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HIGH ORDER METHOD FOR VARIABLE COEFFICIENT INTEGRO-DIFFERENTIAL EQUATIONS AND INEQUALITIES ARISING IN OPTION PRICING

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Abstract. In this article, the implicit-explicit (IMEX) compact schemes are proposed to solve the partial integro-differential equations (PIDEs), and the linear complementarity problems (LCPs) arising in option pricing. A diagonally dominant tri-diagonal system of linear equations is achieved for a fully discrete problem by eliminating the second derivative approximation using the variable itself and its first derivative approximation. The stability of the fully discrete problem is proved using Schur polynomial approach. Moreover, the problem's initial condition is smoothed to ensure the fourth-order convergence of the proposed IMEX compact schemes. Numerical illustrations for solving the PIDEs and the LCPs with constant and variable coefficients are presented. For each case, obtained results are compared with the IMEX finite difference scheme, and it is observed that proposed approach significantly outperforms the finite difference scheme.

Key words. Schur polynomials, implicit-explicit schemes, partial integro-differential equations, jump-diffusion models, option pricing.

1. Introduction

The assumptions of log-normal distribution of underlying assets and constant volatility considered by Black and Scholes [1] to derive the partial differential equation (PDE) have been proven inconsistent with the real market scenario. Consequently, the research community came up with advanced models to elaborate the term like negative skewness, heavy tails, and volatility smile. In one of those efforts, the jumps phenomenon was incorporated into the dynamics of the underlying asset by Merton [2] to surpass the shortcomings of the Black-Scholes model, and the model is termed as Merton's jump-diffusion model. The PIDEs for pricing European options and the LCPs for pricing American options were obtained under Merton's jump-diffusion model. Since the analytical solution for these PIDEs and LCPs does not exist in general, it is inevitable to apply numerical methods to solve these equations.

Let us now briefly review the existing finite difference based numerical methods for solving PIDEs and LCPs. An IMEX finite difference method (FDM) has been developed in [3] for pricing European and Barrier options. The convergence of the proposed IMEX method has also been proved. In [4], a fully implicit FDM has been proposed for solving the PIDEs, and the stability of the method has also been proved. Three time levels implicit FDM were proposed for pricing European and American options in [5] and [6] respectively. All these numerical methods are atmost second order accurate, and it is one of those concerns where further research is required.

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It is observed that substantial increment in the number of grid points of computational stencil may result in high-order accurate FDM, however the implementation of boundary conditions would become tedious in such a case. Moreover in such a case, the discretization matrices with more bandwidth appears in fully discrete problem and it may also suffer from restrictive stability conditions. Therefore, FDMs have been developed using compact stencils at the expense of some complication in their evaluation and these are commonly known as compact schemes. The compact schemes provide high-order accuracy and they are also parsimonious while solving the problems on hypercube computational domains as compared to FDMs. Apart from this advantage, another exception is that compact schemes can be developed in three ways. In the first approach, original equation is considered as an auxiliary equation and each of the derivative of leading term of truncation error is compactly approximated, see [7, 8]. The second approach is known as the operator compact implicit (OCI) method. In this approach, a relationship on three adjacent points between PDE operator and unknown variable is obtained and resulting fourth order accurate relationship is derived by Taylor series expansion, see [9, 10]. In the third approach, Hermitian schemes are considered for spatial discretization of PDEs, see [11, 12]. Although compact schemes have already been proposed for option pricing in [13, 14] and in many other papers using first approach, we consider third approach for developing IMEX compact scheme in this manuscript because of the following reasons:

- It is comparatively easy to develop a compact scheme for solving high-dimensional PDEs using the third approach (see [15] for reference) as compared with the first approach (see [16] for reference).
- Recently, a fourth order accurate compact scheme is developed for space fractional advection-diffusion reaction equations with variable coefficients using the third approach in [17]. It is also explained there that first and second approaches are either not feasible or very tedious for such equations.
- It is straightforward to develop the compact scheme for the variable coefficient problems using the third approach just by discretizing the variable coefficients at each grid point, see [15, 18] for more details. However, it is cumbersome with the first approach because one has to take care of the compact discretization of the coefficient term also, see [8, 19] for detailed discussion.

In this manuscript, an IMEX compact scheme is proposed for solving the following PIDE governing the price of European options under jump-diffusion model (see [5]):

$$\begin{aligned} \frac{\partial u}{\partial \tau}(x,\tau) &= \mathbb{L}u,\ (x,\tau) \in\ (-\infty,\infty)\times(0,T], \\ u(x,0) &= f(x)\ \ \forall\ \ x\in(-\infty,\infty), \end{aligned} \tag{1}$$

where

$$\mathbb{L}u = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x,\tau)+\left(r-\frac{\sigma^2}{2}-\lambda\zeta\right)\frac{\partial u}{\partial x}(x,\tau)-(r+\lambda)u(x,\tau)+\lambda\int_{\mathbb{R}} u(y,\tau)g(y-x)dy, \tag{2}$$

$\tau = T-t$, $x = ln\left(\frac{S}{K}\right)$, $u(x,\tau) = V(Ke^x, T-\tau)$, λ is the intensity of the jump sizes, $\zeta = \int_{\mathbb{R}}(e^x-1)g(x)dx$, and $V(S,0)$ is the option price. In this manuscript, Merton's model is discussed, and the probability density function $g(x)$ for the same can be written as

$$g(x) = \frac{1}{\sqrt{2\pi\sigma_J^2}} e^{-\frac{(x-\mu_J)^2}{2\sigma_J^2}}. \tag{3}$$

Further, IMEX compact scheme is also applied to solve the following LCP governing the price of American options (see [6]):

$$\begin{aligned} \frac{\partial u}{\partial \tau}(x,\tau) - \mathbb{L}u(x,\tau) \geq 0, \quad u(x,\tau) \geq f(x), \\ \left(\frac{\partial u}{\partial \tau}(x,\tau) - \mathbb{L}u(x,\tau)\right)\left(u\left(x,\tau\right) - f(x)\right) = 0. \end{aligned} \tag{4}$$

The initial conditions for European and American options are as follows:

$$\text{Put options:} \quad f(x) = \max(K - Ke^x, 0) \quad \forall \quad x \in \mathbb{R}. \tag{5}$$

$$\text{Call options:} \quad f(x) = \max(Ke^x - K, 0) \quad \forall \quad x \in \mathbb{R}. \tag{6}$$

The asymptotic behaviour for both type of options may be described as

$$\text{European put options:} \lim_{x\to-\infty}[u(x,\tau) - (Ke^{-r\tau} - Ke^x)] = 0 \text{ and } \lim_{x\to\infty} u(x,\tau) = 0,$$

$$\text{European call options:} \lim_{x\to-\infty} u(x,\tau) = 0 \text{ and } \lim_{x\to\infty}[u(x,\tau) - (Ke^x - Ke^{-r\tau})] = 0,$$

$$\text{American put options:} \lim_{x\to-\infty}[u(x,\tau) - (K - Ke^x)] = 0 \text{ and } \lim_{x\to\infty} u(x,\tau) = 0,$$

$$\text{American call options:} \lim_{x\to-\infty} u(x,\tau) = 0 \text{ and } \lim_{x\to\infty}[u(x,\tau) - (Ke^x - K)] = 0.$$

