{\mathbf{u}}(0)).
\end{aligned}$$

Using (7)- (8) shows

$$\begin{aligned}
&\left\|\boldsymbol{\eta}_{\mathbf{u},t}(t)\right\|_0^2 \\
\lesssim &h\left\|\boldsymbol{\psi}\right\|_1 (\left|\boldsymbol{\eta}_{\mathbf{u},t}\right|_1 + \left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 + t^{\alpha-1}\left|\boldsymbol{\eta}_{\mathbf{u}}(0)\right|_1 + \int_0^t K(t-s)\left|\boldsymbol{\eta}_{\mathbf{u},t}(s)\right|_1 \mathrm{d}s) \\
&+ h\left\|\boldsymbol{\phi}\right\|_2 \left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 + t^{\alpha-1}\left\|\mathrm{div}\boldsymbol{\psi}\right\|_0 \left\|\boldsymbol{\eta}_{\mathbf{u}}(0)\right\|_0 + \left\|\mathrm{div}\boldsymbol{\psi}\right\|_0 \int_0^t K(t-s)\left\|\boldsymbol{\eta}_{\mathbf{u},t}(s)\right\|_0 \mathrm{d}s,
\end{aligned}$$

which, together with the elliptic regularity and Lemma 2.3 at the appropriate steps, leads to

$$\begin{aligned}
\left\|\boldsymbol{\eta}_{\mathbf{u},t}(t)\right\|_0 &\lesssim h(\left|\boldsymbol{\eta}_{\mathbf{u},t}\right|_1 + \left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 + t^{\alpha-1}\left|\boldsymbol{\eta}_{\mathbf{u}}(0)\right|_1 + \int_0^t (t-s)^{\alpha-1}\left|\boldsymbol{\eta}_{\mathbf{u},t}(s)\right|_1 \mathrm{d}s) \\
&\quad + t^{\alpha-1}\left\|\boldsymbol{\eta}_{\mathbf{u}}(0)\right\|_0 + \int_0^t (t-s)^{\alpha-1}\left\|\boldsymbol{\eta}_{\mathbf{u},t}(s)\right\|_0 \mathrm{d}s \\
&\lesssim h(\left|\boldsymbol{\eta}_{\mathbf{u},t}\right|_1 + \left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}\right\|_0 + t^{\alpha-1}\left|\boldsymbol{\eta}_{\mathbf{u}}(0)\right|_1 + \int_0^t (t-s)^{\alpha-1}\left|\boldsymbol{\eta}_{\mathbf{u},t}(s)\right|_1 \mathrm{d}s) \\
&\quad + t^{\alpha-1}\left\|\boldsymbol{\eta}_{\mathbf{u}}(0)\right\|_0 + h\int_0^t (t-s)^{\alpha-1}\left\|\boldsymbol{\eta}_{\boldsymbol{\sigma},t}(s)\right\|_0 \mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1}\int_0^s (s-r)^{\alpha-1}\left|\boldsymbol{\eta}_{\mathbf{u},t}(r)\right|_1 \mathrm{d}r\mathrm{d}s.
\end{aligned}$$

As a result,

$$\int_0^t \left\|\boldsymbol{\eta}_{\mathbf{u},t}\right\|_0 \mathrm{d}s \lesssim h^2(\left\|\boldsymbol{\sigma}(0)\right\|_1 + \left\|\varphi_0\right\|_2 + \int_0^t \left\|\boldsymbol{\sigma}_t\right\|_1 \mathrm{d}s + \int_0^t \left\|\mathbf{u}_t\right\|_2 \mathrm{d}s),$$

i.e. the desired result follows. ■

Proof of Theorem 3.3. Using the mixed Ritz-Volterra projection, we rewrite

$$\begin{aligned}
\mathbf{u} - \mathbf{u}_h &= (\mathbf{u} - \tilde{\mathbf{u}}_h) - (\mathbf{u}_h - \tilde{\mathbf{u}}_h) =: \boldsymbol{\eta}_{\mathbf{u}} - \boldsymbol{\zeta}_{\mathbf{u}}, \\
\boldsymbol{\sigma} - \boldsymbol{\sigma}_h &= (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h) - (\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}_h) =: \boldsymbol{\eta}_{\boldsymbol{\sigma}} - \boldsymbol{\zeta}_{\boldsymbol{\sigma}},
\end{aligned}$$

