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ORTHOGONAL SPLINE COLLOCATION FOR POISSON'S EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We apply orthogonal spline collocation with splines of degree $r \geq 3$ to solve, on the unit square, Poisson's equation with Neumann boundary conditions. We show that the H^1 norm error is of order r and explain how to compute efficiently the approximate solution using a matrix decomposition algorithm involving the solution of a symmetric generalized eigenvalue problem.

Key words. Poisson's equation, Neumann boundary conditions, orthogonal spline collocation, convergence analysis, matrix decomposition algorithm.

1. Introduction

In this paper we consider Poisson's equation

$$-\Delta u = f(x_1, x_2), \quad (x_1, x_2) \in \Omega = (0,1) \times (0,1), \tag{1}$$

where $\Delta \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ and u satisfies nonhomogeneous Neumann boundary conditions

$$u_{x_1}(\alpha, x_2) = g_1(\alpha, x_2), \quad \alpha = 0, 1, \quad x_2 \in [0,1], \tag{2}$$

$$u_{x_2}(x_1, \beta) = g_2(x_1, \beta), \quad x_1 \in [0,1], \quad \beta = 0, 1. \tag{3}$$

Using (1)–(3) and integrating with respect to x_1 and x_2, we obtain

$$\int_\Omega f(x_1,x_2)dx_1dx_2 + \int_0^1 [g_1(1,x_2) - g_1(0,x_2)]dx_2 + \int_0^1 [g_2(x_1,1) - g_2(x_1,0)]dx_1 = 0, \tag{4}$$

which is a necessary condition for the existence of u satisfying (1)–(3). To guarantee uniqueness of the solution u of (1)–(3), we impose the condition

$$\int_\Omega u(x_1,x_2)dx_1dx_2 = \gamma, \tag{5}$$

where γ in R is specified.

A finite difference scheme for (1)–(3) in section 4.7.2 of [12], involving an extended system of linear equations, is second order accurate in the discrete maximum norm. A finite difference scheme for (1)–(3), (5), described in Theorem 9 on page 327 in [17], involving a finite difference counterpart of (5), is second order accurate in the discrete H^1 norm. It is also shown in Theorem 2 on page 338 in [17] that this scheme is second order accurate in the discrete maximum norm. [1] is concerned with a Galerkin spectral solver for the Neumann problem for the constant coefficient Helmholtz equation on a rectangle. Finite element schemes for solving the pure Neumann problem on a bounded domain Ω are discussed in [13]. The present paper is a generalization of [4] to nonzero Neumann boundary conditions and splines of arbitrary degree $r \geq 3$. Moreover, in comparison to [4], the scheme in the present paper involves orthogonal spline collocation (OSC) counterpart of

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(5), rather than the least squares solution. Also, using the OSC analog of the Poincaré inequality, we show that the H^1 norm of the error, rather than its H^1 seminorm considered in [4], is of order r. Hence our OSC scheme is more accurate than FD schemes of [12, 17]. The Galerkin spectral solution of [1] is obtained using the matrix decomposition algorithm that involves the computation of eigenvalues and eigenvectors of a symmetric pentadiagonal matrix. In comparison, we obtain our OSC solution using the matrix decomposition algorithm which involves finding eigenvalues and eigenvectors of a generalized symmetric banded eigenvalue problem. For $r = 3$, explicit formulas for eigenvalues and eigenvectors of these eigenvalue problems are given in [4]. Our OSC solution is required to satisfy nonzero Neumann boundary conditions at corners and collocation points on $\partial\Omega$ while in [1], a function, defined on $\overline{\Omega}$ and satisfying nonzero Neumann boundary conditions on $\partial\Omega$, is first determined. Spectral accuracy of the approximate solution in [1] is demonstrated numerically only while we give a theoretical convergence analysis of our OSC scheme. In [1], integrals involving f of (4) are evaluated approximately which is unnecessary for our OSC scheme. Generalization of our OSC scheme to (1)–(3) with (1) replaced by the separable equation

$$\sum_{i=1}^{2}[-a_i(x_i)u_{x_ix_i} + c_i(x_i)u] = f(x_1, x_2), \quad (x_1, x_2) \in \Omega, \tag{6}$$

with variable coefficients

$$a_i(x_i) > 0, \quad c_i(x_i) \geq 0, \quad x_i \in [0, 1],$$

still involves solution of a generalized symmetric banded eigenvalue problem, while the approach in [1] for (6), with $-(a_i u_{x_i})_{x_i}$ replacing $-a_i(x_i)u_{x_ix_i}$, involves solution of a generalized symmetric eigenvalue problem with full matrices. Like scheme in [1], our scheme generalizes to 3 dimensions, in which case the cost of solving eigenvalue problem is negligible in comparison to the total cost of the solution process. Matrix decomposition algorithms for solving finite element Galerkin schemes for separable equations on a rectangle were developed in [14, 15] for Dirichlet and mixed boundary conditions. However, [14, 15] do not provide details of such algorithms for solving a singular linear system arising in the case of the pure Neumann problem (1)–(3). The OSC solution of the Neumann problem on a rectangle was recently used in a pressure Poisson OSC method for solving the Navier-Stokes equation [11].

The paper is outlined as follows. Section 2 gives some preliminary results used in the convergence analysis. Section 3 introduces an OSC solution for (1)–(3), (5). Error bounds are derived in Section 4. A matrix decomposition algorithm to find the OSC solution is described in Section 5. Finally, numerical results are presented in Section 6.

2. Preliminaries

In what follows, $\delta_{x_1} = \{x_1^{(i)}\}_{i=0}^{N_{x_1}}$ and $\delta_{x_2} = \{x_2^{(j)}\}_{j=0}^{N_{x_2}}$ are partitions of $[0, 1]$, such that,

$$0 = x_1^{(0)} < x_1^{(1)} < \cdots < x_1^{(N_{x_1})} = 1, \quad 0 = x_2^{(0)} < x_2^{(1)} < \cdots < x_2^{(N_{x_2})} = 1,$$

and $\delta = \delta_{x_1} \times \delta_{x_2}$. We introduce

$$h_i^{x_1} = x_1^{(i)} - x_1^{(i-1)}, \quad i = 1, \ldots, N_{x_1}, \quad h_j^{x_2} = x_2^{(j)} - x_2^{(j-1)}, \quad j = 1, \ldots, N_{x_2},$$

and we set

$$h_{x_1} = \max_{i=1,\ldots,N_{x_1}} h_i^{x_1}, \quad h_{x_2} = \max_{j=1,\ldots,N_{x_2}} h_j^{x_2}, \quad h = \max(h_{x_1}, h_{x_2}).$$

Throughout the paper we assume that a collection of the partitions δ of $\overline{\Omega}$ is regular (see Section 2 in [2]).

For $r \ge 3$, let P_r be the set of polynomials of degree $\le r$, and let $\mathcal{M}_{x_1}$ and $\mathcal{M}_{x_2}$ be the spaces of C^1 splines of degree $\le r$ with the breaks at the $x_1^{(i)}$ and $x_2^{(j)}$, respectively, defined by

$$\mathcal{M}_{x_1} = \{v \in C^1[0,1] : v|_{[x_1^{(i-1)},x_2^{(i)}]} \in P_r, i = 1,\dots,N_{x_1}\},$$

$$\mathcal{M}_{x_2} = \{v \in C^1[0,1] : v|_{[x_2^{(j-1)},x_2^{(j)}]} \in P_r, j = 1,\dots,N_{x_2}\}.$$

Note that

$$\dim \mathcal{M}_{x_i} = d_i + 2, \quad d_i = (r-1)N_{x_i}, \quad i = 1,2.$$

Additionally, we introduce

$$\mathcal{M}_{x_1}^{\mathcal{N}} = \{v \in \mathcal{M}_{x_1} : v'(0) = v'(1) = 0\}, \qquad \mathcal{M}_{x_2}^{\mathcal{N}} = \{v \in \mathcal{M}_{x_2} : v'(0) = v'(1) = 0\},$$

$$\text{(7)} \qquad \mathcal{M}_r = \mathcal{M}_{x_1} \otimes \mathcal{M}_{x_2}, \quad \mathcal{M}_r^{\mathcal{N}} = \mathcal{M}_{x_1}^{\mathcal{N}} \otimes \mathcal{M}_{x_2}^{\mathcal{N}},$$

where for two spaces V_1 and V_2 of functions on $[0,1]$, $V_1 \otimes V_2$ is the space of functions on $\overline{\Omega}$ consisting of all finite linear combinations of $v_1(x_1)v_2(x_2)$ with $v_1 \in V_1$ and $v_2 \in V_2$.

Let $\{\xi_k\}_{k=1}^{r-1}$ and $\{\omega_k\}_{k=1}^{r-1}$ be respectively the nodes and the corresponding weights of the $(r-1)$-point Gauss-Legendre quadrature for $(0,1)$, and let $\mathcal{G}_{x_1} = \{\xi_{i,k}^{x_1}\}_{i=1,k=1}^{N_{x_1},r-1}$, $\mathcal{G}_{x_2} = \{\xi_{j,l}^{x_2}\}_{j=1,l=1}^{N_{x_2},r-1}$ be the collocation points in $[0,1]$, where

$$\text{(8)} \qquad \xi_{i,k}^{x_1} = x_1^{(i-1)} + h_i^{x_1}\xi_k, \quad \xi_{j,l}^{x_2} = x_2^{(j-1)} + h_j^{x_2}\xi_l.$$

We also introduce $\{\xi_i^{x_1}\}_{i=0}^{d_1+1}$ and $\{\xi_i^{x_2}\}_{j=0}^{d_2+1}$, where

$$\xi_0^{x_1} = 0, \quad \xi_{d_1+1}^{x_1} = 1, \quad \xi_0^{x_2} = 0, \quad \xi_{d_2+1}^{x_2} = 1,$$

$$\xi_{(i-1)(r-1)+k}^{x_1} = \xi_{i,k}^{x_1}, \quad i = 1,\dots,N_{x_1}, \quad k = 1,\dots,r-1,$$

$$\xi_{(j-1)(r-1)+l}^{x_2} = \xi_{j,l}^{x_2}, \quad j = 1,\dots,N_{x_2}, \quad l = 1,\dots,r-1.$$

Note that $\mathcal{G}_{x_1} = \{\xi_i^{x_1}\}_{i=1}^{d_1}$, $\mathcal{G}_{x_2} = \{\xi_j^{x_2}\}_{j=1}^{d_2}$. The set $\mathcal{G}_r$ of collocation points in Ω is defined by

$$\mathcal{G}_r = \{\xi = (\xi^{x_1}, \xi^{x_2}) : \xi^{x_1} \in \mathcal{G}_{x_1}, \xi^{x_2} \in \mathcal{G}_{x_2}\}.$$

For v and z defined on $\mathcal{G}_r$, we introduce

$$\text{(9)} \qquad \langle v, z\rangle = \sum_{i=1}^{N_{x_1}}\sum_{j=1}^{N_{x_2}} h_i^{x_1}h_j^{x_2}\sum_{k=1}^{r-1}\sum_{l=1}^{r-1}\omega_k\omega_l(vz)(\xi_{i,k}^{x_1}, \xi_{j,l}^{x_2}), \quad \|v\| = \sqrt{\langle v, v\rangle}.$$

Note that

$$\text{(10)} \qquad \langle v, z\rangle = \sum_{i=1}^{N_{x_1}} h_i^{x_1}\sum_{k=1}^{r-1}\omega_k\langle v(\xi_{i,k}^{x_1}, \cdot), z(\xi_{i,k}^{x_1}, \cdot)\rangle_{x_2},$$

$$\text{(11)} \qquad \langle v, z\rangle = \sum_{j=1}^{N_{x_2}} h_j^{x_2}\sum_{l=1}^{r-1}\omega_l\langle v(\cdot, \xi_{j,l}^{x_2}), z(\cdot, \xi_{j,l}^{x_2})\rangle_{x_1},$$

where

$$\text{(12)} \qquad \langle p, q\rangle_{x_1} = \sum_{i=1}^{N_{x_1}} h_i^{x_1}\sum_{k=1}^{r-1}\omega_k(pq)(\xi_{i,k}^{x_1}), \quad \langle p, q\rangle_{x_2} = \sum_{j=1}^{N_{x_2}} h_j^{x_2}\sum_{l=1}^{r-1}\omega_l(pq)(\xi_{j,l}^{x_2}).$$

Since $\sum_{k=1}^{r-1} \omega_k = 1$ and $\sum_{i=1}^{N_{x_1}} h_i^{x_1} = \sum_{j=1}^{N_{x_2}} h_j^{x_2} = 1$, it follows from (12) and the first equation in (9) that

$$\langle 1,1\rangle_{x_1} = 1, \quad \langle 1,1\rangle_{x_2} = 1, \quad \langle 1,1\rangle = 1. \tag{13}$$

Lemma 2.1. *Assume $i = 1,2$ and $v \in \mathcal{M}_{x_i}^{\mathcal{N}}$ and $v = 0$ on $\mathcal{G}_{x_i}$. Then $v = 0$ on $[0,1]$.*

Proof. Since δ_{x_1} and δ_{x_2} are arbitrary partitions of $[0,1]$, it suffices to prove the lemma for $i = 1$. We follow the proof of Lemma 2.3 in [10].