For most of the numerical methods referred in this manuscript, the stability has been proved using von-Neumann stability analysis. However in this manuscript, Schur polynomial approach is considered to prove the stability of the proposed IMEX compact scheme. The advantages of using Schur-polynomial approach over von-Neumann technique for proving the stability of any numerical scheme have already been explained in [20] on page 8. Moreover, it is well known fact that the fourth-order convergence rate can not be achieved using proposed IMEX compact scheme because of the non-smooth initial condition (a.k.a. payoff) (5). Several approaches, for example, local mesh refinement (see [21]) and co-ordinate transformation (see [22]) have already been considered in literature to achieve high-order convergence rate even for non-smooth initial conditions. These approaches suffer with certain drawbacks, for example, it may not be possible to define a coordinate transformation for all PIDEs and LCPs. Further, manual inclusion of grid points near singularity to accomplish local mesh refinement becomes tedious in certain cases. In order to avoid these limitations of local mesh refinement and co-ordinate transformation, smoothing operator from [23] is employed to smooth the initial condition which helps us to achieve fourth order convergence rate.

Conclusively, the IMEX compact schemes are proposed in present manuscript for pricing European and American options under jump-diffusion model considering the following concerns:

(1) Stability of the proposed IMEX compact scheme for solving PIDEs using Schur polynomial approach.
(2) Accuracy of proposed IMEX compact scheme for computing the prices of European and American options, and greeks.

(3) Fourth-order convergence rate of the proposed IMEX compact scheme and effect of non-smooth initial condition on the convergence rate.
(4) Location and nature of the spurious oscillations present in the obtained numerical solutions.
(5) Comparison of CPU time taken using IMEX compact scheme and FDM in price computation.

Moreover, following are some new contributions to the literature presented in this manuscript which have already been interpreted in the third and fifth paragraphs of this section:

(1) Stability of proposed numerical scheme using Schur polynomial approach.
(2) Solving variable coefficients equation using Hermitian approach based compact scheme.

The rest of the paper is organized as follows. In Sec. 2, IMEX compact scheme is proposed for pricing European options and stability of the fully discrete problem is proved. Operator splitting technique along with IMEX compact scheme for pricing American options is discussed in Sec. 3. Numerical illustrations are presented to validate the theoretical claims in Sec. 4. Conclusion and some future directions are discussed in Sec. 5.

2. IMEX compact scheme for PIDEs

In this section, compact approximations for first and second derivatives are discussed. The fully discrete problem for the PIDE (1) is obtained, and the stability of fully discrete problem is also proved.

2.1. Compact approximations. The compact approximations for first, second, third, and fourth derivatives have been discussed in [24]) using Hermitian approach. Let δx and $\delta\tau$ denote the equispaced step size in the spatial and temporal domain, respectively and if u_{x_i} and u_{xx_i} represents the first and the second derivative approximations of variable u at grid point x_i, then following expression are taken from [24];

$$\frac{1}{4}u_{x_{i-1}} + u_{x_i} + \frac{1}{4}u_{x_{i+1}} = \frac{1}{\delta x}\left[-\frac{3}{4}u_{i-1} + \frac{3}{4}u_{i+1}\right], \tag{7}$$

$$\frac{1}{10}u_{xx_{i-1}} + u_{xx_i} + \frac{1}{10}u_{xx_{i+1}} = \frac{1}{\delta x^2}\left[\frac{6}{5}u_{i-1} - \frac{12}{5}u_i + \frac{6}{5}u_{i+1}\right]. \tag{8}$$

Moreover, second-order accurate finite difference approximation can be written as

$$\Delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2\delta x}, \quad \Delta_{xx} u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\delta x^2}, \tag{9}$$

where $\Delta_x u_i$ and $\Delta_{xx} u_i$ denote the first and second derivative approximations respectively. Taking u_{x_i} instead of u_i in Eq. (7), we write

$$\frac{1}{4}u_{xx_{i-1}} + u_{xx_i} + \frac{1}{4}u_{xx_{i+1}} = \frac{1}{\delta x}\left[-\frac{3}{4}u_{x_{i-1}} + \frac{3}{4}u_{x_{i+1}}\right]. \tag{10}$$

If $u_{xx_{i-1}}$ and $u_{xx_{i+1}}$ are eliminated from Eqs. (8) and (10), then Eq. (9) gives

$$u_{xx_i} = 2\Delta_{xx} u_i - \Delta_x u_{x_i}. \tag{11}$$

In this way, compact approximation for the second derivative has been expressed in terms of unknown and its first derivative approximation. In Eq. (11), the value of

u_{x_i} is obtained from Eq. (7). As we have non-periodic boundary conditions for the PIDE (1) and the LCP (4), fourth order accurate one-sided compact approximations discussed in [25] are used to compute the derivatives at boundary points. Moreover, a detailed discussion about the resolution characteristics of above discussed compact approximations is presented in [26, 27]. It has been shown in these works that resolution characteristics of compact approximation (11) are better than the compact approximation (8). Thus, it is another advantage of splitting the second derivative approximation apart from the fact that it also gives us a diagonally dominant tri-diagonal system of linear equations for the fully discrete problem which will be shown later in this manuscript.

2.2. Localization to Bounded Domain. In order to solve the PIDE (1) and the LCP (4) numerically, the spatial domain $(-\infty, \infty)$ must be truncated to a finite domain $\Omega = [-L, L]$ for sufficiently large L. It has already been proved in [3] that localization error (arisen by truncating the infinite domain to a finite one) decreases exponentially point-wise. Moreover, an exponential bound on localization error has also been proposed in [28]. Now, we take $\delta x = 2L/N$ and $\delta\tau = T/M$, where N and M are two positive integers. Thus, we define $x_n = -L + n\delta x$ $(n = 0, 1, ..., N)$ and $\tau_m = m\delta\tau$ $(m = 0, 1, ..., M)$. Let us first discuss the temporal and spatial discretization of PIDE (1) by writing it in the following way:

$$\begin{aligned} &\frac{\partial u(x,\tau)}{\partial \tau} = \mathbb{D}u(x,\tau) + \mathbb{I}u(x,\tau), \quad (x,\tau) \in \Omega \times (0, T], \\ &u(-L,\tau) = Ke^{-r\tau} - Ke^{-L} \quad \text{and} \quad u(L,\tau) = 0, \end{aligned} \tag{12}$$

where $\mathbb{D}$ and $\mathbb{I}$ are the following differential and integral operators respectively

$$\begin{aligned} \mathbb{D}u(x,\tau) &= \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x,\tau) + \left(r - \frac{\sigma^2}{2} - \lambda\zeta\right)\frac{\partial u}{\partial x}(x,\tau) - (r+\lambda)u(x,\tau), \\ \mathbb{I}u(x,\tau) &= \lambda \int_{\mathbb{R}} u(y,\tau)g(y-x)dy. \end{aligned} \tag{13}$$