where $(\boldsymbol{\sigma}, \mathbf{u})$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ are solutions of (4) and (9), respectively. Note that $(\boldsymbol{\zeta}_{\boldsymbol{\sigma}}, \boldsymbol{\zeta}_{\mathbf{u}})$ satisfies the following equations

$$a(\boldsymbol{\zeta}_{\boldsymbol{\sigma}}(t), \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{\zeta}_{\mathbf{u}}(t)) - \int_0^t K(s) b(\boldsymbol{\tau}_h, \boldsymbol{\zeta}_{\mathbf{u}}(t-s))\mathrm{d}s = 0, \quad \forall \boldsymbol{\tau}_h \in \Sigma_h, \tag{50a}$$

$$b(\boldsymbol{\zeta}_{\boldsymbol{\sigma}}(t), \mathbf{v}_h) = \rho(\boldsymbol{\zeta}_{\mathbf{u},tt}(t) - \boldsymbol{\eta}_{\mathbf{u},tt}(t), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h. \tag{50b}$$

After integrating (50b) with respect to t, we obtain the following system:

$$a(\boldsymbol{\zeta}_{\boldsymbol{\sigma}}(t), \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{\zeta}_{\mathbf{u}}(t)) - \int_0^t K(t-s) b(\boldsymbol{\tau}_h, \boldsymbol{\zeta}_{\mathbf{u}}(s))\mathrm{d}s = 0, \quad \forall \boldsymbol{\tau}_h \in \Sigma_h, \tag{51a}$$

$$b(\bar{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}(t), \mathbf{v}_h) = \rho(\boldsymbol{\zeta}_{\mathbf{u},t}(t), \mathbf{v}_h) - \rho(\boldsymbol{\eta}_{\mathbf{u},t}(t), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h, \tag{51b}$$

where $\bar{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}(t) := \int_0^t \boldsymbol{\zeta}_{\boldsymbol{\sigma}}(s)\mathrm{d}s$, and we have used in (51b) the fact that $\varphi_{1,h} = Q_h\varphi_1$. Now take $\boldsymbol{\tau}_h = \bar{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}(t), \mathbf{v}_h = -\boldsymbol{\zeta}_{\mathbf{u}}(t)$ and $\mathbf{v}_h = -\boldsymbol{\zeta}_{\mathbf{u}}(s)$ respectively in (51a), (51b) and (51b), and add the three resulting equations, we then get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\left\|\bar{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}(t)\right\|_a^2 + \rho\left\|\boldsymbol{\zeta}_{\mathbf{u}}(t)\right\|_0^2) - \rho\int_0^t K(t-s)(\boldsymbol{\zeta}_{\mathbf{u},t}(t), \boldsymbol{\zeta}_{\mathbf{u}}(s))\mathrm{d}s \\
=&\rho(\boldsymbol{\eta}_{\mathbf{u},t}(t), \boldsymbol{\zeta}_{\mathbf{u}}(t)) - \rho\int_0^t K(t-s)(\boldsymbol{\eta}_{\mathbf{u},t}(t), \boldsymbol{\zeta}_{\mathbf{u}}(s))\mathrm{d}s.
\end{aligned}$$

Integrate from 0 to t to deduce

$$\begin{aligned}
&\rho^{-1}\left\|\bar{\boldsymbol{\zeta}}_{\boldsymbol{\sigma}}(t)\right\|_a^2 + \left\|\boldsymbol{\zeta}_{\mathbf{u}}(t)\right\|_0^2 - 2\int_0^t\int_0^r K(r-s)(\boldsymbol{\zeta}_{\mathbf{u},t}(r), \boldsymbol{\zeta}_{\mathbf{u}}(s))\mathrm{d}s\mathrm{d}r \\
=&\left\|\boldsymbol{\zeta}_{\mathbf{u}}(0)\right\|_0^2 + \int_0^t 2(\boldsymbol{\eta}_{\mathbf{u},t}, \boldsymbol{\zeta}_{\mathbf{u}})\mathrm{d}s - 2\int_0^t\int_0^r K(r-s)(\boldsymbol{\eta}_{\mathbf{u},t}(r), \boldsymbol{\zeta}_{\mathbf{u}}(s))\mathrm{d}s\mathrm{d}r.
\end{aligned} \tag{52}$$