If $v(0) = 0$, then $v = 0$ on $[x_1^{(0)}, x_1^{(1)}]$ since $v \in P_r$ has $r+1$ zeros in $[x_1^{(0)}, x_1^{(1)}]$ (v has double zero at 0 and zeros at $\xi_{1,1}^{x_1}, \ldots, \xi_{1,r-1}^{x_1}$). By a similar argument $v = 0$ on $[x_1^{(1)}, x_1^{(2)}]$ through $[x_1^{(Nx_1-1)}, x_1^{(N_{x_1})}]$.

If $v(0) \neq 0$, then $v' \in P_{r-1}$ has $r-1$ zeros in $[x_1^{(0)}, x_1^{(1)}]$ (v' has zero at 0 and zeros between $\xi_{1,1}^{x_1}, \ldots, \xi_{1,r-1}^{x_1}$). v' has no other zeros in $[x_1^{(0)}, x_1^{(1)}]$ since otherwise v' would be 0 on $[x_1^{(0)}, x_1^{(1)}]$ and hence v would be a constant on $[x_1^{(0)}, x_1^{(1)}]$ which would violate $v(0) \neq 0$ and $v(\xi_{1,1}^{x_1}) = 0$. If $v' > 0$ on $[\xi_{1,r-1}^{x_1}, x_1^{(1)}]$, then $v(x_1^{(1)}) > 0$ since $v(\xi_{1,r-1}^{x_1}) = 0$. If $v' < 0$ on $[\xi_{1,r-1}^{x_1}, x_1^{(1)}]$, then $v(x_1^{(1)}) < 0$. Consequently $v(x_1^{(1)})v'(x_1^{(1)}) > 0$. Since $v(\xi_{1,r-1}^{x_1}) = v(\xi_{2,1}^{x_1}) = 0$ and v' has no zero in $[\xi_{1,r-1}^{x_1}, x_1^{(1)}]$, v' has a zero between $x_1^{(1)}$ and $\xi_{2,1}^{x_1}$. Hence v' has $r-1$ zeros in $[x_1^{(1)}, x_1^{(2)}]$ (v' has also zeros between $\xi_{2,1}^{x_1}, \ldots, \xi_{2,r-1}^{x_1}$). v' has no other zeros in $[x_1^{(1)}, x_1^{(2)}]$. Consequently $v(x_1^{(2)})v'(x_1^{(2)}) > 0$. Repeating this argument, we obtain $v(x_1^{(N_{x_1})})v'(x_1^{(N_{x_1})}) > 0$ which contradicts $v'(1) = 0$. □

Lemma 2.1 implies that $\langle v, z\rangle$ and $\|v\|$ of (9) are respectively an inner product and a norm on $\mathcal{M}_r^{\mathcal{N}}$.

Lemma 2.2. *For $i = 1,2$, we have*

$$\langle p'', q\rangle_{x_i} - (p'q)|_0^1 = \langle q'', p\rangle_{x_i} - (q'p)|_0^1, \quad p, q \in \mathcal{M}_{x_i}.$$

Proof. The desired result follows from (12) and Lemma 3.1 in [10]. □

For $i = 1,2$, Lemma 2.2 implies

$$\langle p'', 1\rangle_{x_i} = p'(1) - p'(0), \quad p \in \mathcal{M}_{x_i}. \tag{14}$$

Lemma 2.3. *We have*

$$\langle -\Delta v, z\rangle = \langle v, -\Delta z\rangle, \quad v, z \in \mathcal{M}_r^{\mathcal{N}}.$$

Proof. The desired result follows from (10), (11), (12) and Lemma 2.2. □

Throughout the paper, we assume that C is a generic positive constant, dependent possibly on r but independent of u and h.

Lemma 2.4. *For $i = 1,2$, we have*

$$C\langle p, p\rangle_{x_i} \le \int_0^1 p^2(x_i)dx_i \le C\langle p, p\rangle_{x_i}, \quad p \in \mathcal{M}_{x_i}^{\mathcal{N}}.$$

Proof. The first inequality is a special case of (3.4) in [10] for $p \in \mathcal{M}_{x_i}$. (3.4) in [10], stated without a complete proof, can be proved using the inverse inequality

$$\max_{x_i \in [x_i^{(j-1)}, x_i^{(j)}]} |p(x_i)| \le C[h_j^{x_i}]^{-1/2} \|p\|_{L^2(x_i^{(j-1)}, x_i^{(j)})}.$$

The second inequality can be proved following the proofs of Lemma 5.4 and Theorem 5.5 in [16]. We provide details in Appendix. □

The next result is the 2d counterpart of Lemma 2.4.

Lemma 2.5. *We have*

$$C\|v\| \le \|v\|_{L^2(\Omega)} \le C\|v\|, \quad v \in \mathcal{M}_r^{\mathcal{N}}.$$

Proof. Using the right-hand side inequality of Lemma 2.4 with $i = 1$, the first equation in (12), the right-hand side inequality of Lemma 2.4 with $i = 2$, (10), and the second equation in (9), we obtain

$$\begin{aligned}\|v\|^2_{L^2(\Omega)} &= \int_0^1 \left[\int_0^1 v^2(x_1,x_2)dx_1\right] dx_2 \\ &\le C\int_0^1 \langle v(\cdot,x_2), v(\cdot,x_2)\rangle_{x_1} dx_2 \\ &= C\sum_{i=1}^{N_{x_1}} h_i^{x_1} \sum_{k=1}^{r-1} \omega_k \int_0^1 v^2(\xi_{i,k}^{x_1}, x_2)dx_2 \\ &\le C\sum_{i=1}^{N_{x_1}} h_i^{x_1} \sum_{k=1}^{r-1} \omega_k \langle v(\xi_{i,k}^{x_1},\cdot), v(\xi_{i,k}^{x_1},\cdot)\rangle_{x_2} = C\|v\|^2,\end{aligned}$$

which proves the right-hand side inequality of the lemma. In a similar way we prove the left-hand side inequality of the lemma. □

We introduce

(15)
$$\|\nabla v\|_{L^2(\Omega)} = \left(\|v_{x_1}\|^2_{L^2(\Omega)} + \|v_{x_2}\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \quad \|v\|_{H^1(\Omega)} = \left(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Lemma 2.6. *We have*

$$\|\nabla v\|^2_{L^2(\Omega)} \le C\langle -\Delta v, v\rangle, \quad v \in \mathcal{M}_r^{\mathcal{N}}.$$

Proof. It follows from Lemma 3.1 in [10] that, for $i = 1, 2$, we have

(16)
$$\int_0^1 (p')^2(x_i)dx_i \le \langle -p'', p\rangle_{x_i}, \quad p \in \mathcal{M}_{x_i}^{\mathcal{N}}.$$

Assume $v \in \mathcal{M}_r^{\mathcal{N}}$. Using the right-hand side inequality of Lemma 2.4 with $i = 2$, the second equation in (12), (16) with $i = 1$, and (11), we have

(17)
$$\begin{aligned}\|v_{x_1}\|^2_{L^2(\Omega)} &= \int_0^1 \left[\int_0^1 v^2_{x_1}(x_1,x_2)dx_2\right] dx_1 \le C\int_0^1 \langle v_{x_1}(x_1,\cdot), v_{x_1}(x_1,\cdot)\rangle_{x_2} dx_1 \\ &= C\sum_{j=1}^{N_{x_2}} h_j^{x_2} \sum_{l=1}^{r-1} \omega_l \int_0^1 v^2_{x_1}(x_1, \xi_{j,l}^{x_2})dx_1 \le C\sum_{j=1}^{N_{x_2}} h_j^{x_2} \sum_{l=1}^{r-1} \omega_l \left\langle -v_{x_1x_1}(\cdot,\xi_{j,l}^{x_2}), v(\cdot,\xi_{j,l}^{x_2})\right\rangle_{x_1} \\ &= C\left\langle -v_{x_1x_1}, v\right\rangle.\end{aligned}$$

In a similar way we obtain

(18)
$$\|v_{x_2}\|^2_{L^2(\Omega} \le C\left\langle -v_{x_2x_2}, v\right\rangle.$$

Equations (17), (18), and the first equation in (15) yield the desired result. □

The following lemma is the OSC analog of the Poincaré inequality.

Lemma 2.7. *We have*

$$\|v\|^2 - \langle v,1\rangle^2 \le \langle -\Delta v, v\rangle, \quad v \in \mathcal{M}_r^{\mathcal{N}}.$$

Proof. For $i = 1,\dots,N_{x_1}$, $j = 1,\dots,N_{x_2}$, $k,l = 1,\dots,r-1$, and $y_1, y_2 \in [0,1]$, using the fundamental theorem of calculus, the triangle inequality, $\xi_{i,k}^{x_1}, \xi_{i,k}^{x_1}, y_1, y_2 \in [0,1]$, the inequality

$$(\alpha+\beta)^2 \le 2(\alpha^2+\beta^2), \quad \alpha,\beta \in R,$$

and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(19)\quad v^2(\xi_{i,k}^{x_1},\xi_{j,l}^{x_2}) &- 2v(\xi_{i,k}^{x_1},\xi_{j,l}^{x_2})v(y_1,y_2) + v^2(y_1,y_2) = [v(\xi_{i,k}^{x_1},\xi_{j,l}^{x_2}) - v(y_1,y_2)]^2 \\ &= \left[\int_{y_1}^{\xi_{i,k}^{x_1}} v_{x_1}(x_1,\xi_{j,l}^{x_2})dx_1 + \int_{y_2}^{\xi_{j,l}^{x_2}} v_{x_2}(y_1,x_2)dx_2\right]^2 \\ &= \left|\int_{y_1}^{\xi_{i,k}^{x_1}} v_{x_1}(x_1,\xi_{j,l}^{x_2})dx_1 + \int_{y_2}^{\xi_{j,l}^{x_2}} v_{x_2}(y_1,x_2)dx_2\right|^2 \\ &\le \left[\int_0^1 \left|v_{x_1}(x_1,\xi_{j,l}^{x_2})\right| dx_1 + \int_0^1 |v_{x_2}(y_1,x_2)|\, dx_2\right]^2 \\ &\le 2\int_0^1 v_{x_1}^2(x_1,\xi_{j,l}^{x_2})dx_1 + 2\int_0^1 v_{x_2}^2(y_1,x_2)dx_2.\end{aligned}$$

Multiplying (19) by $h_i^{x_1}h_j^{x_2}\omega_k\omega_l$, summing over $i = 1,\dots,N_{x_1}$, $j = 1,\dots,N_{x_2}$, $k,l = 1,\dots,r-1$, using (9), the third and first equations in (13), the first equation in (12), (16) with $i = 1$, and (11), we obtain

$$\begin{aligned}(20)\quad &\|v\|^2 - 2v(y_1,y_2)\langle v,1\rangle + v^2(y_1,y_2) \\ &\le 2\sum_{j=1}^{N_{x_2}} h_j^{x_2}\sum_{l=1}^{r-1}\omega_l \int_0^1 v_{x_1}^2(x_1,\xi_{j,l}^{x_2})dx_1 + 2\int_0^1 v_{x_2}^2(y_1,x_2)dx_2. \\ &\le 2\langle -v_{x_1x_1}, v\rangle + 2\int_0^1 v_{x_2}^2(y_1,x_2)dx_2.\end{aligned}$$

Taking $y_1 = \xi_{i,k}^{x_1}$, $y_2 = \xi_{j,l}^{x_2}$, multiplying (20) by $h_i^{x_1}h_j^{x_2}\omega_k\omega_l$, summing over $i = 1,\dots,N_{x_1}$, $j = 1,\dots,N_{x_2}$, $k,l = 1,\dots,r-1$, using (9), the third and second equations in (13), the second equation in (12), (16) with $i = 2$, and (10), we obtain

$$2\|v\|^2 - 2\langle v,1\rangle^2 \le 2\langle -v_{x_1x_1}, v\rangle + 2\sum_{i=1}^{N_{x_1}} h_i^{x_1}\sum_{k=1}^{r-1}\omega_k\int_0^1 v_{x_2}^2(\xi_{i,k},x_2)dx_2 \le 2\langle -\Delta v, v\rangle,$$

which yields the desired result. □

As in [10], for $r > 3$, let $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{r-3} < 1$ be the simple zeros of the polynomial

$$\frac{d^{r-3}}{dt^{r-3}}\left[t^{r-1}(t-1)^{r-1}\right]$$

and let

$$\begin{aligned}\zeta_{i,k}^{x_1} &= x_1^{(i-1)} + h_i^{x_1}\zeta_k, \quad i = 1,\dots,N_{x_1}, \quad k = 1,\dots,r-3, \\ \zeta_{j,l}^{x_2} &= x_2^{(j-1)} + h_j^{x_2}\zeta_l, \quad j = 1,\dots,N_{x_2}, \quad l = 1,\dots,r-3.\end{aligned}$$

According to [9], $\zeta_1 = 1/2$ for $r = 4$, $\zeta_1 = 1/2 - \sqrt{7}/14$, $\zeta_2 = 1/2 + \sqrt{7}/14$ for $r = 5$, $\zeta_1 = 1/2 - \sqrt{3}/6$, $\zeta_2 = 1/2$, $\zeta_3 = 1/2 + \sqrt{3}/6$ for $r = 6$, $\zeta_1 = 1/2 -$

$\sqrt{33(9+4\sqrt{3})}/66$, $\zeta_2 = 1/2 - \sqrt{33(9-4\sqrt{3})}/66$, $\zeta_3 = 1/2 + \sqrt{33(9-4\sqrt{3})}/66$, $\zeta_4 = 1/2 + \sqrt{33(9+4\sqrt{3})}/66$ for $r = 7$. For any $r > 3$, the ζ_j can be computed in Matlab using symbolic differentiation and the roots function.