2.3. Temporal semi-discretization. The time derivative in Eq. (12) can be discretized using IMEX backward differentiation formula (BDF) as follows:

$$\begin{aligned} &\frac{3u^{m+1} - 4u^m + u^{m-1}}{2\delta\tau} = \mathbb{D}u^{m+1} + \lambda\mathbb{I}(Eu^m), \quad m \geq 1, \\ &u^{m+1}(-L) = K(e^{-r\tau_{m+1}} - e^{-L}) \quad \text{and} \quad u^{m+1}(L) = 0, \end{aligned} \tag{14}$$

where $Eu^m = 2u^m - u^{m-1}$. For all $u(.,\tau) \in L^2(\Omega)$, a new variable $\hat{u}(x,\tau)$ can be defined as follows:

$$\hat{u}(x,\tau) = \begin{cases} u(x,\tau) & : (x,\tau) \in \Omega \times [0,T], \\ 0 & : (x,\tau) \in \Omega^c \times [0,T]. \end{cases}$$

Note that the integral operator satisfies the condition $\|\mathbb{I}\hat{u}(.,\tau)\| \leq B\|u(.,\tau)\|$, where B is a constant independent of τ. If u^m is solution of Eq. (12) and $\tilde{u}^m$ is solution of the following perturbed equation:

$$\begin{aligned} &\frac{3\tilde{u}^{m+1} - 4\tilde{u}^m + \tilde{u}^{m-1}}{2\delta\tau} = \mathbb{D}\tilde{u}^{m+1} + \lambda\mathbb{I}(E\tilde{u}^m) + \epsilon^{m+1}, \quad m \geq 1, \\ &\tilde{u}^{m+1}(-L) = K(e^{-r\tau_{m+1}} - e^{-L}) \quad \text{and} \quad \tilde{u}^{m+1}(L) = 0, \end{aligned} \tag{15}$$

then let us define error at any time level τ_m as $e^m := u^m - \tilde{u}^m$. The following theorem in [29] proves the stability of the semi-discrete problem (14) as follows:

Theorem 1. *For $\delta\tau < \frac{1}{2\lambda B + 4Z + 2}$, we have*

$$\|e^k\|^2 \le C\left(\|e^0\|^2 + \|e^1\|^2 + \max_{2\le i\le k}\|\epsilon^i\|^2\right), \quad \forall\ 2 \le k \le \frac{T}{\delta\tau}, \tag{16}$$

where $Z = \left|\frac{\left(r - \frac{\sigma^2}{2} - \lambda\zeta\right)^2 - 2(r+\lambda)\sigma^2}{2\sigma^2}\right|$, and C is a parameter depending on r, σ, and B.

Remark 1. *Throughout this article, various constant notations will be used. These generic constants do not necessarily have the same meaning at each occurrence unless it is clearly specified.*

2.4. The Fully Discrete Problem. The differential and integral operators $\mathbb{D}$ and $\mathbb{I}$ given in Eq. (13) are approximated by discrete operators $\mathbb{D}_\delta$ and $\mathbb{I}_\delta$. Let us write $\mathbb{D}u_n^m \approx \mathbb{D}_\delta u_n^m, \mathbb{I}u_n^m \approx \mathbb{I}_\delta u_n^m$, and therefore $\mathbb{L}u_n^m \approx \mathbb{L}_\delta u_n^m$, where $u_n^m = u(x_n, \tau_m)$. Note that $\mathbb{D}_\delta u_n^m = \frac{\sigma^2}{2} u_{xx_n}^m + \left(r - \frac{\sigma^2}{2} - \lambda\zeta\right) u_{x_n}^m - (r+\lambda)u_n^m$ and using (11) we get

$$\mathbb{D}_\delta u_n^m = \frac{\sigma^2}{2}\left(2\Delta_{xx}u_n^m - \Delta_x u_{x_n}^m\right) + \left(r - \frac{\sigma^2}{2} - \lambda\zeta\right) u_{x_n}^m - (r+\lambda)u_n^m. \tag{17}$$

In order to compute $\mathbb{I}_\delta u_n^m$, the integral operator $\mathbb{I}u(x,\tau)$ is divided in two parts, i.e. $\mathbb{I}u(x,\tau)$ defined on the interval Ω, and $\mathbb{I}u(x,\tau)$ defined on $\mathbb{R}\backslash\Omega$, where $\mathbb{R}$ is set of real numbers. The value of $\mathbb{I}u(x,\tau)$ on the interval Ω is obtained using the approach discussed in [30]. Moreover, the value of $\mathbb{I}u(x,\tau)$ on $\mathbb{R}\backslash\Omega$ is obtained from [5] for European options, and from [6] for American options.

We find U_n^m, the approximate value of u_n^m, which is the solution of following problem:

$$\frac{3U_n^{m+1} - 4U_n^m + U_n^{m-1}}{2\delta\tau} = \mathbb{D}_\delta U_n^{m+1} + \mathbb{I}_\delta(EU_n^m),\ 1 \le m \le M-1,\ 1 \le n \le N-1, \tag{18}$$

with initial condition given in Eq. (5) and boundary conditions given in Eq. (12). Substituting the value of $\mathbb{D}_\delta u_n^m$ in Eq. (18), we get

$$\begin{aligned}\frac{3U_n^{m+1} - 4U^m + U_n^{m-1}}{2\delta\tau} &= \frac{\sigma^2}{2}\left(2\Delta_{xx}U_n^{m+1} - \Delta_x U_{x_n}^{m+1}\right) + \left(r - \frac{\sigma^2}{2} - \lambda\zeta\right) U_{x_n}^{m+1} \\ &\quad - (r+\lambda)U_n^{m+1} + \mathbb{I}_\delta(EU_n^m).\end{aligned} \tag{19}$$

Rearrangement of the terms in above Eq. (19) gives

$$\begin{aligned}\left(3 - 4\frac{\sigma^2}{2}\delta\tau\Delta_{xx} + 2\delta\tau(r+\lambda)\right) U_n^{m+1} &= 2\delta\tau\left[\left(r - \frac{\sigma^2}{2} - \lambda\zeta\right) - \frac{\sigma^2}{2}\Delta_x\right] U_{x_n}^{m+1} \\ &\quad + 4U_n^m - U_n^{m-1} + 2\delta\tau\mathbb{I}_\delta(EU_n^m).\end{aligned} \tag{20}$$

