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A CONFORMING DG METHOD FOR THE BIHARMONIC EQUATION ON POLYTOPAL MESHES

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Abstract. A conforming discontinuous Galerkin finite element method is introduced for solving the biharmonic equation. This method, by its name, uses discontinuous approximations and keeps simple formulation of the conforming finite element method at the same time. The ultra simple formulation of the method will reduce programming complexity in practice. Optimal order error estimates in a discrete H^2 norm is established for the corresponding finite element solutions. Error estimates in the L^2 norm are also derived with a sub-optimal order of convergence for the lowest order element and an optimal order of convergence for all high order of elements. Numerical results are presented to confirm the theory of convergence.

Key words. finite element methods, weak Laplacian, biharmonic equations, polyhedral meshes.

1. Introduction

We consider the biharmonic equation of the form

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is a bounded polytopal domain in $\mathbb{R}^d$.

The weak formulation of the boundary value problem (2) and (3) is seeking $u \in H_0^2(\Omega)$ satisfying

$$(\Delta u, \Delta v) = (f, v) \qquad \forall v \in H_0^2(\Omega). \tag{4}$$

The H^2 conforming finite element method for the problem (1)-(3) keeps the same simple form as in (4): find $u_h \in V_h \subset H_0^2(\Omega)$ such that

$$(\Delta u_h, \Delta v) = (f, v) \quad \forall v \in V_h. \tag{5}$$

The early works in the area of finite elements (in 1960s) are mostly constructions of conforming and nonconforming elements for solving the biharmonic equation, for example, the Argyris element (1968), the Bell element (1969), the Bogner-Fox-Schmit rectangle (1965), the Hsieh-Clough-Tocher element (1965), the Fraeijs de Veubeke-Sander (1964) and the Morley element (1969), cf. [1, 2, 3, 11, 16, 44, 34]. Many more publications on C^1-conforming and nonconforming finite elements can be found in [12, 14, 19, 22, 24, 25, 26, 27, 28, 29, 31, 33, 40, 41, 42, 47, 48, 56, 57, 58, 59, 60, 61, 62]. Some alternative methods are the interior penalty discontinuous Galerkin finite element methods [13, 17, 18, 35, 36, 46], the mixed finite elements of two Laplacians [4, 8, 9, 10, 15, 32, 43, 45], the Hellan-Herrmann-Johnson element [5, 20, 21, 30], and $H(\text{div}\mathbf{div})$ mixed finite elements [6, 7, 23, 55].

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An approach of avoiding construction of H^2-conforming elements is to use discontinuous approximations. Due to the flexibility of discontinuous Galerkin (DG) finite element methods in element constructions and in mesh generations, many finite element methods have been developed using totally discontinuous polynomials. Here we are only interested in interior penalty discontinuous Galerkin (IPDG) methods since the proposed the method shares the same finite element spaces with IPDG method [13, 17, 18, 35, 36, 46]. One obvious disadvantage of discontinuous finite element methods is their rather complicated formulations which are often necessary to guarantee well posedness and convergence of the methods. For example, the symmetric IPDG method for the biharmonic equation with homogenous boundary conditions [13, 17] has the following formulation:

$$\begin{aligned}(\Delta u_h, \Delta v)_{\mathcal{T}_h} \quad &+ \quad \sum \int_e (\{\nabla\Delta u_h\}\cdot[v] + \{\nabla\Delta v\}\cdot[u_h])ds \\ &+ \quad \sum \int_e (\{\Delta u_h\}\cdot[\nabla v] + \{\Delta v\}\cdot[\nabla u_h])ds \\ &+ \quad \sum \int_e (\sigma[u_h]\cdot[v] + \tau[\nabla u_h][\nabla v])ds = (f, v), \end{aligned} \tag{6}$$

where σ and τ are two parameters that need to be tuned.

The purpose of this work is to introduce a conforming DG finite element method for the biharmonic equation which has the following ultra simple formulation without any stabilizing/penalty terms and other mixed terms of lower dimension integrals in (6):

$$(\Delta_w u_h,\ \Delta_w v) = (f,\ v), \tag{7}$$

where Δ_w is called weak Laplacian, an approximation of Δ. The formulation (7) can be viewed as a counterpart of (5) for discontinuous approximations. The conforming DG method was first introduced in [52, 53] for second order elliptic equations. A conforming DG method, by name, means the method using the simple formulation of the conforming finite element method and the spaces of some DG finite element methods. That is, in the finite element equations, there is no penalty term neither any consistence-error control term.

This conforming DG finite element method (7) shares the same finite element space with the IPDG methods but having much simpler formulation. This simple formulation is obtained by defining weak Laplacian Δ_w appropriately. The idea here is to raise the degree of polynomials used to compute the weak Laplacian Δ_w. Using higher degree polynomials in computing the weak Laplacian will not change the size, neither the global sparsity of the stiffness matrix. Optimal order error estimates in a discrete H^2 for $k \geq 2$ and in L^2 norm for $k > 2$ are established for the corresponding finite element solutions. Numerical results are provided, confirming the theory.

The analysis of conforming DG finite element methods is based on that of the weak Galerkin finite element methods. [37] is the first weak Galerkin finite element work on the biharmonic equation. [39] is on the weak Galerkin method with the C^0 finite element spaces, for solving the biharmonic equation. A stabilizer free weak Galerkin method for the biharmonic equation is designed in [54]. This work is based on [54], by eliminating further the auxiliary variables of the function and the normal derivative on the inter-element edges or polygons.

2. A Conforming DG Finite Element Method

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension or polyhedra in three dimension of size h, satisfying a set of conditions defined in [49] and additional conditions specified in [54]. For simplicity, in this work, we assume every polygon is convex, contains a circle of radius ch and has every edge of length ch, in 2D. In 3D, we assume every polyhedron is convex, contains a sphere of radius ch and has every face-polygon satisfying the 2D conditions above. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges or flat faces.

For simplicity, we adopt the following notations,

$$\begin{aligned}(v,w)_{\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h}(v,w)_T = \sum_{T\in\mathcal{T}_h}\int_T vwd\mathbf{x},\\ \langle v,w\rangle_{\partial\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h}\langle v,w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h}\int_{\partial T} vwds.\end{aligned}$$

Let $P_k(K)$ consist all the polynomials degree less or equal to k defined on K.

We define a finite element space V_h for $k \geq 2$ as follows

$$V_h = \left\{ v \in L^2(\Omega) :\ v|_T \in P_k(T)\ \ T \in \mathcal{T}_h \right\}. \tag{8}$$

Let T_1 and T_2 be two polygons/polyhedrons sharing e if $e \in \mathcal{E}_h^0$. Let v and $\mathbf{q}$ be scalar and vector valued functions, the jumps $[v]$ and $[\mathbf{q}]$ are defined as

$$[v] = v|_{T_1}\mathbf{n}_1 + v|_{T_2}\mathbf{n}_2, \quad [\mathbf{q}] = \mathbf{q}|_{T_1}\cdot\mathbf{n}_1 + \mathbf{q}|_{T_2}\cdot\mathbf{n}_2, \tag{9}$$

and the averages $\{v\}$ and $\{\mathbf{q}\}$ are defined as

$$\{v\} = \frac{1}{2}(v|_{T_1} + v|_{T_2}) \quad \{\mathbf{q}\} = \frac{1}{2}(\mathbf{q}|_{T_1} + \mathbf{q}|_{T_2}). \tag{10}$$

If e is on $\partial\Omega$, then

$$\{v\} = 0, \quad \{\mathbf{q}\} = 0, \quad [v] = v\mathbf{n}, \quad [\mathbf{q}] = \mathbf{q}\cdot\mathbf{n}. \tag{11}$$

The new conforming DG finite element method for the biharmonic equation (1)-(3) is defined as follows.