Integration by parts yields

$$\begin{aligned}
&-2\int_0^t\int_0^r K(r-s)(\boldsymbol{\zeta}_{\mathbf{u},t}(r), \boldsymbol{\zeta}_{\mathbf{u}}(s))\mathrm{d}s\mathrm{d}r \\
=&-2\int_0^t\int_0^r \frac{d}{ds}\kappa(r-s)(\boldsymbol{\zeta}_{\mathbf{u},t}(r), \boldsymbol{\zeta}_{\mathbf{u}}(s))\mathrm{d}s\mathrm{d}r \\
=&-\nu\left\|\boldsymbol{\zeta}_{\mathbf{u}}(t)\right\|_0^2 + 2\kappa(t)(\boldsymbol{\zeta}_{\mathbf{u}}(t), \boldsymbol{\zeta}_{\mathbf{u}}(0)) - 2\kappa(0)\|\boldsymbol{\zeta}_{\mathbf{u}}(0)\|_0^2 + 2\int_0^t K(s)(\boldsymbol{\zeta}_{\mathbf{u}}(s), \boldsymbol{\zeta}_{\mathbf{u}}(0))\mathrm{d}s \\
&+2\int_0^t\int_0^r \kappa(r-s)(\boldsymbol{\zeta}_{\mathbf{u},t}(r), \boldsymbol{\zeta}_{\mathbf{u},t}(s))\mathrm{d}s\mathrm{d}r,
\end{aligned}$$

which, together with (52), the Cauchy-Schwarz inequality, the Young inequality, Lemma 2.2 and (23), leads to

$$\begin{aligned}
&(1-\nu)\left\|\boldsymbol{\zeta}_{\mathbf{u}}(t)\right\|_0^2 \\
\leq&\frac{\nu}{q}\left\|\boldsymbol{\zeta}_{\mathbf{u}}(t)\right\|_0^2 + (1+2\nu+q\nu)\left\|\boldsymbol{\zeta}_{\mathbf{u}}(0)\right\|_0^2 + 2\int_0^t \left\|\boldsymbol{\eta}_{\mathbf{u},t}\right\|_0\left\|\boldsymbol{\zeta}_{\mathbf{u}}\right\|_0\mathrm{d}s \\
&+2\int_0^t K(s)\left\|\boldsymbol{\zeta}_{\mathbf{u}}(s)\right\|_0\left\|\boldsymbol{\zeta}_{\mathbf{u}}(0)\right\|_0\mathrm{d}s + 2\int_0^t\left\|\boldsymbol{\eta}_{\mathbf{u},t}(r)\right\|_0\int_0^r K(r-s)\left\|\boldsymbol{\zeta}_{\mathbf{u}}(s)\right\|_0\mathrm{d}s\mathrm{d}r
\end{aligned}$$

with the constant $q > \frac{\nu}{1-\nu} > 0$. Thus,

$$\begin{aligned}
&(1-\nu-\frac{\nu}{q})\,\|\boldsymbol{\zeta}_{\mathbf{u}}(t)\|_0^2 \\
\leq &(1+2\nu+q\nu)\,\|\boldsymbol{\zeta}_{\mathbf{u}}(0)\|_0^2 + 2\int_0^t K(s)\,\|\boldsymbol{\zeta}_{\mathbf{u}}(s)\|_0\,\|\boldsymbol{\zeta}_{\mathbf{u}}(0)\|_0\,\mathrm{d}s \\
&+ 2\int_0^t \|\boldsymbol{\eta}_{\mathbf{u},t}\|_0\,\|\boldsymbol{\zeta}_{\mathbf{u}}\|_0\,\mathrm{d}s + 2\int_0^t \|\boldsymbol{\eta}_{\mathbf{u},t}(r)\|_0 \int_0^r K(r-s)\,\|\boldsymbol{\zeta}_{\mathbf{u}}(s)\|_0\,\mathrm{d}s\mathrm{d}r.
\end{aligned}$$