Let us denote $\mathbf{U}^m = (U_1^m, U_2^m, ..., U_{N-1}^m)^{\mathcal{T}}$ and $\mathbf{U}_x^m = (U_{x_1}^m, U_{x_2}^m, ..., U_{x_{N-1}}^m)^{\mathcal{T}}$, where $(.,.,.)^{\mathcal{T}}$ denotes the transpose of the vector. Then, resulting system of equations corresponding to the difference scheme (20) can be written as

$$A\mathbf{U}^{m+1} = F(\mathbf{U}^m, \mathbf{U}^{m-1}, \mathbf{U}_x^{m+1}), \tag{21}$$

where A is a matrix, and F is a function of matrices and vectors. The presence of $\mathbf{U}_x^{m+1}$ on right hand side of Eq. (21) bind us to employ following correcting to convergence approach discussed in [31]:

Correcting to Convergence Algorithm

1. Start with $\mathbf{U}^m$.
2. Obtain $\mathbf{U}_x^m$ using equation (7).
3. Take $\mathbf{U}_{old}^{m+1} = \mathbf{U}^m$, $\mathbf{U}_{x_{old}}^{m+1} = \mathbf{U}_x^m$.
4. Correct to $\mathbf{U}_{new}^{m+1}$ using equation (20).
5. If $\|\mathbf{U}_{new}^{m+1} - \mathbf{U}_{old}^{m+1}\| < \epsilon$, then $\mathbf{U}_{new}^{m+1} = \mathbf{U}_{old}^{m+1}$.
6. Obtain $\mathbf{U}_{x_{new}}^{m+1}$ using equation (7).
7. Take $\mathbf{U}_{old}^{m+1} = \mathbf{U}_{new}^{m+1}$, $\mathbf{U}_{x_{old}}^{m+1} = \mathbf{U}_{x_{new}}^{m+1}$ and go to step 4.

The stopping criterion for the inner iteration is set at $\epsilon = 10^{-12}$ for the above approach.

Remark 2. *The fully discrete problem (21) is obtained for the constant coefficient case. In case of variable coefficient problems, it is straightforward to develop the compact scheme using the Hermitian approach just by discretizing the variable coefficients at each grid point; see* [15] *on page* 7 *for more details. Specifically, if we take σ as a function of x & τ, in Eq. (20), the coefficients $\left(r - \frac{\sigma^2}{2} - \lambda\zeta\right)$ will also become a matrix. This is the difference while solving a variable coefficient problem with the proposed comapct scheme. However, it is cumbersome with the auxiliary equation based approach because one has to take care of the compact discretization of the coefficient term also; see* [19] *on pages* 10 *and* 11 *for a detailed discussion. This is the major advantage of the Hermitian approach based compact schemes; we can use the same discretized equation for a variable coefficient problem that has been derived for constant coefficient, with slight modifications in the coefficient matrices and vectors. The Examples* 2 *and* 4 *in this manuscript are handled by discretizing the variable coefficients at each grid point.*

Remark 3. *Note that the fully discrete problem given in Eq. (21) for solving the PIDE (12) requires solution vectors at two initial time steps. Therefore, the solution vector at $\tau = 0$ is obtained from the initial condition, and the fully implicit scheme given in* [3] *provides the solution vector at time $\delta\tau$.*

Remark 4. *It can be easily pointed out that for each time step, the number of iterations vary while solving the fully discrete problem (21) using correcting to convergence approach. Therefore, an upper bound is obtained for computational complexity rather than giving an exact number of iterations. Let number of iterations required by above approach be n_m at a fixed time level τ_m and $n_s := \max_{1\le m\le M} n_m$. Using the fact that a tri-diagonal system of equations takes $O(N)$ operations and integral operator exhibits $O(NlogN)$ computational cost (see* [30]*), the upper bound for complexity of the fully discrete problem (21) would be $O\left((n_s + logN)NM\right)$.*

2.5. Consistency.

Theorem 2. *For sufficiently small δx and $\delta\tau$, we have*

$$\frac{\partial u}{\partial \tau} - \mathbb{L}u_n^m - \left(\frac{3u_n^{m+1} - 4u_n^m + u_n^{m-1}}{2\delta\tau} - \mathbb{L}_\delta u_n^m\right) = O(\delta\tau^2 + \delta x^4), \quad \text{for } m \ge 1. \tag{22}$$

2.6. Stability. As discussed in Sec. 1, let us now prove the stability of the proposed IMEX compact scheme (20) using Schur polynomial approach. Consider a single node $U_n^m = p^m e^{In\theta}$, where p^m is amplitude at time level τ_m, $I = \sqrt{-1}$, and $\theta = 2\pi/N$. The integral operator (13) can be rewritten in an equivalent form as $\mathbb{I}u(x,\tau) = \lambda \int_{-L}^{L} u(y+x,\tau)g(y)dy$. Fourth order accurate composite Simpson's rule for $\mathbb{I}u(x,\tau)$ yields

$$\mathbb{I}_\delta U_n^m = \delta x \sum_{k=0}^{N} w_k U_{k+n}^m g_k = \delta x \sum_{k=0}^{N} w_k p^m e^{I\theta(k+n)} g_k = p^m e^{i\theta n} G_k,$$

where $G_k = \delta x \sum_{k=0}^{N} w_k e^{I\theta k} g_k$ and $g_k = g(x_k)$. The following Lemma is proved in [30] for the numerical quadrature G_k.

Lemma 1. *The numerical quadrature G_k satisfies*

$$|G_k| \leq 1 + C\delta x^4,$$

where C is a constant.

For simplicity, $\frac{\sigma^2}{2}$ and $\left(r - \frac{\sigma^2}{2} - \lambda\zeta\right)$ will be denoted by a and b respectively. Consequently, Eq. (20) can be written as follows:

$$\begin{aligned}\left[3 - 4a\delta\tau\Delta_{xx} + 2\delta\tau(r+\lambda)\right] U_n^{m+1} &= 2\delta\tau\left[b - a\Delta_x\right] U_{x_n}^{m+1} + 2\delta\tau\lambda G_k(2U_n^m - u_n^{m-1}) \\ &\quad + 4U_n^m - U_n^{m-1}.\end{aligned} \tag{23}$$

From [32], the following relations are obtained

$$\Delta_x U_n^m = I\frac{\sin(\theta)}{\delta x} U_n^m, \quad \Delta_x^2 U_n^m = \frac{2\cos(\theta) - 2}{\delta x^2} U_n^m, \quad U_{x_n}^m = I\frac{3\sin(\theta)}{\delta x(2+\cos(\theta))} U_n^m. \tag{24}$$

