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FULL DISCRETISATION OF THE TIME DEPENDENT NAVIER-STOKES EQUATIONS WITH ANISOTROPIC SLIP BOUNDARY CONDITION

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Abstract. In this work, we study theoretically and numerically the non-stationary Navier-Stokes's equations under power law slip boundary condition. We establish existence of a unique solution by using a semi-discretization in time combined with the weak convergence approach. Next, we formulate and analyze the discretzation in time and the finite element approximation in space associated to the continuous problem. We derive optimal convergence in time and space provided that the solution is regular enough on the slip zone. Iterative schemes for solving the nonlinear problems is formulated and convergence is studied. Numerical experiments presented confirm the theoretical findings.

Key words. Power law slip boundary condition, Navier-Stokes equations, space-time discretization, monotonicity, error estimates.

1. Introduction

We are concerned with the dicretization of the non-stationary incompressible Navier-Stokes equations

$$
(1) \qquad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - 2\nu \,\mathrm{div} D\mathbf{u} + [\mathbf{u}\cdot\nabla]\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega\times(0,T), \\ \mathrm{div}\,\mathbf{u} &= 0 \quad \text{in } \Omega\times(0,T) \\ \mathbf{u}(\mathbf{x},\mathbf{0}) &= \mathbf{u}_0 \quad \text{in } \Omega\times\{0\}, \end{cases}
$$

where Ω is a open and bounded domain in $\mathbb{R}^d$, with a Lipschitz-continuous boundary $\partial\Omega$. It is assumed that $d = 2, 3$, and $T > 0$ is the final time of observation of the fluid. The unknowns are the velocity $\mathbf{u}$ and the pressure p. $\mathbf{f}$ is the external force acting on the fluid and ν is the kinematic viscosity of the fluid, assume non-negative. $\mathbf{u}_0$ is the initial velocity and we assume for the moment that div $\mathbf{u}_0 = 0$. We recall that the Cauchy stress tensor is $\mathbf{T} = -p\mathbf{I} + 2\nu D\mathbf{u}$, with $\mathbf{I}$, the identity matrix in $\mathbb{R}^{d\times d}$, while the symmetric part of the velocity gradient is $2D\mathbf{u} = \nabla\mathbf{u} + (\nabla\mathbf{u})^T$. We are interested in (1) when the position and the direction of the slip boundary condition are taken into account (see [14, 15]). We then assume that the boundary $\partial\Omega$ is made of two components S and Γ, such that $\overline{\partial\Omega} = \overline{S\cup\Gamma}$, with $S\cap\Gamma = \emptyset$. We assume the homogeneous Dirichlet condition on Γ, that is

$$
(2) \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma\ .
$$

Thus Γ is the porous or artificial boundary where the fluid is prescribed. On S, we assume the impermeability condition

$$
(3) \qquad \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } S\,,
$$

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where $\mathbf{n} : S \longrightarrow \mathbb{R}^d$ is the normal outward unit vector to S. S is an impermeable solid surface along which the fluid may slip. Taking the scalar product of $\mathbf{u}$ and the balance of linear momentum in (1), we obtain

(4)
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2\, dx + 2\nu \int_\Omega |D\mathbf{u}|^2\, dx + \int_S (-\mathbf{Tn})_{\boldsymbol{\tau}} \cdot \mathbf{u}_{\boldsymbol{\tau}} d\sigma = \int_\Omega \mathbf{f}\cdot\mathbf{u} dx\ , \quad \text{for all } t \geq 0\,,$$

with $d\sigma$ being the surface measure associated to S. Also, for any vector $\mathbf{v}$ defined on S, we set $\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$. Thus $(\mathbf{Tn})_{\boldsymbol{\tau}}$ denotes the projection of the normal stress into the corresponding tangent plane. We note that the first term in (4) stands for the change of the kinetic energy, the second and third expressions represent the energy that is dissipated and transformed to other form forms of energy. We are more interested in the energy on the boundary S that can only be fully expressed if $(\mathbf{Tn})_{\boldsymbol{\tau}}$ is given. For that purpose, the most general relation between $\mathbf{u}_{\boldsymbol{\tau}}$ and $(\mathbf{Tn})_{\boldsymbol{\tau}}$ is the implicit constitutive relation [19]

$$\psi(\mathbf{u}_{\boldsymbol{\tau}}, (\mathbf{Tn})_{\boldsymbol{\tau}}) = 0 \tag{5}$$

where ψ is function. The simplest form of (5) that ensure the non-negativity of $\int_S (-\mathbf{Tn})_{\boldsymbol{\tau}} \cdot \mathbf{u}_{\boldsymbol{\tau}} d\sigma$ is the choice

$$(-\mathbf{Tn})_{\boldsymbol{\tau}} = \alpha \mathbf{u}_{\boldsymbol{\tau}} d\sigma \quad , \quad \alpha > 0\ .$$

This is the Navier's slip boundary conditions. If $(\mathbf{Tn})_{\boldsymbol{\tau}} = \mathbf{0}$, then one gets a perfect slip boundary condition, while if $\mathbf{u}_{\boldsymbol{\tau}} = \mathbf{0}$, then there is no slip. We are interested in the power law slip boundary condition given as follows [7]

$$(\mathbf{Tn})_{\boldsymbol{\tau}} + |K\mathbf{u}_{\boldsymbol{\tau}}|^{s-2} K^2 \mathbf{u}_{\boldsymbol{\tau}} = 0 \quad \text{on } S \times (0,T)\,, \tag{6}$$

where $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$ is the Euclidean norm. K is an anisotropic tensor, assumed to be uniformly positive definite, symmetric, and bounded. s is a real, strictly positive number representing the flow behavior index. The tangential shear is a power law function of the tangential velocity. Such a boundary condition arises when the contact surface is lubricated with a thin layer of a non-Newtonian fluid. It is manifest that for $s = 2$ and $K = \mathbf{I}$, one obtains the classical Navier's slip condition. The anisotropic slip law (6) defines from the slip relation introduce in [14, 15]; that is

$$(\mathbf{Tn})_{\boldsymbol{\tau}} + \psi(\mathbf{u}_{\boldsymbol{\tau}})\mathbf{u}_{\boldsymbol{\tau}} = \mathbf{0} \quad \text{on } S \times (0,T)\,, \tag{7}$$

where the function ψ is real valued and satisfies;

(i) ψ is bounded and there exist two positive constants α_1, α_2 such that for any vector $\mathbf{v} \in \mathbb{R}^d$

$$\alpha_1 \leq \psi(\mathbf{v}) \leq \alpha_2\ . \tag{8}$$

(ii) ψ is Lipschitz-continuous with Lipschitz constant λ, that is

$$\forall\, \mathbf{v}, \boldsymbol{w} \in \mathbb{R}^d\ ,\ |\psi(\mathbf{v}) - \psi(\boldsymbol{w})| \leq \lambda\, |\mathbf{v} - \boldsymbol{w}|\ . \tag{9}$$

It is manifest by taking $\psi(\mathbf{u}_{\boldsymbol{\tau}}) = |K\mathbf{u}_{\boldsymbol{\tau}}|^{s-2} K^2$, the conditions (8) and (9) are not verified. Hence (6) does not belongs to the class of anisotropic slip boundary conditions defined by C. Le Roux in [14, 15]. We intend to study the finite element solution of the Navier-Stokes equations with (7), (8) and (9). A similar model but for the stationary case has been analysed in [7] using conforming finite element approach.

We analyse problem (1),(2), (3) and (6) in three steps. First, we discretize in time the weak formulation associated with (1),(2), (3) and (6). Next, since the discrete in time problem is a Navier-Stokes's like problem with an extra nonlinear monotone term, we used the Galerkin's approximation together with monotone operator's theory to claim existence at this juncture. The last step in the procedure is to derive some boundeness of the solution and used some compactness results to recover the solution of the continuous in time formulation. This approach is not new and the interested reader may consult the celebrated book of J.L. Lions (see [16]). Global uniqueness is readily obtained in 2d, but in 3d we are well aware that the uniqueness for 3d Navier-Stokes problems remains an open question. We discretize the weak formulation associated to (1),(2), (3) and (6) in time by Euler's implicit scheme and in space with the conforming finite element in polygonal or polyhedral domain. We prefer this scheme for its simplicity and also because we are able to obtain an analogue of (4) in the discrete setting. We derive existence, and conditional uniqueness, and optimal a priori error estimates. Thirdly, the space-time approximation problem is linearized and convergence of the iterative scheme is established. Finally, some representatives numerical simulations that validate the theoretical findings are exhibited.

For the physical motivations, interpretations and derivation of (6), we refer the interested reader to the works [14, 15, 17, 3, 13, 12, 6], where basic continuum mechanics are revisited. We also mentioned that this boundary condition is present in the context of laminar flows of Newtonian liquids (e.g. water) over complex surfaces, also when a rough or structured boundary surface is anisotropic, e.g. when it has rows of riblets, pillars or periodic patterns, the effective slip condition is anisotropic, i.e., direction-dependent. When the surface is heterogeneous, the effective slip is also position-dependent. This can occur, for example, when the boundary has a varying degree of roughness or when the boundary is a smooth surface with a varying hydrophobic/hydrophilic composition.

The outline of the paper is as follows:

- Section 2 is devoted to the introduction of classical notations and functional setting for a mathematical understanding of the boundary value problem (1),(2), (3) and (6). Section 3 desalts with the construction of the weak solution of (1),(2), (3) and (6).
- In Section 4, we introduce the discrete problem, recall their main properties, and study their a priori errors and derive convergence. We also formulate the iterative scheme for the practical implementation of the nonlinear discrete problem.
- Numerical results and some concluding remarks are reported in Section 5.

2. Preliminaries

In this section, we recall the main notations and results which we will use later on.