Take $t = t^*$ in the above inequality, with $t^* \in [0, t]$ satisfying

$$\|\boldsymbol{\zeta}_{\mathbf{u}}(t^*)\|_0 = \sup_{0\leq s\leq t} \|\boldsymbol{\zeta}_{\mathbf{u}}(s)\|_0\,,$$

and we obtain

$$(1-\nu-\frac{\nu}{q})\,\|\boldsymbol{\zeta}_{\mathbf{u}}(t^*)\|_0 \leq (2+2\nu)\int_0^{t^*} \|\boldsymbol{\eta}_{\mathbf{u},t}\|_0\,\mathrm{d}s + (1+4\nu+q\nu)\,\|\boldsymbol{\zeta}_{\mathbf{u}}(0)\|_0\,.$$

Therefore,

$$\|\boldsymbol{\zeta}_{\mathbf{u}}(t)\|_0 \leq \|\boldsymbol{\zeta}_{\mathbf{u}}(t^*)\|_0 \lesssim \int_0^t \|\boldsymbol{\eta}_{\mathbf{u},t}\|_0\,\mathrm{d}s + \|\boldsymbol{\zeta}_{\mathbf{u}}(0)\|_0\,.$$

Finally, a use of the triangle inequality with Lemmas 3.4 and 3.5 concludes the proof of Theorem 3.3. ■

4. Numerical tests

In this section, we first give a fully discrete scheme based on the semi-discrete hybrid stress quadrilateral finite element scheme (9), then provide some numerical results to verify the spatial accuracy of the proposed method.

4.1. A full discretization.

Let M be a positive integer, and set $\Delta t := \frac{T}{M}$. Take $t_i = i\Delta t$ for $i = 0, 1, 2, \ldots, M$ and $t_{i+\frac{1}{2}} := \frac{t_i + t_{i+1}}{2}$ for $i = 0, 1, 2, \ldots, M-1$. For any function ϕ of t, denote

$$\phi^n := \phi(t_n), \qquad \phi^{n+\frac{1}{2}} := \frac{\phi^n + \phi^{n+1}}{2}, \qquad \phi^{n;\frac{1}{4}} := \frac{\phi^{n+1} + 2\phi^n + \phi^{n-1}}{4},$$

$$\Delta_t \phi^n := \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t}, \quad \Delta_t \phi^{n+\frac{1}{2}} := \frac{\phi^{n+1} - \phi^n}{\Delta t}, \quad \Delta_t^2 \phi^n := \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{(\Delta t)^2}.$$

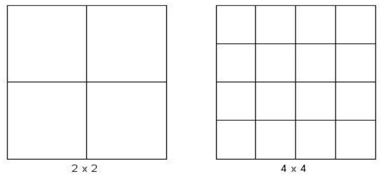
Since the integral term $\int_0^t K(t-s)b(\boldsymbol{\tau}_h, \mathbf{u}_h(s))\mathrm{d}s$ in (9) has the end point singularity, we apply the Navots quadrature formula [26, 27] for it:

$$J_n(\phi) = \sum_{j=0}^{n} \omega_{nj} k(t_n - t_j)\phi(t_j) \approx \int_0^{t_n} (t_n - s)^{\alpha-1} k(t_n - s)\phi(s)\mathrm{d}s, \tag{53}$$

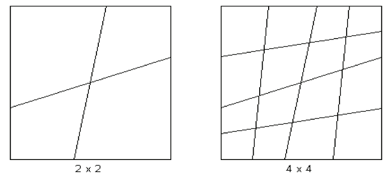
where

$$\omega_{n0} = \frac{\Delta t}{2}(t_n - t_0)^{\alpha-1},\ \omega_{nn} = -(\Delta t)^{\alpha}\zeta(1-\alpha),\ \omega_{nj} = \Delta t(t_n - t_j)^{\alpha-1},\ j = 1, \cdots, n-1,$$

and $\zeta(x)$ is the Riemann Zeta function.