Using equation (24) in difference scheme (23), we get

$$\begin{aligned}&\left[3 - 8a\delta\tau\left(\frac{\cos(\theta)-1}{\delta x^2}\right) + 2\delta\tau(r+\lambda)\right] U_n^{m+1} \\ =&2\delta\tau\left[\left(a\frac{\sin(\theta)}{\delta x} + Ib\right)\frac{3\sin(\theta)}{\delta x(2+\cos(\theta))}\right] U_n^{m+1} \\ &+ 2\delta\tau\lambda G_k(2U_n^m - U^{m-1}) + 4U_n^m - U_n^{m-1},\end{aligned} \tag{25}$$

which implies

$$\begin{aligned}&\left[3 - 2\delta\tau a\left(\frac{4\cos\theta - 4}{\delta x^2} + \frac{3\sin^2\theta}{\delta x^2(2+\cos\theta)}\right) - I2\delta\tau b\frac{3\sin\theta}{\delta x(2+\cos\theta)} + 2\delta\tau(r+\lambda)\right] U_n^{m+1} \\ &\qquad = (4 + 4\delta\tau\lambda G_k)U_n^m - (1 + 2\delta\tau\lambda G_k)U_n^{m-1},\end{aligned}$$

$$\begin{aligned}&\left[3 - 2\delta\tau a\left(\frac{\cos^2\theta + 4\cos\theta - 5}{\delta x^2(2+\cos\theta)}\right) - I2\delta\tau b\frac{3\sin\theta}{\delta x(2+\cos\theta)} + 2\delta\tau(r+\lambda)\right] U_n^{m+1} \\ &\qquad = (4 + 4\delta\tau\lambda G_k)U_n^m - (1 + 2\delta\tau\lambda G_k)U_n^{m-1}.\end{aligned}$$

Using $U_n^m = p^m e^{In\theta}$ in above and divide above equation by $p^{m-1}e^{In\theta}$, following amplification polynomial is obtained:

$$\Theta(p) = \gamma_0 p^2 + \gamma_1 p + \gamma_2, \tag{26}$$

where

$$(27)\quad \begin{aligned} \gamma_0 &= \left[3 - 2\delta\tau\left(a\frac{\cos^2(\theta)+4\cos(\theta)-5}{\delta x^2(2+\cos(\theta))} + Ib\frac{3\sin(\theta)}{\delta x(2+\cos(\theta))} - (r+\lambda)\right)\right],\\ \gamma_1 &= -\left[4 + 4\lambda\delta\tau G_k\right],\\ \gamma_2 &= \left[1 + 2\delta\tau\lambda G_k\right]. \end{aligned}$$

In order to prove the stability, a direct determination of the roots of these polynomial is not advised. Instead, a well established theory based on Schur polynomials is considered to prove the stability of the proposed IMEX compact scheme (20). A detailed discussion on stability analysis for advection-diffusion equations using Schur polynomials is given in [20]. Let us consider p_1 and p_2 as roots of amplification polynomial $\Theta(p)$ and introduce the following polynomials, definitions and theorems from [20]:

$$(28)\quad \Theta^*(p) = \bar{\gamma_2}p^2 + \bar{\gamma_1}p + \bar{\gamma_0} = \gamma_2 p^2 + \gamma_1 p + \bar{\gamma_0},$$

$$(29)\quad \Theta_1(p) = \frac{\Theta^*(0)\Theta(p) - \Theta(0)\Theta^*(p)}{p},$$

where $\bar{\gamma_i},\ i = 0, 1,$ and 2 is complex conjugate of γ_i.

Definition 1. *Polynomial $\Theta(p)$ is called Schur polynomial if $|p_i| < 1\ \forall\ i$. Further, polynomial $\Theta(p)$ is called simple von Neumann polynomial if $|p_i| \leq 1$, and if $p_i = 1$, it must be a simple root.*

Definition 2. *Proposed Compact scheme (20) is stable if its amplification polynomial $\Theta(p)$ is a simple von Neumann polynomial.*

Theorem 3. *$\Theta(p)$ is a Schur polynomial iff $|\Theta^*(0)| > |\Theta(0)|$ and $\Theta_1(p)$ is a Schur polynomial.*

Theorem 4. *$\Theta(p)$ is a simple von Neumann polynomial iff either $|\Theta^*(0)| > |\Theta(0)|$ and $\Theta_1(p)$ is a simple von Neumann polynomial or $\Theta_1(p) \equiv 0$ and first derivative of $\Theta(p)$ (with respect to its independent variable) is a Schur polynomial.*

Note that, the question of whether a n^{th} degree polynomial is a simple von Neumann polynomial is now reduced to that for a first degree polynomial by using above two theorems repeatedly. The results in above theorems are immensnely useful, as compared to first finding explicitly the roots of the characteristic polynomial and then determining their absolute values. Now, the stability of proposed IMEX compact scheme (20) will be proved in the following theorem using above definitions and theorems.

Theorem 5. *Proposed compact scheme (20) is stable if $\lambda < 1$, and*

$$(30)\quad |(\gamma_2 - \bar{\gamma_0})\,\gamma_1| \leq \left||\gamma_0|^2 - \gamma_2^2\right|.$$

Proof. From (28), we have $|\Theta^*(0)|^2 = |\bar{\gamma_0}|^2$. Since $a > 0$ and $\left(\frac{cos^2(\theta)+4cos(\theta)-5}{\delta x^2(2+cos(\theta))}\right) \leq 0$, therefore $|\Theta^*(0)| > 3$. Further, $|\Theta(0)| = |\bar{\gamma_2}|$. If $\lambda < 1$, then Lemma 1 gives $|\Theta(0)| < 3$. Therefore $|\Theta^*(0)| > |\Theta(0)|$. Hence, first part of Theorem 4 is proved. For second part, let us write $\Theta_1(p)$ from (29) as follows:

$$\Theta_1(p) = \frac{\bar{\gamma_0}\Theta(p) - \gamma_2\Theta^*(p)}{p}.$$

Using values of $\Theta(p)$ and $\Theta^*(p)$ from (26) and (28), we get

$$\Theta_1(p) = \left(|\bar{\gamma_0}|^2 - \gamma_2^2\right) p + \gamma_1 \left(\bar{\gamma_0} - \gamma_2\right).$$

It can be observed that $\Theta_1(p)$ is simple von Neumann polynomial if (30) holds. Further, Theorem 4 gives that $\Theta(p)$ is simple von Neumann polynomial and result follows from Definition 2. □

Algorithm 1: Algorithm for American options.

for $m = 0$
  for $n = 1, 2, \cdot, N-1$
    $\frac{U_n^{m+1} - U_n^m}{\delta\tau} = \mathbb{D}_\delta U_n^{m+1} + \mathbb{I}_\delta U_n^m + \Psi_n^m$
  end
  Solve for $n = 1, 2, ..., N-1$
    $U_n^{m+1} = \max\left(f(x_n), U_n^{m+1} - \delta\tau \Psi_n^m\right)$
    $\Psi_n^{m+1} = \frac{U_n^{m+1} - U_n^{m+1}}{\delta\tau} + \Psi_n^m$
end
for $m \geq 1$
  for $n = 1, 2, ..., N-1$
    $\frac{3U_n^{m+1} - 4U_n^m + U_n^{m-1}}{2\delta\tau} = \mathbb{D}_\delta U_n^{m+1} + \mathbb{I}_\delta(EU_n^m) + \Psi_n^m$
  end
  Solve for $n = 1, 2, ..., N-1$
    $U_n^{m+1} = \max\left(f(x_n), U_n^{m+1} - \frac{2\delta\tau}{3}\Psi_n^m\right)$
    $\Psi_n^{m+1} = \frac{3}{2}\frac{U_n^{m+1} - U_n^{m+1}}{2\delta\tau} + \Psi_n^m$
end