We denote by $[L^p(\Omega)]^d$ the space of measurable functions $\mathbf{v}$ such that $|\mathbf{v}|^p$ is integrable. For $\mathbf{v} \in [L^p(\Omega)]^d$, the norm is defined by

$$\| \mathbf{v} \|_{[L^p(\Omega)]^d} = \left(\int_\Omega |\mathbf{v}(\mathbf{x})|^{\mathbf{p}} \mathbf{dx} \right)^{1/p}.$$

We introduce the Sobolev space

$$W^{m,r}(\Omega)^d = \left\{ \mathbf{v} \in [L^r(\Omega)]^d ; \partial^k \mathbf{v} \in [L^r(\Omega)]^d, \forall |k| \leq m \right\},$$

where $k = (k_1, \cdots, k_d)$ is a vector of non negative integers, such that $|k| = k_1 + \cdots + k_d$ and

$$\partial^k \mathbf{v} = \frac{\partial^{|k|}\mathbf{v}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}.$$

This space is equipped with the semi-norm

$$|\mathbf{v}|_{m,r,\Omega} = \left(\sum_{|k|=m} \int_\Omega |\partial^k \mathbf{v}|^r d\mathbf{x} \right)^{1/r},$$

and is a Banach space for the norm

$$\| \mathbf{v} \|_{m,r,\Omega} = \left(\sum_{\ell=0}^{m} |\mathbf{v}|^r_{\ell,r,\Omega} d\mathbf{x} \right)^{1/r}.$$

When $r = 2$, this space is the Hilbert space $H^m(\Omega)^d$. In particular, we consider the following spaces

$$H_0^1(\Omega)^d = \{\mathbf{v} \in H^1(\Omega)^d, \mathbf{v}_{|\partial\Omega} = 0\},$$

equipped with the norm

$$|\mathbf{v}|_{H_0^1(\Omega)^d} = |\mathbf{v}|_{1,\Omega} = \left(\int_\Omega |\nabla \mathbf{v}|^2 d\mathbf{x} \right)^{1/2}.$$

The dual of $H_0^1(\Omega)^d$ is denoted by $H^{-1}(\Omega)^d$.
We also introduce

$$L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q(\mathbf{x}) d\mathbf{x} = 0\}$$

and we define the following scalar product in $L^2(\Omega)^d$

$$(\mathbf{v}, \mathbf{w}) = \int_\Omega \mathbf{v}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x}, \quad \forall \mathbf{v}, \mathbf{w} \in L^2(\Omega)^d.$$

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $[a, b]$ with values in a separable functional space W equipped with a norm $\| \,.\, \|_W$. For all $r \geq 1$ we introduce the space

$$L^r(a, b; W) = \left\{ \mathbf{f} \text{ is measurable on }]a, b[\text{ and } \int_a^b \| \mathbf{f}(t) \|_W^r \, dt < \infty \right\},$$

equipped with the norm

$$\| \mathbf{f} \|_{L^r(a,b;W)} = \left(\int_a^b \| \mathbf{f}(t) \|_W^r \, dt \right)^{1/r}.$$

If $r = \infty$, then

$$L^\infty(a, b; W) = \left\{ \mathbf{f} \text{ is measurable on } (a, b) \text{ and } \sup_{t \in [a,b]} \| \mathbf{f}(t) \|_W < \infty \right\},$$

equipped with the norm

$$\| \mathbf{f} \|_{L^\infty(a,b;W)} = \sup_{t \in [a,b]} \| \mathbf{f}(t) \|_W .$$

Remark 2.1. *$L^r(a,b;W)$ is Banach space if W is a Banach space.*

In addition, we define $C^j(0,T;W)$ as the space of functions C^j in time with values in W. We consider the following spaces:

$$X = \{\mathbf{v} \in H^1(\Omega)^d;\ \text{with } \mathbf{v}_\Gamma = \mathbf{0} \text{ and } (\mathbf{v}\cdot\mathbf{n})_S = \mathbf{0}\},$$
$$M = L_0^2(\Omega),$$
$$V = \{\mathbf{v} \in X;\ \text{div } \mathbf{v} = 0 \text{ in } \Omega\}$$

and

$$V^\perp = \{\mathbf{v} \in X;\ \forall \mathbf{w} \in V,\ (\nabla \mathbf{v}, \nabla \mathbf{w}) = 0\}.$$

Lemma 2.2. *For any $p \geq 1$, when $d = 1$ or 2, or $1 \leq p \leq \dfrac{2d}{d-2}$ when $d \geq 3$, there exist two positive constants S_p and S_p^0 such that (see* [2]*)*

$$\forall \mathbf{v} \in X, \quad \| \mathbf{v} \|_{L^p(\Omega)^d} \leq S_p^0 |\mathbf{v}|_{1,\Omega},$$

and

$$\forall \mathbf{v} \in H^1(\Omega)^d, \quad \| \mathbf{v} \|_{L^p(\Omega)^d} \leq S_p \| \mathbf{v} \|_{1,\Omega}\ .$$

Lemma 2.3. *For $d = 2$, Ladyzhenskaya's inequality states that (see* [22]*);*

$$\|\mathbf{u}\|_{L^4(\Omega)^2} \leq c(\Omega) \|\nabla \mathbf{u}\|^{1/2} \|\mathbf{u}\|^{1/2}\,. \tag{10}$$

For $d = 3$, then Gagliardo-Nirenberg's inequality reads

$$\|\mathbf{u}\|_{L^4(\Omega)} \leq c(\Omega) \|\mathbf{u}\|^{1/4} \|\nabla \mathbf{u}\|^{3/4}\ . \tag{11}$$

Lemma 2.4. *(Korn's inequality (see* [5]*)) There exists a constant c such that*

$$\forall \mathbf{v} \in X, \qquad \int_\Omega |D(\mathbf{u})|^2 d\mathbf{x} \geq \mathbf{c} \int_{\boldsymbol{\Omega}} |\nabla \mathbf{u}|^{\mathbf{2}} \mathbf{dx}.$$

Henceforth, we suppose that:

Assumption 2.5. *We assume that the data $\mathbf{f}, \mathbf{u}_0$ verify:*
i) $\mathbf{f} \in C^0(0,T;L^2(\Omega)^d)$.
ii) $\mathbf{u}_0 \in L^2(\Omega)^d$, div $\mathbf{u}_0 = 0$, $\mathbf{u}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$ *and* $\mathbf{u}_0|_\Gamma = 0$.

In the next lemma, we introduce the discrete Gronwall's lemma.

Lemma 2.6. *(Discrete Gronwall Lemma)* [23, p. 294]. *Let $(y_n)_n, (\tilde{f}_n)_n$ and $(\tilde{g}_n)_n$ three positive sequences that verifies:*

$$\forall n \geq 0, \ \ y_n \leq \tilde{f}_n + \sum_{k=0}^{n-1} \tilde{g}_k y_k.$$

Then we have:

$$\forall n \geq 0, \ \ y_n \leq \tilde{f}_n + \sum_{k=0}^{n-1} \tilde{f}_k \tilde{g}_k \exp\Big(\sum_{j=k}^{n-1} \tilde{g}_j\Big).$$

In the following, we shall use, if necessary , the notation $\psi(t)$ for the function $\mathbf{x} \longrightarrow \psi(\mathbf{x}, \mathbf{t})$.

3. Analysis of the continuous problem

The goal in the section is twofold. First, we formulate the weak problem associated to (1),(2), (3) and (6), and secondly, we construction of the weak solution.

3.1. Variational formulation. This is the first subsection of the second section.

The weak formulation associated to (1),(2), (3) and (6) is standard and reads;

(12)
$$\begin{cases} \text{Find } t \mapsto (\mathbf{u}(t), p(t)) \in X \times M \text{ such that} \\ \dfrac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) + c_{\mathbf{u}}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) + \displaystyle\int_S |K\mathbf{u}_{\boldsymbol{\tau}}(t)|^{s-2} K\mathbf{u}_{\boldsymbol{\tau}}(t) \cdot K\mathbf{v}_{\boldsymbol{\tau}} d\sigma \\ \qquad -(p(t), \text{div } \mathbf{v}) = (\mathbf{f}(t), \mathbf{v}), \quad \forall \mathbf{v} \in X, \\ \\ (\text{div } \mathbf{u}(t), q) = 0, \quad \forall q \in M, \\ \mathbf{u}(0) = \mathbf{0} \text{ in } \Omega, \end{cases}$$

with

$$a(\mathbf{u}, \mathbf{v}) = 2\nu \int_\Omega D\mathbf{u} : D\mathbf{v}\, d\mathbf{x} \qquad \text{and} \qquad \mathbf{c}_{\mathbf{u}}(\mathbf{u}, \mathbf{u}, \mathbf{v}) = ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}),$$

and

$$\boldsymbol{A} : \boldsymbol{B} = \sum_{1 \le i,j \le d} A_{ij} B_{ij}.$$

It is worth mentioning that by deriving (12), we make use of the identity

$$\sum_{1 \le i,j \le d} D_{ij}\mathbf{u} \frac{\partial v_i}{\partial x_j} = \sum_{1 \le i,j \le d} D_{ij}\mathbf{u} D_{ij}\mathbf{v} \ .$$

To study (12) it is convenient to recall the following monotonicity and continuity properties (see [10, 21]): there exists a constant c independent of $\mathbf{x}, \mathbf{y}$ elements of $\mathbb{R}^d$ such that for $1 \le s < 2$;

$$(|\mathbf{y}| + |\mathbf{x}|)^{2-s} \left(|\mathbf{y}|^{s-2}\mathbf{y} - |\mathbf{x}|^{s-2}\mathbf{x}, \mathbf{y} - \mathbf{x}\right) \ge c|\mathbf{x} - \mathbf{y}|^2 \ , \tag{13}$$

and

$$\big|\, |\mathbf{x}|^{s-2}\mathbf{x} - |\mathbf{y}|^{s-2}\mathbf{y} \big| \le c|\mathbf{x} - \mathbf{y}|^{s-1} \ . \tag{14}$$

Now we introduce the mapping $\mathbf{v} \longrightarrow \mathcal{A}(\mathbf{v})$ defined as follows

$$\forall\ \mathbf{w} \in V \ , \ \langle \mathcal{A}(\mathbf{v}), \mathbf{w} \rangle = a(\mathbf{v}, \mathbf{w}) + \int_S |K\mathbf{v}_{\boldsymbol{\tau}}|^{s-2} K\mathbf{v}_{\boldsymbol{\tau}} \cdot K\mathbf{w}_{\boldsymbol{\tau}} d\sigma\,.$$

From (13) and (14) we have the following Lemma (see [7]):

Lemma 3.1. *We have that for $1 < s < 2$, then*

(a) *$\mathcal{A}$ maps V into its dual V', and is bounded on all bounded subsets of V.*

(b) *For all $\mathbf{v}, \mathbf{u}$ elements of V*

$$\|\mathcal{A}(\mathbf{v}) - \mathcal{A}(\mathbf{u})\|_{V'} \le 2\nu\|\mathbf{v} - \mathbf{u}\|_1 + c\|K\|^s_{L^\infty(S)} \|\mathbf{u} - \mathbf{v}\|_1^{s-1} \ .$$

(c) *The mapping $\mathbf{v} \to \mathcal{A}\mathbf{v}$ is strictly monotone from V into V':*

For all $\mathbf{v}, \mathbf{w} \in V$, $\langle \mathcal{A}(\mathbf{v}) - \mathcal{A}(\mathbf{w}), \mathbf{v} - \mathbf{w} \rangle \ge 0$.