Weak Galerkin Algorithm 1. *A numerical approximation for (1)-(3) can be obtained by seeking $u_h \in V_h$ satisfying the following equation:*

$$(\Delta_w u_h,\ \Delta_w v)_{\mathcal{T}_h} = (f,\ v) \quad \forall v \in V_h. \tag{12}$$

Next we will discuss how to compute the weak Laplacian $\Delta_w u_h$ and $\Delta_w v$ in (12). The concept of weak derivative was first introduced in [50, 49] for weak functions in weak Galerkin methods and was modified in [38, 51]. A weak Laplacian operator, denoted by

$$\Delta_w : V_h \to \{q \in L^2(\Omega) : q|_T \in P_j(T)\}$$

for some $j > k$ to be specified late, is defined as the unique solution of the following equation

$$(\Delta_w v,\ \varphi)_T = (v,\ \Delta\varphi)_T - \langle\{v\},\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle\{\nabla v\}\cdot\mathbf{n}, \varphi\rangle_{\partial T} \quad \forall\varphi \in P_j(T). \tag{13}$$

We remark again that $\{v\}$ and $\{\nabla v\}$ are the averages on two neighboring elements. Thus $\Delta_w v$ is defined locally but supported globally on a patch of elements.

Lemma 2.1. *Let $\phi \in H^2(\Omega)$, then on any $T \in \mathcal{T}_h$,*

$$\Delta_w \phi = \mathbb{Q}_h(\Delta\phi), \tag{14}$$

where $\mathbb{Q}_h$ is a locally defined L^2 projections onto $P_j(T)$ on each element $T \in \mathcal{T}_h$.

Proof. It is not hard to see that for any $\tau \in P_j(T)$ we have

$$\begin{aligned}(\Delta_w\phi,\ \tau)_T &= (\phi,\ \Delta\tau)_T + \langle\{\nabla\phi\}\cdot\mathbf{n},\ \tau\rangle_{\partial T} - \langle\{\phi\},\ \nabla\tau\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\phi, \Delta\tau)_T + \langle\nabla\phi\cdot\mathbf{n},\ \tau\rangle_{\partial T} - \langle\phi,\ \nabla\tau\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\Delta\phi,\ \tau)_T = (\mathbb{Q}_h\Delta\phi,\ \tau)_T,\end{aligned}$$

which implies (14). We complete the proof. □

3. Well Posedness

First we define a semi-norm $|\!|\!| \cdot |\!|\!|$ as

$$|\!|\!| v |\!|\!|^2 = (\Delta_w v, \Delta_w v)_{\mathcal{T}_h}. \tag{15}$$

Then we introduce a discrete H^2 norm as follows:

$$\|v\|_{2,h} = \left(\sum_{T\in\mathcal{T}_h} \left(\|\Delta v\|_T^2 + h_T^{-3}\|[v]\|_{\partial T}^2 + h_T^{-1}\|[\nabla v]\|_{\partial T}^2\right)\right)^{\frac{1}{2}}. \tag{16}$$

The following lemma indicates that the two norms $\|\cdot\|_{2,h}$ and $|\!|\!| \cdot |\!|\!|$ are equivalent.

First we need the following trace inequality. For any function $\varphi \in H^1(T)$, the trace inequality holds true (see [49] for details):

$$\|\varphi\|_e^2 \le C\left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{17}$$

Lemma 3.1. *There exist two positive constants C_1 and C_2 such that for any $v \in V_h$, we have*

$$C_1\|v\|_{2,h} \le |\!|\!| v |\!|\!| \le C_2\|v\|_{2,h}, \tag{18}$$

where the two norms are defined in (15) *and* (16).

Proof. For any $v \in V_h$, it follows from the definition of weak Laplacian (13) and integration by parts that

$$\begin{aligned}(\Delta_w v,\ \varphi)_T &= (v,\ \Delta\varphi)_T - \langle\{v\},\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle\{\nabla v\}\cdot\mathbf{n},\ \varphi\rangle_{\partial T}\\ &= -(\nabla v,\ \nabla\varphi)_T + \langle v - \{v\},\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle\{\nabla v\}\cdot\mathbf{n},\ \varphi\rangle_{\partial T}\\ &= (\Delta v,\ \varphi)_T + \langle v - \{v\},\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle(\{\nabla v\} - \nabla v)\cdot\mathbf{n},\ \varphi\rangle_{\partial T}.\end{aligned} \tag{19}$$

By letting $\varphi = \Delta_w v$ in (19) we arrive at

$$\|\Delta_w v\|_T^2 = (\Delta v,\ \Delta_w v)_T + \langle v - \{v\},\ \nabla(\Delta_w v)\cdot\mathbf{n}\rangle_{\partial T} + \langle(\{\nabla v\} - \nabla v)\cdot\mathbf{n},\ \Delta_w v\rangle_{\partial T}$$

It is easy to see that the following equations hold true for v on T with $e \subset \partial T$,

$$\|v - \{v\}\|_e = \|[v]\|_e \quad \text{if } e \subset \partial\Omega, \quad \|v - \{v\}\|_e = \frac{1}{2}\|[v]\|_e \ \text{ if } e \in \mathcal{E}_h^0. \tag{20}$$

and

$$\begin{aligned}\|(\nabla v - \{\nabla v\})\cdot\mathbf{n}\|_e &= \|[\nabla v]\|_e \quad \text{if } e \subset \partial\Omega,\\ |(\nabla v - \{\nabla v\})\cdot\mathbf{n}\|_e &= \frac{1}{2}\|[\nabla v]\|_e \ \text{ if } e \in \mathcal{E}_h^0.\end{aligned} \tag{21}$$