3. IMEX compact scheme for LCPs

Let us now discuss the discretization of the LCP (4) using the proposed IMEX compact scheme. To this end, Ikonen et al. [33] proposed an operator splitting technique for pricing American options under Black-Scholes model and it was extended by Toivanen [34] for jump-diffusion models. In operator splitting technique, a new auxiliary variable ψ is taken to change the inequality (4) in an equation form such that $\psi = u_\tau - \mathbb{L}u$, and the LCP (4) is rewritten as follows:

$$\begin{aligned} &u_\tau - \mathbb{L}u = \psi, \\ &\psi \geq 0, \quad u \geq f, \quad \psi(u-f) = 0. \end{aligned} \tag{31}$$

Application of operator splitting technique in (31) introduces a new variable $\hat{U}_n^m$ and gives the following two discrete equations:

$$\begin{aligned} \frac{3\hat{U}_n^{m+1} - 4U_n^m + U_n^{m-1}}{2\delta\tau} &= \mathbb{D}_\delta \hat{U}_n^{m+1} + \mathbb{I}_\delta(EU_n^m) + \Psi_n^m, \\ \frac{3U_n^{m+1} - 4U_n^m + U_n^{m-1}}{2\delta\tau} &= \mathbb{D}_\delta \hat{U}_n^{m+1} + \mathbb{I}_\delta(EU_n^m) + \Psi_n^{m+1}. \end{aligned} \tag{32}$$

A pair $(\Psi_n^{m+1}, U_n^{m+1})$ is to be found satisfying (32), and the following constraints

$$U_n^{m+1} \geq f(x_n), \quad \Psi_n^{m+1} \geq 0, \quad \Psi_n^{m+1}\left(U_n^{m+1} - f(x_n)\right) = 0. \tag{33}$$

The approach to solve (32)-(33) is given in Algorithm 1. The system of linear equations obtained from Algorithm 1 is solved using correcting to convergence approach discussed in Sec. 2.4.

4. Numerical Illustrations and Discussion

In this section, the applicability of the proposed IMEX compact scheme for pricing European and American options under Merton's jump-diffusion models is demonstrated. It has always been challenging to achieve high-order accuracy for the problems with non-smooth initial conditions. A thorough discussion is manifested in [23] to obtain high-order accuracy for parabolic problems with the initial conditions having low regularity. Accordingly, technique given in [23] is applied in this manuscript to achieve fourth order convergence rate using the proposed IMEX compact schemes, and the detailed explanation about this technique can be found in [30]. Since proposed compact scheme is second order accurate in time variable and fourth order accurate in spatial variable, the parabolic mesh ratio $\left(\frac{\delta\tau}{\delta x^2}\right)$ is fixed as 0.4 in all the computations. The parameters considered for pricing European and American options under Merton jump-diffusion model are listed in Table 1. For all the computation, L is taken 1.5, i.e. the computational domain is $[-1.5, 1.5]$. Here, L is chosen in a way that the interval $[-L, L]$ provides a sufficiently large interval for stock price S after the inverse transformation $S = Ke^x$. The value of L is taken 1.5 & $K = 100$; throughout the examples, it implies that the lower limit for stock price is $Ke^{-L} = 22.31$ and the upper limit for stock price is $Ke^L = 448.17$, which gives a sufficiently large interval for the stock price, i.e. $(22.31, 448.17)$. Moreover, the rate of convergence R^C is computed with the following formula $R^C = \frac{\log\left(\frac{\xi_1}{\xi_2}\right)}{\log\left(\frac{\delta x_1}{\delta x_2}\right)}$, where ξ_1 and ξ_2 are relative errors with respect to the reference solution in discrete l^2 norm $\left(\frac{\|U^{ref}-U^c\|}{\|U^{ref}\|}\right)$ corresponding to the mesh sizes δx_1 and δx_2, respectively. Here, U^{ref} is the reference solution computed on a fine grid with $N = 6144$ and U^c is the solution obtained from the proposed IMEX compact scheme.

Parameters	λ	T	r	K	σ	μ_J	σ_J
Values	0.10	0.25	0.05	100	0.15	-0.90	0.45

Table 1. The values of various parameters to compute the option price and greeks for European and American options under Merton's jump-diffusion models.

Moreover, the discussion on option pricing is always incomplete without analysing the behaviour of the certain function of option price (commonly known as Greeks). Therefore, the Greeks are also studied in this manuscript to understand and analyse the risk. The Greeks may be defined as follows:

- The rate of change of option price w.r.t the change in underlying asset's price is known as Delta (Δ).
- The rate of change in the Δ w.r.t change in the underlying asset price is known as Gamma (Γ).

Therefore, the applicability of proposed IMEX compact scheme is also demonstrated for the computation of Greeks in the first example.

Example 1. *(European put option: Constant volatility case)*

Values	(S, t) = (90,0)	(S, t) = (100,0)	(S, t) = (110,0)
Option price (analytical)	9.28541807	3.14902574	1.40118588
Option price (IMEX scheme)	9.28541752	3.14902381	1.40118482
Delta (analytical)	-0.84671538	-0.35566306	-0.05810123
Delta (IMEX scheme)	-0.84671683	-0.35566292	-0.05810323
Gamma (analytical)	0.03486014	0.04882567	0.01212941
Gamma (IMEX scheme)	0.03486198	0.04882482	0.01212874

TABLE 2. The comparison between the analytical values of option prices, Delta, and Gamma obtained from the series solution presented in [2] and the values achieved using proposed IMEX compact scheme for $N = 6144$ and parameters given in Table 1.

The first example comprises the PIDE (12) along with the parameters in Table (1). For this example, the analytical solution is given by Merton's formula [2]

$$U(S,\tau) = \sum_{i=0}^{\infty} \frac{[\lambda(w+1)(T-\tau)^n \exp(-\lambda(w+1))(T-\tau)]}{n!} BSM(S,T,K,r_i,\sigma_i),$$

where, $w := -1 + \exp(\mu_J + \frac{1}{2}\sigma_J^2), r_i := r - \lambda w + \frac{i \log(w+1)}{T-\tau}, \sigma_i := \sqrt{\sigma^2 + \frac{i\sigma_J^2}{T-\tau}}$. Here, $BSM(S,\tau,K,r_i,\sigma_i)$ is the solution of classical Black–Scholes PDE [1], and

$$BSM(S,\tau,K,r_i,\sigma_i) = K\exp(-r_i(T-\tau))\mathcal{N}(-d_2) - S\mathcal{N}(-d_1),$$

$$d_1(S,\tau) = \frac{\ln\left(\frac{S}{K}\right)\left(r_i + \frac{1}{2}\sigma_i^2\right)(T-\tau)}{\sigma_i\sqrt{T-\tau}}, \quad d_2(S,\tau) = d_1(S,\tau) - \sigma_i\sqrt{T-\tau},$$

$$\mathcal{N}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} \exp\left(\frac{-x^2}{2}\right) dx.$$

Therefore, analytical solution will be used as reference solution U^{ref} for this example. The values of option prices and the Greeks at a particular stock price S are computed from the obtained numerical solution using cubic spline interpolation. For the sake of completeness and comparison of results, the PIDE (12) is also solved using FDM discussed in [5].