Throughout the paper, we assume that the solution u of (1)–(3), (5) is a sufficiently smooth function on $\overline{\Omega}$. Let $W \in \mathcal{M}_r$ be piecewise polynomial interpolant of u defined by

$$
\begin{aligned}
&(W-u)(\zeta^{x_1}_{i,k}, \zeta^{x_2}_{j,l}) = 0, \quad i = 1,\dots,N_{x_1}, \quad j = 1,\dots,N_{x_2},\\
&\frac{\partial^n}{\partial x_1^n}(W-u)(x_1^{(i)}, \zeta^{x_2}_{j,l}) = 0, \quad i = 0,\dots,N_{x_1}, \quad j = 1,\dots,N_{x_2},\\
&\frac{\partial^m}{\partial x_2^m}(W-u)(\zeta^{x_1}_{i,k}, x_2^{(j)}) = 0, \quad i = 1,\dots,N_{x_1}, \quad j = 0,\dots,N_{x_2},\\
&\frac{\partial^{n+m}}{\partial x_1^n x_2^m}(W-u)(x_1^{(i)}, x_2^{(j)}) = 0, \quad i = 0,\dots,N_{x_1}, \quad j = 0,\dots,N_{x_2},
\end{aligned}
\tag{21}
$$

where $n, m = 0, 1$ and $k, l = 1, \dots, r-3$. Throughout the paper, for a function s defined on $[0,1]$ or $\overline{\Omega}$, $C(s)$ denotes a generic positive constant, dependent possibly on r and s, but independent of h. It follows from Lemmas 2.3 and 2.5 in [2] that

$$\|u - W\| \le C(u)h^{r+1}, \tag{22}$$

$$\|\Delta(u - W)\| \le C(u)h^{r}, \tag{23}$$

$$\|u - W\|_{H^1(\Omega)} \le C(u)h^{r}. \tag{24}$$

It follows from the second and last equations in (21) with $x_1^{(i)} = 0, 1$, $n = 1$, $m = 0, 1$, that for $\alpha = 0, 1$, $W_{x_1}(\alpha, \cdot)$ is the piecewise polynomial interpolant of $u_{x_1}(\alpha, \cdot)$ on $[0, 1]$. Hence adapting the proof of (22) to the one-dimensional case, we can show that

$$\|(u - W)_{x_1}(\alpha, \cdot)\|_{x_2} \le C(u)h^{r+1}, \quad \alpha = 0, 1, \tag{25}$$

where $\|p\|_{x_2} = \langle p, p\rangle_{x_2}^{1/2}$.

Lemma 2.8. *If $s \in H^{2r-2}(\Omega)$, then*

$$\left| \int_\Omega s(x_1, x_2)dx_1dx_2 - \langle s, 1\rangle \right| \le C(s)h^{2r-2}.$$

Proof. Consider the functional on $H^{2r-2}(\Omega)$ defined by

$$F(v) = \int_0^1\int_0^1 v(x,y)dxdy - \sum_{k=1}^{r-1}\sum_{l=1}^{r-1}\omega_k\omega_l v(\xi_k, \xi_l), \quad v \in H^{2r-2}(\Omega).$$

Then

$$F(v) = 0, \quad v \in P_{2r-3},$$

where P_{2r-3} denotes the set of polynomials of degree $\le 2r-3$ in x, y. Using the triangle inequality and the Sobolev imbedding theorem, we have

$$|F(v)| \le C \max_{(x,y)\in\overline{\Omega}} |v(x,y)| \le C\|v\|_{H^2(\Omega)} \le C\|v\|_{H^{2r-2}(\Omega)}, \quad v \in H^{2r-2}(\Omega).$$

Hence Bramble-Hilbert lemma (see Theorem 4.1.3 in [6]) gives

$$|F(v)| \le C\left(\sum_{p+q=2r-2}\left\|\frac{\partial^{p+q}v}{\partial x^p\partial y^q}\right\|^2_{L^2(\Omega)}\right)^{1/2}, \quad v \in H^{2r-2}(\Omega). \tag{26}$$

For $s \in H^{2r-2}(\Omega)$ and fixed $i = 1, \ldots, N_{x_1}$, $j = 1, \ldots, N_{x_2}$, we introduce $v_{ij} \in H^{2r-2}(\Omega)$ defined by

$$v_{ij}(x,y) = h_i^{x_1} h_j^{x_2} s(x_1^{(i-1)} + h_i^{x_1} x, x_2^{(j-1)} + h_j^{x_2} y), \quad x, y \in (0,1).$$

For this v_{ij} we have

$$F(v_{ij}) = \int_{x_1^{(i-1)}}^{x_1^{(i)}} \int_{x_2^{(j-1)}}^{x_2^{(j)}} s(x_1,x_2)dx_1dx_2 - h_i^{x_1} h_j^{x_2} \sum_{k=1}^{r-1}\sum_{l=1}^{r-1} \omega_k \omega_l s(\xi_{i,k}^{x_1}, \xi_{j,l}^{x_2}), \tag{27}$$

(28)

$$\begin{aligned}
&\left(\sum_{p+q=2r-2} \left\| \frac{\partial^{p+q} v_{ij}}{\partial x^p \partial y^q} \right\|^2_{L^2(\Omega)} \right)^{1/2} = \left(\sum_{p+q=2r-2} \int_0^1 \int_0^1 \left[\frac{\partial^{p+q} v_{ij}}{\partial x^p \partial y^q}(x,y) \right]^2 dxdy \right)^{1/2} \\
&= h_i^{x_1} h_j^{x_2} \left(\sum_{p+q=2r-2} [h_i^{x_1}]^{2p} [h_j^{x_2}]^{2q} \right. \\
&\quad \left. \times \int_0^1 \int_0^1 \left[\frac{\partial^{p+q} s}{\partial x^p \partial y^q} (x_1^{(i-1)} + h_i^{x_1} x, x_2^{(j-1)} + h_j^{x_2} y) \right]^2 dxdy \right)^{1/2} \\
&\le h^{2r-2} [h_i^{x_1} h_j^{x_2}]^{1/2} \left(\sum_{p+q=2r-2} \int_{x_1^{(i-1)}}^{x_1^{(i)}} \int_{x_2^{(j-1)}}^{x_2^{(j)}} \left[\frac{\partial^{p+q} s}{\partial x^p \partial y^q}(x_1,x_2) \right]^2 dx_1 dx_2 \right)^{1/2}.
\end{aligned}$$

Using the first equation in (9), the triangle inequality, (26) with v replaced by v_{ij}, (27), (28), and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\left| \int_\Omega s(x_1,x_2)dx_1dx_2 - \langle s, 1 \rangle \right| \\
&= \left| \sum_{i=1}^{N_{x_1}} \sum_{j=1}^{N_{x_2}} \left[\int_{x_1^{(i-1)}}^{x_1^{(i)}} \int_{x_2^{(j-1)}}^{x_2^{(j)}} s(x_1,x_2)dx_1dx_2 - h_i^{x_1} h_j^{x_2} \sum_{k=1}^{r-1}\sum_{l=1}^{r-1} \omega_k \omega_l s(\xi_{i,k}^{x_1}, \xi_{j,l}^{x_2}) \right] \right| \\
&\le \sum_{i=1}^{N_{x_1}} \sum_{j=1}^{N_{x_2}} \left| \int_{x_1^{(i-1)}}^{x_1^{(i)}} \int_{x_2^{(j-1)}}^{x_2^{(j)}} s(x_1,x_2)dx_1dx_2 - h_i^{x_1} h_j^{x_2} \sum_{k=1}^{r-1}\sum_{l=1}^{r-1} \omega_k \omega_l s(\xi_{i,k}^{x_1}, \xi_{j,l}^{x_2}) \right| \\
&\le C h^{2r-2} \sum_{i=1}^{N_{x_1}} \sum_{j=1}^{N_{x_2}} [h_i^{x_1} h_j^{x_2}]^{1/2} \left(\sum_{p+q=2r-2} \int_{x_1^{(i-1)}}^{x_1^{(i)}} \int_{x_2^{(j-1)}}^{x_2^{(j)}} \left[\frac{\partial^{p+q} s}{\partial x^p \partial y^q}(x_1,x_2) \right]^2 dx_1 dx_2 \right)^{\frac{1}{2}} \\
&\le C h^{2r-2} \left(\sum_{i=1}^{N_{x_1}} \sum_{j=1}^{N_{x_2}} h_i^{x_1} h_j^{x_2} \right)^{1/2} \\
&\quad \cdot \left(\sum_{i=1}^{N_{x_1}} \sum_{j=1}^{N_{x_2}} \sum_{p+q=2r-2} \int_{x_1^{(i-1)}}^{x_1^{(i)}} \int_{x_2^{(j-1)}}^{x_2^{(j)}} \left[\frac{\partial^{p+q} s}{\partial x^p \partial y^q}(x_1,x_2) \right]^2 dx_1 dx_2 \right)^{1/2} \\
&= C h^{2r-2} \left(\sum_{p+q=2r-2} \left\| \frac{\partial^{p+q} s}{\partial x^p \partial y^q} \right\|^2_{L^2(\Omega)} \right)^{1/2},
\end{aligned}$$

which proves the required inequality. □

We will also need the following 1d counterpart of Lemma 2.8.

Lemma 2.9. *If* $s \in H^{2r-2}(0,1)$, *then*

$$\left|\int_0^1 s(x_1)dx_1 - \langle s,1\rangle_{x_1}\right| \le C(s)h_{x_1}^{2r-2}, \quad \left|\int_0^1 s(x_2)dx_2 - \langle s,1\rangle_{x_2}\right| \le C(s)h_{x_2}^{2r-2}.$$

Proof. Since δ_{x_1} and δ_{x_2} are arbitrary partitions of $[0,1]$, it suffices to prove the first inequality of the lemma. We omit a proof of this inequality since the proof is similar to that of Lemma 2.8 with the functional on $H^{2r-2}(0,1)$ defined now by

$$F(v) = \int_0^1 \int_0^1 v(x)dx - \sum_{k=1}^{r-1} \omega_k v(\xi_k), \quad v \in H^{2r-2}(0,1).$$

□

3. OSC Solution

First, for $i = 1, 2$, we describe an appropriate choice of B-spline basis functions for $\mathcal{M}_{x_i}^{\mathcal{N}}$. Let $\{B_j^{x_i}\}_{j=0}^{d_i+1}$ be B-spline basis functions for $\mathcal{M}_{x_i}$ such that (see, for example, [5])

$$B_0^{x_i}(0) \ne 0, \quad [B_0^{x_i}]'(0) \ne 0, \quad B_1^{x_i}(0) = 0, \quad [B_1^{x_i}]'(0) \ne 0,$$
$$B_j^{x_i}(0) = [B_j^{x_i}]'(0) = 0, \quad j = 2, \ldots, d_i + 1,$$

(29) $$B_j^{x_i}(1) = [B_j^{x_i}]'(1) = 0, \quad j = 0, \ldots, d_i - 1,$$

(30) $$B_{d_i}^{x_i}(1) = 0, \quad [B_{d_i}^{x_i}]'(1) \ne 0, \quad B_{d_i+1}^{x_i}(1) \ne 0, \quad [B_{d_i+1}^{x_i}]'(1) \ne 0.$$

We set

(31) $$\phi_1^{x_i} = [B_1^{x_i}]'(0)B_0^{x_i} - [B_0^{x_i}]'(0)B_1^{x_i},$$
$$\phi_j^{x_i} = B_j^{x_i}, \quad j = 2, \ldots, d_i - 1,$$
$$\phi_{d_i}^{x_i} = [B_{d_i}^{x_i}]'(1)B_{d_i+1}^{x_i} - [B_{d_i+1}^{x_i}]'(1)B_{d_i}^{x_i}.$$

Then $[\phi_1^{x_i}]'(0) = 0$, $[\phi_{d_i}^{x_i}]'(1) = 0$ and $\{\phi_j^{x_i}\}_{j=1}^{d_i}$ is a basis for $\mathcal{M}_{x_i}^{\mathcal{N}}$. We also set

$$\phi_0^{x_i} = B_0^{x_i}, \quad \phi_{d_i+1}^{x_i} = B_{d_i+1}^{x_i}.$$

Then $\{\phi_j^{x_i}\}_{j=0}^{d_i+1}$ is a basis for $\mathcal{M}_{x_i}$.

