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DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR QUASI-STATIC POROELASTICITY PROBLEMS

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Abstract. This paper is devoted to a discontinuous Galerkin (DG) method for nonlinear quasi-static poroelasticity problems. The fully implicit nonlinear numerical scheme is constructed by utilizing DG method for the spatial approximation and the backward Euler method for the temporal discretization. The existence and uniqueness of the numerical solution is proved. Then we derive the optimal convergence order estimates in a discrete H^1 norm for the displacement and in H^1 and L^2 norms for the pressure. Finally, numerical experiments are supplied to validate the theoretical error estimates of our proposed method.

Key words. Nonlinear quasi-static poroelasticity problem, discontinuous Galerkin method, fully implicit nonlinear numerical scheme, optimal convergence order estimate.

1. Introduction

Poroelasticity refers to the movement of Darcy flow within a deformable and porous medium. It is widely used in many practical problems, such as materials science [34], biomechanics [30], and reservoir engineering [16]. Poroelasticity theory is also called Biot's consolidation model when the porous media is linear elastic, homogeneous, isotropic and saturated by incompressible Newtonian fluid. The original Biot's model can retrospect to the contribution of Terzaghi and Biot. Terzaghi [33] analyzes the one-dimensional case and then finds the relevant theory based on the consolidation of a soil column. Biot [7] generalizes Terzaghi's theory and research to the three-dimensional situation. Up to now, many complex mathematical models based on the Biot's model have been proposed and studied, including various nonlinear models, where, for example, the permeability is taken as a nonlinear function of the fluid content [18, 19, 17].

In this paper, we are concerned with a nonlinear quasi-static poroelasticity problem. The linear type of this problem (the permeability is a constant) is studied by Showalter [29] on the well-posedness of solutions of porous elastic systems. And various numerical methods have been applied to the linear model, such as finite volume method [23, 24], finite difference method [12], finite element method [22, 21]. The nonlinear model, where the permeability depending nonlinearly on the dilatation of the medium, is firstly introduced in [15] for the simulation of paper production. In [9], Cao et al. establish the variational formulation of the nonlinear model and firstly discuss the existence and uniqueness of the solution by the modified Rothe's method. In [8], Bociu et al. extend the theoretical results of [9] to the case of more general boundary conditions. Compared with the development of numerical methods of the linear model, the existing results mainly make use of finite element based methods for spatial discretization of the nonlinear model. Cao et al. [9] respectively adopt the linear conforming finite element approximation to the displacement and pressure of the nonlinear model, and derive optimal error estimates. Subsequently, they utilize the conforming finite element method [10] and the hybrid

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finite element method [11] for the static case of the nonlinear problem and obtain a priori error estimates. In [39], Zhang et al. investigate a variant formulation of the steady nonlinear problem, use weak Galerkin-finite element method for the spatial discretization of the displacement, pressure and volumetric stress and derive the optimal convergence order estimates.

DG method is one of the important numerical methods for solving partial differential equations. The method is first proposed by Reed and Hill [27] for solving the neutron transport equation. DG method solves the differential equations by piecewise polynomial functions over a finite element space without any requirement on inter-element continuity. Continuity on inter-element boundaries together with boundary conditions is weakly enforced through the bilinear form. DG method has many advantages that make it very attractive for practical numerical simulations, such as good mesh flexibility, local mass conservation, convenience for *hp*-adaptivity. At present, DG method has been widely used to solve various partial differential equations [3, 32, 25, 36, 37, 38, 20, 26]. For the nonlinear poroelasticity model, Wen et al. [35] consider the four-field mixed formulation by introducing two additional variables, construct a linearized fully discrete DG scheme and analyze a priori error esimates.

In this paper, we propose an interior penalty DG method for solving nonlinear quasi-static poroelasticity problems. It is well-known that the fully implicit nonlinear numerical scheme is stable and can preserve the physical properties of the original problem. Based on this, we consider the original two-field model and establish the fully implicit nonlinear DG scheme by the backward Euler method for time discretization, which is different from [35]. The existence and uniqueness of the numerical solution is proved and the optimal error estimates for the displacement and pressure are obtained. Finally, numerical experiments are given to verify the theoretical results of our proposed method.

The outline of this paper goes as follows. In Section 2, we present the nonlinear quasi-static poroelastic model and the corresponding variational formulation. The fully discrete DG scheme is provided in Section 3, and we prove the existence and uniqueness of the numerical solution. In Section 4, the optimal convergence order estimates for the DG scheme are derived. We supply numerical experiments to validate our theoretical findings in Section 5. And finally, some conclusions are made in Section 6.

2. Mathematical model and variational formulation

In this section, we firstly present the mathematical model of nonlinear quasi-static poroelasticity problems. Then we establish the corresponding variational formulation after introducing some necessary notations and definitions.

Let Ω be a convex polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\partial\Omega = \Gamma_{p,D} \cup \Gamma_{p,N}$ with $\Gamma_{p,D}$ nonempty, and $T > 0$ is the final time. In this paper, we concentrate on the following nonlinear quasi-static poroelasticity problems [9]: Seeking the displacement of porous solid media $\boldsymbol{u}(t) : \Omega \to \mathbb{R}^d$ and the pore pressure of fluid $p(t) : \Omega \to \mathbb{R}$, such that

$$-(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) - \mu\triangle\boldsymbol{u} + \alpha\nabla p = \boldsymbol{f}, \quad \text{in } \Omega,\ t \in (0, T], \tag{1}$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha\nabla \cdot \boldsymbol{u}) - \nabla \cdot (\kappa(\nabla \cdot \boldsymbol{u})\nabla p) = g, \quad \text{in } \Omega,\ t \in (0, T], \tag{2}$$

with the boundary conditions

$$\boldsymbol{u} = \boldsymbol{0}, \quad \text{on } \partial\Omega, \tag{3}$$

$$p = 0, \quad \text{on } \Gamma_{p,D}, \tag{4}$$

$$\kappa(\nabla \cdot \boldsymbol{u})\nabla p \cdot \boldsymbol{n} = \gamma, \quad \text{on } \Gamma_{p,N}, \tag{5}$$

and the initial conditions

$$\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}^0, \quad \text{in } \Omega, \tag{6}$$

$$p(\cdot, 0) = p^0, \quad \text{in } \Omega. \tag{7}$$

Here $\boldsymbol{f}(t) : \Omega \to \mathbb{R}^d$ is the body force, $g(t) : \Omega \to \mathbb{R}$ is the volumetric fluid source (or sink), and $\gamma(t) : \Omega \to \mathbb{R}$ denotes the prescribed discharge on the boundary. λ and μ are Lamé constants, α is the Biot-Willis parameter, and $c_0 \geq 0$ represents the constrained specific storage coefficient. κ denotes the hydraulic conductivity (permeability), which is related to the dilatation $\nabla \cdot \mathbf{u}$, that is, $\kappa = \kappa(\nabla \cdot \mathbf{u})$. $\boldsymbol{n}$ is the unit outward normal vector on $\Gamma_{p,N}$.

Assume that $\kappa(\cdot)$ is a continuous function and that there exist positive constants $\kappa_{\min}$ and $\kappa_{\max}$, such that

$$\kappa_{\min} \leq \kappa(x) \leq \kappa_{\max}, \quad \forall x \in \mathbb{R}.$$

In [9], the existence and uniqueness of the solution of the system (1)-(7) is derived, and especially for the proof of the uniqueness, the result is obtained based on the assumptions: The hydraulic conductivity κ is Lipschitz continuous with Lipschitz constant k_L, i.e.,

$$|\kappa(x) - \kappa(y)| \leq k_L |x - y|, \tag{8}$$

$\nabla p \in L^\infty(\Omega)$ and c_0 is strictly positive.

In what follows, for simplicity, we assume that $\alpha = 1$. If $\alpha \neq 1$, one may reduce the model to the one with $\alpha = 1$ by rescaling the equations. In addition, the letter C (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences throughout this paper.

Let $\mathcal{T}_h$ be a shape-regular triangulation of the domain Ω. Denote by h_K the partition diameter of the element $K \in \mathcal{T}_h$ and $h = \max\limits_{K \in \mathcal{T}_h} h_K$. Additionally, we denote by Γ the set of all edges in $\mathcal{T}_h$, and by e an edge of Γ. Because of the polygonal or polyhedral domain Ω, it is obtained that $\Gamma = \Gamma_I \cup \partial\Omega$, where Γ_I represents the set of all inner edges, and we denote $\Gamma_{I,D} = \Gamma_I \cup \Gamma_{p,D}$.

In this paper, we utilize the standard definition of Sobolev space $H^s(\Omega)$ with $s \geq 0$ (cf. [1]). The associated inner-product and norm in $H^s(\Omega)$ are denoted by $(\cdot,\cdot)_s$ and $\|\cdot\|_s$, respectively. When $s = 0$, $H^0(\Omega)$ coincides with the space of square-integrable functions $L^2(\Omega)$. In this case, the subscript s is suppressed from the notation of inner product and norm. Denote by $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ consisting of the functions with vanishing trace on $\partial\Omega$, and by $\|\cdot\|_\infty$ the norm on $L^\infty(\Omega)$.

Define the spaces $\mathbf{V}$ and Q for the displacement and pressure, respectively, by

$$\mathbf{V} = \{\mathbf{v} \in (L^2(\Omega))^d : \ \mathbf{v}|_K \in (H^1(K))^d, \ \forall K \in \mathcal{T}_h\},$$

$$Q = \{q \in L^2(\Omega) : \ q|_K \in H^1(K), \ \forall K \in \mathcal{T}_h\}.$$

We use discontinuous piecewise polynomials to approximate the displacement and pressure. Denote by $P_k(K)$ the set of polynomials with degree no more than a positive integer $k \geq 1$ on each $K \in \mathcal{T}_h$. Define the discontinuous finite element spaces

$$\mathbf{V}_h = \{\mathbf{v} \in (L^2(\Omega))^d : \ \mathbf{v}|_K \in (P_{k+1}(K))^d, \ \forall K \in \mathcal{T}_h\},$$
$$Q_h = \{q \in L^2(\Omega) : \ q|_K \in P_k(K), \ \forall K \in \mathcal{T}_h\}.$$

Next, we define the jump and average operators that are required for the DG method. Let K_i and K_j $(i > j)$ be two adjacent elements of $\mathcal{T}_h$ which share a common edge e, i.e., an interior edge $e = \partial K_i \cap \partial K_j \subset \Gamma_I$. We assume that the unit normal vector $\mathbf{n}_e$ is oriented from K_i to K_j on e. The jump and average of v on e are given by

$$[v] = v|_{K_i} - v|_{K_j}, \quad \{v\} = \frac{1}{2}(v|_{K_i} + v|_{K_j}).$$

If $e \subset \partial K_i \cap \partial\Omega$, then

$$[v] = v|_e, \quad \{v\} = v|_e.$$

Furthermore, v can be some scalar-, vector-, and matrix-valued functions.