Note that, the accuracy of the proposed IMEX compact scheme is evident from Table 2 since the values of option prices and the Greeks obtained from the proposed IMEX compact scheme are almost equal to the values obtained from analytical solution. Here, the comparison is done for three different stock price $S = 90, 100$, and 110 to analyse the behaviour of proposed IMEX compact scheme in the vicinity of strike price value, which is 100.

Next concern is to validate the fourth-order convergence rate of the proposed IMEX compact scheme. To claim this, the relative ℓ^2-errors with proposed IMEX compact scheme (using smooth and non-smooth payoffs) and with FDM (smoothing of payoff does not affect rate of convergence in this case) are presented in Table 3. It is evident that proposed IMEX compact scheme is fourth order accurate when payoff is smoothed according to the technique given in [23]. However, only second

N	IMEX Compact scheme with smooth payoff		IMEX compact scheme with non-smooth payoff		IMEX finite difference scheme with non-smooth payoff	
	Error	R^C	Error	R^C	Error	R^C
24	2.395e-02	-	1.683e-02	-	9.317e-02	-
48	1.524e-03	3.97	4.187e-03	2.00	2.362e-02	1.97
96	9.631e-05	3.98	9.988e-04	2.05	5.815e-03	2.02
192	5.931e-06	4.02	2.495e-04	2.00	1.425e-03	2.02
384	3.678e-07	4.01	6.121e-05	2.02	3.521e-04	2.01

TABLE 3. The comparison of errors and rate of convergence achieved using proposed IMEX compact scheme with non-smooth and smooth initial conditions (payoffs) with the error and rate of convergence achieved using FDM for European put options with constant volatility.

order convergence rate is achieved with IMEX compact scheme while using non-smooth payoff. Moreover, second order convergence rate is achieved with FDM irrespective of smoothing the payoff.

Now, the location and nature of the spurious oscillations present in the numerical solution will be explained. This very phenomenon arising in the case of FDM has already been noticed and explained by several authors, for example see [35] and references therein. In order to depict this graphically, the difference between analytical and numerical solutions is plotted in Figures 1(a) and 1(b) as a function of time and stock prices without smoothing payoffs using FDM and proposed IMEX compact scheme respectively. It can be observed that maximum error at strike price is comparatively smaller with the proposed IMEX compact scheme, which again proves the better damping properties of compact approximations as compared to finite difference approximations, see [19] for more details. The same difference is again plotted in Figures 1(c) and 1(d) after smoothing the payoffs, and the results obtained from IMEX compact scheme seems better again as compared to the results from FDM. Moreover, it can be observed in Figures 1(c) and 1(d) that spurious oscillations are further damped when smooth payoffs are used, which gives another advantage of using smoothing operator apart from achieving high convergence rate for solving the PIDEs.

Efficiency is one of the desirable property for any numerical method developed. Therefore, the CPU time taken in proposed IMEX compact scheme is compared with the time taken in FDM to solve this example. In order to show this, the error between numerical and analytical solutions, and corresponding CPU times at grid points N=12, 24, 48, 96, 192, 384 are computed and plotted in Figure 2. The numerical computation is done using MATLAB on a computer with Intel(R) Core(TM)i5-8400 CPU @ 2.80GHz. It can be noticed that proposed IMEX compact scheme is taking less time to achieve a particular accuracy as compared to time taken by FDM. Thus, all the claims asserted in Sec.1 have been illustrated and validated via this first example.

Example 2. *(European put option: Local volatility case)*

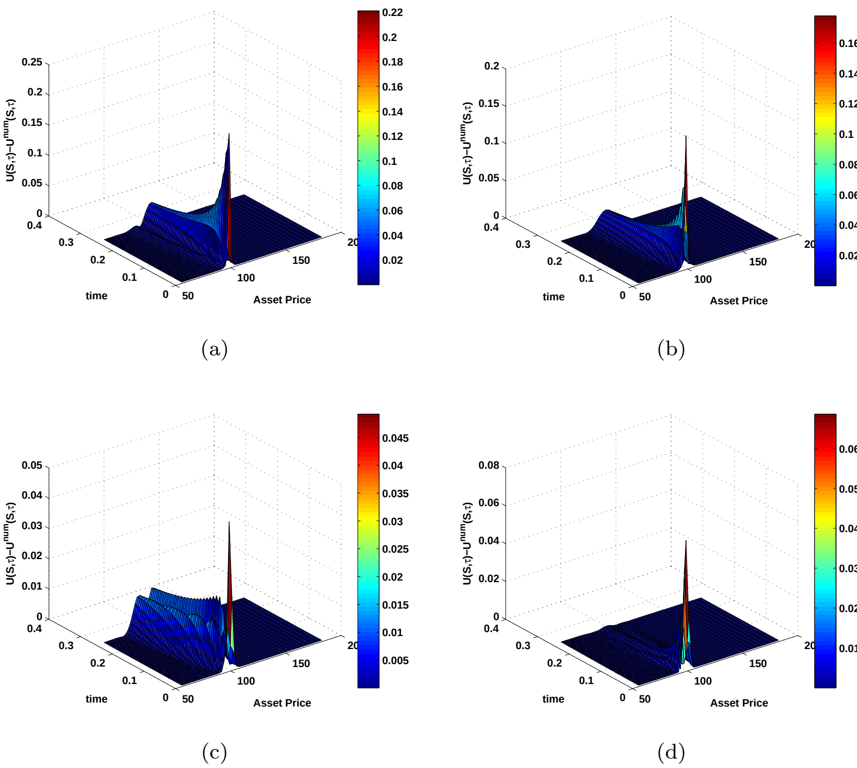

FIGURE 1. The location and nature of the spurious oscillations present in the obtained numerical solutions are depicted via this figure in the error (the difference between the analytical and numerical values) form using: (a) FDM with non-smooth initial condition, (b) IMEX compact scheme with non-smoothing initial condition, (c) FDM with smoothed initial condition, and (d) IMEX compact scheme with smoothed initial condition.

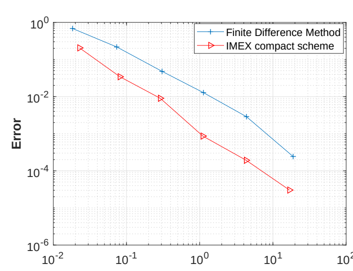

FIGURE 2. Efficiency: The CPU time (in seconds) and the corresponding errors using proposed IMEX compact scheme and FDM.