(d) *$\mathcal{A}$ is hemi-continuous in V, i.e. for all $\mathbf{u}, \mathbf{v}$ in V, the mapping $t \to \mathcal{A}(\mathbf{u} + t\mathbf{v})$ is continuous from $\mathbb{R}$ into $\mathbb{R}$.*

3.2. Existence of the solutions. In this paragraph, we will show the existence of a solution $(\mathbf{u}, p)$ of the problem (12) by using a semi-discretization in time (following [16, 22]). In this objective, we introduce the following space:

$$H = \{\mathbf{v} \in L^2(\Omega)^d; \text{div } \mathbf{v} = 0 \text{ in } \Omega,\ (\mathbf{v} \cdot \mathbf{n})_{\partial\Omega} = \mathbf{0}\}.$$

We have that $V \subset H$ and the injection is compact.

Let $N > 1$ be an integer, define the time step k by $k = \dfrac{T}{N}$ and the subdivision points $t_n = nk$. For each $n \geq 1$, we approximate $\mathbf{f}(t_n)$ by the average defined almost everywhere in Ω by

$$\mathbf{f}^n(\mathbf{x}) = \frac{1}{k} \int_{t_{n-1}}^{t_n} \mathbf{f}(t, \mathbf{x}) dt.$$

We set $\mathbf{u}^0 = \mathbf{0}$ and we introduce the following semi-discrete problem (i.e. exact in space and discrete in time): Find sequences $(\mathbf{u}_n)_n$ and $(p_n)_n$ such that $\mathbf{u}_n \in X$ and $p_n \in M$,

(15)
$$\begin{cases} (\dfrac{\mathbf{u}^n - \mathbf{u}^{n-1}}{k}, \mathbf{v}) + a(\mathbf{u}^n, \mathbf{v}) + c_{\mathbf{u}}(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}) \\ \qquad + \displaystyle\int_S |K\mathbf{u}^n_{\boldsymbol{\tau}}|^{s-2} K\mathbf{u}^n_{\boldsymbol{\tau}} \cdot K\mathbf{v}_{\boldsymbol{\tau}} d\sigma - (p^n, \text{div } \mathbf{v}) = (\mathbf{f}^n, \mathbf{v}), \quad \forall \mathbf{v} \in X, \\ (\text{div } \mathbf{u}^n, q) = 0, \quad \forall q \in M. \end{cases}$$

Given $\mathbf{u}^{n-1}$, (15) is essentially a steady Navier-Stokes, and we have

Theorem 3.2. [7] *At each time step n and for a given $\mathbf{u}^{n-1} \in X$, Problem* (15) *admits at least one solution $(\mathbf{u}^n, p^n) \in X \times M$.*

We next derive some bounds for the expression $\mathbf{u}^n$. The following proposition gives basic uniform *a priori* estimates for each solution of (15).

Proposition 3.3. *Each solution $(\mathbf{u}_n, p_n)$ of Problem* (15) *satisfies the following uniform a priori estimates: For each $n \geq 1$,*

$$\| \mathbf{u}^n \|^2_{L^2(\Omega)^d} + \sum_{i=1}^n \| \mathbf{u}^n - \mathbf{u}^{n-1} \|^2_{L^2(\Omega)^d} + \sum_{i=1}^n k|\mathbf{u}^n|^2_X + \sum_{i=1}^n k \int_S |\mathbf{u}^n_{\boldsymbol{\tau}}|^s d\sigma$$

$$\leq c_1 \sum_{i=1}^n k \| \mathbf{f}^n \|^2_{L^2(\Omega)^d}, \tag{16}$$

where c_1 is a positive constant independent of the time and k.

Proof. We consider the first equation of System (15) multiplied by the time step k, take $\mathbf{v} = \mathbf{u}^n$, use the relation $(a - b, a) = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 + \frac{1}{2}|a - b|^2$ and the fact that the tensor K is bounded, and use the relation $ab \leq \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ for the right hand side $(\mathbf{f}^n, \mathbf{v})$ with a suitable choice of the parameter ε. Hence we get the bound (16). □

Here, it is convenient to transform the sequence $(\mathbf{u}_n)_n$ into function. Since it need to be "differentiated", we define the piecewise linear function in time:

$$\forall t \in [t_{n-1}, t_n],\ \mathbf{u}_k(t) = \mathbf{u}_{n-1} + \frac{t - t_{n-1}}{k}(\mathbf{u}_n - \mathbf{u}_{n-1}), \quad 0 < n \leq N.$$

We define also the step functions

$$\forall t \in]t_{n-1}, t_n],\ \mathbf{f}_k(t) = \mathbf{f}^n, \quad 0 < n \le N$$

and

$$\forall t \in]t_{n-1}, t_n],\ \mathbf{w}_k(t) = \mathbf{u}^n, \quad 0 < n \le N.$$

We have the following convergence theorem:

Proposition 3.4. *There exist functions $\mathbf{u} \in L^2(0,T;V) \cap L^\infty(0,T,H)$ such that a subsequence of k, still denoted by k, satisfies:*

$$\lim_{k\to 0} \mathbf{u}_k = \lim_{k\to 0} \mathbf{w}_k = \mathbf{u} \textit{ weakly * in } L^\infty(0,T;H)$$

and

$$\lim_{k\to 0} \mathbf{u}_k = \lim_{k\to 0} \mathbf{w}_k = \mathbf{u} \textit{ weakly in } L^2(0,T;V).$$

Proof. Relation (16) allows us to deduce that $(\mathbf{u}_k)_k$ is also uniformly bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$. Since we get:

$$\begin{aligned}
&\lim_{k\to 0} \mathbf{u}_k = \mathbf{u} \text{ weakly * in } L^\infty(0,T;H),\\
&\lim_{k\to 0} \mathbf{u}_k = \mathbf{u} \text{ weakly in } L^2(0,T;V)\\
&\lim_{k\to 0} \mathbf{w}_k = \mathbf{w} \text{ weakly * in } L^\infty(0,T;H)\\
&\lim_{k\to 0} \mathbf{w}_k = \mathbf{w} \text{ weakly in } L^2(0,T;V).
\end{aligned}$$

As far as the functions $\mathbf{w}$ is concerned, observe that

$$\forall t \in]t_{n-1}, t_n],\ \mathbf{w}_k(t) - \mathbf{u}_k(t) = \frac{t_n - t}{k}(\mathbf{u}_n - \mathbf{u}_{n-1}), \quad 0 < n \le N.$$

Therefore

$$\| \mathbf{w}_k - \mathbf{u}_k \|^2_{L^2(0,T;L^2(\Omega)^d)} = \frac{k}{3} \sum_{n=1}^{N} \| \mathbf{u}^n - \mathbf{u}^{n-1} \|_{L^2(\Omega)^d} .$$

Then, Relation (16) gives that the term $\sum_{n=1}^{N} \| \mathbf{u}^n - \mathbf{u}^{n-1} \|_{L^2(\Omega)^d}$ is uniformly bounded and we obtain

$$\lim_{k\to 0} \| \mathbf{w}_k - \mathbf{u}_k \|^2_{L^2(0,T;L^2(\Omega)^d)} = 0. \tag{17}$$

The uniqueness of the limit allows us to get $\mathbf{u} = \mathbf{w}$. Since we deduce the convergence results. □

In order to pass to the limit with System (15), we still need to prove the strong convergence of $(\mathbf{u}_k)$ and $(\mathbf{w}_k)$.

Theorem 3.5. *Under the assumptions of Proposition 3.4, there exists $\xi \in L^2(0,T;V')$ and a subsequence still denoted by k such that*

$$\lim_{k\to 0} \frac{d\mathbf{u}_k}{dt} = \frac{d\mathbf{u}}{dt} \textit{ weakly in } L^2(0,T;V'), \tag{18}$$

$$\lim_{k\to 0} \mathbf{u}_k = \mathbf{u} \textit{ strongly in } L^2(0,T;H) \tag{19}$$

$$\lim_{k\to 0} \mathbf{w}_k = \mathbf{u} \textit{ strongly in } L^2(0,T;H). \tag{20}$$

and

$$\lim_{k\to 0} \mathcal{A}(\mathbf{w}_k) = \xi \textit{ weakly in } L^2(0,T;V') \tag{21}$$

Proof. System (15) can be written as following: $\forall \mathbf{v} \in V$,

$$(\frac{d\mathbf{u}_k}{dt}, \mathbf{v}) = (\mathbf{f}_k, \mathbf{v}) - a(\mathbf{w}_k, \mathbf{v}) - c_{\mathbf{u}}(\mathbf{w}_k, \mathbf{w}_k, \mathbf{v}) - \int_S |K\mathbf{w}_{k,\boldsymbol{\tau}}|^{s-2} K\mathbf{w}_{k,\boldsymbol{\tau}} \cdot K\mathbf{v}_{\boldsymbol{\tau}} d\sigma.$$

We use the fact that $\mathcal{A}$ maps V into its dual V' (see Lemma 3.1) and the Lemma 4.6 of [22] (page 327) to get the following result:

$$\Big(\sum_{k=1}^{N} k \parallel \frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{k} \parallel_{V'}^2 \Big) \quad \text{is bounded independently of } k.$$

Then we deduce that $\frac{d\mathbf{u}_k}{dt}$ is bounded in $L^2(0,T;V')$ and there exists a subsequence of k, still denoted by k such that

$$\lim_{k\to 0} \frac{d\mathbf{u}_k}{dt} = \frac{d\mathbf{u}_1}{dt} \text{ weakly in } L^2(0,T;V').$$

Relation (17) gives that $\mathbf{u}_1 = \mathbf{u}$ and then (18). To prove (19), we apply the theorem 2.1, page 271 in [22]. Furthermore, Relation (20) is a simple consequence of (17). Finally, Lemma 3.1 allows us to deduce that there exists a subsequence of k, still denoted by k such that $\mathcal{A}(\mathbf{w}_k)$ satisfies 21 and converge weakly to ξ in $L^2(0,T;V')$. □

Theorem 3.6. *Under assumption 12, Problem* (12) *admits at list one solution in* $X \times M$.