Now, we can define the general DG variational formulation of (1)-(2) as follows: For any $t \in (0, T]$, seek $(\mathbf{u}(t); p(t)) \in \mathbf{V} \times Q$ such that

$$a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) = (\boldsymbol{f}, \mathbf{v}), \ \forall \mathbf{v} \in \mathbf{V}, \tag{9}$$

$$c_0(\partial_t p, q) + b(\partial_t \mathbf{u}, q) + e(\nabla \cdot \mathbf{u}; p, q) = (g, q) + \langle \gamma, q\rangle_{\Gamma_{p,N}}, \ \forall q \in Q, \tag{10}$$

with the initial conditions (6) and (7). Here,

$$\begin{aligned}
a(\mathbf{v}, \mathbf{w}) = \ & (\lambda + \mu) \sum_{K \in \mathcal{T}_h} (\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{w})_K - (\lambda + \mu) \sum_{e \in \Gamma} \int_e \{\nabla \cdot \mathbf{v}\} \boldsymbol{n}_e [\mathbf{w}] ds \\
& + \mu \sum_{K \in \mathcal{T}_h} (\nabla \mathbf{v}, \nabla \mathbf{w})_K - \mu \sum_{e \in \Gamma} \int_e \{\nabla \mathbf{v}\} \cdot \boldsymbol{n}_e [\mathbf{w}] ds \\
& + \epsilon(\lambda + \mu) \sum_{e \in \Gamma} \int_e \{\nabla \cdot \mathbf{w}\} \boldsymbol{n}_e [\mathbf{v}] ds + \epsilon\mu \sum_{e \in \Gamma} \int_e \{\nabla \mathbf{w}\} \cdot \boldsymbol{n}_e [\mathbf{v}] ds \\
& + \sigma_1 \sum_{e \in \Gamma} \int_e h_e^{-1} [\mathbf{v}][\mathbf{w}] ds, \\
b(p, \mathbf{v}) \ = \ & \sum_{K \in \mathcal{T}_h} (p, \nabla \cdot \mathbf{v})_K - \sum_{e \in \Gamma_{I,D}} \int_e \{p\} \boldsymbol{n}_e \cdot [\mathbf{v}] ds \\
= \ & - \sum_{K \in \mathcal{T}_h} (\nabla p, \mathbf{v})_K + \sum_{e \in \Gamma_{I,D}} \int_e \{\mathbf{v}\} \cdot \boldsymbol{n}_e [p] ds, \\
e(\phi; p, q) = \ & \sum_{K \in \mathcal{T}_h} (\kappa(\phi) \nabla p, \nabla q)_K - \sum_{e \in \Gamma_{I,D}} \int_e \{\kappa(\phi) \nabla p\} \cdot \boldsymbol{n}_e [q] ds \\
& + \epsilon \sum_{e \in \Gamma_{I,D}} \int_e \{\kappa(\phi) \nabla q\} \cdot \boldsymbol{n}_e [p] ds + \sigma_2 \sum_{e \in \Gamma_{I,D}} \int_e h_e^{-1} [p][q] ds,
\end{aligned}$$

where σ_1 and σ_2 are two large enough positive constants, and $\epsilon \in \{-1, 0, 1\}$ corresponds to different interior penalty methods, which will be introduced in Section 3.

By using the skills in [13], we obtain the following lemma.

Lemma 2.1. *If $\mathbf{u} \in \mathbf{V} \cap (H_0^1(\Omega))^d$ and $p \in Q \cap H_0^1(\Omega)$ is the solution of* (1)-(2)*, then $\mathbf{u}$ and p satisfies the variational formulation* (9)-(10) *and vice versa.*

3. Fully discrete DG scheme

In this section, we propose the fully implicit nonlinear numerical scheme for the problem (1)-(7). The scheme is constructed by using the DG method for spatial discretization and the first order backward difference method for the temporal approximation.

We introduce a time step size $\tau = \frac{T}{N}$ for some positive integer N and $t_n = n\tau$ for $n = 0, 1, \cdots, N$. By $\mathbf{u}_h^n$ and p_h^n, we denote the approximation of $\mathbf{u}(t_n)$ and $p(t_n)$, respectively, $\boldsymbol{f}^n$, g^n and γ^n, we denote $\boldsymbol{f}(t_n)$ and $g(t_n)$ and $\gamma(t_n)$ respectively. By the backward Euler method to approximate the time derivative in (9), the fully discrete scheme reads: For $n = 1, \cdots, N$, seek $\mathbf{u}_h^n \in \mathbf{V}_h$ and $p_h^n \in Q_h$ such that

$$a(\mathbf{u}_h^n, \mathbf{v}_h) - b(\mathbf{v}_h, p_h^n) = (\boldsymbol{f}^n, \mathbf{v}_h), \ \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{11}$$

$$c_0(\partial_\tau p_h^n, q_h) + b(\partial_\tau \mathbf{u}_h^n, q_h) + e(\nabla \cdot \mathbf{u}_h^n; p_h^n, q_h) = (g^n, q_h) + \langle \gamma^n, q_h \rangle_{\Gamma_{p,N}}, \ \forall q_h \in Q_h, \tag{12}$$

where $\partial_\tau \mathbf{u}_h^n = \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau}$ and $\partial_\tau p_h^n = \frac{p_h^n - p_h^{n-1}}{\tau}$.

Remark: Different values of ϵ correspond to the different DG methods: $\epsilon = -1$ leads to the symmetric interior penalty Galerkin method (SIPG); $\epsilon = 1$ leads to the non-symmetric interior penalty method (NIPG); $\epsilon = 0$ leads to the incomplete interior penalty method (IIPG). Here in the following theoretical analysis and numerical experiments, we mainly focus on SIPG with $\epsilon = -1$.

For the purpose of theoretical analysis, we introduce the following semi-norms and norms:

$$||\nabla \mathbf{v}||^2 = \sum_{K \in \mathcal{T}_h} ||\nabla \mathbf{v}||_{0,K}^2, \ ||\nabla \cdot \mathbf{v}||^2 = \sum_{K \in \mathcal{T}_h} ||\nabla \cdot \mathbf{v}||_{0,K}^2, \ |\mathbf{v}|_*^2 = \sum_{e \in \Gamma} \int_e h_e^{-1} [\mathbf{v}]^2 ds,$$

$$\|\mathbf{v}\|_V^2 = ||\nabla \mathbf{v}||^2 + ||\nabla \cdot \mathbf{v}||^2 + |\mathbf{v}|_*^2,$$

$$\|q\|_Q^2 = \sum_{K \in \mathcal{T}_h} ||\nabla q||_{0,K}^2 + \sum_{e \in \Gamma_{I,D}} \int_e h_e^{-1} [q]^2 ds = |q|_1^2 + |q|_*^2.$$

According to [14], we obtain that $\|\cdot\|_V$ is a norm in the space $\mathbf{V}_h$. From [28] and [2], we supply the following useful lemmas.

Lemma 3.1. [28] *There exist positive constants C, independent of mesh size h, such that*

$$||v||_{0,e} \le C h_K^{-\frac{1}{2}} ||v||_{0,K}, \ \forall v \in P_k(K), \tag{13}$$

$$||\nabla v \cdot \boldsymbol{n}_e||_{0,e} \le C h_K^{-\frac{1}{2}} ||\nabla v||_{0,K}, \ \forall v \in P_k(K), \tag{14}$$

$$||v||_{0,e} \le C h_K^{-\frac{1}{2}} (||v||_{0,K} + h_K ||\nabla v||_{0,K}), \ \forall v \in H^1(K), \tag{15}$$

$$||\nabla v \cdot \boldsymbol{n}_e||_{0,e} \le C h_K^{-\frac{1}{2}} (||\nabla v||_{0,K} + h_K ||\nabla^2 v||_{0,K}), \ \forall v \in H^2(K). \tag{16}$$

Lemma 3.2. [2] *There exist constants $C > 0$, such that*

$$||\mathbf{v}||^2 \le C \|\mathbf{v}\|_V^2, \ \forall \mathbf{v} \in \mathbf{V}_h,$$
$$||q||^2 \le C \|q\|_Q^2, \ \forall q \in Q_h.$$

Next, we prove the boundedness of $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $e(\cdot;\cdot,\cdot)$, and the coercivity of $a(\cdot,\cdot$ and $e(\cdot;\cdot,\cdot)$ in the following two lemmas, respectively.