(a) Square meshes



(b) Irregular meshes

FIGURE 2. Finite element meshes of the spatial domain Ω.

TABLE 1. Numerical results with $\Delta t = 0.005$: Example 4.1 at square meshes.

	$m \times m$	$\alpha = 0.1$	order	$\alpha = 0.5$	order	$\alpha = 0.8$	order
$E^0_{\boldsymbol{u}}$	2×2	$1.6951e-01$	$-$	$1.6953e-01$	$-$	$1.6954e-01$	$-$
	4×4	$3.5000e-02$	2.28	$3.5045e-02$	2.27	$3.5056e-02$	2.27
	8×8	$8.3925e-03$	2.06	$8.4523e-03$	2.05	$8.4671e-03$	2.05
	16×16	$2.1531e-03$	1.96	$2.1924e-03$	1.95	$2.2021e-03$	1.94
	32×32	$5.3469e-04$	2.01	$5.4054e-04$	2.02	$5.4218e-04$	2.02
$E^1_{\boldsymbol{u}}$	2×2	$4.8821e-01$	$-$	$4.8829e-01$	$-$	$4.8832e-01$	$-$
	4×4	$2.3148e-01$	1.08	$2.3117e-01$	1.08	$2.3175e-01$	1.08
	8×8	$1.1379e-01$	1.02	$1.1408e-01$	1.02	$1.1415e-01$	1.02
	16×16	$5.5885e-02$	1.03	$5.6145e-02$	1.02	$5.6210e-02$	1.02
	32×32	$2.5179e-02$	1.15	$2.5266e-02$	1.15	$2.5290e-02$	1.15
$E^0_{\boldsymbol{\sigma}}$	2×2	$5.1353e-01$	$-$	$5.1302e-01$	$-$	$5.1294e-01$	$-$
	4×4	$2.5396e-01$	1.02	$2.5348e-01$	1.02	$2.5341e-01$	1.02
	8×8	$1.2516e-01$	1.02	$1.2485e-01$	1.02	$1.2480e-01$	1.02
	16×16	$6.0875e-02$	1.04	$6.0785e-02$	1.04	$6.0774e-02$	1.04
	32×32	$2.7306e-02$	1.16	$2.7266e-02$	1.16	$2.7261e-02$	1.16

The corresponding fully discrete hybrid stress finite element scheme for the problem (4) reads: For $n = 0, 1, \cdots, M$, find $(\boldsymbol{\sigma}_h^n, \mathbf{u}_h^n) \in \Sigma_h \times V_h$ such that

$$\begin{cases} \rho(\dfrac{2}{\Delta t}\Delta_t \mathbf{u}_h^{\frac{1}{2}}, \mathbf{v}_h) - b(\boldsymbol{\sigma}_h^{\frac{1}{2}}, \mathbf{v}_h) = (\mathbf{f}^{\frac{1}{2}}, \mathbf{v}_h) + (\dfrac{2\varphi_1}{\Delta t}, \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h, \\ a(\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{u}_h^n) - b(\boldsymbol{\tau}_h, J_n(\mathbf{u}_h)) = 0, & \forall \boldsymbol{\tau}_h \in \Sigma_h,\ n \geq 0, \\ \rho(\Delta_t^2 \mathbf{u}_h^n, \mathbf{v}_h) - b(\boldsymbol{\sigma}_h^{n;\frac{1}{4}}, \mathbf{v}_h) = (\mathbf{f}^{n;\frac{1}{4}}, \mathbf{v}_h), & \forall \mathbf{v}_h \in V_h,\ n \geq 1, \\ \mathbf{u}_h^0 = \varphi_{0,h}. \end{cases} \tag{54}$$

Remark 4.1. *We mention that the same implicit second-order temporal discretization as in the scheme above has been used in [44] to obtain a fully discrete hybrid stress method for 2-dimensional linear elastodynamic problems.*

4.2. Numerical results. We consider the problem (1) with $\bar{\Omega} = [0,1] \times [0,1]$, $T = 1$, $\nu = 0.5, \tau = 10, \rho = 1000, \lambda = 1$, $\mu = 2$ and $\alpha = 0.1, 0.5, 0.8$, and give two numerical examples. Figure 2 shows $m \times m$ square meshes and irregular meshes of Ω for $m = 1, 2$. To investigate the spatial accuracy of the proposed method we compute the scheme (54) at a fine time mesh with $\Delta t = 0.005$.