To define an OSC solution of (1)–(3), (5), consider $U \in \mathcal{M}_r$ which can be written as

(32) $$U(x_1,x_2) = \sum_{m=0}^{d_1+1}\sum_{n=0}^{d_2+1} U_{m,n}\phi_m^{x_1}(x_1)\phi_n^{x_2}(x_2) = \widehat{U}(x_1,x_2) + \overline{U}(x_1,x_2),$$

where

(33)
$$\widehat{U}(x_1,x_2) = \sum_{m=0}^{d_1+1}\sum_{n=0,d_2+1} U_{m,n}\phi_m^{x_1}(x_1)\phi_n^{x_2}(x_2) + \sum_{m=0,d_1+1}\sum_{n=1}^{d_2} U_{m,n}\phi_m^{x_1}(x_1)\phi_n^{x_2}(x_2),$$

and

(34) $$\overline{U}(x_1,x_2) = \sum_{m=1}^{d_1}\sum_{n=1}^{d_2} U_{m,n}\phi_m^{x_1}(x_1)\phi_n^{x_2}(x_2) \in \mathcal{M}_r^{\mathcal{N}}.$$

The coefficients $U_{m,n}$ in (33) are determined (see Section 5 for details) by interpolating (2), (3), that is, we require that,

(35) $$\frac{\partial^i U_{x_1}}{\partial x_2^i}(\alpha, x_2^j) = \frac{\partial^i g_1}{\partial x_2^i}(\alpha, x_2^j), \quad j = 0, \ldots, N_{x_2}, \quad i = 0, 1,$$

$$U_{x_1}(\alpha, \zeta^{x_2}_{j,l}) = g_1(\alpha, \zeta^{x_2}_{j,l}), \quad j = 1, \ldots, N_{x_2}, \quad l = 1, \ldots, r-3, \tag{36}$$

$$\frac{\partial^j U_{x_2}}{\partial x_1^j}(x_1^{(i)}, \beta) = \frac{\partial^j g_2}{\partial x_1^j}(x_1^{(i)}, \beta), \quad i = 0, \ldots, N_{x_1}, \quad j = 0, 1, \tag{37}$$

$$U_{x_2}(\zeta^{x_1}_{i,k}, \beta) = g_2(\zeta^{x_1}_{i,k}, \beta), \quad i = 1, \ldots, N_{x_1}, \quad k = 1, \ldots, r-3, \tag{38}$$

where $\alpha, \beta = 0, 1$. It is natural to determine $\overline{U} \in \mathcal{M}_r^{\mathcal{N}}$ such that

$$-\Delta \overline{U}(\xi) = \widehat{f}(\xi), \quad \xi \in \mathcal{G}_r, \tag{39}$$

where

$$\widehat{f}(\xi) = f(\xi) + \Delta \widehat{U}(\xi), \quad \xi \in \mathcal{G}_r. \tag{40}$$

Lemma 2.3 implies that the operator $-\Delta$ is self-adjoint on $\mathcal{M}_r^{\mathcal{N}}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ of (9). Clearly $1 \in \mathcal{M}_r^{\mathcal{N}}$ and $-\Delta 1(\xi) = 0$, $\xi \in \mathcal{G}_r$. Moreover, it follows from Lemma 2.6 that if $v \in \mathcal{M}_r^{\mathcal{N}}$ and $-\Delta v(\xi) = 0$, $\xi \in \mathcal{G}_r$, then v is constant. Hence the necessary and sufficient condition for the existence of $\overline{U} \in \mathcal{M}_r^{\mathcal{N}}$ satisfying (39) is $\langle \widehat{f}, 1 \rangle = 0$. Since, in general, $\langle \widehat{f}, 1 \rangle \neq 0$, (39) will not have a solution $\overline{U} \in \mathcal{M}_r^{\mathcal{N}}$. We introduce

$$\overline{f} = \widehat{f} - \langle \widehat{f}, 1 \rangle \tag{41}$$

and look for $\overline{U} \in \mathcal{M}_r^{\mathcal{N}}$ such that

$$-\Delta \overline{U}(\xi) = \overline{f}(\xi), \quad \xi \in \mathcal{G}_r. \tag{42}$$

It follows from (41) and the third equation in (13) that $\langle \overline{f}, 1 \rangle = 0$ and hence (42) has a solution $\overline{U} \in \mathcal{M}_r^{\mathcal{N}}$. Assume $\widehat{U}$ of (33) is determined by (35)–(38), and $\overline{U} \in \mathcal{M}_r^{\mathcal{N}}$ is a solution of (42). Then, for any $c \in R$, $\overline{U} + c \in \mathcal{M}_r^{\mathcal{N}}$ and (42) holds with $\overline{U} + c$ replacing $\overline{U}$. Hence

$$U = \widehat{U} + \overline{U} + c, \quad c \in R, \tag{43}$$

can be viewed as a general OSC solution of (1)–(3). If u satisfies (1)–(3) and (5), then U of (43) with c such that

$$\langle U, 1 \rangle = \gamma \tag{44}$$

is the OSC solution of (1)–(3), (5).

4. Convergence Analysis

Lemma 4.1. *Set*

$$\widetilde{\mathcal{M}}_r^{\mathcal{N}} = \{v \in \mathcal{M}_r : v_{x_1}(\alpha, x_2) = v_{x_2}(x_1, \beta) = 0, \alpha, \beta = 0, 1, x_1, x_2 \in [0, 1]\}.$$

Then $\mathcal{M}_r^{\mathcal{N}} = \widetilde{\mathcal{M}}_r^{\mathcal{N}}$, where $\mathcal{M}_r^{\mathcal{N}}$ is defined in (7).

Proof. It follows from the second equation in (7) that $\mathcal{M}_r^{\mathcal{N}} \subset \widetilde{\mathcal{M}}_r^{\mathcal{N}}$. To show $\widetilde{\mathcal{M}}_r^{\mathcal{N}} \subset \mathcal{M}_r^{\mathcal{N}}$, consider $v \in \widetilde{\mathcal{M}}_r^{\mathcal{N}}$. Since $v \in \mathcal{M}_r$, we have

$$v(x_1, x_2) = \sum_{m=0}^{d_1+1} \sum_{n=0}^{d_2+1} c_{m,n} \phi_m^{x_1}(x_1) \phi_n^{x_2}(x_2),$$

where the $c_{m,n} \in R$. Using $v_{x_1}(0, x_2) = 0$, $x_2 \in [0, 1]$, $[\phi_0^{x_1}]'(0) \neq 0$, and $[\phi_m^{x_1}]'(0) = 0$, $m = 1, \ldots, d_1 + 1$, we get

$$\sum_{n=0}^{d_2+1} c_{0,n} \phi_n^{x_2}(x_2) = 0, \quad x_2 \in [0, 1].$$

Since the set $\{\phi_n^{x_2}\}_{n=0}^{d_2+1}$ is linearly independent, we have $c_{0,n} = 0$, $n = 0, \dots, d_2+1$. Using, in a similar way,

$$v_{x_1}(1, x_2) = 0, \quad x_2 \in [0,1], \quad v_{x_2}(x_1, \beta) = 0, \quad x_1 \in [0,1], \quad \beta = 0, 1,$$

we obtain

$$v(x_1, x_2) = \sum_{m=1}^{d_1} \sum_{n=1}^{d_2} c_{m,n} \phi_m^{x_1}(x_1) \phi_n^{x_2}(x_2) \in \mathcal{M}_r^{\mathcal{N}}.$$

□

Theorem 4.1. *Assume U in $\mathcal{M}_r$ is given by (32), where $\widehat{U}$ of (33) is determined by (35)–(38), and $\overline{U}$ in $\mathcal{M}_r^{\mathcal{N}}$ satisfies (42). Then*

$$\|\nabla(u - U)\|_{L^2(\Omega)} \le C(u)h^r.$$

Proof. Assume $W \in \mathcal{M}_r$ is the interpolant of u defined in (21) and set

$$v = W - U - c, \quad c = \langle W - U, 1\rangle. \tag{45}$$

Using (45) and the third equation in (13), we have

$$\langle v, 1\rangle = \langle W - U, 1\rangle - c\langle 1, 1\rangle = c - c = 0. \tag{46}$$

It follows from the second and last equations in (21) with $x_1^{(i)} = 0, 1$, $n = 1$, $m = 0, 1$, the third and last equations in (21) with $x_2^{(j)} = 0, 1$, $m = 1$, $n = 0, 1$, (35), (36), (37), (38), (2), and (3) that

$$W_{x_1}(\alpha, x_2) = U_{x_1}(\alpha, x_2), \quad W_{x_2}(x_1, \beta) = U_{x_2}(x_1, \beta), \quad \alpha, \beta = 0, 1, \quad x_1, x_2 \in [0, 1]. \tag{47}$$

Using (47) and Lemma 4.1, we have $W - U \in \mathcal{M}_r^{\mathcal{N}}$. Hence it follows from (45) and $c \in \mathcal{M}_r^{\mathcal{N}}$ that $v \in \mathcal{M}_r^{\mathcal{N}}$. Next, using (45), (32), (42), (41), (40), (1), (46), the Cauchy-Schwarz inequality, and (23), we have

$$\begin{aligned} \langle -\Delta v, v\rangle &= \langle -\Delta W + \Delta U, v\rangle = \left\langle -\Delta W + \Delta\widehat{U} + \Delta\overline{U}, v\right\rangle = \left\langle -\Delta W + \Delta\widehat{U} - \overline{f}, v\right\rangle \\ &= \left\langle -\Delta W + \Delta\widehat{U} - \widehat{f} + \langle\widehat{f}, 1\rangle, v\right\rangle = \langle -\Delta W - f, v\rangle + \langle\langle\widehat{f}, 1\rangle, v\rangle \\ &= \langle\Delta(u - W), v\rangle + \langle\widehat{f}, 1\rangle\langle 1, v\rangle \le \|\Delta(u - W)\|\|v\| \le C(u)h^r\|v\|. \end{aligned} \tag{48}$$

Lemma 2.7 and (46) imply $\|v\| \le \langle -\Delta v, v\rangle^{1/2}$ and hence (48) yields

$$\langle -\Delta v, v\rangle^{1/2} \le C(u)h^r. \tag{49}$$

It follows from Lemma 2.6 and (49) that $\|\nabla v\|_{L^2(\Omega)} \le C(u)h^r$ which, on using (45), gives

$$\|\nabla(W - U)\|_{L^2(\Omega)} \le C(u)h^r. \tag{50}$$

Using the triangle inequality, the second equation in (15), (24), and (50), we have

$$\|\nabla(u - U)\|_{L^2(\Omega)} \le \|u - W\|_{H^1(\Omega)} + \|\nabla(W - U)\|_{L^2(\Omega)} \le C(u)h^r.$$

□

Since $\|\nabla v\|_{L^2(\Omega)}$ is not a norm on $H^1(\Omega)$, we next bound $\|u - U\|_{H^1(\Omega)}$.