Proof. To show the existence of a solution of Problem (12), we use all the above results to pass to the limit with the system (15) wich can be written as: $\forall \mathbf{v} \in V$,

$$(\frac{d\mathbf{u}_k}{dt}, \mathbf{v}) + \langle \mathcal{A}(\mathbf{w}_k), \mathbf{v} \rangle + c_{\mathbf{u}}(\mathbf{w}_k, \mathbf{w}_k, \mathbf{v}) = (\mathbf{f}_k, \mathbf{v}), \tag{22}$$

where

$$\langle \mathcal{A}(\mathbf{w}_k), \mathbf{v} \rangle = a(\mathbf{w}_k, \mathbf{v}) + \int_S |K\mathbf{w}_{k,\boldsymbol{\tau}}|^{s-2} K\mathbf{w}_{k,\boldsymbol{\tau}} \cdot K\mathbf{v}_{\boldsymbol{\tau}} d\sigma.$$

We will proceed as [22] (page 320) and [16] (page 158). To show the convergence of the term $c_{\mathbf{u}}(\mathbf{w}_k, \mathbf{w}_k, \mathbf{v})$ by using the strong convergence (20), we need to take the test function $\mathbf{v}$ in a space more regular than V. We take $\mathbf{v} \in V_s = V \cap H^{3/2}(\Omega)^d$ and we pass to the limit in Equation (22) where all the terms except $\langle \mathcal{A}(\mathbf{w}_k), \mathbf{v} \rangle$ are treated in [22] (page 320). Thus, Relation (21) allows us to get in $L^2(0,T;V_s')$,

$$\frac{d\mathbf{u}}{dt} + \xi + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f}. \tag{23}$$

Proposition 3.4 and Theorem 3.5 allows us to deduce that Equation (23) is valid in $L^2(0,T;V')$. Furthermore, as we have

$$\mathbf{u}' \in L^2(0,T;V') \quad \text{and} \quad \mathbf{u}' = \mathbf{f} - \frac{d\mathbf{u}}{dt} - \xi - (\mathbf{u} \cdot \nabla)\mathbf{u},$$

We get by using [22] (Lemma 1.1, page 250) that $\mathbf{u}$ is almost everywhere equal to a continuous function from $[0,T]$ into V' and we get $\mathbf{u}(0) = 0$.
To get the existence of the solution, we still have to show that $\xi = \mathcal{A}(\mathbf{u})$ by following [16]. This will be done by using the properties of $\mathcal{A}$ (see Lemma 3.1). First, we have:

$$X_k = \int_0^T \langle \mathcal{A}(\mathbf{w}_k(t)) - \mathcal{A}(\mathbf{v}(t)), \mathbf{w}_k(t) - \mathbf{v}(t) \rangle \, dt \geq 0$$

Or, Equation (22), $\mathbf{w}_k(0) = \mathbf{0}$ and the relation $c_{\mathbf{u}}(\mathbf{w}_k(t), \mathbf{w}_k(t), \mathbf{w}_k(t)) = 0$ give us the following relation:

$$\int_0^T \langle \mathcal{A}(\mathbf{w}_k(t)), \mathbf{w}_k(t) \rangle = \int_0^T (\mathbf{f}(t), \mathbf{w}_k(t))dt - \frac{1}{2}||\mathbf{w}_k(T)||^2_{L^2(\Omega)^d}.$$

Then we obtain

$$X_k = \int_0^T (\mathbf{f}(t), \mathbf{w}_k(t))dt - \frac{1}{2}||\mathbf{w}_k(T)||^2_{L^2(\Omega)^d} - \int_0^T \langle \mathcal{A}(\mathbf{w}_k(t)), \mathbf{v}(t) \rangle dt$$
$$- \int_0^T \langle \mathcal{A}(\mathbf{v}(t))), \mathbf{w}_k(t) - \mathbf{v}(t) \rangle dt.$$

As we have

$$\liminf ||\mathbf{w}_k(T)||^2_{L^2(\Omega)^d} \geq ||\mathbf{u}(T)||^2_{L^2(\Omega)^d},$$

then we obtain

$$\liminf X_k \leq \int_0^T (\mathbf{f}(t), \mathbf{u}(t))dt - \frac{1}{2}||\mathbf{u}(T)||^2_{L^2(\Omega)^d}$$
$$- \int_0^T \langle \mathcal{A}(\mathbf{u}(t)), \mathbf{v}(t) \rangle dt - \int_0^T \langle \mathcal{A}(\mathbf{v}(t))), \mathbf{u}(t) - \mathbf{v}(t) \rangle dt.$$

By taking into account (23) multiplied by $\mathbf{u}$ and integrated in Ω and between 0 and T we get,

$$0 \leq \int_0^T \langle \xi(t) - \mathcal{A}(\mathbf{v}(t)), \mathbf{u}(t) - \mathbf{v}(t) \rangle dt.$$

Now, by taking $\mathbf{v} = \mathbf{u} - \lambda \mathbf{w}$ where $\lambda > 0$ and $\mathbf{w} \in L^2(0, T; V)$, it holds that:

$$\int_0^T \langle \xi(t) - \mathcal{A}(\mathbf{u}(t) - \lambda \mathbf{w}(t)), \mathbf{w}(t) \rangle dt \geq 0, \quad \forall \mathbf{w} \in L^2(0, T; V).$$

By taking $\lambda \to 0$ and by using the hemi-continuity of $\mathcal{A}$ we get

$$\int_0^T \langle \xi(t) - \mathcal{A}(\mathbf{u}(t)), \mathbf{w}(t) \rangle dt \geq 0, \quad \forall \mathbf{w} \in L^2(0, T; V)$$

which get that $\xi = \mathcal{A}(\mathbf{u})$.
Thus, Relation 23 allows us to get the following equation satisfied by the velocity in $L^2(0, T; V')$:

$$\frac{d\mathbf{u}}{dt} + \mathcal{A}(\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f}. \tag{24}$$

To prove the existence of the pressure, we define the function $L_t \in X'$ for all $\mathbf{v} \in X$:

$$L_t(v) = \int_0^t \big((f(t), \mathbf{v}) - \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) - \langle \mathcal{A}(\mathbf{u}(t)), \mathbf{v} \rangle - \langle (\mathbf{u}(t) \cdot \nabla)\mathbf{u}(t), \mathbf{v} \rangle \, dt.$$

Relation 24 allows us to deduce that for all $\mathbf{v} \in V$, $L_t(\mathbf{v}) = 0$. Hence, see [8] (Chap. I, Lemme 2.1), for each $t \in [0, T]$, there exists a function $P(t) \in M$ such that:

$$L_t(\mathbf{v}) = -(\text{div } \mathbf{v}, P(t)), \quad \forall \mathbf{v} \in X, \tag{25}$$

and

$$\| P(t) \|_{L^2(\Omega)} \leq \sup_{\mathbf{v} \in X} \frac{|L_t(\mathbf{v})|}{\| \mathbf{v} \|_X}.$$

Now, differentiating (25) with respect to t, and setting $p = \partial_t P(t)$, we obtain the first equation of System (1). Hence the existence of the pressure p. □

Theorem 3.7. *If the Assumption 12 holds, every solution of* (12) *verifies the bound*

$$\| \mathbf{u} \|_{L^\infty(0,T;L^2(\Omega)^d)} + \| \mathbf{u} \|_{L^2(0,T;X)} \le C \| \mathbf{f} \|_{L^2(0,T;L^2(\Omega)^d)}, \tag{26}$$

where C *is a positive constant independent of* $\mathbf{u}$.

Proof. Let $(\mathbf{u},p)$ be a solution of (12). To prove the bound (26), we take $\mathbf{v}=\mathbf{u}$ in the first to equation of problem (12), use lemma 2.2 and the relation $ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \| \mathbf{u}(t) \|^2_{L^2(\Omega)^d} + 2\nu c|\mathbf{u}(t)|^2_X &\le S_2^0 \| \mathbf{f}(t) \|_{L^2(\Omega)^d} |\mathbf{u}(t)|_X \\ &\le \frac{(S_2^0)^2\varepsilon}{2} \| \mathbf{f}(t) \|^2_{L^2(\Omega)^d} + \frac{1}{2\varepsilon}|\mathbf{u}(t)|^2_X.\end{aligned}$$

For $\varepsilon = \dfrac{1}{2\nu c}$, we obtain:

$$\frac{1}{2}\frac{d}{dt} \| \mathbf{u}(t) \|^2_{L^2(\Omega)^d} + \nu c|\mathbf{u}(t)|^2_X \le \frac{(S_2^0)^2}{4\nu c} \| \mathbf{f}(t) \|^2_{L^2(\Omega)^d}.$$

First, we integrate with respect to t, between 0 and T, to derive the following bound:

$$\| \mathbf{u} \|^2_{L^2(0,T;X)} \le \frac{(S_2^0)^2}{4\nu^2c^2} \| \mathbf{f} \|^2_{L^2(0,T;L^2(\Omega)^d)}.$$

Second, we integrate with respect to t, between 0 and t, and then take the maximum on t to obtain,

$$\frac{1}{\nu} \| \mathbf{u} \|^2_{L^\infty(0,T;L^2(\Omega)^d)} \le \frac{(S_2^0)^2}{4\nu^2c^2} \| \mathbf{f} \|^2_{L^2(0,T;L^2(\Omega)^d)}.$$

We deduce,

$$\frac{1}{\nu} \| \mathbf{u} \|^2_{L^\infty(0,T;L^2(\Omega)^d)} + \| \mathbf{u} \|^2_{L^2(0,T;X)} \le \frac{(S_2^0)^2}{2\nu^2c^2} \| \mathbf{f} \|^2_{L^2(0,T;L^2(\Omega)^d)}.$$

Hence the desired result. □

Remark 3.8. *It is important to mention that uniqueness rely on Ladyzhenskaya's inequality (10) when d=2. In 3 d, this argument does not hold anymore, and we are well aware that the uniqueness for 3d Navier-Stokes problems remains an open question. Only limited results on uniqueness for local in time (see* [4]*), or for special domains (see* [20]*) are available. For general 3 d domains, uniqueness can only be obtained for a restricted class of solutions (see* [22]*). In particular, within the class of solutions* $(\mathbf{u},p)$ *of (12) such that* $\mathbf{u} \in L^q(0,T;L^4(\Omega)^d)$ *for some* $q \ge 8$. *In this case, we use Gagliardo-Nirenberg's inequality (11).*

4. Space-Time discretization and a priori errors

In this section, we propose a time and space discretizations of the problem (12) and we prove a corresponding *a priori* error estimation. We use the semi-implicit Euler method for the time discretization and the finite element method for the space discretization. For the time discretization, we introduce a partition of the interval $[0,T]$ into N subintervals $[t_{n-1},t_n]$ of length k (the time step). For the space discretization, we suppose that the domain Ω is a polygon (respectively polyhedron) for $d=2$ (respectively $d=3$). Let $h>0$ be a discretization parameter in space and for each h, let $\mathcal{T}_h$ be a corresponding regular (or non-degenerate) family of triangles (respectively tetrahedra) for $d=2$ (respectively for $d=3$), in the usual sense that:

- $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_h$;

- the intersection of two different elements of $\mathcal{T}_h$, if not empty, is a vertex or a whole edge (or a whole face of both of them for $d = 3$);
- the ratio of the diameter of an element κ in $\mathcal{T}_h$ to the diameter of its inscribed sphere is bounded by a constant independent of h.