From the trace inequality (17), (20)-(21) and the inverse inequality we have

$$\begin{aligned}\|\Delta_w v\|_T^2 &\le \|\Delta v\|_T\|\Delta_w v\|_T + \|v-\{v\}\|_{\partial T}\|\nabla(\Delta_w v)\|_{\partial T}\\ &\quad +\|(\{\nabla v\}-\nabla v)\cdot\mathbf{n}\|_{\partial T}\|\Delta_w v\|_{\partial T}\\ &\le C(\|\Delta v\|_T + h_T^{-3/2}\|[v]\|_{\partial T} + h_T^{-1/2}\|[\nabla v]\|_{\partial T})\|\Delta_w v\|_T,\end{aligned}$$

which implies

$$\|\Delta_w v\|_T \le C\left(\|\Delta v\|_T + h_T^{-3/2}\|[v]\|_{\partial T} + h_T^{-1/2}\|[\nabla v]\|_{\partial T}\right),$$

and consequently

$$|\!|\!|v|\!|\!| \le C_2\|v\|_{2,h}.$$

Next we will prove

$$\sum h_T^{-3}\|[v]\|_{\partial T}^2 \le C|\!|\!|v|\!|\!|.$$

It follows from (19) that for any $\varphi\in P_j(T)$,

$$\begin{aligned}(\Delta_w v,\ \varphi)_T &= (\Delta v,\ \varphi)_T + \langle v-\{v\},\ \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T}\\ &\quad + \langle(\{\nabla v\}-\nabla v)\cdot\mathbf{n},\ \varphi\rangle_{\partial T}.\end{aligned} \tag{22}$$

By Lemma 3.1 in [54], there exist a φ_0 such that for $e\subset\partial T$,

$$\begin{aligned}(\Delta v,\varphi_0)_T &= 0,\\ (\{\nabla v\}-\nabla v)\cdot\mathbf{n},\ \varphi_0\rangle_{\partial T} &= 0,\\ \langle v-\{v\},\nabla\varphi_0\cdot\mathbf{n}\rangle_{\partial T\setminus e} &= 0,\\ \langle v-\{v\},\nabla\varphi_0\cdot\mathbf{n}\rangle_{\partial T} &= \|v-\{v\}\|_e^2,\\ \|\varphi_0\|_T &\le Ch_T^{3/2}\|v-\{v\}\|_e.\end{aligned} \tag{23}$$

Letting $\varphi=\varphi_0$ in (22) yields

$$\|v-\{v\}\|_e^2 = (\Delta_w v,\ \varphi_0)_T \le \|\Delta_w v\|_T\|\varphi_0\|_T \le Ch_T^{3/2}\|\Delta_w v\|_T\|v-\{v\}\|_e,$$

where ϕ_0 is defined in (23). It implies

$$\|v-\{v\}\|_e \le Ch_T^{3/2}\|\Delta_w v\|_T.$$

Taking the summation of the above equation over $T\in\mathcal{T}_h$ and using (20), one has

$$\sum_{T\in\mathcal{T}_h} h_T^{-3}\|[v]\|_{\partial T}^2 \le C|\!|\!|v|\!|\!|^2. \tag{24}$$

Similarly, by Lemma 3.2 in [54], we can have

$$\sum_{T\in\mathcal{T}_h} h_T^{-1}\|[\nabla v]\|_{\partial T}^2 \le C|\!|\!|v|\!|\!|^2. \tag{25}$$

Finally, by letting $\varphi=\Delta v$ in (22) we arrive at

$$\begin{aligned}\|\Delta v\|_T^2 &= (\Delta v,\ \Delta_w v)_T - \langle v-\{v\},\ \nabla(\Delta_w v)\cdot\mathbf{n}\rangle_{\partial T}\\ &\quad -\langle(\{\nabla v\}-\nabla v)\cdot\mathbf{n},\ \Delta_w v\rangle_{\partial T}.\end{aligned}$$

Using the trace inequality (17), the inverse inequality and (24)-(25), one has

$$\|\Delta v\|_T^2 \le C\|\Delta_w v\|_T\|\Delta v\|_T,$$

which gives

$$\sum_{T\in\mathcal{T}_h}\|\Delta v\|_T^2 \le C|\!|\!|v|\!|\!|^2.$$

We complete the proof. □

Lemma 3.2. *The finite element scheme (12) has a unique solution.*

Proof. It suffices to show that the solution of (12) is trivial if $f = g = \phi = 0$. It follows that

$$(\Delta_w u_h, \Delta_w u_h)_{\mathcal{T}_h} = 0.$$

Then the norm equivalence (18) implies $\|u_h\|_{2,h} = 0$, i.e.

$$\sum_{T\in\mathcal{T}_h} \|\Delta u_h\|_T^2 = 0, \quad \sum_{T\in\mathcal{T}_h} h_T^{-3}\|[u_h]\| = 0, \quad \sum_{T\in\mathcal{T}_h} h_T^{-1}\|[\nabla u_h]\| = 0.$$

Therefore, u_h is a smooth harmonic function on Ω and $u_h = 0$ on $\partial\Omega$. Thus we have $u_h = 0$, which completes the proof. □

4. An Error Equation

Let $e_h = u - u_h$. Next we derive an error equation that e_h satisfies.

Lemma 4.1. *For any $v \in V_h$, we have*

$$(\Delta_w e_h, \Delta_w v)_{\mathcal{T}_h} = \ell_1(u, v) + \ell_2(u, v), \tag{26}$$

where

$$\begin{aligned}
\ell_1(u, v) &= \langle \nabla(\mathbb{Q}_h\Delta u - \Delta u)\cdot\mathbf{n}, v - \{v\}\rangle_{\partial T_h},\\
\ell_2(u, v) &= \langle \Delta u - \mathbb{Q}_h\Delta u, (\nabla v - \{\nabla v\})\cdot\mathbf{n}\rangle_{\partial T}.
\end{aligned}$$

Proof. Testing (1) by $v \in V_h$ and using the fact that $\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta u)\cdot\mathbf{n}, \{v\}\rangle_{\partial T} = 0$ and $\sum_{T\in\mathcal{T}_h}\langle\Delta u, \{\nabla v\}\cdot\mathbf{n}\rangle_{\partial T} = 0$ and integration by parts, we arrive at

$$\begin{aligned}
(f, v) &= (\Delta^2 u, v)_{\mathcal{T}_h}\\
&= (\Delta u, \Delta v)_{\mathcal{T}_h} - \langle\Delta u, \nabla v\cdot\mathbf{n}\rangle_{\partial T_h} + \langle\nabla(\Delta u)\cdot\mathbf{n}, v\rangle_{\partial T_h}\\
&= (\Delta u, \Delta v)_{\mathcal{T}_h} - \langle\Delta u, (\nabla v - \{\nabla v\})\cdot\mathbf{n}\rangle_{\partial T_h}\\
&\quad + \langle\nabla(\Delta u)\cdot\mathbf{n}, v - \{v\}\rangle_{\partial T_h}.
\end{aligned} \tag{27}$$

Next we investigate the term $(\Delta u, \Delta v)_{\mathcal{T}_h}$ in the above equation. Using (14), integration by parts and the definition of weak Laplacian (13), we have

$$\begin{aligned}
&(\Delta u, \Delta v)_{\mathcal{T}_h} = (\mathbb{Q}_h\Delta u, \Delta v)_{\mathcal{T}_h}\\
=\ & (v, \Delta(\mathbb{Q}_h\Delta u))_T + \langle\nabla v\cdot\mathbf{n},\ \mathbb{Q}_h\Delta u\rangle_{\partial T} - \langle v, \nabla(\mathbb{Q}_h\Delta u)\cdot\mathbf{n}\rangle_{\partial T}\\
=\ & (\Delta_w v,\ \mathbb{Q}_h\Delta u)_T - \langle v - \{v\}, \nabla(\mathbb{Q}_h\Delta u)\cdot\mathbf{n}\rangle_{\partial T} + \langle(\nabla v - \{\nabla v\})\cdot\mathbf{n}, \mathbb{Q}_h\Delta u\rangle_{\partial T}\\
=\ & (\Delta_w u,\ \Delta_w v)_T - \langle v - \{v\}, \nabla(\mathbb{Q}_h\Delta u)\cdot\mathbf{n}\rangle_{\partial T} + \langle(\nabla v - \{\nabla v\})\cdot\mathbf{n}, \mathbb{Q}_h\Delta u\rangle_{\partial T}.
\end{aligned}$$