Theorem 4.2. *Assume U in $\mathcal{M}_r$ is given by (43), where $\widehat{U}$ of (33) satisfies (35), (36), (37), (38), $\overline{U}$ in $\mathcal{M}_r^{\mathcal{N}}$ satisfies (42), and c is determined so that (44) holds. Then*

$$\|u - U\|_{H^1(\Omega)} \le C(u)h^r. \tag{51}$$

Proof. Assume $W \in \mathcal{M}_r$ is the interpolant of u defined in (21) and set

$$v = W - U. \tag{52}$$

It follows from (52), (47), and Lemma 4.1 that $v \in \mathcal{M}_r^{\mathcal{N}}$. Using the right-hand side inequality of Lemma 2.5 and Lemma 2.7, we have

$$\|v\|_{L^2(\Omega)}^2 \le C\|v\|^2 \le C\langle v, 1\rangle^2 + C\langle -\Delta v, v\rangle. \tag{53}$$

Following the derivation of (48) and using (40), we obtain

$$\langle -\Delta v, v\rangle \le C(u)h^r\|v\| + |\langle f + \Delta\widehat{U}, 1\rangle||\langle 1, v\rangle|. \tag{54}$$

The left-hand side inequality of Lemma 2.5 and the inequality

$$\alpha\beta \le \epsilon\alpha^2 + \frac{1}{4\epsilon}\beta^2, \quad \alpha, \beta \in R, \quad \epsilon > 0,$$

give

$$C(u)h^r\|v\| \le C(u)h^r\|v\|_{L^2(\Omega)} \le \epsilon\|v\|_{L^2(\Omega)}^2 + \frac{C(u)}{\epsilon}h^{2r}. \tag{55}$$

Using (53), (54), (55), and taking ϵ sufficiently small, we obtain

$$\|v\|_{L^2(\Omega)}^2 \le C\langle v, 1\rangle^2 + C|\langle f + \Delta\widehat{U}, 1\rangle||\langle 1, v\rangle| + C(u)h^{2r}. \tag{56}$$

We bound the first two terms on the right-hand side in (56). Using (52), (44), (5), the triangle inequality, the Cauchy-Schwarz inequality, the third equation in (13), Lemma 2.8 with u replacing s, (22), and $r \ge 3$, we have

$$\begin{aligned} |\langle v, 1\rangle| &= |\langle W, 1\rangle - \gamma| \le |\langle W - u, 1\rangle| + \left|\langle u, 1\rangle - \int_\Omega u(x_1, x_2)dx_1dx_2\right| \\ &\le \|W - u\| + C(u)h^{2r-2} \le C(u)h^{r+1}(1 + h^{r-3}) \le C(u)h^{r+1}. \end{aligned} \tag{57}$$

It follows from (11), (10), (14), (12), (32), $\overline{U} \in \mathcal{M}_r^{\mathcal{N}}$, Lemma 4.1, (47), (2), and (3) that

$$\begin{aligned} \langle \Delta\widehat{U}, 1\rangle &= \langle \widehat{U}_{x_1x_1}, 1\rangle + \langle \widehat{U}_{x_2x_2}, 1\rangle \\ &= \sum_{j=1}^{N_{x_2}} h_j^{x_2} \sum_{l=1}^{r-1} \omega_l \langle \widehat{U}_{x_1x_1}(\cdot, \xi_{j,l}^{x_2}), 1\rangle_{x_1} + \sum_{i=1}^{N_{x_1}} h_i^{x_1} \sum_{k=1}^{r-1} \omega_k \langle \widehat{U}_{x_2x_2}(\xi_{i,k}^{x_1}, \cdot), 1\rangle_{x_2} \\ &= \sum_{j=1}^{N_{x_2}} h_j^{x_2} \sum_{l=1}^{r-1} \omega_l [\widehat{U}_{x_1}(1, \xi_{j,l}^{x_2}) - \widehat{U}_{x_1}(0, \xi_{j,l}^{x_2})] \\ &\quad + \sum_{i=1}^{N_{x_1}} h_i^{x_1} \sum_{k=1}^{r-1} \omega_k [\widehat{U}_{x_2}(\xi_{i,k}^{x_1}, 1) - \widehat{U}_{x_2}(\xi_{i,k}^{x_1}, 0)] \\ &= \langle \widehat{U}_{x_1}(1, \cdot), 1\rangle_{x_2} - \langle \widehat{U}_{x_1}(0, \cdot), 1\rangle_{x_2} + \langle \widehat{U}_{x_2}(\cdot, 1), 1\rangle_{x_1} - \langle \widehat{U}_{x_2}(\cdot, 0), 1\rangle_{x_1} \\ &= \langle U_{x_1}(1, \cdot), 1\rangle_{x_2} - \langle U_{x_1}(0, \cdot), 1\rangle_{x_2} + \langle U_{x_2}(\cdot, 1), 1\rangle_{x_1} - \langle U_{x_2}(\cdot, 0), 1\rangle_{x_1} \\ &= \langle W_{x_1}(1, \cdot), 1\rangle_{x_2} - \langle W_{x_1}(0, \cdot), 1\rangle_{x_2} + \langle W_{x_2}(\cdot, 1), 1\rangle_{x_1} - \langle W_{x_2}(\cdot, 0), 1\rangle_{x_1} \\ &= S_1 + S_2 + S_3, \end{aligned} \tag{58}$$

where

$$S_1 = \langle (W - u)_{x_1}(1, \cdot), 1\rangle_{x_2} - \langle (W - u)_{x_1})(0, \cdot), 1\rangle_{x_2}, \tag{59}$$

$$S_2 = \langle (W - u)_{x_2}(\cdot, 1), 1\rangle_{x_1} - \langle (W - u)_{x_2}(\cdot, 0), 1\rangle_{x_1}, \tag{60}$$

$$S_3 = \langle g_1(1, \cdot), 1\rangle_{x_2} - \langle g_1(0, \cdot), 1\rangle_{x_2} + \langle g_2(\cdot, 1), 1\rangle_{x_1} - \langle g_2(\cdot, 0), 1\rangle_{x_1}. \tag{61}$$

Using (59), the triangle and Cauchy-Schwarz inequalities, (25), and the second equation in (13), we have

$$|S_1| \leq \sum_{\alpha=0,1} \|(W-u)_{x_1}(\alpha,\cdot)\|_{x_2}\|1\|_{x_2} \leq C(u)h^{r+1}. \tag{62}$$

In a similar way, for S_2 of (60), we obtain

$$|S_2| \leq C(u)h^{r+1}. \tag{63}$$

It follows from (58), (4), (61), the triangle inequality, Lemma 2.8 with f replacing s, Lemma 2.9 with $g_1(\alpha, x_2)$, $g_2(x_1, \beta)$ replacing $s(x_2)$, $s(x_1)$, respectively, (62), and (63), that

$$\begin{aligned}
\left|\langle f,1\rangle + \langle \Delta\widehat{U},1\rangle\right| &= \left|\langle f,1\rangle + S_1 + S_2 + S_3 - \int_\Omega f(x_1,x_2)dx_1dx_2\right. \\
&\quad \left. - \int_0^1 [g_1(1,x_2) - g_1(0,x_2)]dx_2 - \int_0^1 [g_2(x_1,1) - g_2(x_1,0)]dx_1\right| \\
&\leq \left|\langle f,1\rangle - \int_\Omega f(x_1,x_2)dx_1dx_2\right| + \sum_{\alpha=0,1}\left|\langle g_1(\alpha,\cdot),1\rangle_{x_2} - \int_0^1 g_1(\alpha,x_2)dx_2\right| \\
&\quad + \sum_{\beta=0,1}\left|\langle g_2(\cdot,\beta),1\rangle_{x_1} - \int_0^1 g_2(x_1,\beta)dx_1\right| + |S_1| + |S_2| \\
&\leq C(u)h^{r+1}(1+h^{r-3}) \leq C(u)h^{r+1}.
\end{aligned} \tag{64}$$

Using (56), (57), and (64), we have $\|v\|_{L^2(\Omega)} \leq Ch^r$ which, on using (52), (15), and (50), gives

$$\|W-U\|_{H^1(\Omega)} \leq C(u)h^r. \tag{65}$$

Using the triangle inequality, (24), and (65), we have

$$\|u-U\|_{H^1(\Omega)} \leq \|u-W\|_{H^1(\Omega)} + \|W-u\|_{H^1(\Omega)} \leq C(u)h^r.$$

□

5. Implementation

In this section we explain how to compute the OSC solution U defined in Section 3. We will use the following observation: The matrix-vector form of

$$\phi_{i,j} = \sum_{m=1}^{M'} c_{i,m}^{(1)} \sum_{n=1}^{N'} c_{j,n}^{(2)}\psi_{m,n}, \quad i=1,\ldots,I', \quad j=1,\ldots,J', \tag{66}$$

is

$$\Phi = (C_1 \otimes C_2)\Psi, \tag{67}$$

where

$$C_1 = [c_{i,m}^{(1)}]_{i=1,m=1}^{I',M'}, \quad C_2 = [c_{j,n}^{(2)}]_{j=1,n=1}^{J',N'},$$

and

$$\Phi = [\phi_{1,1},\ldots,\phi_{1,J'},\ldots,\phi_{I',1},\ldots,\phi_{I',J'}],$$
$$\Psi = [\psi_{1,1},\ldots,\psi_{1,N'},\ldots,\psi_{M',1},\ldots,\psi_{M',N'}].$$

With i,j being row indices and m,n being columns indices, we introduce the matrices

$$\overline{B}_{x_1} = [\phi_m^{x_1}(\xi_i^{x_1})]_{i=0,m=0}^{d_1+1,d_1+1}, \quad \overline{B}_{x_2} = [\phi_n^{x_2}(\xi_j^{x_2})]_{j=0,n=0}^{d_2+1,d_2+1},$$
$$\widetilde{A}_{x_1} = [-[\phi_m^{x_1}]''(\xi_i^{x_1})]_{i=1,m=0}^{d_1,d_1+1}, \quad \widetilde{A}_{x_2} = [-[\phi_n^{x_2}]''(\xi_j^{x_2})]_{j=1,n=0}^{d_2,d_2+1}.$$

where $\{\phi_m^{x_1}\}_{m=0}^{d_1+1}$ and $\{\phi_n^{x_2}\}_{n=0}^{d_2+1}$, introduced in Section 3, are bases for $\mathcal{M}_{x_1}$ and $\mathcal{M}_{x_2}$, respectively. For $k = 1, 2$, we assume that:
$\widetilde{B}_{x_k}$ is the matrix obtained by deleting the first and last rows of $\overline{B}_{x_k}$;
$\widehat{A}_{x_k}$, $\widehat{B}_{x_k}$ are the matrices formed by the first and last columns of $\widetilde{A}_{x_k}$, $\widetilde{B}_{x_k}$, respectively;
A_{x_k}, B_{x_k} are the matrices obtained by deleting the first and last columns of $\widetilde{A}_{x_k}$, $\widetilde{B}_{x_k}$, respectively.

Lemma 2.3 in [10] implies that $\overline{B}_{x_1}$ and $\overline{B}_{x_2}$ are nonsingular.

First we explain how to obtain the coefficients $U_{m,n}$ in $\widehat{U}$ of (33). For $\beta = 0$ and $i = 1, \ldots, N_{x_1}$, (37) and (38) give

$$\frac{\partial^j U_{x_2}}{\partial x_1^j}(x_1^{(i-1)}, 0) = \frac{\partial^j g_2}{\partial x_1^j}(x_1^{(i-1)}, 0), \quad \frac{\partial^j U_{x_2}}{\partial x_1^j}(x_1^{(i)}, 0) = \frac{\partial^j g_2}{\partial x_1^j}(x_1^{(i)}, 0), \quad j = 0, 1,$$
$$U_{x_2}(\zeta_{i,k}^{x_1}, 0) = g_2(\zeta_{i,k}^{x_1}, 0), \quad k = 1, \ldots, r-3,$$

which we use to determine $U_{x_2}(x_1, 0)$, $x_1 \in [x_1^{(i-1)}, x_1^{(i)}]$, in terms of B-splines for $[x_1^{(i-1)}, x_1^{(i)}]$. This allows us to compute $U_{x_2}(\xi_i^{x_1}, 0)$, $i = 0, \ldots, d_i + 1$. It follows from (32) that

$$U_{x_2}(x_1, 0) = [\phi_0^{x_2}]'(0) \sum_{m=0}^{d_1+1} U_{m,0}\phi_m^{x_1}(x_1), \quad x_1 \in [0, 1],$$

and hence we obtain

$$\sum_{m=0}^{d_1+1} \phi_m^{x_1}(\xi_i^{x_1}) U_{m,0} = U_{x_2}(\xi_i^{x_1}, 0)/[\phi_0^{x_2}]'(0), \quad i = 0, \ldots, d_1 + 1,$$

which is a linear system in

$$[U_{0,0}, U_{1,0}, \ldots, U_{d_1,0}, U_{d_1+1,0}]^T$$

with the matrix $\overline{B}_{x_1}$. In a similar way, for $\beta = 1$, we obtain a linear system in

$$[U_{0,d_2+1}, U_{1,d_2+1}, \ldots, U_{d_1,d_2+1}, U_{d_1+1,d_2+1}]^T$$

with the matrix $\overline{B}_{x_1}$. In a similar way, using (35) and (36), we obtain the coefficients $\{U_{m,n}\}_{n=0}^{d_2+1}$, $m = 0, d_1 + 1$ in (33). It is not obvious at this point why $U_{m,n}$, $m = 0, d_1 + 1$, $n = 0, d_2 + 1$, obtained using (37), (38) are the same as those obtained using (35), (36). However, note, for example, that (37) with $j = 1$, $i = 0$, $\beta = 0$, and (32) yield

$$(g_2)_{x_1}(0, 0) = U_{x_2x_1}(0, 0) = [\phi_0^{x_1}]'(0)[\phi_0^{x_2}]'(0)U_{0,0},$$

while (35) with $i = 1$, $j = 0$, $\alpha = 0$, and (32) yield

$$(g_1)_{x_2}(0, 0) = U_{x_1x_2}(0, 0) = [\phi_0^{x_1}]'(0)[\phi_0^{x_2}]'(0)U_{0,0}.$$

Moreover, (3) and (2) imply

$$(g_2)_{x_1}(0, 0) = u_{x_2x_1}(0, 0) = u_{x_1x_2}(0, 0) = (g_1)_{x_2}(0, 0),$$

and hence $U_{0,0}$ obtained using (37) is the same as that obtained using (35).