Lemma 3.3. *There exist positive constants C, such that*

$$a(\mathbf{v}, \mathbf{w}) \le C \|\mathbf{v}\|_V \|\mathbf{w}\|_V, \ \forall \mathbf{v}, \mathbf{w} \in \mathbf{V}_h,$$
$$b(\mathbf{v}, p) \le C \|\mathbf{v}\|_V ||p||, \ \forall \mathbf{v} \in \mathbf{V}_h, p \in Q_h,$$
$$e(\phi; p, q) \le C \|p\|_Q \|q\|_Q, \ \forall p, q \in Q_h.$$

Proof. Using Cauchy-Schwarz inequality and Lemma 3.1, we have

$$\begin{aligned}\sum_{e\in\Gamma}\int_e\{\nabla\cdot\mathbf{v}\}\boldsymbol{n}_e[\mathbf{w}]ds &\le (\sum_{e\in\Gamma}\int_e h_e(\{\nabla\cdot\mathbf{v}\}\boldsymbol{n}_e)^2ds)^{\frac{1}{2}}(\sum_{e\in\Gamma}\int_e h_e^{-1}[\mathbf{w}]^2ds)^{\frac{1}{2}}\\ &\le (\sum_{K\in\Gamma}||\nabla\cdot\mathbf{v}||_{0,K}^2)^{\frac{1}{2}}(\sum_{e\in\Gamma}\int_e h_e^{-1}[\mathbf{w}]^2ds)^{\frac{1}{2}}\\ &\le C||\nabla\cdot\mathbf{v}|||\mathbf{w}|_*. \qquad (17)\end{aligned}$$

Similarly, we obtain

$$\sum_{e\in\Gamma}\int_e\{\nabla\cdot\mathbf{w}\}\boldsymbol{n}_e[\mathbf{v}]ds \le C||\nabla\cdot\mathbf{w}|||\mathbf{v}|_*,$$

$$\sum_{e\in\Gamma}\int_e\{\nabla\mathbf{v}\}\cdot\boldsymbol{n}_e[\mathbf{w}]ds \le C||\nabla\mathbf{v}|||\mathbf{w}|_*,$$

$$\sum_{e\in\Gamma}\int_e\{\nabla\mathbf{w}\}\cdot\boldsymbol{n}_e[\mathbf{v}]ds \le C||\nabla\mathbf{w}|||\mathbf{v}|_*,$$

$$\sum_{e\in\Gamma}\int_e h_e^{-1}[\mathbf{v}][\mathbf{w}]ds \le C|\mathbf{v}|_*|\mathbf{w}|_*.$$

By virtue of the inequality $ab+cd\le(a^2+c^2)(b^2+d^2)$, we provide

$$\begin{aligned}a(\mathbf{v},\mathbf{w}) \le& C(\lambda+\mu)(||\nabla\cdot\mathbf{v}||||\nabla\cdot\mathbf{w}||+||\nabla\cdot\mathbf{v}|||\mathbf{w}|_*+||\nabla\cdot\mathbf{w}|||\mathbf{v}|_*)\\ &+C\mu(||\nabla\mathbf{v}||||\nabla\mathbf{w}||+||\nabla\mathbf{v}|||\mathbf{w}|_*++||\nabla\mathbf{w}|||\mathbf{v}|_*)+C\sigma_1|\mathbf{v}|_*|\mathbf{w}|_*\\ \le& C(||\nabla\cdot\mathbf{v}||^2+||\nabla\mathbf{v}||^2+|\mathbf{v}|_*^2)(||\nabla\cdot\mathbf{w}||^2+||\nabla\mathbf{w}||^2+|\mathbf{w}|_*^2)\\ \le& C|||\mathbf{v}|||_V|||\mathbf{w}|||_V. \qquad (18)\end{aligned}$$

In the same way, we render

$$b(\mathbf{v},p)\le C||p||||\nabla\cdot\mathbf{v}||+C||p|||\mathbf{v}|_*\le C||p|||||\mathbf{v}|||_V, \qquad (19)$$

and

$$\begin{aligned}e(\phi;p,q) &\le C\kappa_{\max}||\nabla p||||\nabla q||+Ck_2||\nabla p|||q|_*+C\kappa_{\max}||\nabla q|||p|_*+C\sigma|p|_*|q|_*\\ &\le C|||p|||_Q|||q|||_Q, \qquad (20)\end{aligned}$$

which completes the proof. □

Lemma 3.4. *For any $\mathbf{v}\in\mathbf{V}_h$ and $p\in Q_h$, the following estimates hold*

$$a(\mathbf{v},\mathbf{v})\ge C_a|||\mathbf{v}|||_V^2,$$

and

$$e(\phi;p,p)\ge C_e|||p|||_Q^2,$$

where C_a and C_e are two positive constants independent of mesh size h.

Proof. Because of the formulation of $a(\cdot,\cdot)$, we supply

$$\begin{aligned}a(\mathbf{v},\mathbf{v}) =& (\lambda+\mu)||\nabla\cdot\mathbf{v}||^2+\mu||\nabla\mathbf{v}||^2-2(\lambda+\mu)\sum_{e\in\Gamma}\int_e\{\nabla\cdot\mathbf{v}\}\boldsymbol{n}_e[\mathbf{v}]ds\\ &-2\mu\sum_{e\in\Gamma}\int_e\{\nabla\mathbf{v}\}\cdot\boldsymbol{n}_e[\mathbf{v}]ds+\sigma_1\sum_{e\in\Gamma}\int_e {h_e}^{-1}[\mathbf{v}]^2ds. \qquad (21)\end{aligned}$$

Together with the trace inequality and the inequality $ab\le\frac{a^2}{2\delta}+\frac{\delta}{2}b^2$, we provide

$$\sum_{e\in\Gamma}\int_e\{\nabla\mathbf{v}\}\cdot\boldsymbol{n}_e[\mathbf{v}]ds\le C||\nabla\mathbf{v}|||\mathbf{v}|_*\le\frac{1}{2\delta}||\nabla\mathbf{v}||^2+\frac{\delta C^2}{2}|\mathbf{v}|_*^2,$$

$$\sum_{e\in\Gamma}\int_e\{\nabla\cdot\mathbf{v}\}\boldsymbol{n}_e[\mathbf{v}]ds \le C||\nabla\cdot\mathbf{v}|||\mathbf{v}|_* \le \tfrac{1}{2\delta}||\nabla\cdot\mathbf{v}||^2+\tfrac{\delta C^2}{2}|\mathbf{v}|_*^2,$$

and

$$\begin{aligned} a(\mathbf{v},\mathbf{v})\ge\ & (\lambda+\mu)||\nabla\cdot\mathbf{v}||^2+\mu||\nabla\mathbf{v}||^2-\tfrac{\lambda+\mu}{\delta}||\nabla\cdot\mathbf{v}||^2-\delta C^2(\lambda+\mu)|\mathbf{v}|_*^2\\ & -\tfrac{\mu}{\delta}||\nabla\mathbf{v}||^2-\delta\mu C^2|\mathbf{v}|_*^2+\sigma_1|\mathbf{v}|_*^2. \end{aligned}$$

Taking $C_a=\min\{(\lambda+\mu)(1-\frac{1}{\delta}),\mu(1-\frac{1}{\delta}),\sigma_1-\delta C^2(\lambda+\mu)-\mu C^2\delta\}$, we have

$$a(\mathbf{v},\mathbf{v})\ge C_a|||\mathbf{v}|||_V^2. \tag{22}$$

Similarly to $e(\cdot;\cdot,\cdot)$, it follows that

$$\begin{aligned} e(\phi;q,q)\ &\ge\kappa_{\min}||\nabla q||^2-2C\kappa_{\max}||\nabla q|||q|_*+\sigma_2|q|_*^2\\ &\ge\kappa_{\min}||\nabla q||^2-\tfrac{1}{\delta}||\nabla q||^2-\delta(C\kappa_{\max})^2|q|_*^2+\sigma_2|q|_*^2. \end{aligned}$$

Taking $C_e=\min\{\kappa_{\min}-\frac{1}{\delta},\sigma_2-\delta(C\kappa_{\max})^2\}$, we get

$$e(\phi;q,q)\ge C_e|||q|||_Q^2, \tag{23}$$

which finishes the proof. □

Now, we are ready to prove the existence and uniqueness of the solution of the numerical scheme (11)-(12) in the next two theorems, separately.

Theorem 3.5. *Given $\boldsymbol{f}\in L^2(0,T;(L^2(\Omega))^d)$, $g\in L^2(0,T;L^2(\Omega))$ and $\gamma\in L^2(0,T;L^2(\Gamma_{p,N}))$, the initial conditions $\mathbf{u}^0\in(H_0^1(\Omega))^d$ and $p^0\in H_0^1(\Omega)$, then for any $(\mathbf{v}_h;q_h)\in\mathbf{V}_h\times Q_h$, there is a solution $(\mathbf{u}_h^n;p_h^n)\in\mathbf{V}_h\times Q_h$ satisfying* (11) *and* (12)*.*

Proof. The proof can be divided into two steps. Step 1: for $n=1,\cdots,N$, assuming that $\mathbf{u}_h^{n-1}$, p_h^{n-1}, $\boldsymbol{f}^n$, g^n and γ^n are known, we prove that we can find $(\mathbf{u}_h^n;p_h^n)\in\mathbf{V}_h\times Q_h$ satisfying (11) and (12). Similarly to the proof of Lemma 5 in [8], here, we define a map $G:\mathbf{V}_h\times Q_h\to\mathbf{V}_h\times Q_h$ such that, for $(\mathbf{u}_h^n;p_h^n)\in\mathbf{V}_h\times Q_h$,

$$\begin{aligned} &(G\begin{bmatrix}\mathbf{u}_h^n\\ p_h^n\end{bmatrix},\begin{bmatrix}\mathbf{v}_h\\ q_h\end{bmatrix})\\ =&a(\mathbf{u}_h^n,\mathbf{v}_h)-b(p_h^n,\mathbf{v}_h)+c_0(p_h^n,q_h)+b(\mathbf{u}_h^n,q_h)+\tau e(\nabla\cdot\mathbf{u}_h^n;p_h^n,q_h)\\ &-\tau(g^n,q_h)-\tau\langle\gamma^n,q_h\rangle_{\Gamma_{p,N}}-c_0(p_h^{n-1},q_h)-b(\mathbf{u}_h^{n-1},q_h)-(\boldsymbol{f}^n,\mathbf{v}_h), \end{aligned} \tag{24}$$

for all $(\mathbf{v}_h;q_h)\in\mathbf{V}_h\times Q_h$. Notice that G defines an operator on $\mathbf{V}_h$ and Q_h simultaneously. Employing Lemma 3.4 and Cauchy-Schwarz inequality, we note the following estimate

$$\begin{aligned} &(G\begin{bmatrix}\mathbf{u}_h^n\\ p_h^n\end{bmatrix},\begin{bmatrix}\mathbf{u}_h^n\\ p_h^n\end{bmatrix})\\ \ge&C_a|||\mathbf{u}_h^n|||_V^2+c_0||p_h^n||^2+C_e\tau|||p_h^n|||_Q^2-\tau||g^n||||p_h^n||\\ &-\tau||\gamma^n||_{L^2(\Gamma_{p,N})}||p_h^n||-c_0||p_h^{n-1}||||p_h^n||-|||\mathbf{u}_h^{n-1}|||_V||p_h^n||-||\boldsymbol{f}^n||||\mathbf{u}_h^n||\\ \ge&C_a|||\mathbf{u}_h^n|||_V^2+c_0||p_h^n||^2+C_e\tau|||p_h^n|||_Q^2-\tau C(\delta)||g^n||^2-\tau\delta||p_h^n||^2\\ &-\tau C(\delta)||\gamma^n||_{L^2(\Gamma_{p,N})}^2-\tau\delta||p_h^n||^2-c_0C(\delta)||p_h^{n-1}||^2-\frac{1}{2}\delta||p_h^n||^2\\ &-CC(\delta)|||\mathbf{u}_h^{n-1}|||_V^2-\frac{1}{2}\delta||p_h^n||^2-C(\delta)||\boldsymbol{f}^n||^2-\delta||\mathbf{u}_h^n||^2\\ =&(C_a-\delta)|||\mathbf{u}_h^n|||_V^2+(c_0-2\tau\delta-\delta)||p_h^n||^2+C_e\tau|||p_h^n|||_Q^2\\ &-\tau C(\delta)||g^n||^2-\tau C(\delta)||\gamma^n||_{L^2(\Gamma_{p,N})}^2-CC(\delta)|||\mathbf{u}_h^{n-1}|||_V^2-C(\delta)||\boldsymbol{f}^n||^2. \end{aligned} \tag{25}$$