Combining the above two equations gives

$$\begin{aligned}
(f, v) &= (\Delta^2 u, v)_{\mathcal{T}_h}\\
&= (\Delta_w u,\ \Delta_w v)_{\mathcal{T}_h} - \langle v - \{v\}, \nabla(\mathbb{Q}_h\Delta u - \Delta u)\cdot\mathbf{n}\rangle_{\partial T_h}\\
&\quad - \langle(\nabla v - \{\nabla v\})\cdot\mathbf{n}, \Delta u - \mathbb{Q}_h\Delta u\rangle_{\partial T},
\end{aligned} \tag{28}$$

which implies that

$$(\Delta_w u,\ \Delta_w v)_{\mathcal{T}_h} = (f, v) + \ell_1(u, v) + \ell_2(u, v).$$

The error equation follows from subtracting (12) from the above equation,

$$(\Delta_w e_h,\ \Delta_w v)_{\mathcal{T}_h} = \ell_1(u, v) + \ell_2(u, v).$$

We have proved the lemma. □

5. An Error Estimate in H^2

We start this section by defining some approximation operator. Let Q_h be the element-wise defined L^2 projection onto $P_k(T)$ on each element T.

Lemma 5.1. *Let $k \geq 2$ and $w \in H^{\max\{k+1,4\}}(\Omega)$. There exists a constant C such that the following estimates hold true:*

$$\left(\sum_{T\in\mathcal{T}_h} h_T\|\Delta w - \mathbb{Q}_h\Delta w\|_{\partial T}^2\right)^{\frac{1}{2}} \leq Ch^{k-1}\|w\|_{k+1}, \tag{29}$$

$$\left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w - \mathbb{Q}_h\Delta w)\|_{\partial T}^2\right)^{\frac{1}{2}} \leq Ch^{k-1}(\|w\|_{k+1} + h\delta_{k,2}\|w\|_4). \tag{30}$$

Here $\delta_{i,j}$ is the usual Kronecker's delta with value 1 when $i = j$ and value 0 otherwise.

The above lemma can be proved by using the trace inequality (17) and the definition of $\mathbb{Q}_h$. The proof can also be found in [37].

Lemma 5.2. *Let $w \in H^{\max\{k+1,4\}}(\Omega)$, and $v \in V_h$. There exists a constant C such that*

$$|\ell_1(w,v)| \leq Ch^{k-1}(\|w\|_{k+1} + h\delta_{k,2}\|w\|_4)|\!|\!|v|\!|\!|. \tag{31}$$

$$|\ell_2(w,v)| \leq Ch^{k-1}|w|_{k+1}|\!|\!|v|\!|\!|. \tag{32}$$

Proof. Using the Cauchy-Schwartz inequality, (29), (30), (20), (21) and (18), we have

$$\begin{aligned}
\ell_1(w,v) &= \left|\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta w - \mathbb{Q}_h\Delta w)\cdot\mathbf{n}, v - \{v\}\rangle_{\partial T}\right| \\
&\leq \left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w - \mathbb{Q}_h\Delta w)\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|v - \{v\}\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w - \mathbb{Q}_h\Delta w)\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|[v]\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\leq Ch^{k-1}(\|w\|_{k+1} + h\delta_{k,2}\|w\|_4)|\!|\!|v|\!|\!|, \qquad (33)
\end{aligned}$$

and

$$\begin{aligned}
\ell_2(w,v) = &\left|\sum_{T\in\mathcal{T}_h}\langle\Delta w - \mathbb{Q}_h\Delta w, (\nabla v - \{\nabla v\})\cdot\mathbf{n}\rangle_{\partial T}\right| \\
\leq &\left(\sum_{T\in\mathcal{T}_h} h_T\|\Delta w - \mathbb{Q}_h\Delta w\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|[\nabla v]\|_{\partial T}^2\right)^{\frac{1}{2}} \\
\leq &\ Ch^{k-1}\|w\|_{k+1}|\!|\!|v|\!|\!|. \qquad (34)
\end{aligned}$$

We complete the proof. □

Lemma 5.3. *Let $w \in H^{\max\{k+1,4\}}(\Omega)$, then*

$$|\!|\!|w - Q_h w|\!|\!| \leq Ch^{k-1}|w|_{k+1}. \tag{35}$$

Proof. For any $T \in \mathcal{T}_h$, it follows from (13), integration by parts, (17) and inverse inequality that for $w \in P_j(T)$,

$$\begin{aligned}
\|\Delta_w(w - Q_h w)\|_T^2 &= (\Delta_w(w - Q_h w), \Delta_w(w - Q_h w))_T \\
&= (w - Q_h w, \Delta(\Delta_w(w - Q_h w)))_T \\
&\quad -\langle \{w - Q_h w\}, \nabla(\Delta_w(w - Q_h w)) \cdot \mathbf{n}\rangle_{\partial T} \\
&\quad +\langle \{\nabla w - \nabla Q_h w\} \cdot \mathbf{n}, \Delta_w(w - Q_h w)\rangle_{\partial T} \\
&\leq C(h_T^{-2}\|w - Q_h w\|_T + h_T^{-3/2}\|w - Q_h w\|_{\partial T} \\
&\quad +h_T^{-1/2}\|\nabla w - \nabla Q_h w\|_{\partial T})\|\Delta_w(w - Q_h w)\|_T \\
&\leq Ch^{k-1}|w|_{k+1,T}\|\Delta_w(w - Q_h w)\|_T,
\end{aligned}$$

which implies

$$\|\Delta_w(w - Q_h w)\|_T \leq Ch^{k-1}|w|_{k+1,T}.$$

Taking the summation over $T \in \mathcal{T}_h$, we have proved the lemma. □

Theorem 5.1. *Let $u_h \in V_h$ be the finite element solution arising from (12). Assume that the exact solution $u \in H^{\max\{k+1,4\}}(\Omega)$. Then, there exists a constant C such that*

$$\|\!|u - u_h|\!\| \leq Ch^{k-1}\left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right). \tag{36}$$