Next we discuss the matrix-vector form of (42). Using (33) and $\xi = (\xi_i^{x_1}, \xi_j^{x_2})$, $i = 1, \ldots, d_1$, $j = 1, \ldots, d_2$, for $\Delta\widehat{U}(\xi)$ of (40) we have

$$(68) \qquad \Delta\widehat{U}(\xi) = \Delta\widehat{U}(\xi_i^{x_1}, \xi_j^{x_2}) = \sum_{m=0}^{d_1+1} [\phi_m^{x_1}]''(\xi_i^{x_1}) \sum_{n=0,d_2+1} \phi_n^{x_2}(\xi_j^{x_2}) U_{m,n}$$

$$+\sum_{m=0}^{d_1+1}\phi_m^{x_1}(\xi_i^{x_1})\sum_{n=0,d_2+1}[\phi_n^{x_2}]''(\xi_j^{x_2})U_{m,n}$$

$$+\sum_{m=0,d_1+1}[\phi_m^{x_1}]''(\xi_i^{x_1})\sum_{n=1}^{d_2}\phi_n^{x_2}(\xi_j^{x_2})U_{m,n}$$

$$+\sum_{m=0,d_1+1}\phi_m^{x_1}(\xi_i^{x_1})\sum_{n=1}^{d_2}[\phi_n^{x_2}]''(\xi_j^{x_2})U_{m,n}.$$

It follows from (66), (67) that (68) can be computed for all $\xi\in\mathcal{G}_r$ by multiplying vector

$$[U_{0,0},U_{0,d_2+1},U_{1,0},U_{1,d_2+1},\dots,U_{d_1+1,0},U_{d_1+1,d_2+1}]^T$$

by the matrices

$$-\widetilde{A}_{x_1}\otimes\widehat{B}_{x_2},\quad -\widetilde{B}_{x_1}\otimes\widehat{A}_{x_2},$$

and the vector

$$[U_{0,1},U_{0,2},\dots,U_{0,d_2},U_{d_1+1,1},U_{d_1+1,2},\dots,U_{d_1+1,d_2}]^T$$

by the matrices

$$-\widehat{A}_{x_1}\otimes B_{x_2},\quad -\widehat{B}_{x_1}\otimes A_{x_2}.$$

Substituting (34) into (42) and taking $\xi=(\xi_i^{x_1},\xi_j^{x_2})$, $i=1,\dots,d_1$, $j=1,\dots,d_2$, we obtain

$$-\sum_{m=1}^{d_1}\sum_{n=1}^{d_2}[\phi_m^{x_1}]''(\xi_i^{x_1})\phi_n^{x_2}(\xi_j^{x_2})U_{m,n}-\sum_{m=1}^{d_1}\sum_{n=1}^{d_2}\phi_m^{x_1}(\xi^{x_1})[\phi_n^{x_2}]''(\xi_j^{x_2})U_{m,n}=\overline{f}(\xi_i^{x_1},\xi_j^{x_2}).$$

Hence it follows from (66), (67) that the matrix-vector form of (42) is

$$(A_{x_1}\otimes B_{x_2}+B_{x_1}\otimes A_{x_2})\,\mathbf{u}=\mathbf{f}, \tag{69}$$

where

$$\mathbf{u}=[U_{1,1},\dots,U_{1,d_2},\dots,U_{d_1,1},\dots,U_{d_1,d_2}]^T, \tag{70}$$

$$\mathbf{f}=[f_{1,1},\dots,f_{1,d_2},\dots,f_{d_1,1},\dots,f_{d_1,d_2}]^T,\quad f_{i,j}=\overline{f}(\xi_i^{x_1},\xi_j^{x_2}).$$

Next we present an efficient method for finding a solution of (69). Following (3.4) in [3], we introduce the matrices

$$F_1=B_{x_1}^TW_1B_{x_1},\quad G_1=B_{x_1}^TW_1A_{x_1}, \tag{71}$$

where the diagonal matrix W_1 is defined by

$$W_1=\operatorname{diag}\left(h_1^{x_1}\omega_1,\dots,h_1^{x_1}\omega_{r-1},\dots,h_{N_{x_1}}^{x_1}\omega_1,\dots,h_{N_{x_1}}^{x_1}\omega_{r-1}\right).$$

Adapting the proof of Lemma 3.1 in [3] to the zero Neumann boundary conditions, we can show that F_1 is symmetric and positive definite and G_1 is symmetric and nonnegative definite. Since F_1 is symmetric and positive definite and since G_1 is symmetric, Theorem 4.12 in [8] implies that there are a real diagonal $\Lambda=\operatorname{diag}(\lambda_j)_{j=1}^{d_1}$ and a real nonsingular Z such that

$$Z^TG_1Z=\Lambda,\quad Z^TF_1Z=I_{d_1}, \tag{72}$$

where, here and in what follows, I_k is the identity matrix of size k. Since G_1 is nonnegative definite it follows from the first equation in (72) that all λ_j are nonnegative. Rank $A_{x_1}=d_1-1$ since A_{x_1} is singular and since the first d_1-1 columns of A_{x_1} are linearly independent (these columns are the same as the first d_1-1 columns of nonsingular $\widetilde{A}_{x_1}$ in (83).) It follows from the second equation in

(71) that Rank G_1 = Rank A_{x_1} and it follows from the first equation in (72) that Rank G_1 = Rank Λ. Hence Rank $\Lambda = d_1 - 1$ which implies that exactly one of the λ_j, say λ_k, is 0.

The matrices Z and Λ of (72) can be obtained as follows (see [3]). Note a factorization $F_1 = LL^T$, where $L = B_{x_1}^T W_1^{1/2}$, in place of the Cholesky factorization of F_1. Since G_1 is symmetric and nonnegative definite, then so is

$$C = L^{-1} G_1 L^{-T} = W_1^{1/2} A_{x_1} B_{x_1}^{-1} W_1^{-1/2}. \tag{73}$$

Hence there are orthogonal Q and diagonal Λ such that

$$Q^T C Q = \Lambda.$$

Then $Z = B_{x_1}^{-1} W_1^{-1/2} Q$ and Λ satisfy two equations in (72) since

$$Z^T G_1 Z = Q^T W_1^{-1/2} B_{x_1}^{-T} B_{x_1}^T W_1 A_{x_1} B_{x_1}^{-1} W_1^{-1/2} Q = Q^T C Q = \Lambda,$$
$$Q^T W_1^{-1/2} B_{x_1}^{-T} B_{x_1}^T W_1 B_{x_1} B_{x_1}^{-1} W_1^{-1/2} Q = Q^T Q = I_{d_1}.$$

The matrices F_1 and G_1 of (71) can be viewed as banded. Hence the matrices Z and Λ of (72) can also be obtained using the method of [7] which computes a banded matrix $\widetilde{C}$ orthogonally similar to the full matrix C of (73).

Using (72), (71), and properties of the matrix tensor product, we obtain

$$\left(Z^T B_{x_1}^T W_1 \otimes I_{d_2}\right)(A_{x_1} \otimes B_{x_2} + B_{x_1} \otimes A_{x_2})(Z \otimes I_{d_2}) = (\Lambda \otimes B_{x_2} + I_{d_1} \otimes A_{x_2}).$$

Hence, (69) is equivalent to

$$(\Lambda \otimes B_{x_2} + I_{d_1} \otimes A_{x_2}) \mathbf{v} = \mathbf{g}, \tag{74}$$

where

$$\mathbf{g} = \left(Z^T B_{x_1}^T W_1 \otimes I_{d_2}\right) \mathbf{f}, \quad \mathbf{v} = (Z \otimes I_{d_2})^{-1} \mathbf{u}. \tag{75}$$

We set

$$\mathbf{v} = [\mathbf{v}_1, \dots, \mathbf{v}_{d_1}], \quad \mathbf{v}_m = [v_{m,1}, \dots, v_{m,d_2}],$$
$$\mathbf{g} = [\mathbf{g}_1, \dots, \mathbf{g}_{d_1}], \quad \mathbf{g}_m = [g_{m,1}, \dots, g_{m,d_2}].$$

Then (74) is equivalent to

$$A_{x_2} + \lambda_j B_{x_2} \mathbf{v}_j = \mathbf{g}_j, \quad j = 1, \dots, d_2, \quad j \neq k,$$

$$A_{x_2} \mathbf{v}_k = \mathbf{g}_k. \tag{76}$$

Note that A_{x_2} is singular and $A_{x_2} + \lambda_j B_{x_2}$, $j \neq k$, is nonsingular. Since (69) is consistent, (76) has infinitely many solutions. To find a solution of (69), it suffices to find a solution of (76). To this end note that (76) is the matrix vector form of the 1d collocation problem

$$-v''(\xi^{x_2}) = g(\xi^{x_2}), \quad \xi^{x_2} \in \mathcal{G}_{x_2}, \quad v \in \mathcal{M}_{x_2}^{\mathcal{N}}, \tag{77}$$

where

$$g(\xi_j^{x_2}) = \mathbf{g}_{k,j}, \quad j = 1, \dots, d_2. \tag{78}$$

Since (77) has a solution, using (77) and (14), we have

$$\langle g, 1 \rangle_{x_2} = \langle -v'', 1 \rangle_{x_2} = 0. \tag{79}$$

Consider the 1d collocation problem

$$-w''(\xi^{x_2}) = g(\xi^{x_2}), \quad \xi^{x_2} \in \mathcal{G}_{x_2}, \quad w \in \mathcal{M}_{x_2}^{\mathcal{N},\mathcal{D}}, \tag{80}$$

where g is that of (78) and

$$\mathcal{M}_{x_2}^{\mathcal{N},\mathcal{D}} = \{v \in \mathcal{M}_{x_2} : v'(0) = 0, v(1) = 0\}.$$

Note that $\{\psi_j^{x_2}\}_{j=1}^{d_2}$, where

$$\psi_j^{x_2} = \phi_j^{x_2}, \quad j = 1, \dots, d_2 - 1, \quad \psi_{d_2}^{x_2} = B_{d_2}^{x_2}, \tag{81}$$

is a basis for $\mathcal{M}_{x_2}^{\mathcal{N},\mathcal{D}}$. Substituting

$$w(x_2) = \sum_{j=1}^{d_2} w_j \psi_j(x_2) \tag{82}$$

into (80) and using (78) we see that the matrix vector form of (80) is

$$\widetilde{A}_{x_2} \mathbf{w} = \mathbf{g}_k, \tag{83}$$

where $\mathbf{w} = [w_1, \dots, w_{d_2}]^T$ and nonsingular $\widetilde{A}_{x_2}$ differs from A_{x_2} only in the last column. Using (79), (80), (14), (82), and (81), we obtain

$$0 = \langle g, 1 \rangle_{x_2} = \langle -w'', 1 \rangle_{x_2} = -w'(1) = -w_{d_2} [B_{d_2}^{x_2}]'(1).$$

Since $[B_{d_2}^{x_2}]'(1) \neq 0$, we have $w_{d_2} = 0$. Hence (83) gives

$$A_{x_2} \mathbf{w} = \widetilde{A}_{x_2} \mathbf{w} = \mathbf{g}_k,$$

which show that the solution $\mathbf{w}$ of the nonsingular system (83) is a solution of (76).

We arrive at the following matrix decomposition algorithm to obtain a solution of (69).

Algorithm

1. Compute Z and Λ satisfying (72).
2. Compute $\mathbf{g} = \left(Z^T B_{x_1}^T W_1 \otimes I_{d_2}\right) \mathbf{f}$.
3. Solve $(A_{x_2} + \lambda_j B_{x_2}) \mathbf{v}_j = \mathbf{g}_j$, $j \neq k$.
4. Solve $\widetilde{A}_{x_2} \mathbf{w} = \mathbf{g}_k$ and set $\mathbf{v}_k = \mathbf{w}$.
5. Compute $\mathbf{u} = (Z \otimes I_{d_2}) \mathbf{v}$.