Taking $C_a - \delta > 0$ and $c_0 - 2\tau\delta - \delta > 0$, then the mapping $G : \mathbf{V}_h \times Q_h \to \mathbf{V}_h \times Q_h$ has the property that

$$(G \begin{bmatrix} \mathbf{u}_h^n \\ p_h^n \end{bmatrix}, \begin{bmatrix} \mathbf{u}_h^n \\ p_h^n \end{bmatrix}) \geq 0, \tag{26}$$

when

$$\begin{aligned}&(C_a - \delta)\|\mathbf{u}_h^n\|_V^2 + (c_0 - 2\tau\delta - \delta)||p_h^n||^2 + C_e\tau\|p_h^n\|_Q^2 \\ \geq &\tau C(\delta)||g^n||^2 + \tau C(\delta)||\gamma^n||^2_{L^2(\Gamma_{p,N})} + C(\delta)\|\mathbf{u}_h^{n-1}\|_V^2 + C(\delta)||\boldsymbol{f}^n||^2.\end{aligned}$$

Hence the monotonicity of G is obtained. The continuity of G on $\mathbf{V}_h \times Q_h$ follows straightforwardly from (24). Noting that $\mathbf{V}_h \times Q_h$ is finite dimensional, together with the two properties of G defined on $\mathbf{V}_h \times Q_h$, we utilize a well-known corollary of Brouwer's fixed point theorem [31], which guarantees that there exists a solution (we may also denote it by $(\mathbf{u}_h^n; p_h^n)$) such that for any $(\mathbf{v}_h; q_h) \in \mathbf{V}_h \times Q_h$,

$$(G \begin{bmatrix} \mathbf{u}_h^n \\ p_h^n \end{bmatrix}, \begin{bmatrix} \mathbf{v}_h \\ q_h \end{bmatrix}) = 0. \tag{27}$$

Step 2: based on the assumption $\boldsymbol{f} \in L^2(0,T;(L^2(\Omega))^d)$, $g \in L^2(0,T;L^2(\Omega))$ and $\gamma \in L^2(0,T;L^2(\Gamma_{p,N}))$, and the initial conditions $\mathbf{u}^0 \in (H_0^1(\Omega))^d$ and $p^0 \in H_0^1(\Omega)$, together with Step 1, we adopt the iteration method to produce a weak solution $(\mathbf{u}_h^n; p_h^n) \in \mathbf{V}_h \times Q_h$, for each $n = 1, \cdots, N$, satisfying (11) and (12). The proof is completed. □

Theorem 3.6. *Under the condition of* (8)*, the numerical solution* $(\mathbf{u}_h^n; p_h^n)$ *of* (11)-(12) *is unique.*

Proof. Assuming that both $(\mathbf{u}_h^n, p_h^n)$ and $(\bar{\mathbf{u}}_h^n, \bar{p}_h^n)$ are the solutions of (11)-(12), let $\mathbf{w}_h^n = \mathbf{u}_h^n - \bar{\mathbf{u}}_h^n$, $z_h^n = p_h^n - \bar{p}_h^n$. Then for any $\mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in Q_h$,

$$a(\mathbf{w}_h^n, \mathbf{v}_h) - b(z_h^n, \mathbf{v}_h) = 0, \tag{28}$$

$$c_0(z_h^n, q_h) + b(\mathbf{w}_h^n, q_h) + \tau e(\nabla\cdot\mathbf{u}_h^n; p_h^n, q_h) - \tau e(\nabla\cdot\bar{\mathbf{u}}_h^n; \bar{p}_h^n, q_h) = 0. \tag{29}$$

Taking $\mathbf{v}_h = \mathbf{w}_h^n$, $q_h = z_h^n$ in (28) and (29), we obtain

$$\begin{gathered} a(\mathbf{w}_h^n, \mathbf{w}_h^n) - b(z_h^n, \mathbf{w}_h^n) = 0, \\ c_0(z_h^n, z_h^n) + b(\mathbf{w}_h^n, z_h^n) + \tau e(\nabla\cdot\bar{\mathbf{u}}_h^n; z_h^n, z_h^n) = \tau e(\nabla\cdot\bar{\mathbf{u}}_h^n - \nabla\cdot\mathbf{u}_h^n; p_h^n, z_h^n). \end{gathered}$$

It follows that

$$a(\mathbf{w}_h^n, \mathbf{w}_h^n) + c_0(z_h^n, z_h^n) + \tau e(\nabla\cdot\bar{\mathbf{u}}_h^n; z_h^n, z_h^n) = \tau e(\nabla\cdot\bar{\mathbf{u}}_h^n - \nabla\cdot\mathbf{u}_h^n; p_h^n, z_h^n),$$

together with Cauchy-Schwarz inequality and Young's inequality, then

$$\begin{aligned}
&e(\nabla\cdot\bar{\mathbf{u}}_h^n-\nabla\cdot\mathbf{u}_h^n;p_h^n,z_h^n)\\
=&\sum_{K\in\mathcal{T}_h}((\kappa(\nabla\cdot\bar{\mathbf{u}}_h^n)-\kappa(\nabla\cdot\mathbf{u}_h^n))\nabla p_h^n,\nabla z_h^n)_K\\
&+\sum_{e\in\Gamma_{I,D}}\int_e\{(\kappa(\nabla\cdot\bar{\mathbf{u}}_h^n)-\kappa(\nabla\cdot\mathbf{u}_h^n))\nabla p_h^n\}\cdot\boldsymbol{n}_e[z_h^n]ds\\
&+\sum_{e\in\Gamma_{I,D}}\int_e\{(\kappa(\nabla\cdot\bar{\mathbf{u}}_h^n)-\kappa(\nabla\cdot\mathbf{u}_h^n))\nabla z_h^n\}\cdot\boldsymbol{n}_e[p_h^n]ds\\
\le& k_L||\nabla p_h^n||_\infty||\nabla\cdot\bar{\mathbf{u}}_h^n-\nabla\cdot\mathbf{u}_h^n||||\nabla z_h^n||\\
&+k_L||\nabla p_h^n||_\infty||\nabla\cdot\bar{\mathbf{u}}_h^n-\nabla\cdot\mathbf{u}_h^n||\|z_h^n\|_Q\\
&+k_L\sup_{e\in\Gamma_{I,D}}(h_e^{-1}||[p_h^n]||_\infty)||\nabla\cdot\bar{\mathbf{u}}_h^n-\nabla\cdot\mathbf{u}_h^n||\|z_h^n\|_Q\\
\le& Ck_L\|\mathbf{w}_h^n\|_V\|z_h^n\|_Q.
\end{aligned}$$

By virtue of Lemmas 3.3 and 3.4, it can be concluded that

$$a(\mathbf{w}_h^n,\mathbf{w}_h^n)+c_0(z_h^n,z_h^n)+\tau C_e\|z_h^n\|_Q^2\le\tau k_LC\|\mathbf{w}_h^n\|_V\|z_h^n\|_Q.$$

Using Young inequality with $1-C\tau>0$, we write

$$C\|\mathbf{w}_h^n\|_V^2+c_0||z_h^n||^2+C\tau(1-C\tau)\|z_h^n\|_Q^2\le 0.$$

Hence, $\|\mathbf{w}_h^n\|_V=0$ and $||z_h^n||=0$, which implies that $\mathbf{u}_h^n=\bar{\mathbf{u}}_h^n$, $p_h^n=\bar{p}_h^n$. □

4. Error analysis

In this section, we shall derive the optimal convergence order estimates for the discrete time DG scheme.

Let

$$\begin{aligned}
\mathbf{u}(t)-\mathbf{u}_h(t)&=[\mathbf{u}(t)-\mathbf{R}_h\mathbf{u}(t)]+[\mathbf{R}_h\mathbf{u}(t)-\mathbf{u}_h(t)]=\boldsymbol{\rho}(t)+\boldsymbol{\theta}(t),\\
p(t)-p_h(t)&=[p(t)-R_hp(t)]+[R_hp(t)-p_h(t)]=\eta(t)+\xi(t),
\end{aligned}$$

where the elliptic projection $(\mathbf{R}_h\mathbf{u};R_hp)\in\mathbf{V}_h\times Q_h$ is defined as follows, for $(\mathbf{u};p)\in(H^{k+2}(\Omega)\cap H_0^1(\Omega))^d\times(H^{k+1}(\Omega)\cap H_0^1(\Omega))$,

$$a(\boldsymbol{\rho},\mathbf{v})-b(\mathbf{v},\eta)=0,\quad\forall\mathbf{v}\in\mathbf{V}_h,\tag{30}$$

$$e(\nabla\cdot\mathbf{u};\eta,q)=0,\quad\forall q\in Q_h.\tag{31}$$

Now, we introduce the following lemmas which can be found in [28].