Proof. Let $\epsilon_h = Q_h u - u_h$. Then it is straightforward to obtain

$$\begin{aligned}
\|\!|e_h|\!\|^2 &= (\Delta_w e_h, \Delta_w e_h)_{\mathcal{T}_h} \\
&= (\Delta_w e_h, \Delta_w(u - u_h))_{\mathcal{T}_h} \\
&= (\Delta_w e_h, \Delta_w(Q_h u - u_h))_{\mathcal{T}_h} + (\Delta_w e_h, \Delta_w(u - Q_h u))_{\mathcal{T}_h} \\
&= (\Delta_w e_h, \Delta_w \epsilon_h)_{\mathcal{T}_h} + (\Delta_w e_h, \Delta_w(u - Q_h u))_{\mathcal{T}_h}.
\end{aligned} \tag{37}$$

We will bound the term $(\Delta_w e_h, \Delta_w \epsilon_h)_{\mathcal{T}_h}$ on right hand side of (37) first. Letting $v = \epsilon_h \in V_h$ in (26) and using (31)-(32) and (35), we have

$$\begin{aligned}
|(\Delta_w e_h, \Delta_w \epsilon_h)_{\mathcal{T}_h}| &\leq |\ell_1(u, \epsilon_h)| + |\ell_2(u, \epsilon_h)| \\
&\leq Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4)\|\!|\epsilon_h|\!\| \\
&\leq Ch^{k-1}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4)(\|\!|u - Q_h u|\!\| + \|\!|u - u_h|\!\|) \\
&\leq Ch^{2(k-1)}(\|u\|_{k+1}^2 + h^2\delta_{k,2}\|u\|_4^2) + \frac{1}{4}\|\!|e_h|\!\|^2.
\end{aligned} \tag{38}$$

To bound the second term on right hand side of (37), we have by (35),

$$\begin{aligned}
|(\Delta_w e_h, \Delta_w(u - Q_h u))_{\mathcal{T}_h}| &\leq C\|\!|u - Q_h u|\!\|\|\!|e_h|\!\| \\
&\leq Ch^{2(k-1)}\|u\|_{k+1}^2 + \frac{1}{4}\|\!|e_h|\!\|^2.
\end{aligned} \tag{39}$$

Combining the estimates (38) and (39) with (37), we arrive

$$\|\!|e_h|\!\| \leq Ch^{k-1}\left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4\right),$$

which completes the proof. □

6. Error Estimates in L^2 Norm

In this section, we will obtain an error bound for the finite element solution u_h in L^2 norm.

The dual problem considered has the following form,

$$\Delta^2 w = e_h \quad \text{in } \Omega, \tag{40}$$

$$w = 0 \quad \text{on } \partial\Omega, \tag{41}$$

$$\nabla w \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{42}$$

Assume that the H^4 regularity holds,

$$\|w\|_4 \le C\|e_h\|. \tag{43}$$

Theorem 6.1. *Let $u_h \in V_h$ be the finite element solution arising from (12). Assume that the exact solution $u \in H^{k+1}(\Omega)$ and (43) holds true. Then, there exists a constant C such that*

$$\|u - u_h\| \le Ch^{k+1-\delta_{k,2}}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4). \tag{44}$$

Proof. Testing (40) by e_h and using the fact that $\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta w)\cdot\mathbf{n},\{e_h\}\rangle_{\partial T} = 0$ and $\sum_{T\in\mathcal{T}_h}\langle\Delta w,\{\nabla e_h\}\cdot\mathbf{n}\rangle_{\partial T} = 0$ and integration by parts, we arrive at

$$\begin{aligned}
\|e_h\|^2 &= (\Delta^2 w, e_h)_{\mathcal{T}_h}\\
&= (\Delta w, \Delta e_h)_{\mathcal{T}_h} - \langle\Delta w, \nabla e_h\cdot\mathbf{n}\rangle_{\partial T_h} + \langle\nabla(\Delta w)\cdot\mathbf{n}, e_h\rangle_{\partial T_h}\\
&= (\Delta w, \Delta e_h)_{\mathcal{T}_h} - \langle\Delta w, (\nabla e_h - \{\nabla e_h\})\cdot\mathbf{n}\rangle_{\partial T_h}\\
&+ \langle\nabla(\Delta w)\cdot\mathbf{n}, e_h - \{e_h\}\rangle_{\partial T_h}.\\
&= (\mathbb{Q}_h\Delta w, \Delta e_h)_{\mathcal{T}_h} + (\Delta w - \mathbb{Q}_h\Delta w, \Delta e_h)_{\mathcal{T}_h}\\
&- \langle\Delta w, (\nabla e_h - \{\nabla e_h\})\cdot\mathbf{n}\rangle_{\partial T_h} + \langle\nabla(\Delta w)\cdot\mathbf{n}, e_h - \{e_h\}\rangle_{\partial T_h}.
\end{aligned}$$

It follows from integration by parts, the definition of weak Laplacian (13) and (14),

$$\begin{aligned}
&(\mathbb{Q}_h\Delta w, \Delta e_h)_{\mathcal{T}_h}\\
&= (e_h, \Delta(\mathbb{Q}_h\Delta w))_T + \langle\nabla e_h\cdot\mathbf{n},\ \mathbb{Q}_h\Delta w\rangle_{\partial T} - \langle e_h, \nabla(\mathbb{Q}_h\Delta w)\cdot\mathbf{n}\rangle_{\partial T}\\
&= (\Delta_w e_h,\ \mathbb{Q}_h\Delta w)_T - \langle e_h - \{e_h\}, \nabla(\mathbb{Q}_h\Delta w)\cdot\mathbf{n}\rangle_{\partial T}\\
&\quad + \langle(\nabla e_h - \{\nabla e_h\})\cdot\mathbf{n}, \mathbb{Q}_h\Delta w\rangle_{\partial T}\\
&= (\Delta_w w,\ \Delta_w e_h)_T - \langle e_h - \{e_h\}, \nabla(\mathbb{Q}_h\Delta w)\cdot\mathbf{n}\rangle_{\partial T}\\
&\quad + \langle(\nabla e_h - \{\nabla e_h\})\cdot\mathbf{n}, \mathbb{Q}_h\Delta w\rangle_{\partial T}.
\end{aligned}$$

Combining the two equations above implies

$$\begin{aligned}
\|e_h\|^2 &= (\Delta_w w,\ \Delta_w e_h)_{\mathcal{T}_h} + (\Delta w - \mathbb{Q}_h\Delta w, \Delta e_h)_{\mathcal{T}_h}\\
&\quad + \ell_1(w, e_h) + \ell_2(w, e_h).
\end{aligned}$$

By simple manipulation and (26), we have

$$\begin{aligned}
(\Delta_w w,\ \Delta_w e_h)_{\mathcal{T}_h} &= (\Delta_w Q_h w,\ \Delta_w e_h)_{\mathcal{T}_h} + (\Delta_w(w - Q_h),\ \Delta_w e_h)_{\mathcal{T}_h}\\
&= \ell_1(u, Q_h w) + \ell_2(u, Q_h w) + (\Delta_w(w - Q_h),\ \Delta_w e_h)_{\mathcal{T}_h}.
\end{aligned}$$