Example 4.1. *The initial displacement and velocity are respectively given by*

$$\varphi_0 = \mathbf{0}, \qquad \varphi_1 = (-\sin(\pi x)\sin(\pi y), -\sin(\pi x)\sin(\pi y))^T,$$

TABLE 2. Numerical results with $\Delta t = 0.005$: Example 4.1 at irregular meshes.

	$m \times m$	$\alpha = 0.1$	order	$\alpha = 0.5$	order	$\alpha = 0.8$	order
$E_{\boldsymbol{u}}^0$	2×2	$1.9495e-01$	–	$1.9466e-01$	–	$1.9459e-01$	–
	4×4	$4.0467e-02$	2.27	$4.0369e-02$	2.27	$4.0349e-02$	2.27
	8×8	$9.3777e-03$	2.11	$9.3819e-03$	2.11	$9.3844e-03$	2.10
	16×16	$2.3507e-03$	2.00	$2.3778e-03$	1.98	$2.3848e-03$	1.98
	32×32	$5.9230e-04$	1.99	$5.9687e-04$	1.99	$5.9819e-04$	2.00
$E_{\boldsymbol{u}}^1$	2×2	$4.9868e-01$	–	$4.9835e-01$	–	$4.9827e-01$	–
	4×4	$2.4356e-01$	1.03	$2.4330e-01$	1.03	$2.4324e-01$	1.03
	8×8	$1.1873e-01$	1.04	$1.1872e-01$	1.04	$1.1873e-01$	1.03
	16×16	$5.7982e-02$	1.03	$5.8115e-02$	1.03	$5.8151e-02$	1.03
	32×32	$2.6110e-02$	1.15	$2.6157e-02$	1.15	$2.6171e-02$	1.15
$E_{\boldsymbol{\sigma}}^0$	2×2	$5.4327e-01$	–	$5.4268e-01$	–	$5.4259e-01$	–
	4×4	$2.6904e-01$	1.01	$2.6838e-01$	1.02	$2.6828e-01$	1.02
	8×8	$1.3183e-01$	1.03	$1.3140e-01$	1.03	$1.3134e-01$	1.03
	16×16	$6.3934e-02$	1.04	$6.3791e-02$	1.04	$6.3772e-02$	1.04
	32×32	$2.8657e-02$	1.16	$2.8605e-02$	1.16	$2.8598e-02$	1.16

TABLE 3. Numerical results with $\Delta t = 0.005$: Example 4.2 at square meshes.

	$m \times m$	$\alpha = 0.1$	order	$\alpha = 0.5$	order	$\alpha = 0.8$	order
$E_{\boldsymbol{u}}^0$	4×4	$2.0947e-01$	–	$2.0824e-01$	–	$2.0800e-01$	–
	8×8	$4.2435e-02$	2.30	$4.1621e-02$	2.32	$4.1460e-02$	2.33
	16×16	$8.4842e-03$	2.32	$8.2846e-03$	2.33	$8.2486e-03$	2.33
	32×32	$1.8741e-03$	2.18	$1.8612e-03$	2.15	$1.8594e-03$	2.15
$E_{\boldsymbol{u}}^1$	4×4	$4.9362e-01$	–	$4.9116e-01$	–	$4.9067e-01$	–
	8×8	$2.3758e-01$	1.06	$2.3464e-01$	1.06	$2.3407e-01$	1.07
	16×16	$1.1013e-01$	1.11	$1.0867e-01$	1.11	$1.0840e-01$	1.11
	32×32	$4.8630e-02$	1.18	$4.8247e-02$	1.17	$4.8180e-02$	1.17
$E_{\boldsymbol{\sigma}}^0$	4×4	$5.3569e-01$	–	$5.3283e-01$	–	$5.3238e-01$	–
	8×8	$2.6617e-01$	1.01	$2.6288e-01$	1.02	$2.6238e-01$	1.02
	16×16	$1.2535e-01$	1.09	$1.2384e-01$	1.09	$1.2363e-01$	1.09
	32×32	$5.5765e-02$	1.17	$5.5294e-02$	1.16	$5.5225e-02$	1.16

and the body force $\mathbf{f} = \mathbf{0}$. *Numerical results are listed in Tables 1 and 2.*