Lemma 4.1. *Let $\mathbf{u}\in(H^{k+2}(K))^d$, there exists a constant C independent of $\mathbf{u}$, h_K and the function $I_h\mathbf{u}\in(P_{k+1}(K))^d$, such that*

$$\|\mathbf{u}-I_h\mathbf{u}\|_V\le Ch_K^{k+1}||\mathbf{u}||_{k+2}.$$

Lemma 4.2. *Let $p\in H^{k+1}(\Omega)\cap H_0^1(\Omega)$ with any integer $k\ge 1$, then*

$$\begin{aligned}
\|\eta\|+h\|\eta\|_Q&\le Ch^{k+1}\|p\|_{k+1},\\
\|\eta_t\|+h\|\eta_t\|_Q&\le Ch^{k+1}\|p_t\|_{k+1}.
\end{aligned}$$

Lemma 4.3. *Let $\mathbf{u}\in(H^{k+2}(\Omega)\cap H_0^1(\Omega))^d$, $p\in H^{k+1}(\Omega)\cap H_0^1(\Omega)$ with any integer $k\ge 1$, then*

$$\begin{aligned}
\|\boldsymbol{\rho}\|_V&\le Ch^{k+1}(\|\mathbf{u}\|_{k+2}+\|p\|_{k+1}),\\
\|\boldsymbol{\rho}_t\|_V&\le Ch^{k+1}(\|\mathbf{u}_t\|_{k+2}+\|p_t\|_{k+1}).
\end{aligned}$$

Proof. For any $\mathbf{v}_h \in V_h$, according to (30) and Lemma 3.3, we obtain

$$\begin{aligned}
&a(\mathbf{v}_h - \mathbf{R}_h\mathbf{u}, \mathbf{v}_h - \mathbf{R}_h\mathbf{u})\\
=&a(\mathbf{v}_h - \mathbf{u}, \mathbf{v}_h - \mathbf{R}_h\mathbf{u}) + a(\mathbf{u} - \mathbf{R}_h\mathbf{u}, \mathbf{v}_h - \mathbf{R}_h\mathbf{u})\\
=&a(\mathbf{v}_h - \mathbf{u}, \mathbf{v}_h - \mathbf{u}) + b(\mathbf{v}_h - \mathbf{R}_h\mathbf{u}, \eta)\\
\le&C|||\mathbf{v}_h - \mathbf{u}|||_V|||\mathbf{v}_h - \mathbf{R}_h\mathbf{u}|||_V + C|||\mathbf{v}_h - \mathbf{R}_h\mathbf{u}|||_V||\eta||.
\end{aligned}$$

Applying Lemma 3.4, there is

$$|||\mathbf{v}_h - \mathbf{R}_h\mathbf{u}|||_V \le C|||\mathbf{v}_h - \mathbf{u}|||_V + C||\eta||. \tag{32}$$

Together with Lemma 4.1 and Lemma 4.2, it follows that

$$\begin{aligned}
|||\boldsymbol{\rho}|||_V &= |||\mathbf{u} - \mathbf{R}_h\mathbf{u}|||_V\\
&\le |||\mathbf{u} - \mathbf{v}_h|||_V + |||\mathbf{v}_h - \mathbf{R}_h\mathbf{u}|||_V\\
&\le \inf_{\mathbf{v}_h \in V_h} |||\mathbf{u} - \mathbf{v}_h|||_V + C||\eta||\\
&\le |||\mathbf{u} - I_h\mathbf{u}|||_V + C||\eta||\\
&\le Ch^{k+1}(||\mathbf{u}||_{k+2} + ||p||_{k+1}).
\end{aligned} \tag{33}$$

Take the derivative of both sides of (30) with respect to t, we get

$$a(\boldsymbol{\rho}_t, \mathbf{v}) - b(\mathbf{v}, \eta_t) = 0.$$

Similarly to the proof of the estimate (33), we can provide the second estimate of this lemma, which completes the proof. □

Next, we present the error equations of the fully-discrete DG scheme. According to (9)-(10) and (11)-(12), together with (30)-(31), we obtain the following error equations, for any $\mathbf{v}_h \in \mathbf{V}_h, q_h \in Q_h$,

(34)
$$a(\boldsymbol{\theta}^n, \mathbf{v}_h) - b(\mathbf{v}_h, \xi^n) = 0,$$

$$c_0(\partial_\tau\xi^n, q_h) + b(\partial_\tau\boldsymbol{\theta}^n, q_h) + e(\nabla\cdot\mathbf{u}_h^n; \xi^n, q_h) = -c_0(\partial_t p^n - \partial_\tau p^n, q_h) - c_0(\partial_\tau\eta^n, q_h)$$
(35)
$$- b(\partial_t\mathbf{u}^n - \partial_\tau\mathbf{u}^n, q_h) - b(\partial_\tau\boldsymbol{\rho}^n, q_h) + e(\nabla\cdot\mathbf{u}_h^n - \nabla\cdot\mathbf{u}^n; p^n, q_h).$$

Now we are ready for the optimal convergence order estimates.

Theorem 4.4. *For $k \ge 1$, let $(\mathbf{u}(t); p(t)) \in L^\infty(0, T; (H^{k+2}(\Omega) \cap H_0^1(\Omega))^d) \times L^\infty(0, T; H^{k+1}(\Omega) \cap H_0^1(\Omega))$ be the solution of the problem (1)-(2), and $(\mathbf{u}_h^n; p_h^n) \in \mathbf{V}_h \times Q_h$ be the solution of the problem (11)-(12), respectively. Assume that $\mathbf{u}_t \in L^2(0, T; (H^{k+2}(\Omega))^d)$, $p_t \in L^2(0, T; H^{k+1}(\Omega))$, $\mathbf{u}_{tt} \in L^2(0, T; (H_0^1(\Omega))^d)$, $p_{tt} \in L^2(0, T; H_0^1(\Omega))$, then*

$$\begin{aligned}
&|||\mathbf{u}(t_n) - \mathbf{u}_h^n|||_V^2 + ||p(t_n) - p_h^n||^2\\
\le&C\tau^2\Big(\int_0^{t_n} \|p_{tt}\|^2 ds + \int_0^{t_n} \|\mathbf{u}_{tt}\|^2 ds\Big)\\
&+ Ch^{2k+2}\Big(||\mathbf{u}(t)||_{k+2}^2 + ||p(t)||_{k+1}^2 + \int_0^{t_n} \|\mathbf{u}_t\|_{k+2}^2 ds + \int_0^{t_n} ||p_t||_{k+1}^2 ds\Big).
\end{aligned}$$

Proof. Taking $\mathbf{v}_h = \partial_\tau \boldsymbol{\theta}^n$, $q_h = \xi^n$ in (34) and (35), and adding these two equations, we present

$$\begin{aligned}
&a(\boldsymbol{\theta}^n, \partial_\tau \boldsymbol{\theta}^n) + c_0(\partial_\tau \xi^n, \xi^n) + e(\nabla \cdot \mathbf{u}_h^n; \xi^n, \xi^n) \\
=& - c_0(\partial_\tau \eta^n, \xi^n) - c_0(\partial_t p^n - \partial_\tau p^n, \xi^n) - b(\partial_\tau \boldsymbol{\rho}^n, \xi^n) \\
&- b(\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n, \xi^n) + e(\nabla \cdot \mathbf{u}_h^n - \nabla \cdot \mathbf{u}^n; p^n, \xi^n) \\
=& J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

Next, we estimate the terms $J_1 - J_5$, respectively. As for J_1, since

$$\|\partial_\tau \eta^n\|^2 = \|\frac{1}{\tau}(\eta^n - \eta^{n-1})\|^2 = \frac{1}{\tau^2}\int_\Omega (\int_{t_{n-1}}^{t_n} \eta_t ds)^2 d\Omega \le \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \|\eta_t\|^2 ds, \tag{36}$$

together with Young's inequality, we supply

$$|J_1| = | - c_0(\partial_\tau \eta^n, \xi^n)| \le C(\|\xi^n\|^2 + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \|\eta_t\|^2 ds).$$

With respect to J_2, combining

$$\|\partial_t p^n - \partial_\tau p^n\|^2 = \int_\Omega (\frac{1}{\tau}\int_{t_{n-1}}^{t_n} (s - t_{n-1}) p_{tt} ds)^2 d\Omega \le C\tau \int_{t_{n-1}}^{t_n} \|p_{tt}\|^2 ds, \tag{37}$$

with Young's inequality, we get

$$|J_2| = | - c_0(\partial_t p^n - \partial_\tau p^n, \xi^n)| \le C(\|\xi^n\|^2 + \tau \int_{t_{n-1}}^{t_n} \|p_{tt}\|^2 ds).$$

Using Cauchy-Schwarz inequality, Young's inequality and Lemma 3.1, we obtain the following estimate about J_3

$$\begin{aligned}
|J_3| &= | - b(\partial_\tau \boldsymbol{\rho}^n, \xi^n)| \\
&\le ||\xi^n|| ||\nabla \cdot \partial_\tau \boldsymbol{\rho}^n|| + ||\xi^n||| \partial_\tau \boldsymbol{\rho}^n|_* \\
&\le C(\|\xi^n\|^2 + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \|\boldsymbol{\rho}_t\|_V^2 ds).
\end{aligned}$$

The term J_4 can be bounded by utilizing integration by parts,

$$\begin{aligned}
J_4 =& - b(\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n, \xi^n) \\
=& - \sum_{K \in \mathcal{T}_h} \int_K \xi^n \nabla \cdot (\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n) dK + \sum_{e \in \Gamma_{I,D}} \int_e \{\xi^n\} \boldsymbol{n}_e \cdot [\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n] ds \\
=& \sum_{K \in \mathcal{T}_h} \int_K \nabla \xi^n \cdot (\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n) dK - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n) \cdot \boldsymbol{n}_K \xi^n ds \\
&+ \sum_{e \in \Gamma_{I,D}} \int_e \{\xi^n\} \boldsymbol{n}_e \cdot [\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n] ds \\
=& \sum_{K \in \mathcal{T}_h} \int_K \nabla \xi^n \cdot (\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n) dK - \sum_{e \in \Gamma_{I,D}} \int_e \{(\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n)\} \cdot \boldsymbol{n}_e [\xi^n] ds.
\end{aligned}$$

Making use of Cauchy-Schwarz inequality, Young's inequality and Lemma 3.1, together with

$$\|\partial_t \mathbf{u}^n - \partial_\tau \mathbf{u}^n\|^2 \le C\tau \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{tt}\|^2 ds,$$

we obtain

$$|J_4| \leq \frac{C_e}{4}\|\xi^n\|_Q^2 + C\tau \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{tt}\|^2 ds.$$

For J_5, we provide

$$\begin{aligned} |J_5| \leq & |\sum_{K\in\mathcal{T}_h} ((\kappa(\nabla\cdot\mathbf{u}_h^n) - \kappa(\nabla\cdot\mathbf{u}^n))\nabla p^n, \nabla\xi^n)_K| \\ & + |\sum_{e\in\Gamma_{I,D}} \int_e \{(\kappa(\nabla\cdot\mathbf{u}_h^n) - \kappa(\nabla\cdot\mathbf{u}^n))\nabla p^n\}\cdot\boldsymbol{n}_e[\xi^n]ds| \\ & + |\sum_{e\in\Gamma_{I,D}} \int_e \{(\kappa(\nabla\cdot\mathbf{u}_h^n) - \kappa(\nabla\cdot\mathbf{u}^n))\nabla\xi^n\}\cdot\boldsymbol{n}_e[p^n]ds| \\ = & w_1 + w_2 + w_3. \end{aligned}$$