Let $X_h \subset X$ and $M_h \subset M$ be a "stable" pair of finite-element spaces for discretizing the velocity $\mathbf{u}$ and the pressure p, stable in the sense that it satisfies a uniform discrete inf-sup condition: there exists a constant $\beta^\star \geq 0$, independent of h, such that,

$$\forall q_h \in M_h, \sup_{\mathbf{v}_h \in X_h} \frac{1}{|\mathbf{v}_h|_{H^1(\Omega)^d}} \int_\Omega q_h \text{div } \mathbf{v}_h dx \geq \beta^\star \parallel q_h \parallel_{L^2(\Omega)},$$

where h denotes the maximal diameter of the elements of $\mathcal{T}_h$.
Let $\mathbb{P}_\kappa$ denote the space of polynomials with total degree less than or equal to κ. We choose the "mini-element" (see D. Arnold, F. Brezzi and M. Fortin in [1]) discretization, where in each element κ, the pressure p is a polynomial of $\mathbb{P}_1$ and each component of the velocity $\mathbf{u}$ is the sum of a polynomial of $\mathbb{P}_1$ and a "bubble" function b_κ (for each element κ, the bubble function is equal to the product of the barycentric coordinates associated with the vertices of κ).

Therefore, the finite-element spaces for the velocity and the pressure are :

$$X_h = \left\{ \mathbf{v}_h \in C^0(\overline{\Omega})^d \cap X;\ \forall \kappa \in \mathcal{T}_h,\ \mathbf{v}_{h|_\kappa} \in \mathcal{P}(\kappa) \right\},$$

and

$$M_h = \left\{ q_h \in C^0(\overline{\Omega}) \cap M;\ \forall \kappa \in \mathcal{T}_h,\ \mathbf{v}_{h|_\kappa} \in \mathbb{P}_1(\kappa) \right\},$$

where

$$\mathcal{P}(\kappa) = [\mathbb{P}_1 \oplus Span(b_\kappa)]^d.$$

We introduce the following discrete spaces:

$$V_h = \left\{ \mathbf{v}_h \in X_h;\ \forall q_h \in M_h,\ \int_\Omega q_h(x) \text{ div } \mathbf{v}_h(x) d\mathbf{x} = \mathbf{0} \right\}.$$

There exists an approximation operator $P_h \in \mathcal{L}(H_0^1(\Omega)^d; X_h)$ such that (see V. Girault and P.-A. Raviart in [8]):

$$\forall v \in H_0^1(\Omega)^d,\ \ \forall q_h \in M_h,\ \ \int_\Omega q_h \text{div } (P_h(v) - v) d\mathbf{x} = \mathbf{0},$$

and for $k = 0$ or 1,

$$\forall v \in [H^{1+k}(\Omega) \cap H_0^1(\Omega)]^d,\ \ \parallel P_h(v) - v \parallel_{L^2(\Omega)^d} \leq\ C_1 h^{1+k} |v|_{H^{1+k}(\Omega)^d},$$

and for all $r \geq 2, k = 0$ or 1,

$$\forall v \in [W^{1+k,r}(\Omega) \cap H_0^1(\Omega)]^d,\ \ |P_h(v) - v|_{W^{1,r}(\Omega)^d}\ \leq\ C_2 h^k |v|_{W^{1+k,r}(\Omega)^d}.$$

In order to introduce the discrete scheme, we define the following form: for all $\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h \in X_h$,

$$d_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w_h}) = c_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) + \frac{1}{2}(\text{div } (\mathbf{u}_h)\ \mathbf{v}_h, \mathbf{w}_h).$$

Remark 4.1. *For all $\mathbf{u}_h$, $\mathbf{v}_h$ in X_h, the form $d_{\mathbf{u}}$ verifies the following stability relation*

$$d_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{v}_h) = 0$$

4.1. Non linear discrete scheme. In this section, we introduce the following space time discretization of (12), analyse its well-posedness and the convergence by means of estimating the expressions $\mathbf{u}(t_n) - \mathbf{u}_h^n$ and $p(t_n) - p_h^n$.

For every $n \in \{1, ..., N\}$, knowing $\mathbf{u}_h^{n-1} \in X_h$, we compute $(\mathbf{u}_h^n, p_h^n) \in (X_h, M_h)$ such that for all $(\mathbf{v}_h, q_h) \in (X_h, M_h)$

$$
(27) \quad \left\{
\begin{aligned}
&\frac{1}{k}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}_h) + a(\mathbf{u}_h^n, \mathbf{v}_h) + d_{\mathbf{u}}(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{v}_h) - (p_h^n, \text{div } \mathbf{v}_h) \\
&\qquad\qquad + \int_S |K\mathbf{u}_{h\boldsymbol{\tau}}^n|^{s-2} K\mathbf{u}_{h\boldsymbol{\tau}}^n \cdot K\mathbf{v}_{\boldsymbol{\tau}} d\sigma = (\mathbf{f}^n, \mathbf{v}_h), \\
&(q_h, \text{div } \mathbf{u}_h^n) = 0,
\end{aligned}
\right.
$$

where $\mathbf{u}_h^0 = \mathbf{0}$ and $\mathbf{f}^n = \mathbf{f}(t_n)$

Theorem 4.2 (Existence of a discrete solution)**.**
At each time step n and for a given $\mathbf{u}_h^{n-1} \in X_h$, Problem (27) admits at least one solution $(\mathbf{u}_h^n, p_h^n) \in X_h \times M_h$ which verifies, for $m = 1, ..., N$, the following bound

$$
||\mathbf{u}_h^m||_{L^2(\Omega)^d} + 2\nu c \sum_{n=1}^{m} k |\mathbf{u}_h^n|_{1,\Omega}^2 + \sum_{n=1}^{m} k \int_S |K\mathbf{u}_{h\boldsymbol{\tau}}^n|^s d\sigma \le C_1 \sum_{n=1}^{N} k ||\mathbf{f}^n||_{L^2(\Omega)^d}^2.
$$

where C_1 is a positive constant independent of h and k.

Proof. For the prove of the existence of at least one solution of System (27), we refer to [7] where the prove is based on the Brouwer's fixed point Theorem.
In addition, by taking $\mathbf{v}_h = \mathbf{u}_h^n$ in the first equation of Problem (27) and by using the relations $(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}|a-b|^2$ and $ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ (for the right hand side and for $\varepsilon = 2\nu c$), we obtain

$$
\begin{aligned}
\frac{1}{2k}||\mathbf{u}_h^n||_{L^2(\Omega)^d}^2 - \frac{1}{2k}||\mathbf{u}_h^{n-1}||_{L^2(\Omega)^d}^2 + \frac{1}{2k}||\mathbf{u}_h^n - \mathbf{u}_h^{n-1}||_{L^2(\Omega)^d}^2 \\
+ \nu c |\mathbf{u}_h^n|_{1,\Omega}^2 + \int_S |K\mathbf{u}_{h\boldsymbol{\tau}}^n|^s d\sigma \le \frac{(S_2^0)^2}{4\nu c} ||\mathbf{f}^n||_{L^2(\Omega)^d}^2.
\end{aligned}
$$

We multiply the last inequality by $2k$ and we sum over $n = 1, \ldots, m$ to obtain the desired results. □

We refer to [7] for the following uniqueness result:

Proposition 4.3. *Let $(\mathbf{u}_h^n, p_h^n) \in V_h \times M_h$ be the solution of* (27)*. There exists a positive constant c depending only on Ω such that if for ν and $\mathbf{f}$, the relation*

$$
\nu^2 \ge c\|\mathbf{f}^n\|_{L^2(\Omega)^d},
$$

is satisfied, then the solution of (27) *is unique.*

Remark 4.4. *For every $\mathbf{u}_h^0 \in V_h$, the sequence $(\mathbf{u}_h^n, p_h^n) \in V_h \times M_h$ generated by* (27) *satisfies the energy law* (4) *. In particular, we easily established that the kinetic energy*

$$
\mathcal{E}(\mathbf{u}) = \int_\Omega |\mathbf{u}|^2 \, dx
$$

decreases with n.

Next, we will establish an *a priori* error estimate between the exact and the numerical solutions.