Algorithm produces a solution $\mathbf{u}$ of (69), which, through (70), gives $\overline{U}$ of (34). For γ of (5) and known $\widehat{U}$ of (33) we determine c in U of (43) so that (44) holds. Using the third equation in (13) and the first equation in (9), we obtain

$$c = \gamma - \langle \widehat{U} + \overline{U}, 1 \rangle = \gamma - \sum_{i=1}^{N_{x_1}} \sum_{j=1}^{N_{x_2}} h_i^{x_1} h_j^{x_2} \sum_{k=1}^{r-1} \sum_{l=1}^{r-1} \omega_k \omega_l (\widehat{U} + \overline{U})(\xi_{i,k}, \xi_{j,l}).$$

It follows from (33), (34), (66), and (67) that the computation of all

$$(\overline{U} + \widehat{U})(\xi_{i,k}, \xi_{j,l}) = \sum_{m=0}^{d_1+1} \sum_{n=0}^{d_2+1} U_{m,n} \phi_m^{x_1}(\xi_{i,k}) \phi_n^{x_2}(\xi_{j,l})$$

involves multiplying a vector by $\widetilde{B}_{x_1} \otimes \widetilde{B}_{x_2}$. In this way we compute the OSC solution U of (43) and (44) satisfying (51), where u is the solution of (1)–(3), (5).

6. Numerical Results

We solved the problem (1)–(3), (5) with the exact solution

$$u(x_1, x_2) = (x_1 + \cos(4\pi x_1))(x_2 + \cos(4\pi x_2))$$

for which $\gamma = 1/4$. For the values of $N = 4, 9, 16, 25$, we used the uniform partitions δ_1 and δ_2 with stepsize $h = 1/N$. For $r = 3, 4, 5, 6$, in Tables 1 and 2, we computed the H^1 and L^2 norm errors, respectively. The integrals in x_i, $i = 1, 2$, in the H^1 norm errors were computed using the composite Gauss-Legendre quadrature with $r + 2$ nodes in each subinterval of the partitions δ_1 and δ_2 to make the quadrature

TABLE 1. H^1 norm errors and rates.

$r = 3$			$r = 4$		
N	error	rate	N	error	rate
5	1.0722e+00		5	1.4803e-01	
10	1.3305e-01	3.0106e+00	10	9.4390e-03	3.9711e+00
15	3.9103e-02	3.0200e+00	15	1.8710e-03	3.9914e+00
20	1.6441e-02	3.0117e+00	20	5.9270e-04	3.9959e+00
$r = 5$			$r = 6$		
N	error	rate	N	error	rate
5	1.7781e-02		5	1.8201e-03	
10	5.7001e-04	4.9632e+00	10	2.9192e-05	5.9623e+00
15	7.5410e-05	4.9886e+00	15	2.5751e-06	5.9882e+00
20	1.7924e-05	4.9944e+00	20	4.5907e-07	5.9942e+00

TABLE 2. L^2 norm errors and rates.

$r = 3$			$r = 4$		
N	error	rate	N	error	rate
5	4.8915e-02		5	4.5198e-03	
10	2.7439e-03	4.1560e+00	10	1.4346e-04	4.9775e+00
15	5.2469e-04	4.0800e+00	15	1.8884e-05	5.0010e+00
20	1.6395e-04	4.0436e+00	20	4.4790e-06	5.0019e+00
$r = 5$			$r = 6$		
N	error	rate	N	error	rate
5	3.7758e-04		5	3.0498e-05	
10	6.1031e-06	5.9511e+00	10	2.4616e-07	6.9530e+00
15	5.3926e-07	5.9841e+00	15	1.4495e-08	6.9851e+00
20	9.6191e-08	5.9923e+00	20	1.9389e-09	6.9926e+00

errors negligible. We also computed the corresponding convergence rates using the formula

$$\text{rate} = \frac{\log(\epsilon_{N_1}/\epsilon_{N_2})}{\log(N_2/N_1)},$$

where $N_1 < N_2$ are two consecutive values of N. As expected, the H^1 norm rates in Table 1 are close to r. The L^2 norm rates in Table 2 appear to be close to $r + 1$. Bounding theoretically the L^2 norm errors remains an open problem.

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Appendix

Proof of the second inequality in Lemma 2.4. It suffices to give a proof for $i = 1$.

Assume that

$$\phi_{i,k} \in \mathcal{M}_{x_1}^{\mathcal{N}}, \quad i = 1, \ldots, N_{x_1}, \quad k = 1, \ldots, r-1,$$

is defined by

$$\phi_{i,k}(\xi_{j,l}^{x_1}) = \delta_{i,j}\delta_{k,l}, \quad j = 1, \ldots, N_{x_1}, \quad l = 1, \ldots, r-1,$$

where $\delta_{i,j}$ is the Kronecker delta. (The roles of our i, k are the same as in [10] while the the roles of i, k in [16] are interchanged.) The existence of $\phi_{i,k}$ follows from Lemma 2.1. Following the proof of Lemma 2.1 we have

$$\phi_{i,k}(x_1^{(i-1)})\phi'_{i,k}(x_1^{(i-1)}) > 0, \quad i = 2, \ldots, N_{x_1}, \quad k = 1, \ldots, r-1. \tag{A.1}$$

Introduce

$$\psi_v(x) = \left[1 - \widetilde{L}'_{r-1}(0)x\right]\widetilde{L}_{r-1}(x), \quad \psi_d(x) = x\widetilde{L}_{r-1}(x), \quad x \in [0, 1], \tag{A.2}$$

where $\widetilde{L}_{r-1}$ is the Legendre polynomial of degree $r-1$ on $[0, 1]$ such that $\widetilde{L}_{r-1}(0) = 1$. Note that

$$\widetilde{L}_{r-1}(x) = (-1)^{r-1}L_{r-1}(-1 + 2x), \quad x \in [0, 1], \tag{A.3}$$

where L_{r-1} is the Legendre polynomial of degree $r-1$ on $[-1, 1]$ such that $L_{r-1}(1) = 1$ and hence $L_{r-1}(-1) = (-1)^{r-1}$ (see, e.g., (1.3.22c) in [18]). Using (A.2) it is easy to verify that

$$\psi_v(0) = \psi'_d(0) = 1, \quad \psi'_v(0) = \psi_d(0) = 0, \quad \psi_v(\xi_k) = \psi_d(\xi_k) = 0, \quad k = 1, \ldots, r-1.$$

As in (5.12) of [16], for $i = 1, \ldots, N_{x_1}$, $k = 1, \ldots, r-1$, and $I_m = [x_1^{(m-1)}, x_1^{(m)}]$, we have

$$\phi_{i,k}|_{I_m} = \phi_{i,k}(x_1^{(m-1)})\psi_{v,m} + h_m^{x_1}\phi'_{i,k}(x_1^{(m-1)})\psi_{d,m}, \quad m = 1, \ldots, i-1, \tag{A.4}$$

where

(A.5)
$$\psi_{v,m}(x) = \psi_v[(x - x_1^{(m-1)})/h_m^{x_1}], \quad \psi_{d,m}(x) = \psi_d[(x - x_1^{(m-1)})/h_m^{x_1}], \quad x \in I_m.$$

Using (A.4) and (A.5), we obtain (see (5.13) and (5.14) in [16])

(A.6) $$\phi_{i,k}(x_1^{(m)}) = \phi_{i,k}(x_1^{(m-1)})\psi_v(1) + h_m^{x_1}\phi'_{i,k}(x_1^{(m-1)})\psi_d(1), \quad m = 1, \ldots, i-1,$$

(A.7)
$$\phi'_{i,k}(x_1^{(m)}) = [h_m^{x_1}]^{-1}\phi_{i,k}(x_1^{(m-1)})\psi'_v(1) + \phi'_{i,k}(x_1^{(m-1)})\psi'_d(1), \quad m = 1, \ldots, i-1.$$

(A.3) and $L'_{r-1}(\pm 1) = (\pm 1)^r (r-1)r/2$ (see, e.g., (1.3.22c) in [18]) give

$$\widetilde{L}'_{r-1}(0) = -(r-1)r, \quad \widetilde{L}'_{r-1}(1) = (-1)^{r-1}(r-1)r.$$

Hence it follows from (A.2) and (A.3) that

$$\psi_v(1) = \psi'_d(1) = (-1)^{r-1}[1 + (r-1)r], \quad \psi'_v(1) = (-1)^{r-1}(r-1)r[2 + (r-1)r],$$

which gives

(A.8) $$|\psi_v(1)| = |\psi'_d(1)| = 1 + (r-1)r \equiv \gamma_r,$$

and along with $\psi_d(1) = (-1)^{r-1}$ also implies that

(A.9) $$\psi_v(1), \psi_d(1), \psi'_v(1), \psi'_d(1) \text{ have the same sign.}$$

Using (A.6), (A.7), (A.1), $\phi'_{i,k}(x_1^{(0)}) = 0$, (A.9), and (A.8), we have, for $m = 1, \ldots, i-1$,

$$|\phi_{i,k}(x_1^{(m)})| = |\phi_{i,k}(x_1^{(m-1)})\psi_v(1)| + |h_m^{x_1}\phi'_{i,k}(x_1^{(m-1)})\psi_d(1)| \geq \gamma_r|\phi_{i,k}(x_1^{(m-1)})|,$$

$$|\phi'_{i,k}(x_1^{(m)})| = |[h_m^{x_1}]^{-1}\phi_{i,k}(x_1^{(m-1)})\psi'_v(1)| + |\phi'_{i,k}(x_1^{(m-1)})\psi'_d(1)| \geq \gamma_r|\phi'_{i,k}(x_1^{(m-1)})|.$$

Hence, for $m = 1, \ldots, i-1$,

(A.10)
$$|\phi_{i,k}(x_1^{(m-1)})| \leq \gamma_r^{-1}|\phi_{i,k}(x_1^{(m)})| \leq \gamma_r^{-2}|\phi_{i,k}(x_1^{(m+1)})| \leq \ldots \leq \gamma_r^{-(i-m)}|\phi_{i,k}(x_1^{(i-1)})|,$$

$$|\phi'_{i,k}(x_1^{(m-1)})| \leq \gamma_r^{-1}|\phi'_{i,k}(x_1^{(m)})| \leq \gamma_r^{-2}|\phi'_{i,k}(x_1^{(m+1)})| \leq \ldots \leq \gamma_r^{-(i-m)}|\phi'_{i,k}(x_1^{(i-1)})|.$$

The last equation and regularity of partitions give

(A.11) $$\begin{aligned} h_m^{x_1}|\phi'_{i,k}(x_1^{(m-1)})| &\leq \frac{h_m^{x_1}}{h_i^{x_1}} h_i^{x_1}\gamma_r^{-(i-m)}|\phi'_{i,k}(x_1^{(i-1)})| \\ &\leq C h_i^{x_1}\gamma_r^{-(i-m)}|\phi'_{i,k}(x_1^{(i-1)})|, \quad m = 1, \ldots, i-1. \end{aligned}$$

It follows from (A.4), the triangle inequality, (A.10), and (A.11) that

(A.12)
$$\begin{aligned} \|\phi_{i,k}\|_{L^\infty(I_m)} &\leq |\phi_{i,k}(x_1^{(m-1)})|\|\psi_{v,m}\|_{L^\infty(I_m)} + h_m^{x_1}|\phi'_{i,k}(x_1^{(m-1)})|\|\psi_{d,m}\|_{L^\infty(I_m)} \\ &\leq \max\{|\phi_{i,k}(x_1^{(m-1)})|, h_m^{x_1}|\phi'_{i,k}(x_1^{(m-1)})|\}\left(\|\psi_v\|_{L^\infty(I)} + \|\psi_d\|_{L^\infty(I)}\right) \\ &\leq C\max\{|\phi_{i,k}(x_1^{(i-1)})|, h_i^{x_1}|\phi'_{i,k}(x_1^{(i-1)})|\}\gamma_r^{-(i-m)}, \quad m = 1, \ldots, i-1. \end{aligned}$$

To bound $|\phi_{i,k}(x_1^{(i-1)})|$ and $h_i|\phi'_{i,k}(x_1^{(i-1)})|$, note that

(A.13) $$\phi_{i,k}|_{I_i} = \phi_{i,k}(x_1^{(i-1)})\psi_{v,i} + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi_{d,i} + \psi_{i,k},$$

where

$$\psi_{i,k}(x) = \psi_k[(x - x_1^{(i-1)})/h_i^{x_1}], \quad x \in I_i,$$

$\psi_k|_I \in P_r$ and

$$\psi_k(0) = \psi'_k(0) = 0, \quad \psi_k(\xi_l) = \delta_{k,l}, \quad l = 1, \ldots, r-1.$$

Assume $k \neq 1, r-1$. Then ψ_k' has $r-1$ zeros in $[0,1]$ (ψ_k' has zero at 0 and zeros between 0, ξ_1, …, ξ_{k-1}, ξ_{k+1},…, ξ_{r-1}). ψ_k' has no other zeros in $[0,1]$. $\psi_k > 0$ on (ξ_{k-1}, ξ_{k+1}) and ψ_k changes sign $r-k-1$ times at ξ_{k+1}, …, ξ_{r-1}. Hence

(A.14) $\quad \psi_k(1) > 0$ when $r-k-1$ is even, $\quad \psi_k(1) < 0$ when $r-k-1$ is odd.