Using Cauchy-Schwarz inequality, Young's inequality, Lemma 3.1 and (8), we render

$$\begin{aligned} w_1 & \leq k_L||\nabla p^n||_\infty||\nabla\cdot\mathbf{u}_h^n - \nabla\cdot\mathbf{u}^n|||\nabla\xi^n|| \\ & \leq C(\|\boldsymbol{\rho}^n\|_V^2 + \|\boldsymbol{\theta}^n\|_V^2) + \frac{C_e}{12}\|\xi^n\|_Q^2, \end{aligned}$$

$$\begin{aligned} w_2 \leq & \sum_{e\in\Gamma_{I,D}} \int_e k_L||\nabla p^n||_\infty\{\nabla\cdot\mathbf{u}_h^n - \nabla\cdot\mathbf{u}^n\}[\xi^n]ds \\ & \leq k_L||\nabla p^n||_\infty \sum_{e\in\Gamma_{I,D}} h_e^{\frac{1}{2}}||\{\nabla\cdot\mathbf{u}_h^n - \nabla\cdot\mathbf{u}^n\}||_{0,e} h_e^{-\frac{1}{2}}||[\xi^n]||_{0,e} \\ & \leq k_L||\nabla p^n||_\infty||\nabla\cdot\mathbf{u}_h^n - \nabla\cdot\mathbf{u}^n||\|\xi^n\|_Q \\ & \leq C(\|\boldsymbol{\rho}^n\|_V^2 + \|\boldsymbol{\theta}^n\|_V^2) + \frac{C_e}{12}\|\xi^n\|_Q^2, \end{aligned}$$

and

$$\begin{aligned} w_3 \leq & k_L \sup_{e\in\Gamma_{I,D}} (h_e^{-1}||[p^n]||_\infty)||\nabla\cdot\mathbf{u}_h^n - \nabla\cdot\mathbf{u}^n||(\sum_{e\in\Gamma_{I,D}} \int_e h_e\{\nabla\xi^n\}^2 ds)^{\frac{1}{2}} \\ \leq & C(\|\boldsymbol{\rho}^n\|_V^2 + \|\boldsymbol{\theta}^n\|_V^2) + \frac{C_e}{12}\|\xi^n\|_Q^2. \end{aligned}$$

Combined with the estimates w_1-w_3, we have

$$|J_5| \leq C(\|\boldsymbol{\rho}^n\|_V^2 + \|\boldsymbol{\theta}^n\|_V^2) + \frac{C_e}{4}\|\xi^n\|_Q^2.$$

Synthesizing the above estimates, together with

$$c_0(\partial_\tau\xi^n, \xi^n) = \frac{c_0}{\tau}(\xi^n - \xi^{n-1}, \xi^n) \geq \frac{c_0}{2\tau}(\|\xi^n\|^2 - \|\xi^{n-1}\|^2),$$

$$a(\partial_\tau\boldsymbol{\theta}^n, \boldsymbol{\theta}^n) \geq \frac{1}{2\tau}(a(\boldsymbol{\theta}^n, \boldsymbol{\theta}^n) - a(\boldsymbol{\theta}^{n-1}, \boldsymbol{\theta}^{n-1})),$$

and

$$e(\nabla\cdot\mathbf{u}_h^n; \xi^n, \xi^n) \geq C_e\|\xi^n\|_Q^2,$$

it follows that

$$\begin{aligned}&\frac{1}{2\tau}(a(\boldsymbol{\theta}^n,\boldsymbol{\theta}^n)-a(\boldsymbol{\theta}^{n-1},\boldsymbol{\theta}^{n-1}))+\frac{c_0}{2\tau}(\|\xi^n\|^2-\|\xi^{n-1}\|^2)+\frac{C_e}{2}|\!|\!|\xi^n|\!|\!|_Q^2\\ \le& C\|\xi^n\|^2+C(\tau\int_{t_{n-1}}^{t_n}\|p_{tt}\|^2ds+\tau\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{tt}\|^2ds)\\ &+C(\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\eta_t\|^2ds+\frac{1}{\tau}\int_{t_{n-1}}^{t_n}|\!|\!|\boldsymbol{\rho}_t|\!|\!|_V^2ds+|\!|\!|\boldsymbol{\rho}^n|\!|\!|_V^2+|\!|\!|\boldsymbol{\theta}^n|\!|\!|_V^2).\end{aligned}$$

By virtue of the iterative method and $\boldsymbol{\theta}^0=0$, $\xi^0=0$, we find

$$\begin{aligned}&C_a|\!|\!|\boldsymbol{\theta}^n|\!|\!|_V^2+c_0\|\xi^n\|^2+C_e\tau\sum_{j=1}^n|\!|\!|\xi^j|\!|\!|_Q^2\\ \le& C\tau\sum_{j=1}^n\|\xi^j\|^2+C\tau\sum_{j=1}^n(|\!|\!|\boldsymbol{\rho}^j|\!|\!|_V^2+|\!|\!|\boldsymbol{\theta}^j|\!|\!|_V^2)\\ &+C(\int_0^{t_n}\|\eta_t\|^2ds+\tau^2\int_0^{t_n}\|p_{tt}\|^2ds+\int_0^{t_n}|\!|\!|\boldsymbol{\rho}_t|\!|\!|_V^2ds+\tau^2\int_0^{t_n}\|\mathbf{u}_{tt}\|^2ds).\end{aligned}$$

Using Gronwall inequality, Lemma 4.2 and Lemma 4.3 with $1-C\tau>0$, we write

$$\begin{aligned}&|\!|\!|\boldsymbol{\theta}^n|\!|\!|_V^2+\|\xi^n\|^2+\tau\sum_{j=1}^n|\!|\!|\xi^j|\!|\!|_Q^2\\ \le& C\tau^2(\int_0^{t_n}\|p_{tt}\|^2ds+\int_0^{t_n}\|\mathbf{u}_{tt}\|^2ds)\\ &+Ch^{2k+2}(||\mathbf{u}(t)||_{k+2}^2+||p(t)||_{k+1}^2+\int_0^{t_n}\|\mathbf{u}_t\|_{k+2}^2ds+\int_0^{t_n}||p_t||_{k+1}^2ds).\end{aligned}\tag{38}$$

Because of the triangle inequality, (38), Lemma 4.2 and Lemma 4.3, we finish the proof. □

Theorem 4.5. *Under the assumption of Theorem 4.4, we have*

$$\begin{aligned}|\!|\!|p(t_n)-p_h^n|\!|\!|_Q^2\le& C\tau^2(\int_0^{t_n}\|p_{tt}\|^2ds+\int_0^{t_n}\|\mathbf{u}_{tt}\|^2ds)+Ch^{2k}||p(t)||_{k+1}^2\\ &+Ch^{2k+2}(||\mathbf{u}(t)||_{k+2}^2+\int_0^{t_n}\|\mathbf{u}_t\|_{k+2}^2ds+\int_0^{t_n}||p_t||_{k+1}^2ds).\end{aligned}$$

Proof. First, differentiating the both sides of (34) with respect to t, and applying the backward Euler method to approximate the time derivative, we get

$$a(\partial_\tau\boldsymbol{\theta}^n,\mathbf{v}_h)-b(\mathbf{v}_h,\partial_\tau\xi^n)=0.\tag{39}$$

Selecting $\mathbf{v}_h=\partial_\tau\boldsymbol{\theta}^n$, $q_h=\partial_\tau\xi^n$ in (39) and (35), and adding, we obtain

$$\begin{aligned}&a(\partial_\tau\boldsymbol{\theta}^n,\partial_\tau\boldsymbol{\theta}^n)+c_0(\partial_\tau\xi^n,\partial_\tau\xi^n)+e(\nabla\cdot\mathbf{u}_h^n;\xi^n,\partial_\tau\xi^n)\\ =&-c_0(\partial_\tau\eta^n,\partial_\tau\xi^n)-c_0(\partial_tp^n-\partial_\tau p^n,\partial_\tau\xi^n)-b(\partial_\tau\boldsymbol{\rho}^n,\partial_\tau\xi^n)\\ &-b(\partial_t\mathbf{u}^n-\partial_\tau\mathbf{u}^n,\partial_\tau\xi^n)+e(\nabla\cdot\mathbf{u}_h^n-\nabla\cdot\mathbf{u}^n;p^n,\partial_\tau\xi^n)\\ =&H_1+H_2+H_3+H_4+H_5.\end{aligned}$$

Next, we derive the estimates of H_1-H_5, respectively. Similarly to the derivation of Theorem 4.4, we render

$$|H_1|\le\frac{c_0}{6}\|\partial_\tau\xi^n\|^2+\frac{C}{\tau}\int_{t_{n-1}}^{t_n}\|\eta_t\|^2ds,$$

$$|H_2| \leq \frac{c_0}{6}\|\partial_\tau \xi^n\|^2 + C\tau \int_{t_{n-1}}^{t_n} \|p_{tt}\|^2 ds,$$

$$|H_3| \leq \frac{c_0}{6}\|\partial_\tau \xi^n\|^2 + \frac{C}{\tau} \int_{t_{n-1}}^{t_n} \|\boldsymbol{\rho}_t\|_V^2 ds,$$

$$|H_4| \leq \frac{\delta}{2}\|\partial_\tau \xi^n\|_Q^2 + C\tau \int_{t_{n-1}}^{t_n} \|\mathbf{u}_{tt}\|^2 ds,$$

$$|H_5| \leq C(\|\boldsymbol{\rho}^n\|_V^2 + \|\boldsymbol{\theta}^n\|_V^2) + \frac{\delta}{2}\|\partial_\tau \xi^n\|_Q^2.$$