Theorem 4.5. *Let $(\mathbf{u}, p)$ be the solution of Problem (12) and $(\mathbf{u}_h^n, p_h^n)$ be the solution of problem (27). If $\mathbf{u} \in L^\infty\left(0,T;H^2(\Omega)^d\right)$, $\mathbf{u}_{\boldsymbol{\tau}} \in L^\infty\left(0,T;H^2(S)^d\right)$, $\mathbf{u}' \in L^2\left(0,T;H^1(\Omega)^d\right)$, $\mathbf{u}'' \in L^\infty\left(0,T;L^2(\Omega)^d\right)$ and $p \in L^\infty\left(0,T;H^1(\Omega)\right)$, there exists a positive constant C independent of h and k such that for all $m \in \{1,\dots,N\}$ we have,*

$$\frac{1}{2} \parallel \mathbf{u}_h^m - P_h\mathbf{u}(t_m) \parallel^2_{L^2(\Omega)^d} + \frac{\nu}{2}\sum_{n=1}^{m} k|\mathbf{u}_h^n - P_h\mathbf{u}(t_n)|^2_{H^1(\Omega)^d} \le C(h^2+k^2). \tag{28}$$

Proof. We choose the test function $\mathbf{v} = \mathbf{v}_h^n = \mathbf{u}_h^n - P_h\mathbf{u}(t_n)$ in the first equations of (12) and (27) multiplied by k. We take $t = t_n$ in the first equation of (27) and replace

$$k\mathbf{u}'(t_n) = \mathbf{u}(t_{n+1}) - \mathbf{u}(t_n) - k^2\mathbf{u}''(\xi), \quad \text{for } \xi \in [t_n.t_{n+1}].$$

Then we subtract the obtained continuous and discrete equations, and insert $\pm P_h\mathbf{u}(t_{n-1})$ and $\pm P_h\mathbf{u}(t_n)$ to obtain :

(29)

$$\begin{aligned}
&\frac{1}{2} \parallel \mathbf{v}_h^n \parallel^2_{L^2(\Omega)^2} - \frac{1}{2} \parallel \mathbf{v}_h^n \parallel^2_{L^2(\Omega)^2} + \frac{1}{2} \parallel \mathbf{v}_h^n - \mathbf{v}_h^{n-1} \parallel^2_{L^2(\Omega)^2} + 2\nu k(D(\mathbf{v}_h^n), D(\mathbf{v}_h^n))\\
&+k\int_S \left(|K\mathbf{u}_{h\boldsymbol{\tau}}^n|^{s-2}\, k\mathbf{u}_{h\boldsymbol{\tau}}^n - |K\mathbf{u}_{\boldsymbol{\tau}}(t_n)|^{s-2}\, K\mathbf{u}_{\boldsymbol{\tau}}(t_n)\right) K\mathbf{v}_{h\boldsymbol{\tau}}^n \, d\sigma\\
&\quad = k(p_h^n - p(t_n), \text{div } (\mathbf{v}_h^n)) - k^2(\frac{d^2\mathbf{u}}{dt^2}(\xi), \mathbf{v}_h^n)\\
&\quad\quad -2\nu k(D(P_h\mathbf{u}(t_n) - \mathbf{u}(t_n)), D\mathbf{v}_h^n) - \int_{t_{n-1}}^{t_n} (P_h\mathbf{u}'(t) - \mathbf{u}'(t), \mathbf{v}_h^n)dt\\
&\quad\quad -k(\mathbf{u}_h^n\nabla)\cdot\mathbf{u}_h^n - (\mathbf{u}(t_n)\nabla)\cdot\mathbf{u}(t_n), \mathbf{v}_h^n) - \frac{1}{2}k\,(\text{div } (\mathbf{u}_h^n)\mathbf{u}_h^n, \mathbf{v}_h^n)\,.
\end{aligned}$$

The last term on the left hand side of (29) can be written as follows

(30)

$$\begin{aligned}
&k\int_S \left(|K\mathbf{u}_{h\boldsymbol{\tau}}^n|^{s-2}\, k\mathbf{u}_{h\boldsymbol{\tau}}^n - |K\mathbf{u}_{\boldsymbol{\tau}}(t_n)|^{s-2}\, K\mathbf{u}_{\boldsymbol{\tau}}(t_n)\right) (K\mathbf{u}_{h\boldsymbol{\tau}}^n - KP_h\mathbf{u}_{\boldsymbol{\tau}}(t_n))\, d\sigma\\
&= k\int_S \left(|K\mathbf{u}_{h\boldsymbol{\tau}}^n|^{s-2}\, K\mathbf{u}_{h\boldsymbol{\tau}}^n - |K\mathbf{u}_{\boldsymbol{\tau}}(t_n)|^{s-2}\, K\mathbf{u}_{\boldsymbol{\tau}}(t_n)\right) (K\mathbf{u}_{h\boldsymbol{\tau}}^n - K\mathbf{u}_{\boldsymbol{\tau}}(t_n))\, d\sigma\\
&\quad +k\int_S \left(|K\mathbf{u}_{h\boldsymbol{\tau}}^n|^{s-2}\, K\mathbf{u}_{h\boldsymbol{\tau}}^n - |K\mathbf{u}_{\boldsymbol{\tau}}(t_n)|^{s-2}\, K\mathbf{u}_{\boldsymbol{\tau}}(t_n)\right) (K\mathbf{u}_{\boldsymbol{\tau}}(t_n) - KP_h\mathbf{u}_{\boldsymbol{\tau}}(t_n))\, d\sigma.
\end{aligned}$$

We replace (30) in (29) to obtain

(31)

$$\begin{aligned}
&\frac{1}{2} \parallel \mathbf{v}_h^n \parallel^2_{L^2(\Omega)^2} - \frac{1}{2} \parallel \mathbf{v}_h^{n-1} \parallel^2_{L^2(\Omega)^2} + \frac{1}{2} \parallel \mathbf{v}_h^n - \mathbf{v}_h^{n-1} \parallel^2_{L^2(\Omega)^2} + 2\nu k(D(\mathbf{v}_h^n), D(\mathbf{v}_h^n))\\
&+k\int_S \left(|K\mathbf{u}_{h\boldsymbol{\tau}}^n|^{s-2}\, K\mathbf{u}_{h\boldsymbol{\tau}}^n - |K\mathbf{u}_{\boldsymbol{\tau}}(t_n)|^{s-2}\, K\mathbf{u}_{\boldsymbol{\tau}}(t_n)\right) (K\mathbf{u}_{h\boldsymbol{\tau}}^n - K\mathbf{u}_{\boldsymbol{\tau}}(t_n))\, d\sigma\\
&= k(p_h^n - p(t_n), \text{div } (\mathbf{v}_h^n)) - k^2(\frac{d^2\mathbf{u}}{dt^2}(\xi), \mathbf{v}_h^n)\\
&\quad -2\nu k(D(P_h\mathbf{u}(t_n) - \mathbf{u}(t_n)), D\mathbf{v}_h^n) - \int_{t_{n-1}}^{t_n} (P_h\mathbf{u}'(t) - \mathbf{u}'(t), \mathbf{v}_h^n)dt\\
&\quad -k(\mathbf{u}_h^n\nabla)\cdot\mathbf{u}_h^n - (\mathbf{u}(t_n)\nabla)\cdot\mathbf{u}(t_n), \mathbf{v}_h^n) - \frac{1}{2}k\,(\text{div } (\mathbf{u}_h^n)\mathbf{u}_h^n, \mathbf{v}_h^n)\\
&\quad -k\int_S \left(|K\mathbf{u}_{h\boldsymbol{\tau}}^n|^{s-2}\, K\mathbf{u}_{h\boldsymbol{\tau}}^n - |K\mathbf{u}_{\boldsymbol{\tau}}(t_n)|^{s-2}\, K\mathbf{u}_{\boldsymbol{\tau}}(t_n)\right) (K\mathbf{u}_{\boldsymbol{\tau}}(t_n) - KP_h\mathbf{u}_{\boldsymbol{\tau}}(t_n))\, d\sigma.
\end{aligned}$$

From (13), we deduce that the third term of the left hand side satisfies the following inequality:

$$k\int_S \Big(|K\mathbf{u}^n_{h\boldsymbol{\tau}}|^{s-2}\,K\mathbf{u}^n_{h\boldsymbol{\tau}} - |K\mathbf{u}_{\boldsymbol{\tau}}(t_n)|^{s-2}\,K\mathbf{u}_{\boldsymbol{\tau}}(t_n)\Big)\,(K\mathbf{u}^n_{h\boldsymbol{\tau}} - K\mathbf{u}_{\boldsymbol{\tau}}(t_n))\,d\sigma\ \geq 0\,.$$

Let us now bound each term of the right hand side of Equation (31).
The first one can be bounded as follows:

$$\begin{aligned} k\big|(p^n_h - p(t_n), \text{div }(\mathbf{v}^n_h))\big| &= k\big|(r_h p(t_n) - p(t_n), \text{div }(\mathbf{v}^n_h))\big| \\ &\leq \frac{c_1^2}{2\varepsilon_1}h^2 k \parallel p \parallel^2_{L^\infty(0,T;H^1(\Omega))} + \frac{\varepsilon_1}{2}k|\mathbf{v}^n_h|^2_{H^1(\Omega)^d}. \end{aligned}$$

We can treat the second term of the right hand side of Equation (31) as following:

$$k^2\big|(\frac{d^2\mathbf{u}}{dt^2}(\xi), \mathbf{v}^n_h)\big| \leq \frac{c_2^2}{2\varepsilon_2}k^3||\mathbf{u}''||^2_{L^\infty(0,T;L^2(\Omega)^d)} + \frac{\varepsilon_2}{2}k|\mathbf{v}^n_h|^2_{H^1(\Omega)^d}.$$

The third term can be treated as following:

$$2\nu k\big|(D(P_h\mathbf{u}(t_n) - \mathbf{u}(t_n)), D\mathbf{v}^n_h)\big| \leq \frac{c_3^2}{2\varepsilon_3}h^2 k \parallel \mathbf{u} \parallel^2_{L^\infty(0,T;H^2(\Omega)^d)} + \frac{\varepsilon_3}{2}k|\mathbf{v}^n_h|^2_{H^1(\Omega)^d}.$$

It is easy to check that the fourth term satisfies the following inequality:

$$\big|\int_{t_{n-1}}^{t_n}(P_h\mathbf{u}'(t) - \mathbf{u}'(t), \mathbf{v}^n_h)dt\big| \leq \frac{c_4}{2\varepsilon_4}h^2||\mathbf{u}'||^2_{L^2(t_{n-1},t_n;H^1(\Omega)^d)} + \frac{\varepsilon_4}{2}k|\mathbf{v}^n_h|^2_{H^1(\Omega)^d}$$