It has always been a challenge to develop the high-order compact schemes for variable coefficient problems (already discussed in Sec 1). This example is taken to validate the applicability of proposed IMEX compact scheme for variable coefficient PIDEs. This example comprises the PIDE (12) along with the parameters in Table (1), except that local volatility $\sigma(x,\tau)$ is taken instead of constant value of σ, and $\sigma(x,\tau)$ is defined as follows:

$$\sigma(x,\tau) = 0.15 + 0.15\,(0.5 + 2(T-\tau))\,\frac{\left((Ke^x/100) - 1.2\right)^2}{\left(Ke^x/100\right)^2 + 1.44}. \tag{34}$$

The discretization of the variable coefficient PIDE is accomplished in a similar way except that the local volatility $\sigma(x,\tau)$ is also discretized at each grid point (x_n,τ_m). It can be observed from Table 4 that the option price values are almost equal to the reference values which in turn proves the accuracy of the proposed IMEX compact scheme.

The fourth-order convergence rate of the proposed IMEX compact scheme is also validated for this example in Table 5. Moreover, the effect of smoothing the initial conditions is also presented in that Table.

Option price	(S, t) = (90,0)	(S, t) = (100,0)	(S, t) = (110,0)
Reference values [36]	9.317323	3.183681	1.407745
Proposed IMEX compact scheme	9.317318	3.183678	1.407741

TABLE 4. Comparison of the option prices obtained from proposed IMEX compact scheme with reference values for European put options with local volatility σ using $N = 6144$.

N	IMEX Compact scheme with smooth payoff		IMEX compact scheme with non-smooth payoff		IMEX finite difference scheme with non-smooth payoff	
	Error	R^C	Error	R^C	Error	R^C
24	3.128e-02	-	3.827e-02	-	6.317e-02	-
48	1.986e-03	3.97	9.414e-03	2.02	1.562e-02	2.01
96	1.231e-04	4.01	2.329e-03	2.01	3.843e-03	2.02
192	7.861e-06	3.96	5.744e-04	2.01	9.725e-04	1.98
384	4.798e-07	4.03	1.439e-04	1.99	2.410e-04	2.01

TABLE 5. The comparison of errors and rate of convergence achieved using proposed IMEX compact scheme with non-smooth and smooth payoffs with the error and rate of convergence achieved using FDM for European put options with local volatility.

Example 3. *(American put option: Constant volatility case)*

This example is considered to extend the applicability of the proposed IMEX compact scheme for pricing American options by solving the LCP (32)-(33) along with the parameters in Table 1. The results from Table 6 validates that proposed IMEX compact scheme is also accurate for pricing American options. We would like to mention that only third order convergence rate could be achieved in Table 7 for American options using proposed IMEX compact scheme even for smoothed initial condition. The reason could be the lack of regularity of the problem due to the free boundary feature of LCP (4), which needs further research to be resolved (see [37] for the detailed discussion on free boundary problems).

Option price	(S, t) = (90,0)	(S, t) = (100,0)	(S, t) = (110,0)
Reference values [36]	10.003866	3.241207	1.419790
Proposed IMEX compact scheme	10.003857	3.241201	1.419785

TABLE 6. Comparison of the option prices obtained from proposed IMEX compact scheme with reference values for American put options with constant volatility using $N = 6144$.

N	IMEX Compact scheme with smooth payoff		IMEX compact scheme with non-smooth payoff		IMEX finite difference scheme with non-smooth payoff	
	Error	R^C	Error	R^C	Error	R^C
24	4.716e-02	-	5.753e-02	-	8.518e-02	-
48	5.843e-03	3.01	1.417e-02	2.02	2.121e-02	2.00
96	6.851e-04	3.09	3.383e-03	2.06	5.192e-03	2.03
192	8.132e-05	3.07	8.545e-04	1.98	1.252e-03	2.05
384	9.837e-06	3.04	1.985e-04	2.09	3.011e-04	2.05

TABLE 7. The comparison of errors and rate of convergence achieved using proposed IMEX compact scheme with non-smooth and smooth payoffs with the error and rate of convergence achieved using FDM for American put options with constant volatility.

Option price	(S, t) = (90,0)	(S, t) = (100,0)	(S, t) = (110,0)
Reference values [36]	10.008881	3.275957	1.426403
Proposed IMEX compact scheme	10.008876	3.275951	1.426402

TABLE 8. Comparison of the option prices obtained from proposed IMEX compact scheme with reference values for American put options with local volatility using $N = 6144$.

Example 4. *(American put option: Local volatility case)*

N	IMEX Compact scheme with smooth payoff		IMEX compact scheme with non-smooth payoff		IMEX finite difference scheme with non-smooth payoff	
	Error	R^C	Error	R^C	Error	R^C
24	5.281e-02	-	8.517e-02	-	1.821e-01	-
48	6.243e-03	3.08	2.187e-02	1.96	4.391e-02	2.05
96	7.431e-04	3.07	5.155e-03	2.08	1.063e-02	2.04
192	9.142e-05	3.02	1.312e-03	1.97	2.712e-03	1.97
384	1.142e-05	3.00	3.172e-04	2.04	7.026e-04	1.94

TABLE 9. The comparison of errors and rate of convergence achieved using proposed IMEX compact scheme with non-smooth and smooth payoffs with the error and rate of convergence achieved using FDM for American put options with local volatility.

This final example discusses the solution of variable coefficient LCP (obtained by taking $\sigma(x,\tau)$ instead of constant σ in Eq. (4)) along with the parameters in Table 1 using proposed IMEX compact scheme. The results from Table 8 validates the accuracy of the proposed IMEX compact scheme for the variable coefficient LCP and rate of convergence for this problem can be found in Table 9.

5. Concluding remarks and future directions

In this article, the IMEX compact schemes have been proposed for solving the PIDEs and LCPs arising in pricing of European and American options respectively under Merton's jump-diffusion model. The very known concerns applicable to any numerical method, for example, stability, accuracy, efficiency, convergence rate, and spurious oscillation in the solution (if present) were discussed in detail via numerical illustrations. Based on the proposed method and the advantages of compact schemes discussed in Sec. 1, following future directions can be observed:

- Since proposed compact scheme suits well for solving high-dimensional complex problems, it can be easily extended for option pricing under more advanced framework, for example, stochastic volatility jump-diffusion models (a.k.a. Bates model [38]) and the stochastic volatility with contemporaneous jump model (a.k.a. SVCJ model [39]).
- In literature, another established approach to solve the high-dimensional problems is the alternating direction implicit (ADI) finite difference schemes. Similarly, the ADI compact schemes can also be developed using the techniques presented in this manuscript to solve the high-dimensional problems proposed in [38, 39]. The comparison of results with ADI schemes and without ADI technique would help the community to choose appropriate method for a particular problem.
- It should be noted that no special property of option pricing theory is used to develop the IMEX compact scheme for solving the PIDEs and LCPs. Therefore, proposed IMEX compact scheme can also be applied with little or no modification to solve such problems arising in other areas, for example, PIDEs in Mathematical Physics, and LCPs in computational Mechanics.

Conflict of interest statement and data availability

Not Applicable.

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