Consider the left-hand term of the equation

$$\begin{aligned} e(\nabla\cdot\mathbf{u}_h^n;\xi^n,\partial_\tau\xi^n) =& \sum_{K\in\mathcal{T}_h}(\kappa(\nabla\cdot\mathbf{u}_h^n)\nabla\xi^n,\nabla\partial_\tau\xi^n)_K \\ & - \sum_{e\in\Gamma_{I,D}}\int_e\{\kappa(\nabla\cdot\mathbf{u}_h^n)\nabla\xi^n\}\cdot\boldsymbol{n}_e[\partial_\tau\xi^n]ds \\ & - \sum_{e\in\Gamma_{I,D}}\int_e\{\kappa(\nabla\cdot\mathbf{u}_h^n)\nabla\partial_\tau\xi^n\}\cdot\boldsymbol{n}_e[\xi^n]ds \\ & + \sigma_2\sum_{e\in\Gamma_{I,D}}\int_e h_e^{-1}[\xi^n][\partial_\tau\xi^n]ds \\ =& T_1+T_2+T_3+T_4. \end{aligned}$$

Using the inequality $a(a-b) \geq \frac{1}{2}(a^2-b^2)$, the terms T_1 and T_4 are bounded

$$T_1 \geq \frac{\kappa_{\min}}{2\tau}(||\nabla\xi^n||^2 - ||\nabla\xi^{n-1}||^2),$$

$$T_4 \geq \frac{\sigma_2}{2\tau}(|\xi^n|_*^2 - |\xi^{n-1}|_*^2).$$

Following the same way as the proof of (17), and making use of Young's inequality, we have the bounds for the terms T_2 and T_3

$$\begin{aligned} |T_2| \leq& \frac{\kappa_{\max}C_1}{\tau}||\nabla\xi^n|||\xi^n-\xi^{n-1}|_* \\ \leq& \frac{1}{\tau}(\frac{1}{4}||\nabla\xi^n||^2 + \frac{1}{4}||\nabla\xi^{n-1}||^2 + (\kappa_{\max}C_1)^2(|\xi^n|_*^2 + |\xi^{n-1}|_*^2)), \\ |T_3| \leq& \frac{\kappa_{\max}C_2}{\tau}||\nabla\xi^n-\nabla\xi^{n-1}|||\xi^n|_* \\ \leq& \frac{1}{\tau}(\frac{1}{4}||\nabla\xi^n||^2 + \frac{1}{4}||\nabla\xi^{n-1}||^2 + (\kappa_{\max}C_2)^2(|\xi^n|_*^2 + |\xi^{n-1}|_*^2)). \end{aligned}$$

Combined with the above estimates, together with

$$a(\partial_\tau\boldsymbol{\theta}^n,\partial_\tau\boldsymbol{\theta}^n) \geq C_a\|\partial_\tau\boldsymbol{\theta}^n\|_V^2,$$

and

$$c_0(\partial_\tau\xi^n,\partial_\tau\xi^n) = c_0||\partial_\tau\xi^n||^2,$$

we obtain

$$\begin{aligned}
&\frac{1}{\tau}(\frac{\kappa_{\min}}{2}-\frac{1}{2})||\nabla\xi^n||^2-\frac{1}{\tau}(\frac{\kappa_{\min}}{2}+\frac{1}{2})||\nabla\xi^{n-1}||^2+\frac{1}{\tau}(\frac{\sigma_2}{2}-\kappa_{\max}^2(C_1^2+C_2^2))|\xi^n|_*^2\\
&-\frac{1}{\tau}(\frac{\sigma_2}{2}+\kappa_{\max}^2(C_1^2+C_2^2))|\xi^{n-1}|_*^2+C_a\|\partial_\tau\boldsymbol{\theta}^n\|_V^2\\
&+\frac{c_0}{2}||\partial_\tau\xi^n||^2-\frac{\delta}{\tau}(\|\xi^n\|_Q^2+\|\xi^{n-1}\|_Q^2)\\
\le& C(\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\eta_t\|^2ds+\tau\int_{t_{n-1}}^{t_n}\|p_{tt}\|^2ds\\
&+\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\boldsymbol{\rho}_t\|_V^2ds+\tau\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{tt}\|^2ds+\|\boldsymbol{\rho}^n\|_V^2+\|\boldsymbol{\theta}^n\|_V^2).
\end{aligned}$$

Let $C_3=\frac{\sigma_2}{2}-\kappa_{\max}^2(C_1^2+C_2^2)$, $C_4=\frac{\sigma_2}{2}+\kappa_{\max}^2(C_1^2+C_2^2)+2\delta$ and $C_5=\min\{\frac{\kappa_{\min}}{2}-\frac{1}{2},C_3\}$. Assume that σ_2 is sufficiently large and choose δ such that $\max\{C_5-1,0\}<\delta<C_5$ and $C_4>\delta$. Then, we get

$$\begin{aligned}
&\frac{C_5-\delta}{\tau}\|\xi^n\|_Q^2-\frac{C_4-\delta}{\tau}\|\xi^{n-1}\|_Q^2+C_a\|\partial_\tau\boldsymbol{\theta}^n\|_V^2+C||\partial_\tau\xi^n||^2\\
\le& C(\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\eta_t\|^2ds+\tau\int_{t_{n-1}}^{t_n}\|p_{tt}\|^2ds\\
&+\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\boldsymbol{\rho}_t\|_V^2ds+\tau\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{tt}\|^2ds+\|\boldsymbol{\rho}^n\|_V^2+\|\boldsymbol{\theta}^n\|_V^2).
\end{aligned}$$

Let $\beta=\frac{C_5-\delta}{C_4-\delta}$, we have $0<\beta<\frac{1}{C_4-\delta}<1$. Therefore

$$\begin{aligned}
&\frac{\beta}{\tau}\|\xi^n\|_Q^2-\frac{1}{\tau}\|\xi^{n-1}\|_Q^2+\beta C_a\|\partial_\tau\boldsymbol{\theta}^n\|_V^2+\beta C||\partial_\tau\xi^n||^2\\
\le& C(\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\eta_t\|^2ds+\tau\int_{t_{n-1}}^{t_n}\|p_{tt}\|^2ds\\
&+\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\boldsymbol{\rho}_t\|_V^2ds+\tau\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{tt}\|^2ds+\|\boldsymbol{\rho}^n\|_V^2+\|\boldsymbol{\theta}^n\|_V^2).
\end{aligned}\tag{40}$$

Multiplying (40) by $\tau\beta^{n-1}$ and noticing that $0<\beta^n<\beta^{n-1}<1$, we have

$$\begin{aligned}
&\beta^n\|\xi^n\|_Q^2-\beta^{n-1}\|\xi^{n-1}\|_Q^2+\beta^nC_a\tau\|\partial_\tau\boldsymbol{\theta}^n\|_V^2+\beta^nC\tau||\partial_\tau\xi^n||^2\\
\le& C\beta^{n-1}(\int_{t_{n-1}}^{t_n}\|\eta_t\|^2ds+\tau^2\int_{t_{n-1}}^{t_n}\|p_{tt}\|^2ds+\int_{t_{n-1}}^{t_n}\|\boldsymbol{\rho}_t\|_V^2ds\\
&+\tau^2\int_{t_{n-1}}^{t_n}\|\mathbf{u}_{tt}\|^2ds+\tau\|\boldsymbol{\rho}^n\|_V^2+\tau\|\boldsymbol{\theta}^n\|_V^2).
\end{aligned}$$

By virtue of the iterative method, using $\beta^n<\beta^l$ $(l=1,...,n-1)$ and $\xi^0=0$, we obtain

$$\begin{aligned}
&\beta^n\|\xi^n\|_Q^2+\beta^nC_a\tau\sum_{j=1}^n\|\partial_\tau\boldsymbol{\theta}^j\|_V^2+\beta^nC\tau\sum_{j=1}^n||\partial_\tau\xi^j||^2\\
\le& C\beta^{l-1}(\int_0^{t_n}\|\eta_t\|^2ds+\tau^2\int_0^{t_n}\|p_{tt}\|^2ds+\int_0^{t_n}\|\boldsymbol{\rho}_t\|_V^2ds+\tau^2\int_0^{t_n}\|\mathbf{u}_{tt}\|^2ds\\
&+\tau\sum_{j=1}^n\|\boldsymbol{\rho}^j\|_V^2+\tau\sum_{j=1}^n\|\boldsymbol{\theta}^j\|_V^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|\xi^n\|_Q^2+\tau\sum_{j=1}^{n}\|\partial_\tau\boldsymbol{\theta}^j\|_V^2+\tau\sum_{j=1}^{n}||\partial_\tau\xi^j||^2\\
\le&C(\int_0^{t_n}\|\eta_t\|^2ds+\tau^2\int_0^{t_n}\|p_{tt}\|^2ds+\int_0^{t_n}\|\boldsymbol{\rho}_t\|_V^2ds\\
&+\tau^2\int_0^{t_n}\|\mathbf{u}_{tt}\|^2ds+\tau\sum_{j=1}^{n}\|\boldsymbol{\rho}^j\|_V^2+\tau\sum_{j=1}^{n}\|\boldsymbol{\theta}^j\|_V^2).
\end{aligned}$$

Using Lemma 4.2, Lemma 4.3 and (38), we write

$$\begin{aligned}
&\|\xi^n\|_Q^2+\tau\sum_{j=1}^{n}\|\partial_\tau\boldsymbol{\theta}^j\|_V^2+\tau\sum_{j=1}^{n}||\partial_\tau\xi^j||^2\\
\le&C\tau^2(\int_0^{t_n}\|p_{tt}\|^2ds+\int_0^{t_n}\|\mathbf{u}_{tt}\|^2ds)\\
&+Ch^{2k+2}(||\mathbf{u}(t)||_{k+2}^2+||p(t)||_{k+1}^2+\int_0^{t_n}\|\mathbf{u}_t\|_{k+2}^2ds+\int_0^{t_n}||p_t||_{k+1}^2ds).
\end{aligned}$$

Combined with the triangle inequality and Lemma 4.2, we finish the proof. □

5. Numerical experiments

In this section, we present a numerical example to validate our theoretical results. It should be stated that the computations here are carried out on the high performance computers of State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences. Each computing node has two 18-core Intel Xeon Gold 6140 processors at 2.3 GHz and 192 GB memory. The linear equations formed by the coefficient matrix from (11) and (12) are solved by the package PETSc [4, 5, 6].