We will treat now the last term of the right hand side of Equation (31) denoted by T_7. It can be bounded by using Lemma 3.1, the relation $||\mathbf{u}_{\boldsymbol{\tau}} - P_h\mathbf{u}_{\boldsymbol{\tau}}||_{L^2(S)^d} \leq ch^2||\mathbf{u}_{\boldsymbol{\tau}}||_{H^2(S)^d}$ and Theorem 4.2 as following :

$$\begin{aligned} &|T_7| \\ =&\Big|k\int_S \Big(|K\mathbf{u}^n_{h\boldsymbol{\tau}}|^{s-2}\,K\mathbf{u}^n_{h\boldsymbol{\tau}} - |K\mathbf{u}_{\boldsymbol{\tau}}(t_n)|^{s-2}\,K\mathbf{u}_{\boldsymbol{\tau}}(t_n)\Big)\,(K\mathbf{u}_{\boldsymbol{\tau}}(t_n) - KP_h\mathbf{u}_{\boldsymbol{\tau}}(t_n))\,d\sigma\Big| \\ \leq& c_5 h^2 k||\mathbf{u}^n_h - \mathbf{u}(t_n)||^{s-1}_{1,\Omega}\ ||\mathbf{u}_{\boldsymbol{\tau}}(t_n)||_{H^2(S)^d} \end{aligned}$$

We use the fact that $\mathbf{u} \in L^\infty(0,T;H^1(\Omega)^d)$ and the Young inequality $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ for $p = \dfrac{2}{s-1}$ and $q = \dfrac{p}{p-1}$ to deduce the following:

$$|T_7| \leq c_6 h^2\big(k||\mathbf{u}^n_h - \mathbf{u}(t_n)||^2_{1,\Omega} + k||\mathbf{u}_{\boldsymbol{\tau}}||^q_{L^\infty(0,T;H^2(S)^d)}\big).$$

Let us now treat the fifth and sixth terms of the right hand side of Equation (31) denoted by L_n. By using Remark 4.1, by inserting $\pm P_h\mathbf{u}(t_n)$, $\pm\mathbf{u}(t_n)$, $\pm P_h\mathbf{u}(t_{n-1})$, we get by using the Green formula:

$$\begin{aligned} L_n &= -k(\mathbf{u}^n_h\nabla)\cdot\mathbf{u}^n_h - (\mathbf{u}(t_n)\nabla)\cdot\mathbf{u}(t_n), \mathbf{v}^n_h) - \frac{1}{2}k\,(\text{div }(\mathbf{u}^n_h)\mathbf{u}^n_h, \mathbf{v}^n_h) \\ &= -k\big[(\mathbf{u}^n_h\nabla)\cdot(P_h\mathbf{u}(t_n) - \mathbf{u}(t_n)), \mathbf{v}^n_h) - \frac{1}{2}k\,(\text{div }(\mathbf{u}^n_h)(P_h\mathbf{u}(t_n) - \mathbf{u}(t_n)), \mathbf{v}^n_h)\,\big] \\ &\quad -k\big[((\mathbf{u}^n_h - \mathbf{u}(t_n))\nabla)\cdot\mathbf{u}(t_n), \mathbf{v}^n_h) - \frac{k}{2}\,(\text{div }(\mathbf{u}^n_h - \mathbf{u}(t_n))\mathbf{u}(t_n), \mathbf{v}^n_h)\,\big] \\ &= -k\big[(\mathbf{u}^n_h\nabla)\cdot P_h\mathbf{u}(t_n) - \mathbf{u}(t_n)), \mathbf{v}^n_h) - \frac{k}{2}\,(\text{div }(\mathbf{u}^n_h)(P_h\mathbf{u}(t_n) - \mathbf{u}(t_n)), \mathbf{v}^n_h)\,\big] \\ &\quad -\frac{k}{2}\big[((\mathbf{u}^n_h - \mathbf{u}(t_n))\nabla)\cdot\mathbf{u}(t_n), \mathbf{v}^n_h) + \frac{k}{2}\big[((\mathbf{u}^n_h - \mathbf{u}(t_n))\nabla)\cdot\mathbf{v}^n_h, \mathbf{u}(t_n))\big]. \end{aligned}$$

Thus, as $\mathbf{u} \in L^\infty(0,T;H^2(\Omega)^d)$, L_n can be bounded as following:

$$\begin{aligned} |L_n| \quad \leq \quad & c_7 h^2 \big(\frac{1}{2\varepsilon_5} k ||\mathbf{u}_h^n||^2_{1,\Omega} + \frac{\varepsilon_5}{2} k ||\mathbf{v}_h^n||^2_{1,\Omega} \big) \\ & + c_8 \big(\frac{1}{2\varepsilon_6} k ||\mathbf{u}_h^n - \mathbf{u}(t_n)||^2_{L^2(\Omega)^d} + \frac{\varepsilon_6}{2} k ||\mathbf{v}_h^n||^2_{1,\Omega} \big) . \end{aligned}$$

Finally, by collecting all the above inequalities, summing over $n = 1, \ldots, m$ (for all integer $m \leq N$) and using Lemma 4.2, we get after a suitable choice of ε_i the following inequality:

$$\frac{1}{2} \parallel \mathbf{v}_h^m \parallel^2_{L^2(\Omega)^2} + 2\nu c |\mathbf{v}_h^n|^2_{1,\Omega} \leq C_1 \big(k^2 + h^2 \big) + C_2 \sum_{n=1}^{n} k ||\mathbf{v}_h^n||^2_{L^2(\Omega)^d} .$$

Relation (28) can be deduced by a simple application of the discrete Gronwall's Lemma (Lemma 2.6). □

By using a triangle inequality, we get immediately the following *a priori* error estimation:

Corollary 4.6. *Under the assumption of Theorem 4.5, the exact solution* $(\mathbf{u}, p)$ *of Problem* (12) *and the discrete solution* $(\mathbf{u}_h^n, p_h^n)$ *of problem* (27) *satisfy the following a priori error estimate:*

$$\frac{1}{2} \parallel \mathbf{u}_h^m - \mathbf{u}(t_m) \parallel^2_{L^2(\Omega)^2} + \frac{\nu}{2} \sum_{n=1}^{m} k |\mathbf{u}_h^n - \mathbf{u}(t_n)|^2_{H^1(\Omega)^2} \leq C(h^2 + k^2),$$

where C *is a positive constant independent of* h *and* k.

4.2. Iterative scheme. To compute the solution of the non-linear problem (27), we introduce the following iterative problem:
For every $n \in \{1, ..., N\}$, knowing $\mathbf{u}_h^{n-1} \in X_h$, we compute $(\mathbf{w}_h^i, p_h^i) \in (X_h, M_h)$ such that for all $(\mathbf{v}_h, q_h) \in (X_h, M_h)$

$$(32) \quad \left\{ \begin{aligned} & \frac{1}{k} (\mathbf{w}_h^i - \mathbf{u}_h^{n-1}, \mathbf{v}_h) + a(\mathbf{w}_h^i, \mathbf{v}_h) + c_{\mathbf{u}}(\mathbf{w}_h^{i-1}, \mathbf{w}_h^i, \mathbf{v}_h) - (p_h^i, \operatorname{div} \mathbf{v}_h) \\ & \qquad\qquad + \int_S |K \mathbf{w}_{h\boldsymbol{\tau}}^{i-1}|^{s-2} K \mathbf{w}_{h\boldsymbol{\tau}}^i \cdot K \mathbf{v}_{\boldsymbol{\tau}} d\sigma = (\mathbf{f}^n, \mathbf{v}_h) , \\ & (q_h, \operatorname{div} \mathbf{w}_h^i) = 0, \end{aligned} \right.$$

where $\mathbf{w}_h^0 = \mathbf{u}_h^{n-1}$.
After the convergence of the scheme (32), we get the numerical solution $\mathbf{u}_h^n$ of the discrete system (27).
For the study of the existence and uniqueness of the solution, and the convergence of the iterative scheme (32) we refer to [7]. So, we state that

Theorem 4.7. *For each* $\mathbf{u}_h^{n-1} \in X_h$, *Problem* (32) *admits a unique solution* $(\mathbf{w}_h^i, p_h^i) \in X_h \times M_h$ *which satisfies the following bound:*

$$\|\mathbf{w}_h^i\|_{1,\Omega} + \|p_h^i\|_{L^2(\Omega)^d} \leq C,$$

where C *is a positive constant independent of* h *and* k.

The formulation of the iterative scheme (32) is validated by the following convergence result

Theorem 4.8. [7] *Let* $(\mathbf{w}_h^i, p_h^i) \in V_h \times M_h$ *the solution of* (32). *Let* $(\mathbf{u}_h^n, p_h^n)$ *solution of* (27). *Assume that there exists* C *positive constant independent of* h *and* k *such that* $\|\mathbf{f}\|_{L^2(\Omega)^d} \leq \dfrac{\nu^2}{C}$. *Then*

$$\lim_{i\to\infty} \mathbf{w}_h^i = \mathbf{u}_h^n \ in\, V.$$

Remark 4.9. *We note that for each time step* $n \in 1, \ldots N$, *we solve the iterative scheme* (32) *until we get the convergence.*

Remark 4.10. *We also note that a simple idea is to consider the linear scheme : for every* $n \in \{1, ..., N\}$, *knowing* $\mathbf{u}_h^{n-1} \in X_h$, *we compute* $(\mathbf{u}_h^n, p_h^n) \in (X_h, M_h)$ *such that for* $(\mathbf{v}_h^n, q_h^n) \in (X_h, M_h)$

$$\left\{\begin{array}{l} \dfrac{1}{k}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}_h) + a(\mathbf{u}_h^n, \mathbf{v}_h) + c_{\mathbf{u}}(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h) - (p_h^n, \text{div } \mathbf{v}_h) \\ \qquad\qquad\qquad + \displaystyle\int_S |K\mathbf{u}_{h\boldsymbol{\tau}}^{n-1}|^{s-2} K\mathbf{u}_{h\boldsymbol{\tau}}^n \cdot K\mathbf{v}_{\boldsymbol{\tau}} d\sigma = (\mathbf{f}^n, \mathbf{v}_h), \\ (q_h, \text{div } \mathbf{u}_h^n) = 0, \end{array}\right. \tag{33}$$

where $\mathbf{u}_h^0 = \mathbf{0}$ *and* $\mathbf{f}^n = \mathbf{f}(t_n)$. *Among the reasons we did not give priority to the above approach let us mention:*
(i) the derivation of a priori bounds using 33 is nontrivial.
(ii) The implementation of (32) *is easy and fast.*
Having said that, we intend to investigate the formulation (33) *in a near future.*

5. Numerical simulations and concluding remarks

To validate the theoretical results, we perform several numerical simulations using the FreeFem++ code (see [11]) in two-dimensions. We begin with a simple test case where we show the numerical results associated to the scheme (27). Note that the solution of the non-linear discrete scheme (27) is approximated by the iterative scheme (32). Next we consider a more complex case, the "Driven cavity flow" and show the corresponding numerical results.
All the numerical simulations showed in this section are performed by considering for a given mesh step h and for a given time step n, the iterative problem (32) with $\mathbf{w}_h^0 = \mathbf{u}_h^{n-1}$. The stopping criterion is one of the two cases:

(1) Classical one:

$$\frac{||\mathbf{w}_h^i - \mathbf{w}_h^{i-1}||_1}{||\mathbf{w}_h^i||_1} \leq 10^{-5}. \tag{34}$$

(2) New one:

$$\frac{||\mathbf{w}_h^i - \mathbf{w}_h^{i-1}||_1}{||\mathbf{w}_h^i||_1} \leq C \min(h, k), \tag{35}$$

where C is a positive constant (but in this work, we take $C = 1$).