$\psi_k'(\xi_{k+1}) < 0$ and ψ_k' changes sign $r-k-2$ times in $[\xi_{k+1}, 1]$. Hence

(A.15) $\quad \psi_k'(1) < 0$ when $r-k-2$ is even, $\quad \psi_k'(1) > 0$ when $r-k-2$ is odd.

(A.14) and (A.15) hold also for $k = 1, r-1$. For example, ψ_1' has $r-1$ zeros in $[0,1]$ (ψ_1' has a zero at 0 and zeros between 0, ξ_2, …, ξ_{r-1}). ψ_1' has no other zeros in $[0,1]$. $\psi_1 > 0$ on $(0, \xi_2)$ and ψ_1 changes sign $r-2$ times at ξ_2, …, ξ_{r-1}. Hence $\psi_1(1) > 0$ when $r-2$ is even and $\psi_1(1) < 0$ when $r-2$ is odd. $\psi_1'(\xi_2) < 0$ and ψ_1' changes sign $r-3$ times in $[\xi_2, 1]$. Hence $\psi_1'(1) < 0$ when $r-3$ is even and $\psi_1'(1) > 0$ when $r-3$ is odd. It follows from (A.14) and (A.15) that (cf. (5.4) in [16])

$$\psi_k(1)\psi_k'(1) > 0, \quad k = 1, \dots, r-1. \tag{A.16}$$

Assume $i \neq 1, N_{x_1}$ and $k \neq 1, r-1$. Then, as in the proof of Lemma 2.1, $\phi_{i,k}'$ has $r-1$ zeros in $[x_1^{(i-1)}, x_1^{(i)}]$ ($\phi_{i,k}'$ has zeros between $x_1^{(i-1)}$, $\xi_{i,1}^{x_1}$, …, $\xi_{i,k-1}^{x_1}$, $\xi_{i,k+1}^{x_1}$, …, $\xi_{i,r-1}^{x_1}$, $x_1^{(i)}$). $\phi_{i,k}'$ has no other zeros in $[x_1^{(i-1)}, x_1^{(i)}]$. $\phi_{i,k} > 0$ on $(\xi_{i,k-1}^{x_1}, \xi_{i,k+1}^{x_1})$ and $\phi_{i,k}$ changes sign $r-k-1$ times at $\xi_{i,k+1}^{x_1}$, …, $\xi_{i,r-1}^{x_1}$. Hence

(A.17) $\phi_{i,k}(x_1^{(i)}) > 0$ when $r-k-1$ is even, $\quad \phi_{i,k}(x_1^{(i)}) < 0$ when $r-k-1$ is odd.

$\phi_{i,k}'(\xi_{i,k+1}^{x_1}) < 0$ and $\phi_{i,k}'$ changes sign $r-k-1$ times in $[\xi_{i,k+1}^{x_1}, x_1^{(i)}]$. Hence

(A.18) $\phi_{i,k}'(x_1^{(i)}) < 0$ when $r-k-1$ is even, $\quad \phi_{i,k}'(x_1^{(i)}) > 0$ when $r-k-1$ is odd.

(A.17) and (A.18) hold also for $i \neq 1, N_{x_1}$ and $k = 1, r-1$. For example, $\phi_{i,1}'$ has $r-1$ zeros in $[x_1^{(i-1)}, x_1^{(i)}]$ ($\phi_{i,1}'$ has zeros between $x_1^{(i-1)}$, $\xi_{i,2}^{x_1}$, …, $\xi_{i,r-1}^{x_1}$, $x_1^{(i)}$). $\phi_{i,1}'$ has no other zeros in $[x_1^{(i-1)}, x_1^{(i)}]$. $\phi_{i,1} > 0$ on $(x_{i-1}^{x_1}, \xi_{i,2}^{x_1})$ and $\phi_{i,1}$ changes sign $r-2$ times at $\xi_{i,2}^{x_1}$, …, $\xi_{i,r-1}^{x_1}$. Hence $\phi_{i,k}(x_1^{(i)}) > 0$ when $r-2$ is even and $\phi_{i,k}(x_1^{(i)}) < 0$ when $r-2$ is odd. $\phi_{i,1}'(\xi_{i,2}^{x_1}) < 0$ and $\phi_{i,1}'$ changes sign $r-2$ times in $[\xi_{i,2}^{x_1}, x_1^{(i)}]$. Hence $\phi_{i,1}'(x_1^{(i)}) < 0$ when $r-2$ is even and $\phi_{i,1}'(x_1^{(i)}) > 0$ when $r-2$ is odd. (A.17) and (A.18) hold also for $i = 1$ and $k \neq 1, r-1$. Then $\phi_{1,k}'$ has $r-1$ zeros in $[x_1^{(0)}, x_1^{(1)}]$ ($\phi_{1,k}'$ has a zero at $x_1^{(0)}$ and zeros between $x_1^{(0)}$, $\xi_{1,1}^{x_1}$, …, $\xi_{1,k-1}^{x_1}$, $\xi_{1,k+1}^{x_1}$,…, $\xi_{1,r-1}^{x_1}$). $\phi_{1,k}'$ has no other zeros in $[x_1^{(0)}, x_1^{(1)}]$. $\phi_{1,k} > 0$ on $(\xi_{1,k-1}^{x_1}, \xi_{1,k+1}^{x_1})$ and $\phi_{1,k}$ changes sign $r-k-1$ times at $\xi_{1,k+1}^{x_1}$, …, $\xi_{1,r-1}^{x_1}$. Hence $\phi_{1,k}(x_1^{(i)}) > 0$ when $r-k-1$ is even and $\phi_{1,k}(x_1^{(i)}) < 0$ when $r-k-1$ is odd. $\phi_{1,k}'(\xi_{1,k+1}^{x_1}) < 0$ and $\phi_{1,k}'$ changes sign $r-k-1$ times in $[\xi_{1,k+1}^{x_1}, x_1^{(1)}]$. Hence $\phi_{1,k}'(x_1^{(1)}) < 0$ when $r-k-1$ is even and $\phi_{1,k}'(x_1^{(i)}) > 0$ when $r-k-1$ is odd. (A.17) and (A.18) hold also for $i = 1$ and $k = 1, r-1$. For example, $\phi_{1,1}'$ has $r-1$ zeros in $[x_1^{(0)}, x_1^{(1)}]$ ($\phi_{1,1}'$ has a zero at $x_1^{(0)}$ and zeros between $x_1^{(0)}$, $\xi_{1,2}^{x_1}$, …, $\xi_{1,r-1}^{x_1}$). $\phi_{1,1}'$ has no other zeros in $[x_1^{(0)}, x_1^{(1)}]$. $\phi_{1,1} > 0$ on $(x_1^{(0)}, \xi_{1,2}^{x_1})$ and $\phi_{1,1}$ changes sign $r-2$ times at $\xi_{1,2}^{x_1}$, …, $\xi_{1,r-1}^{x_1}$. Hence $\phi_{1,1}(x_1^{(1)}) > 0$ when $r-2$ is even and $\phi_{1,1}(x_1^{(1)}) < 0$ when $r-2$ is odd. $\phi_{1,1}'(\xi_{1,2}^{x_1}) < 0$ and $\phi_{1,1}'$ changes sign $r-2$ times in $[\xi_{1,2}^{x_1}, x_1^{(1)}]$. Hence $\phi_{1,1}'(x_1^{(1)}) < 0$ when $r-2$ is even and $\phi_{1,1}'(x_1^{(i)}) > 0$ when $r-2$ is odd. (A.17) holds also for $i = N_{x_1}$ and $k = 1, \dots, r-1$. It follows from (A.14) and (A.17) that (cf. (5.21) in [16])

$$\psi_k(1)\phi_{i,k}(x_1^{(i)}) > 0, \quad i = 1, \dots, N_{x_1}, \quad k = 1, \dots, r-1. \tag{A.19}$$

It follows from (A.15), (A.18), and $\phi'_{N_{x_1},k}(x_1^{(N_{x_1})}) = 0$, that (cf. (5.22) in [16])

$$\psi'_k(1)\phi'_{i,k}(x_1^{(i)}) \le 0, \quad i = 1, \dots, N_{x_1}, \quad k = 1, \dots, r-1. \tag{A.20}$$

Using (A.20), (A.13), and $\psi'_k(1) \neq 0$, we have

$$\begin{aligned} 0 \ge h_i^{x_1}\phi'_{i,k}(x_1^{(i)})\psi'_k(1) &= [\phi_{i,k}(x_1^{(i-1)})\psi'_v(1) + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi'_d(1) + \psi'_k(1)]\psi'_k(1) \\ &> [\phi_{i,k}(x_1^{(i-1)})\psi'_v(1) + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi'_d(1)]\psi'_k(1). \end{aligned} \tag{A.21}$$

It follows from (A.9) and (A.1) that $\phi_{i,k}(x_1^{(i-1)})\psi'_v(1) + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi'_d(1)$ and $\phi_{i,k}(x_1(i-1))\psi_v(1) + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi_d(1)$ have the same sign for $i = 2, \dots, N_{x_1}$. It also follows from (A.9) and $\phi_{1,k}(x_1^{(0)}) \neq 0$ that $\phi_{1,k}(x_1^{(0)})\psi'_v(1)$ and $\phi_{1,k}(x_1^{(0)})\psi_v(1)$ have the same sign. Hence (A.21), (A.16), and $\phi'_{1,k}(x_1^{(0)}) = 0$ give

$$[\phi_{i,k}(x_1^{(i-1)})\psi_v(1) + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi_d(1)]\psi_k(1) < 0. \tag{A.22}$$

Using (A.19) and (A.13), we have

$$0 < \phi_{i,k}(x_1^{(i)})\psi_k(1) = [\phi_{i,k}(x_1^{(i-1)})\psi_v(1) + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi_d(1) + \psi_k(1)]\psi_k(1). \tag{A.23}$$

Equations (A.22) and (A.23) give

$$\begin{aligned} &|[\phi_{i,k}(x_1^{(i-1)})\psi_v(1) + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi_d(1)]\psi_k(1)| \\ &= -[\phi_{i,k}(x_1^{(i-1)})\psi_v(1) + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi_d(1)]\psi_k(1) < \psi_k^2(1), \end{aligned}$$

which yields

$$|[\phi_{i,k}(x_1^{(i-1)})\psi_v(1) + h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi_d(1)]| \le |\psi_k(1)|. \tag{A.24}$$

It follows from (A.9) and (A.1) that $\phi_{i,k}(x_1^{(i-1)})\psi_v(1)$ and $h_i^{x_1}\phi'_{i,k}(x_1^{(i-1)})\psi_d(1)$ have the same sign for $i = 2, \dots, N_{x_1}$. Hence (A.24) and $\phi'_{1,k}(x_1^{(0)}) = 0$ give

$$|\phi_{i,k}(x_1^{(i-1)})| \le |\psi_k(1)|/|\psi_v(1)|, \quad h_i^{x_1}|\phi'_{i,k}(x_1^{(i-1)})| \le |\psi_k(1)|/|\psi_d(1)|. \tag{A.25}$$

It follows from (A.12) and (A.25) that

$$\|\phi_{i,k}\|_{L^\infty(I_m)} \le C\gamma_r^{-(i-m)}, \quad m = 1, \dots, i-1,$$

and by a symmetry of argument

$$\|\phi_{i,k}\|_{L^\infty(I_m)} \le C\gamma_r^{-(m-i)}, \quad m = i+1, \dots, N_{x_1}.$$

For $m = i$ using (A.13) and (A.25), we have

$$\|\phi_{i,k}\|_{L^\infty(I_i)} \le C.$$

Hence

$$\|\phi_{i,k}\|_{L^\infty(I_m)} \le C\gamma_r^{-|m-i|}, \quad m = 1, \dots, N_{x_1}.$$

The rest of the proof of the second inequality in Lemma 2.4 follows very closely the proof of the second inequality of Theorem 5.5 in [16].

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