In our example, we consider the system (1) and (2) on a two dimensional domain $\Omega=(0,1)^2$, with the Dirichlet boundary conditions (3) and (4) for $\mathbf{u}$ and p on the entire boundary. With respect to the parameters of the system, the similar settings with the ones in [9] are adopted and shown as follows. The Kozeny-Carmen-type hydraulic conductivity $\kappa(s)$ satisfies

$$\kappa(s)=\begin{cases}\dfrac{\kappa_0}{\eta}\dfrac{\phi^3(s)}{(1-\phi(s))^2}, & \text{if } \frac{\phi_0}{\phi_0-1}<s_{\min}<s<s_{\max}<1,\\ \kappa_{\min}, & \text{if } s\le s_{\min},\\ \kappa_{\max}, & \text{if } s_{\max}\le s,\end{cases}$$

where

$$\kappa_{\min}=\frac{\kappa_0}{\eta}\frac{(\phi_0+(1-\phi_0)s_{\min})^3}{(1-s_{\min})^2(1-\phi_0)^2},$$

and

$$\kappa_{\max}=\frac{\kappa_0}{\eta}\frac{(\phi_0+(1-\phi_0)s_{\max})^3}{(1-s_{\max})^2(1-\phi_0)^2}.$$

Here, $s_{\min}$ and $s_{\max}$ are some constants, $\kappa_0>0$ represents the intrinsic permeability, $\eta>0$ stands for the viscosity, and ϕ, the porosity, is given by

$$\phi(s)=\phi_0+(1-\phi_0)s$$

for some reference porosity $0<\phi_0<1$ and dilatation s ($\frac{\phi_0}{\phi_0-1}<s<1$). The parameters are taken as $\lambda=1$, $\mu=1$, $c_0=1$, $\alpha=1$, $T=$ 1e-3, $\kappa_0=1$, $\eta=1$,

$\phi_0 = \frac{1}{2}$, $s_{\min} = -\frac{3}{4}$ and $s_{\max} = \frac{3}{4}$. All the right-hand side terms and boundary conditions are selected according to the analytical solution

$$\boldsymbol{u} = \begin{pmatrix} \frac{1}{6}\exp(-t)\sin(\pi x)\sin(\pi y) \\ \frac{1}{6}\exp(-t)\sin(\pi x)\sin(\pi y) \end{pmatrix},$$

$$p = t\sin(\pi x)\sin(\pi y).$$

For the discontinuous finite element spaces, we use a uniform triangular mesh $\mathcal{T}_h$ and choose $k = 1$. At each time step, the nonlinear equations are solved by the fixed point iteration method. The iteration is stopped when the difference between two successive approximations is less than the preset tolerance 1e-10.

Tables 1-7 show the convergence rates for error with the fixed time step τ = 1e-5 and different penalty parameters σ_1 and σ_2. From Tables 4-7, it can be seen that the optimal convergence rates for error can be obtained by our proposed method, which verifies the theoretical results in Theorems 4.4 and 4.5. Additionally, Tables 1-5 illustrate that, under our selection of equation parameters, the penalty parameter σ_1 mainly affects the optimal convergence order for $\mathbf{u}$ in the L^2 norm when the penalty parameter σ_2 is a "large enough" positive constant. And in our practical tests, the numerical results of which are not listed in this paper, if σ_2 is "small", for example, σ_1 = 1e+4 and σ_2 = 5e+2, under our choice of equation parameters, the error results in the four norms are not convergent until h reaches $\frac{1}{64}$ or less.

Table 1. Error convergence rates with σ_1 = 1e+2 and σ_2 = 1e+4.

h	$\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|$	R	$\|\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|\|_V$	R	$\|\|p(t_n) - p_h^n\|\|$	R	$\|\|\|p(t_n) - p_h^n\|\|\|_Q$	R
$\frac{1}{8}$	1.10e-04	-	7.89e-03	-	3.32e-05	-	5.14e-04	-
$\frac{1}{16}$	1.43e-05	2.95	1.99e-03	1.99	1.00e-05	1.73	2.33e-04	1.14
$\frac{1}{32}$	1.99e-06	2.85	5.00e-04	1.99	2.56e-06	1.97	1.11e-04	1.07
$\frac{1}{64}$	3.30e-07	2.59	1.25e-04	2.00	6.37e-07	2.01	5.48e-05	1.02
$\frac{1}{128}$	6.90e-08	2.26	3.13e-05	2.00	1.58e-07	2.02	2.73e-05	1.00

Table 2. Error convergence rates with σ_1 = 2e+2 and σ_2 = 1e+4.

h	$\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|$	R	$\|\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|\|_V$	R	$\|\|p(t_n) - p_h^n\|\|$	R	$\|\|\|p(t_n) - p_h^n\|\|\|_Q$	R
$\frac{1}{8}$	1.10e-04	-	7.87e-03	-	1.36e-05	-	4.55e-04	-
$\frac{1}{16}$	1.40e-05	2.97	1.99e-03	1.99	4.86e-06	1.49	2.22e-04	1.03
$\frac{1}{32}$	1.81e-06	2.95	4.98e-04	2.00	1.31e-06	1.90	1.10e-04	1.02
$\frac{1}{64}$	2.51e-07	2.85	1.25e-04	2.00	3.29e-07	1.99	5.46e-05	1.01
$\frac{1}{128}$	4.18e-08	2.59	3.12e-05	2.00	8.13e-08	2.02	2.73e-05	1.00

Table 3. Error convergence rates with σ_1 = 5e+2 and σ_2 = 1e+4.

h	$\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|$	R	$\|\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|\|_V$	R	$\|\|p(t_n) - p_h^n\|\|$	R	$\|\|\|p(t_n) - p_h^n\|\|\|_Q$	R
$\frac{1}{8}$	1.10e-04	-	7.87e-03	-	8.55e-06	-	4.46e-04	-
$\frac{1}{16}$	1.40e-05	2.97	1.98e-03	1.99	2.25e-06	1.92	2.19e-04	1.02
$\frac{1}{32}$	1.76e-06	2.99	4.97e-04	2.00	6.39e-07	1.82	1.09e-04	1.01
$\frac{1}{64}$	2.25e-07	2.97	1.24e-04	2.00	1.64e-07	1.97	5.46e-05	1.00
$\frac{1}{128}$	3.01e-08	2.90	3.11e-05	2.00	4.02e-08	2.02	2.73e-05	1.00

TABLE 4. Error convergence rates with $\sigma_1 = 1e{+}3$ and $\sigma_2 = 1e{+}4$.

h	$\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|$	R	$\|\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|\|_V$	R	$\|\|p(t_n) - p_h^n\|\|$	R	$\|\|\|p(t_n) - p_h^n\|\|\|_Q$	R
$\frac{1}{8}$	1.10e-04	-	7.87e-03	-	1.07e-05	-	4.49e-04	-
$\frac{1}{16}$	1.40e-05	2.98	1.98e-03	1.99	1.83e-06	2.54	2.19e-04	1.04
$\frac{1}{32}$	1.75e-06	2.99	4.97e-04	2.00	4.88e-07	1.91	1.09e-04	1.00
$\frac{1}{64}$	2.21e-07	2.99	1.24e-04	2.00	1.25e-07	1.97	5.45e-05	1.00
$\frac{1}{128}$	2.81e-08	2.97	3.11e-05	2.00	3.08e-08	2.01	2.73e-05	1.00

TABLE 5. Error convergence rates with $\sigma_1 = 1e{+}4$ and $\sigma_2 = 1e{+}4$.

h	$\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|$	R	$\|\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|\|_V$	R	$\|\|p(t_n) - p_h^n\|\|$	R	$\|\|\|p(t_n) - p_h^n\|\|\|_Q$	R
$\frac{1}{8}$	1.10e-04	-	7.87e-03	-	1.35e-05	-	4.54e-04	-
$\frac{1}{16}$	1.40e-05	2.98	1.98e-03	1.99	1.92e-06	2.82	2.19e-04	1.05
$\frac{1}{32}$	1.75e-06	2.99	4.97e-04	2.00	4.43e-07	2.11	1.09e-04	1.01
$\frac{1}{64}$	2.19e-07	3.00	1.24e-04	2.00	1.10e-07	2.01	5.45e-05	1.00
$\frac{1}{128}$	2.75e-08	3.00	3.11e-05	2.00	2.76e-08	2.00	2.73e-05	1.00

TABLE 6. Error convergence rates with $\sigma_1 = 1e{+}3$ and $\sigma_2 = 5e{+}3$.

h	$\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|$	R	$\|\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|\|_V$	R	$\|\|p(t_n) - p_h^n\|\|$	R	$\|\|\|p(t_n) - p_h^n\|\|\|_Q$	R
$\frac{1}{8}$	1.10e-04	-	7.87e-03	-	1.07e-05	-	4.49e-04	-
$\frac{1}{16}$	1.40e-05	2.98	1.98e-03	1.99	1.83e-06	2.54	2.19e-04	1.04
$\frac{1}{32}$	1.75e-06	2.99	4.97e-04	2.00	4.88e-07	1.91	1.09e-04	1.01
$\frac{1}{64}$	2.21e-07	2.99	1.24e-04	2.00	1.25e-07	1.97	5.46e-05	1.00
$\frac{1}{128}$	2.81e-08	2.97	3.11e-05	2.00	3.08e-08	2.02	2.73e-05	1.00

TABLE 7. Error convergence rates with $\sigma_1 = 5e{+}3$ and $\sigma_2 = 5e{+}3$.

h	$\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|$	R	$\|\|\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|\|\|_V$	R	$\|\|p(t_n) - p_h^n\|\|$	R	$\|\|\|p(t_n) - p_h^n\|\|\|_Q$	R
$\frac{1}{8}$	1.10e-04	-	7.87e-03	-	1.32e-05	-	4.54e-04	-
$\frac{1}{16}$	1.40e-05	2.98	1.99e-03	1.99	1.89e-06	2.80	2.19e-04	1.05
$\frac{1}{32}$	1.75e-06	3.00	4.97e-04	2.00	4.43e-07	2.09	1.09e-04	1.01
$\frac{1}{64}$	2.19e-07	3.00	1.24e-04	2.00	1.10e-07	2.00	5.45e-05	1.00
$\frac{1}{128}$	2.75e-08	3.00	3.11e-05	2.00	2.77e-08	2.00	2.73e-05	1.00

6. Conclusions

In this paper, we investigate the nonlinear quasi-static poroelasticity problem by using the DG method. Different from the existing references, we establish the fully implicit nonlinear numerical scheme based on the IPDG method in space and the backward Euler method in time. Then the well-posedness of the fully discrete numerical scheme is studied, and the a priori error estimates are performed and validated by the numerical experiments.

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