5.1. Navier-Stokes in a square domain. We consider the square $\Omega = (0, 1)^2$ and the time interval $[0, T]$ where $T = 1$. Each edge of $\partial\Omega$ is divided into M equal segments so that Ω is divided into $2M^2$ triangles, so the mesh step is given by $h = \dfrac{1}{M}$. The time step is taken equal to the mesh step $k = h$, so the *a priori* error

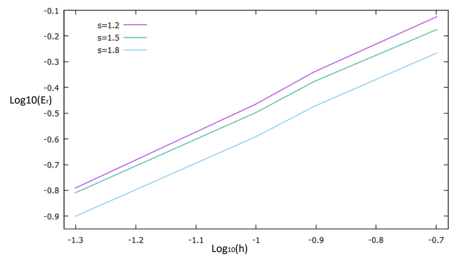

FIGURE 1. The *a priori error* estimate with respect to the mesh step h in logarithmic scale for $s = 1.2, 1.5, 1.8$.

given in Theorem 4.5 can be bounded by h (as $k = h$).
We set $K = \mathbf{I}$, $\nu = 1$ and

$$\mathbf{f} = (1 + (x^2 + y^2)\sin(t(x^2 + y^2)), txy\cos(txy))^T.$$

We take the anisotropic slip boundary condition (6), on right and top boundaries of Ω, that is on $S = \Big\{(x, y)|\ x = 1 \,\text{or}\, y = 1\Big\} = \overline{\Gamma_2 \cup \Gamma_3}$, and the homogeneous Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on the bottom and left boundaries $\Gamma = \overline{\Gamma_1 \cup \Gamma_4}$. First we will compare the stopping criteria (34) and (35) for $s = 1.2, 1.5, 1.8$ and for $M = 10, 20, \ldots, 100$. We remark that for each $n \in \{1, 2, \ldots, M\}$, Algorithm (32) is stopped for $i \leq 15$ with the classical stopping criteria (34) and for $i \leq 2$ with the new stopping criteria (35) (with $c = 1$).

We will see next the impact of this new stopping criteria (35) (for $i = 2$) on the rate of the *a priori* error estimates.

We have $s \in (1, 2)$, and we are interested in computing the rate of convergence of the finite element solution $(\mathbf{u}_h^n, p_h^n)$ given by Theorem 4.5. As we do not have the exact solution $\mathbf{u}$ corresponding to this case, we will approximate it by computing the numerical solution of the iterative system (32) for $M = 200$ which will be designated as the reference solution say, $(\mathbf{u}_r, p_r)$ and which depends on $s \in (1, 2)$. Next, we compute the numerical solution given by (32) for $M = 5, 8, 10, 20, 25$. Figures 1 show the graphs of the relative error

$$E_r = \frac{\sum_{n=1}^{M} k|\mathbf{u}_h^n - \mathbf{u}_r(t_n)|^2_{H^1(\Omega)^2}}{\sum_{n=1}^{M} k|\mathbf{u}_r(t_n)|^2_{H^1(\Omega)^2}} + \frac{\sum_{n=1}^{M} k|p_h^n - p_r(t_n)|^2_{L^2(\Omega)}}{\sum_{n=1}^{M} k|p_r(t_n)|^2_{L^2(\Omega)}} \tag{36}$$

with respect to h in logarithmic scale for $s = 1.2, 1.5, 1.8$ and with the new stopping criteria (35). These lines have slopes $1.1, 1.04, 1.05$ respectively. We can deduce for this particular case that the numerical slope is around one and is independent of s. In all these cases, the numerical slopes satisfy the Theorem 4.5 (as $k = h$).

Next we take $M = 40$ and $s = 1.5$. Figure 2 shows for $T = 1$ the values of $\mathbf{u}_{\boldsymbol{\tau}}$ in the tow borders $x = 1$ and $y = 1$ where we have the slip boundary condition (6).

5.2. Driven Cavity. In this paragraph, we consider the numerical results for the Lid Driven cavity with the Navier-Stokes flows under the power law slip boundary condition. This is very popular example that can be seen in [9, 18]. The fluid is confined in the domain $\Omega = (0, 1)^2$ with the velocity $\mathbf{u}$ satisfying $\mathbf{u} = 0$ on $\Gamma = \Big\{(x, y) :\ x = 0 \text{ or } y = 0\Big\}$, the relation (6) on the right (part S of $\partial\Omega$), and

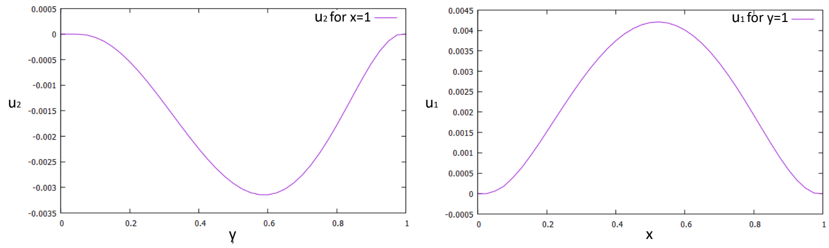

FIGURE 2. The curve of $u_{\boldsymbol{\tau}}$ ($M = 40$ and $s = 1.5$): $u_{\boldsymbol{\tau}} = u_2$ for $x = 1$ with respect to y (on the left) and $u_{\boldsymbol{\tau}} = u_1$ for $y = 1$ with respect to x (on the right).

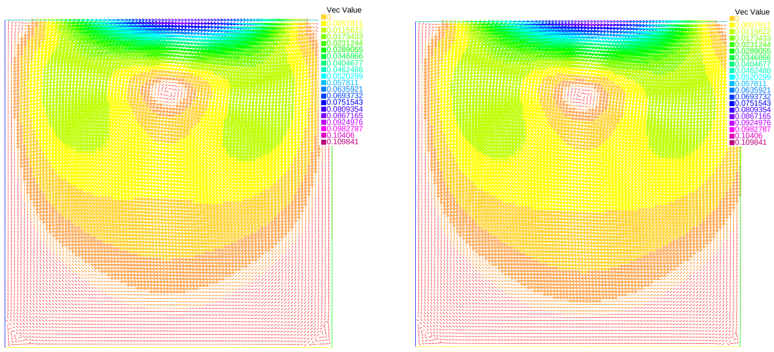

FIGURE 3. The numerical velocity for $s = 1.2$ (left) and $s = 1.5$ (right).

the following Dirichlet condition on the top:

$$\mathbf{u}_{\boldsymbol{\tau}} = \big((1 + \tanh(t))\, x^2 (x-1)^2, 0\big)^T \qquad \text{on } y = 1\ ,\ 0 < x < 1\,. \tag{37}$$

One observes that when t is big enough, $\mathbf{u}_{\boldsymbol{\tau}}$ attains its maximum $(1/8, 0)^T$ at the center $x = 1/2$. This boundary condition has the effect of avoiding local singularities at the top-right and top-left corners as it ensures both the velocity and velocity gradient vanish at the corners. In all the numerical results considered in this section, we set $K = I$, $\mathbf{f} = \mathbf{0}$ and $\nu = 1$.

Here also, each edge is divided into $M = 100$ segments of equal length. Thus the corresponding mesh contains $2M^2$ elements.

Figure 3 and Figure 4 show the numerical velocity for $s = 1.2, 1.5$ and 1.8 respectively. One notes the formation of a vortex which is typical for this problem.

Furthermore, we show in Figure 5 the first and second components of the velocity on the line $y = 1/2$ for $s = 1.2, 1.5$ and 1.8. Again, one observes the dependence on s of the velocity on the line $y = 1/2$.

5.3. Concluding remarks. The objective of this work it to propose a novel formulation for the non stationary Navier-Stokes equations when the tangential shear is a nonlinear function of the tangential velocity. The variational problem is shown to be well-posed in the same functional framework as for the standard boundary conditions. Next, we consider the space-time discretizations, that we analyze thoroughly and proving that optimal convergence rates can be attained if the weak

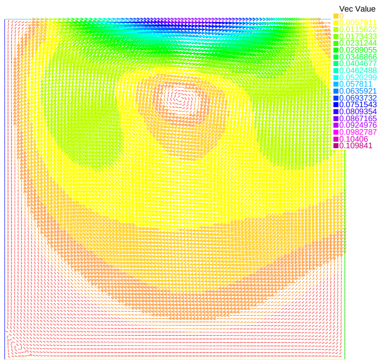

FIGURE 4. The numerical velocity for $s = 1.8$.

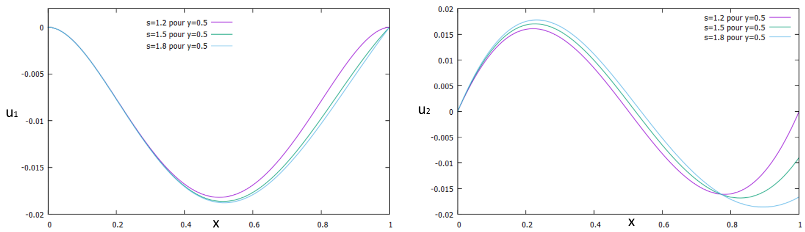

FIGURE 5. The numerical velocity (u_1, u_2) for $s = 1.2, 1.5$ and 1.8, on the line $y = 0.5$.

solution has enough regularity on the slip zone. Iterative method is proposed for the numerical realization of the nonlinear discrete problem. Finally, two dimensional experiments are performed: the first one illustrates the convergence properties of the method, while the second one captures the properties of the driven cavity. This work should further be extended, in particular by (i) considering complicated domains and application to computational hemodynamics, (ii) improving linear solvers by considering ADMM, (iii) extending the present work to take into account non-Newtonian models.

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