Example 4.2. *The initial displacement and velocity are respectively given by* $\varphi_0 = \mathbf{0}, \varphi_1 = \mathbf{0}$, *and the body force*

$$\mathbf{f}(x,y,t) = \left(-(x^2-x)^2(4y^3-6y^2+2y)e^{-t}, -(y^2-y)^2(4x^3-6x^2+2x)e^{-t}\right)^T.$$

Numerical results are listed in Tables 3 and 4.

We compute the following relative errors for the displacement and stress approximations:

$$E_{\boldsymbol{u}}^0 = \frac{\|\tilde{\boldsymbol{u}}(T) - \boldsymbol{u}_h(T)\|_0}{\|\tilde{\boldsymbol{u}}(T)\|_0}, \quad E_{\boldsymbol{u}}^1 = \frac{|\tilde{\boldsymbol{u}}(T) - \boldsymbol{u}_h(T)|_1}{|\tilde{\boldsymbol{u}}(T)|_1}, \quad E_{\boldsymbol{\sigma}}^0 = \frac{\|\tilde{\boldsymbol{\sigma}}(T) - \boldsymbol{\sigma}_h(T)\|_0}{\|\tilde{\boldsymbol{\sigma}}(T)\|_0},$$

TABLE 4. Numerical results with $\Delta t = 0.005$: Example 4.2 at irregular meshes.

	$m \times m$	$\alpha = 0.1$	order	$\alpha = 0.5$	order	$\alpha = 0.8$	order
$E^0_{\boldsymbol{u}}$	4×4	2.2129e − 01	–	2.2012e − 01	–	2.1989e − 01	–
	8×8	4.4849e − 02	2.30	4.4048e − 02	2.32	4.3890e − 02	2.32
	16×16	8.9932e − 03	2.32	8.7861e − 03	2.33	8.7485e − 03	2.33
	32×32	1.9849e − 03	2.18	1.9715e − 03	2.16	1.9697e − 03	2.15
$E^1_{\boldsymbol{u}}$	4×4	5.1368e − 01	–	5.1132e − 01	–	5.1085e − 01	–
	8×8	2.4682e − 01	1.06	2.4402e − 01	1.07	2.4347e − 01	1.07
	16×16	1.1467e − 01	1.11	1.1321e − 01	1.11	1.1294e − 01	1.11
	32×32	5.0579e − 02	1.18	5.0202e − 02	1.17	5.0136e − 02	1.17
$E^0_{\boldsymbol{\sigma}}$	4×4	5.4168e − 01	–	5.3883e − 01	–	5.3839e − 01	–
	8×8	2.6873e − 01	1.01	2.6549e − 01	1.02	2.6500e − 01	1.02
	16×16	1.2659e − 01	1.09	1.2502e − 01	1.09	1.2480e − 01	1.09
	32×32	5.6236e − 02	1.17	5.5751e − 02	1.17	5.5681e − 02	1.16

where $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{\sigma}}$ are the referential solutions obtained by the full discretization (54) at a fine mesh with $h = 2^{-6}, \Delta t = 0.005$. From Tables 1, 2, 3 and 4 we have the following observations:

- For both of the examples with different choices of α, $E^0_{\boldsymbol{u}}$ is of $O(h^2)$ spatial convergence rate, and $E^1_{\boldsymbol{u}}, E^0_{\boldsymbol{\sigma}}$ are both of $O(h)$ spatial convergence rate.

These are conformable to the theoretical results in Theorems 3.2 and 3.3.

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