Combining the two equations above implies

$$\begin{aligned}
\|e_h\|^2 &= \ell_1(u, Q_h w) + \ell_2(u, Q_h w) + (\Delta_w e_h,\ \Delta_w(w - Q_h w))_{\mathcal{T}_h}\\
&\quad + (\Delta w - \mathbb{Q}_h\Delta w, \Delta e_h)_{\mathcal{T}_h} + \ell_1(w, \epsilon_h) + \ell_2(w, \epsilon_h)\\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

Next, we will estimate all the terms on the right hand side of the above equation. Using the Cauchy-Schwartz inequality, (29)-(30), (20) and (33), we have

$$
\begin{aligned}
I_1 &= \ell_1(u, Q_h w) = \left| \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u)\cdot \mathbf{n}, Q_h w - \{Q_h w\}\rangle_{\partial T} \right| \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T^3 \|\nabla(\Delta u - \mathbb{Q}_h\Delta u)\|_{\partial T}^2 \right)^{\frac12} \left(\sum_{T\in\mathcal{T}_h} h_T^{-3} \|[Q_h w]\|_{\partial T}^2 \right)^{\frac12} \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T^3 \|\nabla(\Delta u - \mathbb{Q}_h\Delta u)\|_{\partial T}^2 \right)^{\frac12} \left(\sum_{T\in\mathcal{T}_h} h_T^{-3} \|[Q_h w - w]\|_{\partial T}^2 \right)^{\frac12} \\
&\le C h^{k+1-\delta_{k,2}} \left(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4 \right) \|w\|_4,
\end{aligned}
$$

Similarly, by the Cauchy-Schwartz inequality, (29)-(30), (21) and (34), we have

$$
\begin{aligned}
I_2 &= \ell_2(u, Q_h w) = \left| \sum_{T\in\mathcal{T}_h} \langle \Delta u - \mathbb{Q}_h \Delta u, (\nabla Q_h w - \{\nabla Q_h w\})\cdot \mathbf{n}\rangle_{\partial T} \right| \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|\Delta u - \mathbb{Q}_h\Delta u\|_{\partial T}^2 \right)^{\frac12} \left(\sum_{T\in\mathcal{T}_h} h_T \|[\nabla Q_h w - \nabla w]\|_{\partial T}^2 \right)^{\frac12} \\
&\le C h^{k+1-\delta_{k,2}} \|u\|_{k+2} \|w\|_4.
\end{aligned}
$$

The estimates (36) and (35) give

$$
I_3 \le C h^{k+1} \|u\|_{k+1} \|w\|_4.
$$

To estimate I_4, we need to bound $\|\Delta e_h\|_T$. By (18), (35), (36) and the definition of Q_h, we have

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h} \|\Delta e_h\|_T^2 &\le \sum_{T\in\mathcal{T}_h} \|\Delta \epsilon_h\|_T^2 + \sum_{T\in\mathcal{T}_h} \|\Delta(u - Q_h u)\|_T^2 \\
&\le C(h^{k-1}\|u\|_{k+1} + |\!|\!|\epsilon_h|\!|\!|^2) \\
&\le C(h^{k-1}\|u\|_{k+1} + |\!|\!|e_h|\!|\!|^2 + |\!|\!|Q_h u - u|\!|\!|^2) \\
&\le C h^{k-1}\|u\|_{k+1}.
\end{aligned}
$$

The above estimate and the definition of $\mathbb{Q}_h$ imply

$$
I_4 \le C h^{k+1} \|u\|_{k+1} \|w\|_4.
$$

Using the Cauchy-Schwartz inequality, (20), (29), (18), (35) and (36), we have

$$\begin{aligned} I_5 &= \ell_1(w,e_h) = \left|\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta w-\mathbb{Q}_h\Delta w)\cdot\mathbf{n}, e_h-\{e_h\}\rangle_{\partial T}\right| \\ &\le \left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w-\mathbb{Q}_h\Delta w)\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|e_h-\{e_h\}\|_{\partial T}^2\right)^{\frac12} \\ &\le \left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta w-\mathbb{Q}_h\Delta w)\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|[e_h]\|_{\partial T}^2\right)^{\frac12} \\ &\le Ch^2\|w\|_4\left(|||\epsilon_h||| + (\sum_{T\in\mathcal{T}_h} h_T^{-3}\|[Q_h u-u]\|_{\partial T}^2)^{\frac12}\right) \\ &\le Ch^2\|w\|_4\left(|||e_h||| + |||Q_h u-u||| + (\sum_{T\in\mathcal{T}_h} h_T^{-3}\|[Q_h u-u]\|_{\partial T}^2)^{\frac12}\right) \\ &\le Ch^{k+1}\|u\|_{k+1}\|w\|_4. \end{aligned}$$

Similarly, we obtain

$$I_6 \le Ch^{k+1}\|u\|_{k+1}\|w\|_4.$$

Combining all the estimates above yields

$$\|e_h\|^2 \le Ch^{k+1-\delta_{k,2}}\|u\|_{k+1}\|w\|_4.$$

It follows from the above inequality and the regularity assumption (43).

$$\|e_h\| \le Ch^{k+1-\delta_{k,2}}\|u\|_{k+1}.$$

We have completed the proof. □

7. Numerical Experiments

We solve the following biharmonic equation by P_k conforming DG finite element methods:

$$\Delta^2 u = f, \quad (x,y)\in\Omega=(0,1)^2, \tag{45}$$

with the boundary conditions $u=g_1$ and $\nabla u\cdot\mathbf{n}=g_2$ on $\partial\Omega$. We choose f, g_1 and g_2 so that the exact solution is

$$u = e^{x+y}.$$

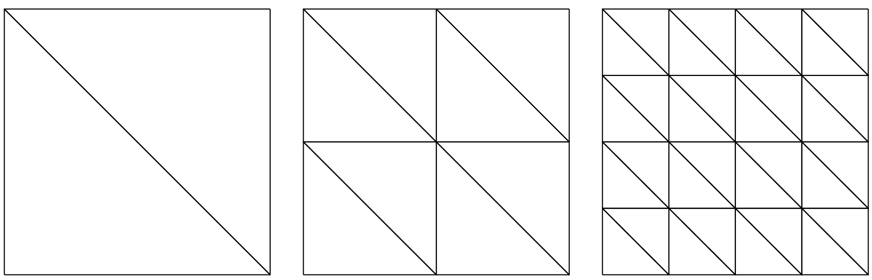

FIGURE 1. The first three levels of grids used in the computation of Table 1.

TABLE 1. The error and the order of convergence for (45) on triangular grids (Figure 1)

level	$\|u_h - u\|_0$	rate	$\|u_h - u\|_{1,h}$	rate	$\|\|\|u_h - u\|\|\|$	rate
	by the P_2 conforming DG finite element					
4	0.3653E-03	2.0	0.3281E-02	1.9	0.1229E+01	0.9
5	0.9566E-04	1.9	0.8733E-03	1.9	0.6312E+00	1.0
6	0.2480E-04	1.9	0.2268E-03	1.9	0.3199E+00	1.0
	by the P_3 conforming DG finite element					
2	0.2291E-03	4.4	0.3275E-02	3.1	0.1612E+00	2.0
3	0.1143E-04	4.3	0.3889E-03	3.1	0.4577E-01	1.8
4	0.7148E-06	4.0	0.4743E-04	3.0	0.1243E-01	1.9

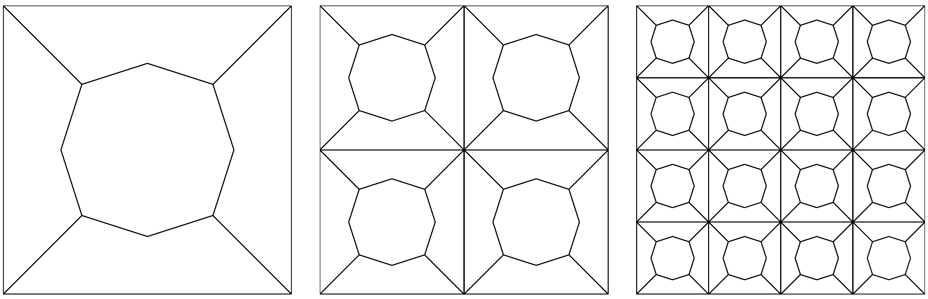

FIGURE 2. The first three polygonal grids for the computation of Table 2.

TABLE 2. Error profiles and convergence rates for (45) on polygonal grids (Figure 2)

level	$\|u_h - u\|_0$	rate	$\|u_h - u\|_{1,h}$	rate	$\|\|\|u_h - u\|\|\|$	rate
	by the P_2 conforming DG finite element					
4	0.3171E-03	1.9	0.4537E-02	1.9	0.3286E+01	1.0
5	0.8671E-04	1.9	0.1175E-02	1.9	0.1647E+01	1.0
6	0.2428E-04	1.8	0.3023E-03	2.0	0.8243E+00	1.0
	by the P_3 conforming DG finite element					
1	0.3402E-02	0.0	0.3868E-01	0.0	0.3702E+01	0.0
2	0.2027E-03	4.1	0.4895E-02	3.0	0.9408E+00	2.0
3	0.1476E-04	3.8	0.6244E-03	3.0	0.2368E+00	2.0

In the first computation, the first three levels of grids are plotted in Figure 1. The error and the order of convergence for the method are listed in Tables 1. Here on triangular grids, we compute the weak Laplacian $\Delta_w v$ by P_{k+2} polynomials. The numerical results confirm the convergence theory.

In the next computation, we use a family of polygonal grids (with pentagons and of 8-side polygons) shown in Figure 2. We note that the polygons in meshes of Figure 2 are not convex. In the analysis, we assume convex polygons for simplicity only. The theory can be extended to cover such non-convex but good-shape polygons. We let the polynomial degree $j = k + 3$ for the weak Laplacian on such polygonal meshes. The rate of convergence is listed in Table 2. The convergence history confirms the theory.

References

[1] J. H. Argyris, I. Fried and D. W. Scharpf, The TUBA family of plate elements for the matrix displacement method, The Aeronautical Journal of the Royal Aeronautical Society 72 (1968), 514–517.

[2] K. Bell, A refined triangular plate bending element, Internal. J. Numer. methods Engrg, 1 (1969), 101–122.

[3] F. K. Bogner, R. L. Fox and L. A. Schmit, The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulas, Proceedings of the Conference on Matrix Methods in Structural Mechanics, Wright Patterson A.F.B. Ohio, 1965.

[4] F. Brezzi and P. A. Raviart, Mixed Finite Element Methods for 4th Order Elliptic Equations, in Topics in Numerical Analysis, Vol.III, J. J. H. MILLER, Ed., Academic Press, 1978.

[5] L. Chen, J. Hu and X. Huang, Multigrid methods for Hellan-Herrmann-Johnson mixed method of Kirchhoff plate bending problems, J. Sci. Comput. 76 (2018), no. 2, 673-696.

[6] L. Chen and X. Huang, Finite elements for divdiv conforming symmetric tensors in three dimensions, Math. Comp. 91 (2022), no. 335, 1107–1142.

[7] L. Chen and X. Huang, Finite elements for div- and divdiv-conforming symmetric tensors in arbitrary dimension, SIAM J. Numer. Anal. 60 (2022), no. 4, 1932–1961.

[8] X.-l. Cheng, W. Han and H.-c. Huang, Some mixed finite element methods for biharmonic equation, Journal of Computational and Applied Mathematics 126 (2000) 91-109.

[9] P. G. Ciarlet, Conforming and Nonconforming Finite Element Methods for Solving the Plate Problem, in Numerical Solution of Differential Equations, G. A.WATSON, Ed., Springer, 1974.

[10] P.G. Ciarlet and P.A. Raviart, A mixed Finite Element Method for the Biharmonic Equation, in Mathematical Aspects of Finite Elements in Partial Differential Equations, C. DEBOOR, Ed., Academic Press, 1974.

[11] R.W. Clough and J.L. Tocher, Finite element stiffness matrices for analysis of plates in bending, in: Proceedings of the Conference on Matrix Methods in Structural Mechanics, Wright Patterson A.F.B. Ohio, 1965.

[12] M. Cui and S. Zhang, On the Uniform Convergence of the Weak Galerkin Finite Element Method for a Singularly-Perturbed Biharmonic Equation, J. Sci. Comput. 82 (2020), no. 1, Art. 5, 15 pp.

[13] Z. Dong, Discontinuous Galerkin methods for the biharmonic problem on polygonal and polyhedral meshes, Int. J. Numer. Anal. Model. 16 (2019), no. 5, 825–846.

[14] J. Douglas Jr., T. Dupont, P. Percell and R. Scott, A family of C^1 finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems, RAIRO Anal. Numer. 13 (1979), no. 3, 227–255.

[15] R. Falk, Approximation of the Biharmonic Equation by a Mixed Finite Element Method, SIAM J. Numer. Anal., 15 (1978), no. 3, 556–567.

[16] B. Fraeijs de Veubeke, A conforming finite element for plate bending, in: O.C. Zienkiewicz and G.S. Holister (Eds.), Stress Analysis, Wiley, New York, 1965, 145–197.

[17] E. Georgoulis and P. Houston, Discontinuous Galerkin methods for the biharmonic problem, IMA J. Numer. Anal., 29 (2009), 573-594

[18] E. Georgoulis, P. Houston and J. Virtanen, An a posteriori error indicator for discontinuous Galerkin approximations of fourth-order elliptic problems, IMA J. Numer. Anal., 31 (2011), 281-298.

[19] H. Han, Z. Huang and S. Zhang, An Iterative Method based on equation decomposition for the fourth-order singular perturbation problem, Numerical Methods for Partial Differential Equations, 29 (2013), no. 3, 961–978.

[20] K. Hellan, Analysis of elastic plates in flexure by a simplified finite element method, Acta Polytechnica Scandinavia, Civil Engineering Series, 46, 1967.

[21] L. R. Herrmann, Finite element bending analysis for plates, Journal of the Engineering Mechanics Division, 93(EM5) (1967) 49–83.

[22] J. Hu, Y. Huang and S. Zhang, The lowest order differentiable finite element on rectangular grids, SIAM Num. Anal., 49 (2011), No 4, 1350–1368.

[23] J. Hu, R. Ma and M. Zhang, A family of mixed finite elements for the biharmonic equations on triangular and tetrahedral grids, Sci. China Math. 64 (2021), no. 12, 2793–2816.

[24] J. Hu and Z.-c. Shi, A lower bound of the L2 norm error estimate for the Adini element of the biharmonic equation, SIAM J. Numer. Anal., 51 (2013), 2651-2659.

[25] J. Hu, S. Tian and S. Zhang, A family of 3D H2-nonconforming tetrahedral finite elements for the biharmonic equation, Sci. China Math. 63 (2020), no. 8, 1505–1522.

[26] J. Hu and S. Zhang, The minimal conforming H^k finite element spaces on R^n rectangular grids, Math. Comp. 84 (2015), no. 292, 563–579.

[27] J. Hu and S. Zhang, A cubic H3-nonconforming finite element, Commun. Appl. Math. Comput. 1 (2019), no. 1, 81–100.

[28] J. Hu and S. Zhang, An error analysis method SPP-BEAM and a construction guideline of nonconforming finite elements for fourth order elliptic problems, J. Comput. Math. 38 (2020), no. 1, 195–222.

[29] J. Huang, X. Huang and S. Zhang, A superconvergence of the Morley element via postprocessing, Recent advances in scientific computing and applications, 189-196, Contemp. Math., 586, Amer. Math. Soc., Providence, RI, 2013.

[30] C. Johnson, On the convergence of a mixed finite-element method for plate bending problems, Numer. Math. 21 (1973), 43–62.

[31] P. Lascaux and P. Lesaint, Some Nonconforming Finite Elements for the Plate Bending Problem, RAIRO Anal. Numer., 1 (1975), 9–53.

[32] P. Monk, A Mixed Finite Element Method for the Biharmonic Equation, SIAM J. Numer. Anal., 24(1987), no. 4, 737–749.

[33] J. Morgan and R. Scott, A nodal basis for C^1 piecewise polynomials of degree $n \geq 5$, Math Comp 29 (1975), 736–740.

[34] L. Morley, The triangular equilibrium element in the solution of plate bending problems, Aero. Quart., 19 (1968), 149–169.

[35] I. Mozolevski, E. Suli and P. Bosing, hp-version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation, J. Sci. Comput., 30 (2007), 465-491.

[36] I. Mozolevski and E. Suli, A priori error analysis for the hp-version of the discontinuous Galerkin finite element method for the biharmonic equation, Comput. Methods Appl. Math., 3 (2003), 596-607.

[37] L. Mu, J. Wang, and X. Ye, A weak Galerkin finite element method for biharmonic equations on polytopal meshes, Numer. Meth. PDE, 30 (2014), 1003-1029.

[38] L. Mu, X. Wang and X. Ye, A modified weak Galerkin finite element method for the Stokes equations, J. Comput. Appl. Math., 275 (2015), 79-90.

[39] L. Mu, J. Wang, X. Ye and S. Zhang, A C0-weak Galerkin finite element method for the biharmonic equation, J. Sci. Comput. 59 (2014), no. 2, 473-495.

[40] L. Mu, X. Ye and S. Zhang, Development of a P2 element with optimal L2 convergence for biharmonic equation, Numer. Methods Partial Differential Equations, 35 (2019), no. 4, 1497–1508.

[41] P. Percell, On cubic and quartic Clough-Tocher finite elements, SIAM J. Numer. Anal. 13 (1976), 100–103.

[42] M.J.D. Powell and M.A. Sabin, Piecewise quadratic approximations on triangles, ACM Transactions on Mathematical Software, 3-4 (1977), 316–25.

[43] R. Rannacher, On nonconforming an mixed finite element methods for plate bending problems. The linear case RAIRO. Analyse numérique, tome 13, no 4 (1979), 369-387.

[44] G. Sander, Bornes supérieures et inférieures dans l'analyse matricielle des plaques en flexion-torsion, Bull. Sco. Roy. Sci. Liège., 33 (1964), 456–494.

[45] R. Scholz, A Mixed Method for 4th Order Problems using Linear Finite Elements, RAIRO Anal. Numer., 12 (1978), 85–90.

[46] E. Suli and I. Mozolevski, hp-version interior penalty DGFEMs for the biharmonic equation, Comput. Methods Appl. Mech. Engrg., 196 (2007), 1851–1863.

[47] M. Wang and J. Xu, The Morley element for fourth order elliptic equations in any dimensions, Numer. Math. 103 (2006), no. 1, 155–169.

[48] M. Wang and J. Xu, Minimal finite element spaces for 2m-th-order partial differential equations in Rn. Math. Comp. 82 (2013), no. 281, 25–43.
[49] J. Wang and X. Ye, J. Wang and X. Ye, A Weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp., 83 (2014), 2101-2126.
[50] J. Wang and X. Ye, A weak Galerkin finite element method for second-order elliptic problems. J. Comput. Appl. Math. 241 (2013), 103-115.
[51] X. Wang, N. Malluwawadu, F Gao and T. McMillan, A modified weak Galerkin finite element method, J. Comput. Appl. Math., 217 (2014), 319-327.
[52] X. Ye and S. Zhang, A conforming discontinuous Galerkin finite element method, Int. J. Numer. Anal. Model. 17 (2020) no. 1, 110–117.
[53] X. Ye and S. Zhang, A conforming discontinuous Galerkin finite element method: Part II, Int. J. Numer. Anal. Model. 17 (2020), no. 2, 281–296.
[54] X. Ye and S. Zhang, A stabilizer free weak Galerkin method for the biharmonic equation on polytopal meshes, SIAM J. Numer. Anal. 58 (2020), no. 5, 2572–2588.
[55] X. Ye and S. Zhang, A family of H-div-div mixed triangular finite elements for the biharmonic equation. Results Appl. Math. 15 (2022), Paper No. 100318, 16 pp.
[56] X. Ye, S. Zhang and Z Zhang, A new P1 weak Galerkin method for the biharmonic equation, J. Comput. Appl. Math. 364 (2020), 12337, 10 pp.
[57] A. Ženišek, Alexander Polynomial approximation on tetrahedrons in the finite element method, J. Approximation Theory 7 (1973), 334–351.
[58] A. Ženišek, A general theorem on triangular finite $C^{(m)}$-elements, Rev. Francaise Automat. Informat. Recherche Operationnelle Sr. Rouge 8 (1974), no. R-2, 119–127.
[59] S. Zhang, An optimal order multigrid method for biharmonic, C^1 finite-element equations, Numer. Math. 56 (1989), 613–624.
[60] S. Zhang, A C1-P2 finite element without nodal basis, M2AN 42 (2008), 175–192.
[61] S. Zhang, A family of 3D continuously differentiable finite elements on tetrahedral grids, Applied Numerical Mathematics, 59 (2009), no. 1, 219–233.
[62] S. Zhang, A family of differentiable finite elements on simplicial grids in four space dimensions, Math. Numer. Sin. 38 (2016), no. 3, 309–324.

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